

Family of regular interiors for nonrotating black holes with $T_0^0 = T_1^1$

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We find the general solution for the spacetimes describing the interior of static black holes with an equation of state of the type $T_0^0 = T_1^1$ (T being the stress-energy tensor). This form is the one expected from taking into account different quantum effects associated with strong gravitational fields. We recover all the particular examples found in the literature. We remark that all the solutions found follow the natural scheme of an interior core linked smoothly with the exterior solution by a transient region. We also discuss their local energy properties and give the main ideas involved in a possible generalization of the scheme in order to include other realistic types of sources.

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I. INTRODUCTION

Any static black hole (BH) arises from the gravitational collapse of some object. Under the premises in this work, the object has not (yet) shrunken indefinitely and has not given rise to a spacetime singularity. It is then natural to consider two regions: one *exterior* to the object, and the object *itself*. The exterior region, as is well known, can be described by a spacetime belonging to the Reissner-Nordström [1,2] solution. In the absence of electric, or magnetic, charge it is simply Schwarzschild's spacetime. Furthermore, one can add a cosmological constant following recent observational results [3,4]. Then, the spacetime belongs to the Kottler-Trefftz solution family [5,6]. In this case, the global properties of the spacetime clearly change, e.g., the spacetime is no longer asymptotically flat (see, e.g., [1,7]).

Going a step further, we consider the body itself as composed of two main regions. One is its *surface* and the other the rest of the body, i.e., the *interior* region. One may expect that some mechanism—having to do, e.g., with quantum gravity—will be able to stop the collapse of the body. Therefore we will think of the interior of the body as being described by some spacetime product of the present knowledge available on the merge of quantum field theory and gravitation. A widely studied issue in this direction is that of quantum vacuum effects [8–14] and the resulting spacetime turns out to be a de Sitter (dS) or anti-de Sitter (AdS) one [15–24]. There are other alternatives, as, e.g., those of [25,26].

In all these cases considered so far, either no distinction has been made between the interior and the exterior of the body (see e.g., [20–26]) or there appears a *singular* distribution of matter at the surface of the body (see, e.g., [18,19]). This distribution is singular in the sense that it is a matter surface density—called a singular shell. However, contrary to the case of electromagnetic charge densities, a matter surface density has neither been observed nor has it been predicted by any theory. It is thus more natural to assume that the matter on the surface of the body is distributed across the

body, and leave for a subsequent study the issue of whether this region is thin or thick, in comparison with the region dominated by quantum vacuum effects, through one of the solutions referred to before. Finally, the only work considering *all* the features of the structure of a regular static BH with a clear physical source is [27]. However, Nariai spacetime was absent, as well as an implementation of previous attempts and a complete study of (local) energy conditions. Thus, in our opinion, it is worth carrying out a unification of the different results obtained so far, as well as extending them in order to cover some important issues that were overlooked in those analysis. Here we provide, for the first time to our knowledge, the general solution of the scheme discussed above. In particular, we carry out an implementation of all those previous works which, for one reason or another did not comply with all the requirements already specified. We also perform a study of the local energy conditions in all these cases.

Finally, it is also important to introduce other kinds of solutions for the interior region, aside from the ones referred to before, which arise from results, or just hints, coming from the contribution of the quantum vacuum to gravity. To summarize, these are the points that will be dealt with, successively, in the body of this paper comprising the next ten sections. They are clearly identified by their titles and will need no further specification here. Section XII is devoted to some final remarks, and in Sec. XIII we provide the conclusions of the work. A brief survey can be found in [28].

Throughout this work we will use units such that $G = 1$, $c = 1$, Einstein's equations are written in the form $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, where $G_{\alpha\beta}$ is the Einstein tensor—we follow the conventions of [1]—and $T_{\alpha\beta}$ is the energy-momentum tensor. A prime will denote derivation with respect to the coordinate r .

II. SPACETIMES WITH A SPHERICALLY SYMMETRIC QUANTUM VACUUM AS A SOURCE

Because of the imposed limitation of nonaccepting singular mass shells, the spacetimes describing the interior of the body are not allowed to be of a well known kind as dS or

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AdS spacetimes. Indeed, the interior solution cannot be everywhere a spatial isotropic solution, as dS or AdS, because this would lead to a sudden change in the pressures exerted by the body to the exterior, and would lead to the appearance of a singular mass shell [29]. Now, the type of generalization depends on the underlying physics one is able to assume. The isotropic case is suitable in order to link it with the expected contributions of a dominating quantum vacuum, especially those associated with vacuum polarization. As we are dealing with spacetimes which are spherically symmetric, a natural generalization is to assume that the body may be described by a solution which is *invariant* to any nonrotating observer, with a free radial motion, instead of a solution which is invariant to any observer. This generalization of the energy-matter content of the body is called *spherically symmetric quantum vacuum* (SSQV), after [20], see also [22], and requires the imposition of $T_0^0 = T_1^1$ for any nonrotating observer. This is the type of energy-matter content that is considered in [18–28,30] and will be the one used in the first part of this work, until we get to Sec. X. In particular, SSQVs and nonlinear electrodynamics have given some relevant results on the issue of regular BHs, see, e.g., [23–25,30].

We shall now characterize the families of spacetimes that are suitable to become SSQVs. Any static, spherically symmetric spacetime can be conveniently described by

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + G^2(r)d\Omega^2, \quad (1)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$. There are certainly other ways to represent these spacetimes, which avoid the problems occurring near the possible horizons, or by putting $R^2 d\Omega^2$, provided $G' \equiv dG(r)/dr \neq 0$ (see, e.g., [31,32]). For a local observer at rest with respect to the coordinate grid of Eq. (1), a standard computation of $T_{\alpha\beta}$ yields (ρ is the energy density, p the radial pressure, and p_2, p_3 the tangential pressures, measured by this observer)

$$8\pi\rho = \frac{1}{G^2}[1 - F(G'^2 + 2GG'') - GG'F'], \quad (2)$$

$$8\pi p = \frac{1}{G^2}[-1 + FG'^2 + GG'F'], \quad (3)$$

$$8\pi p_2 = 8\pi p_3 = \frac{F''}{2} + \frac{FG''}{G} + \frac{F'G'}{G}. \quad (4)$$

Imposing $\rho + p = 0$ in Eqs. (2) and (3), we get

$$FG'' = 0. \quad (5)$$

We now use that G cannot be zero in any open region. Two alternatives appear: $F = 0$ or $G'' = 0$. If $F = 0$, expression (1) is useless. It is first necessary to change the coordinate system of Eq. (1) by $dT \equiv dt + (1 - F)/F dr$, while keeping the rest unchanged. Then one can impose $F = 0$. The result is

$$ds^2 = 2dTdr + 2dr^2 + G^2(r)d\Omega^2. \quad (6)$$

In the orthonormalized cobasis given by $\Theta^0 = dT/\sqrt{2}$, $\Theta^1 = dT/\sqrt{2} + \sqrt{2}dr$, $\Theta^2 = Gd\theta$, and $\Theta^3 = G \sin\theta d\varphi$, the Ricci tensor takes the form

$$\begin{aligned} \text{Ricci} = & \frac{G''}{G}(-\Theta^0 \otimes \Theta^0 + \Theta^0 \otimes \Theta^1 + \Theta^1 \otimes \Theta^0 - \Theta^1 \otimes \Theta^1) \\ & + \frac{1}{G^2}(\Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3). \end{aligned} \quad (7)$$

On the other hand, for a SSQV we must have $\rho + p = 0$ (the conditions on $T_{\alpha\beta}$ being directly translated into conditions for $R_{\alpha\beta}$)

$$\begin{aligned} \text{Ricci} = & R_{00}(-\Theta_N^0 \otimes \Theta_N^0 + \Theta_N^1 \otimes \Theta_N^1) \\ & + R_{22}(\Theta_N^2 \otimes \Theta_N^2 + \Theta_N^3 \otimes \Theta_N^3), \end{aligned} \quad (8)$$

where $\{\Theta_N^\Omega\}$ is some orthonormalized cobasis, not necessarily coincident with the one used in the computation of Eq. (7). Therefore, we must look for an orthonormalized cobasis for which the Ricci tensor (7) becomes of the type (8). Clearly this is the same as finding out whether we can have linear expressions $\Theta_N^0 \equiv A\Theta^0 + B\Theta^1$, and $\Theta_N^1 \equiv C\Theta^0 + D\Theta^1$, with $-\Theta_N^0 \cdot \Theta_N^0 = \Theta_N^1 \cdot \Theta_N^1 = \Theta_N^0 \cdot \Theta_N^1 + 1 = 1$. However, Eq. (8) is invariant under these changes. The only solution that makes Eqs. (7) and (8) compatible is then

$$G'' = 0. \quad (9)$$

If $F \neq 0$, we also have $G'' = 0$. Thus $G'' = 0$ constitutes the *proper* characterization of any possibility.

Now, from $G'' = 0$ two distinct alternatives appear

$$G = \gamma, \quad \text{or} \quad G = \alpha r + \gamma, \quad (10)$$

where $\alpha (\neq 0)$ and γ are constant. Only the latter has been considered in detail in the literature of regular BHs. We will study it in the sequel.

Other expressions for the spacetimes describing SSQVs. In order to include the possible horizons, we write the metrics (1) under the common form

$$ds^2 = -(1 - H)dT^2 + 2HdTdr + (1 + H)dr^2 + \gamma^2 d\Omega^2, \quad (11)$$

$$ds^2 = -(1 - H)dT^2 + 2HdTdr + (1 + H)dr^2 + r^2 d\Omega^2, \quad (12)$$

where $H \equiv 1 - F$, and the coordinate change is given by $dT = dt + (1 - F)/F dr$. We will use these forms in the sequel. We have also used the fact that the case $g = \alpha r + \gamma$ is physically equivalent to the case $G = r$. This is intuitively seen because α merely represents the scale of units used for r and γ is an arbitrary (constant) origin. In terms of coordinate changes we have: The metric (1) for $G = \alpha r + \gamma$ is $ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + (\alpha r + \gamma)^2 d\Omega^2$. Recalling that $\alpha \neq 0$, one can define a new radial coordinate $\tilde{r} \equiv \alpha r + \gamma$. The metric becomes then $ds^2 = -F[(\tilde{r} - \gamma)/\alpha]dt^2 + \alpha^{-2}F^{-1}[(\tilde{r} - \gamma)/\alpha]d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$. Now, under a reparametrization of the t coordinate by $dt \equiv \alpha d\tilde{t}$ we get $ds^2 = -\alpha^2 F[(\tilde{r} - \gamma)/\alpha]d\tilde{t}^2 + \alpha^{-2}F^{-1}[(\tilde{r} - \gamma)/\alpha]d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$. Whence one can conclude that any member of Eq. (1) with $G = \alpha r + \gamma$ with $\alpha \neq 0$ is equivalent to another member of Eq. (1)

with $\alpha=1$ and $\gamma=0$. Since we are studying the general description of SSQVs it is enough to consider the representation $\alpha=1$, $\gamma=0$ and *arbitrary* $F(r)$ to include *any* case of SSQV.

Furthermore, it is easy to show that Eq. (12) can be written in the Kerr-Schild form [33]:

$$ds^2 = ds_\eta^2 + 2H(r)\ell \otimes \ell, \quad (13)$$

where ds_η^2 stands for the flat spacetime metric, H is an arbitrary function of r , and ℓ is a geodesic radial null one-form, $\ell = (1/\sqrt{2})(dt \pm dr)$. Thus the SSQVs in Eq. (12) can be thought of as the family of maximal spherically symmetric spacetimes expanded by a geodesic radial null one-form from flat spacetime (GRNSS spaces).

To summarize, there are *only* two, nonequivalent, families of SSQVs. The case with $G'=0$ is characteristic of the Nariai solution [34,35]. The Nariai solution is a solution of Einstein's equations for the same pattern as the de Sitter solution, i.e., $T_{\alpha\beta} = \Lambda_0 g_{\alpha\beta}$, Λ_0 being the cosmological constant. The difference lies in the "radial" coordinate. In the Nariai case there is no proper center for the spherical symmetry. Therefore we shall call the spacetimes with $G'=0$ generalized Nariai metrics. Finally, the other case corresponds to the GNRSS spaces which constitute a distinguished family of the class of Kerr-Schild metrics.

III. GEOMETRICAL PROPERTIES OF THE SOLUTIONS

A. Generalized Nariai metrics

Using an orthonormal cobasis defined as $\Theta^0 = (1-H/2)dT - (H/2)dr$, $\Theta^1 = (+H/2)dr + (H/2)dT$, $\Theta^2 = \gamma d\theta$, $\Theta^3 = \gamma \sin \theta d\varphi$, we see that the Riemann tensor has as independent components

$$R_{0101} = -H''/2, \quad R_{2323} = 1/\gamma^2. \quad (14)$$

The Ricci tensor is characterized by $R_{00} = -R_{11} = -H''/2$, $R_{22} = R_{33} = 1/\gamma^2$. The scalar curvature is $R = H'' + 2/\gamma^2$, and the Einstein tensor has the following nonzero components:

$$G_{00} = -G_{11} = 1/\gamma^2, \quad G_{22} = G_{33} = -H''/2. \quad (15)$$

The isotropic solution, the one to be found at the core, yields $H = (1/\gamma^2)r^2 + br + c$, where b and c are arbitrary constants. Without losing generality, we can set $b, c = 0$ (as they are clearly gauge freedoms for any spacetime in the family). Thanks to the presence of the Nariai solution inside this family, $T_{\alpha\beta}^{\text{Nariai}} = \Lambda_0 g_{\alpha\beta}$, the factor $1/\gamma^2$ can be identified with Λ_0 . Thus, the only isotropic quantum vacuum belonging to this family is the Nariai solution.

B. The GNRSS metrics

First we note that these spacetimes fulfill the relation

$$p_2 = -\rho - \frac{\rho' r}{2}. \quad (16)$$

As a consequence, for a regular source $p_2 \rightarrow -\rho$ as $r \rightarrow 0$ is approached. Therefore in any regular solution the spacetime becomes more and more isotropic as $r \rightarrow 0$. Thus the contribution from the quantum vacuum becomes more and more dominant as r tends to 0 and the initial idea of distributing the singular mass shell across some region of the body is completed.

On the other hand, one can choose a similar cobasis as in the Nariai-like case, just by replacing γ with r . The Riemann tensor has the following independent components:

$$R_{0101} = -\frac{H''}{2}, \quad R_{0202} = R_{0303} = -\frac{H'}{2r}, \quad (17)$$

$$R_{1212} = R_{1313} = \frac{H'}{2r}, \quad R_{2323} = \frac{H}{r^2}.$$

The Ricci tensor for these spacetimes has the following nonzero components: $R_{00} = -R_{11} = -(1/2)[H'' + (2H'/r)]$, $R_{22} = R_{33} = (1/r)[H' + (H/r)]$. And the scalar curvature is given by $R = H'' + (4H'/r) + (2H/r^2)$. Finally, the nonzero components of the Einstein's are $G_{00} = -G_{11} = (1/r)[H' + (H/r)]$, $G_{22} = G_{33} = -(1/2)[H'' + (2H'/r)]$. Other expressions that will be used later are

$$G_{00} = -G_{11} = \frac{1}{r^2}(Hr)', \quad G_{22} = G_{33} = G_{00} - \frac{G'_{00}r}{2}. \quad (18)$$

In this case, the isotropic (regular) GNRSS metric is the de Sitter solution, given by $H(r) = (\Lambda_0/3)r^2$.

Exterior metrics and GNRSS metrics. It turns out that *all* of the possible exterior metrics, see Sec. I, also belong to the GNRSS family. The function H is $H(r) = (\Lambda_{\text{ext}}/3)r^2 + 2m/r - q^2/r^2$, where Λ_{ext} stands for the external cosmological constant, m is the ADM mass of the BH, and q its electromagnetic charge.

This coincidence will be very useful in the following section.

IV. JUNCTION OF THE INTERIOR AND EXTERIOR SOLUTIONS

The junction, or matching, of two spherically symmetric spacetimes is well-known (see, e.g., [29,36–38]). The general form of a hypersurface that clearly adjusts itself to the spherical symmetry of any of these spacetimes is as follows:

$$\Sigma: \begin{cases} \theta = \lambda_\theta, \\ \varphi = \lambda_\varphi, \\ r = r(\lambda), \\ t = t(\lambda), \end{cases} \quad (19)$$

where $\{\lambda, \lambda_\theta, \lambda_\varphi\}$ are the parameters of the hypersurface. One must thus identify both hypersurfaces in some way. The identification of $(\lambda_i)_1$ with $(\lambda_i)_2$ (1 and 2 label each of the

spacetimes) is the most natural one, due to the symmetry of the above scheme. In the sequel, 1 labels the exterior spacetime and 2 the interior one.

In order to match the exterior solution with the interior one, one basically demands the coincidence of the first and second fundamental forms of Σ at each spacetime—the other way is to accept the presence of *singular* mass shells, which would not require the coincidence of the second fundamental form, but we will dismiss such an unphysical option. In order to include the possibility of matching the interior and the exterior at null hypersurfaces (e.g., at an event horizon) one can follow the formalism in [38]. It is worth recalling that the exterior region is described by a member of the GNRSS family. Thus we have to consider two possibilities: matching a generalized Nariai metric with a GNRSS one and two GNRSS metrics with each other.

A. The junction of a generalized Nariai metric and a GNRSS one

In this case one easily gets $r_1(\lambda) = \gamma = \text{const}$, $t_1(\lambda) = \text{const}$, see [39] for full details. This result tells us that the junction between a generalized Nariai spacetime and a member of the GNRSS family is impossible. It would only happen for a (two-dimensional) surface. Therefore any member of the Nariai class cannot be regarded as a good candidate in order to represent the interior structure of a regular, static BH.

B. The junction of two GNRSS spacetimes

In this case one gets that two members of the GNRSS family match with each other if and only if, see, e.g., [39], either $r_1(\lambda) = r_2(\lambda) = R = \text{const}$, $\dot{t}_1 = \dot{t}_2$, $[H] = [H'] = 0$, or $r_1 + t_1 = r_2 + t_2 = \text{const}$. The last condition, however, describes the motion of a null hypersurface and is not an acceptable solution in order to describe the matter inside a static BH. Therefore we reach the conclusion that: The only acceptable hypersurfaces fulfilling the matching conditions, that preserve the spherical symmetry, between two spacetimes of the GNRSS family, are those satisfying $r_1(\lambda) = r_2(\lambda) = R = \text{const}$, $\dot{t}_1 = \dot{t}_2$, $[H] = [H'] = 0$.

Without losing generality, one can choose $t_1 = t_2 = \lambda$ because of the global existence of the Killing vector ∂_t . Moreover, we realize that the chosen coordinates are privileged ones, in which the matching is explicitly C^1 . The hypersurface, Σ , will be timelike, null, or spacelike according to $H < 1$, $H = 1$, or $H > 1$, respectively.

To summarize, if vacuum polarization is to be the dominant quantum effect, the most simple way to construct a regular BH is to build it upon GNRSS spacetimes.

V. REGULAR INTERIORS OF THE GNRSS TYPE

As mentioned elsewhere, the exterior region can appropriately be characterized by a member of the Kottler-Trefftz class, which is a subclass of the GNRSS family. Then, the matching conditions between the exterior and the interior regions are

$$H_2(R) = H_1(R) = \frac{2m}{R} - \frac{q^2}{R^2} + \frac{\Lambda_1}{3}R^2, \quad (20)$$

$$H_2'(R) = H_1'(R) = -\frac{2m}{R^2} + \frac{2q^2}{R^3} + \frac{2\Lambda_1}{3}R. \quad (21)$$

Moreover, the aim here is to focus on those interior solutions which are everywhere regular. From the expressions of the Riemann tensor and the metric, one sees that this may only be accomplished if

$$H_2(0) = 0, \quad H_2'(0) = 0. \quad (22)$$

Thus we finally encounter four conditions in order to have a regular interior solution.

From now on, we will consider H_2 to be a smooth function of the variable $\tilde{r} \equiv r/R$, a most natural hypothesis in view of the regular character prescribed for the interior solution. In this case, the origin conditions tell us that

$$H_2(\tilde{r}) = \sum_{n=2}^{\infty} b_n \tilde{r}^n. \quad (23)$$

Now, one has to impose the two other conditions. Obviously it is the same to consider $H_2(\tilde{r})$ or $H_2(\tilde{r}-1)$ in the whole procedure. However, we will first work with $H_2(\tilde{r}-1)$ in order to implement the junction conditions directly. From the preceding result, one immediately has

$$H_2 = \sum_{n=0}^{\infty} a_n (\tilde{r}-1)^n, \quad (24)$$

and the junction conditions tell us that

$$a_0 = H_1(1) = \frac{2m}{R} - \frac{q^2}{R^2} + \frac{\Lambda_1}{3}R^2, \quad (25)$$

$$a_1 = \dot{H}_1(1) = 2 \left(-\frac{m}{R} + \frac{q^2}{R^2} + \frac{\Lambda_1}{3}R^2 \right), \quad (26)$$

where $H_1(\tilde{r}) = (2m/R)(1/\tilde{r}) - (q^2/R^2)(1/\tilde{r}^2) + (\Lambda_1 R^2/3)\tilde{r}^2$ and a dot denotes derivation with respect to \tilde{r} . The following step is to impose regularity of the solution, Eq. (22). We get

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0, \quad \sum_{n=1}^{\infty} (-1)^n n a_n = 0, \quad (27)$$

which, by virtue of the matching conditions, yield

$$\sum_{n=0}^{\infty} (-1)^n a_{n+2} = -\frac{4m}{R} + \frac{3q^2}{R^2} + \frac{\Lambda_1}{3}R^2, \quad (28)$$

$$\sum_{n=0}^{\infty} (-1)^n (n+2) a_{n+2} = 2 \left(-\frac{m}{R} + \frac{q^2}{R^2} + \frac{\Lambda_1}{3}R^2 \right). \quad (29)$$

It is clear that there are infinitely many possible candidates for these interiors.

Isotropization. Let us further analyze how they behave near the origin. Taking into account the expression of H_2 in powers of \tilde{r} and using Eq. (18), we get

$$G_{11} = -\frac{1}{R^2} \sum_{l=2}^{\infty} (l+1) b_l \tilde{r}^{l-2},$$

$$G_{22} = -\frac{1}{R^2} \sum_{l=2}^{\infty} \binom{l+1}{2} b_l \tilde{r}^{l-2}. \quad (30)$$

It is then clear that G_{11} and G_{22} are different from each other. Yet we have the very relevant property that, for *any* of these spacetimes, it holds

$$\lim_{\tilde{r} \rightarrow 0} G_{11} = \lim_{\tilde{r} \rightarrow 0} (-G_{00}) = \lim_{\tilde{r} \rightarrow 0} G_{22} = \lim_{\tilde{r} \rightarrow 0} G_{33} = -\frac{3b_2}{R^2}. \quad (31)$$

Whence, we see that a general isotropization of the Einstein tensor—and consequently of the energy-momentum one— independent of the model is actually accomplished. In terms of a_l we get

$$G_{11} = -\frac{1}{R^2} \sum_{M=0}^{\infty} A_M \tilde{r}^M, \quad (32)$$

$$A_M = (-1)^M (M+3) \sum_{l=M+2}^{\infty} (-1)^l \binom{l}{l-2-M} a_l,$$

and

$$G_{22} = -\frac{1}{R^2} \sum_{M=0}^{\infty} \frac{M+2}{2} A_M \tilde{r}^M. \quad (33)$$

So that

$$\lim_{\tilde{r} \rightarrow 0} G_{11} = \lim_{\tilde{r} \rightarrow 0} (-G_{00}) = \lim_{\tilde{r} \rightarrow 0} G_{22} = \lim_{\tilde{r} \rightarrow 0} G_{33} = -\frac{A_0}{R^2},$$

$$A_0 = 3 \sum_{l=2}^{\infty} (-1)^l \binom{l}{l-2} a_l.$$

Finally, making no further assumptions on the coefficients of H_2 , we can isolate two of them in terms of the rest. For simplicity, we shall isolate a_2 and a_3 . The result is

$$a_2 = -\frac{10m}{R} + \frac{7q^2}{R^2} + \frac{\Lambda_1}{3} R^2 + \sum_{l=4}^{\infty} (-1)^l (l-3) a_l,$$

$$a_3 = -\frac{6m}{R} + \frac{4q^2}{R^2} + \sum_{l=4}^{\infty} (-1)^l (l-2) a_l.$$

With this in hand we can write, respectively, the previous expression for the central value in terms of a_l or b_l , $l \geq 4$,

$$G_{11}(0) = G_{22}(0) = -\frac{3}{R^2} \left[\frac{8m}{R} - \frac{5q^2}{R^2} + \frac{\Lambda_1}{3} R^2 \right. \\ \left. + \sum_{l=4}^{\infty} (-1)^l \frac{(l-3)(l-2)}{2} a_l \right],$$

$$= -\frac{3}{R^2} \left[\frac{8m}{R} + \frac{\Lambda_1}{3} R^2 + \sum_{l=4}^{\infty} (l-3) b_l \right].$$

VI. EXAMPLES

We will consider six examples. Two constitute the well-known proposals of [18,19,40] and [20,21]. Two more come from a proposal in [23–26], for electrically charged bodies, and the proposal given in [30], for magnetically charged ones. More specifically, we will here derive their corresponding analogues, within the present scheme (what is actually more than simply rewriting those cases). The remaining two examples constitute a family of brand new candidates, which naturally arise from the preceding expressions. We will start with this last pair.

A. Two arbitrary powers

Let us just make the choice that only two specific powers of $H(r)$, say M and N , be present. In order to fulfill the regularity conditions, both must satisfy $M, N \geq 2$. However, if we wish to obtain a de Sitter-like behavior at, and near, the origin, we must necessarily impose that one, and only one, of them, say M , be equal to 2. Thus $H_2(\tilde{r})$ reads $H_2(\tilde{r}) = b_2 \tilde{r}^2 + b_N \tilde{r}^N$, for $N > 2$, with

$$b_2 = \frac{2m}{R} \left(\frac{N+1}{N-2} \right) - \frac{q^2}{R^2} \left(\frac{N+2}{N-2} \right) + \frac{\Lambda_1 R^2}{3},$$

$$b_N = \frac{2}{(N-2)R} \left(-3m + \frac{2q^2}{R} \right).$$

$G_{11}(\tilde{r})$ and $G_{22}(\tilde{r})$ read [recall Eq. (18)]

$$G_{11}(\tilde{r}) = -\Lambda_1 + \frac{6m}{R^3} \left(\frac{N+1}{N-2} \right) (\tilde{r}^{N-2} - 1) \\ + \frac{q^2}{R^4} \left[\frac{3(N+2) - 4(N+1)\tilde{r}^{N+2}}{N-2} \right],$$

$$G_{22}(\tilde{r}) = -\Lambda_1 + \frac{6m}{R^3} \left(\frac{N+1}{N-2} \right) \left(\frac{N}{2} \tilde{r}^{N-2} - 1 \right) \\ + \frac{q^2}{R^4} \left[\frac{3(N+2) - 2N(N+1)\tilde{r}^{N+2}}{N-2} \right].$$

Whence one readily sees that their finite value at the origin coincides, as expected,

$$G_{11}(0) = G_{22}(0) = -\Lambda_1 - \frac{6m}{R^3} \left(\frac{N+1}{N-2} \right) + \frac{3q^2}{R^4} \left(\frac{N+2}{N-2} \right). \quad (34)$$

Lowest powers. This example corresponds to the case in which H_2 is a polynomial of lowest degree. This amounts to setting $a_l = 0$, $l \geq 4$, in the general expressions. Its interest lies in its being the simplest possible situation. The result is

$$H_2(\tilde{r}) = \left(\frac{8m}{R} - \frac{5q^2}{R^2} + \frac{\Lambda_1 R^2}{3} \right) \tilde{r}^2 + \left(-\frac{6m}{R} + \frac{4q^2}{R^2} \right) \tilde{r}^3, \quad (35)$$

and

$$G_{11} = -\frac{24m}{R^3} + \frac{15q^2}{R^4} - \Lambda_1 + \frac{8}{R^3} \left(3m - \frac{2q^2}{R} \right) \tilde{r}, \quad (36)$$

$$G_{22} = -\frac{24m}{r^3} + \frac{15q^2}{R^4} - \Lambda_1 + \frac{12}{R^3} \left(3m - \frac{2q^2}{R} \right) \tilde{r}. \quad (37)$$

Notice that G_{11} tends to $-\Lambda_1 - q^2/R^4$ as \tilde{r} tends to 1, the same value as $G_{11}^{\text{ext}}(\tilde{r}=1)$, in accordance with Israel's conditions [29].

B. Israel and Poisson's model

In Ref. [18], see also [19], a plausible candidate for the energy-matter content of the interiors of regular noncharged BHs was proposed. The authors proposed that a singular layer of noninflationary material should exist between the de Sitter core and the external Schwarzschild metric. However, the usual spirit of matching a stellar interior with a vacuum exterior was lost, the reason being the unavoidable presence of a singular layer acting as a matter surface density. Indeed, in [40] it was argued that their approach could be improved by imposing a smooth transition from the hypersurface to the de Sitter core. Yet this step was not implemented. In any case, it was the only available candidate to continue the studies of quantum regular BHs at that time. The task here will be to see whether this geometrical and physical model can be recovered from our analysis.

In order to do that, we search for a solution within our family which be as close as possible to this particular solution. What amounts to looking for a de Sitter core for small values of \tilde{r} and a quantum contribution of the type of the square of the characteristic curvature of Schwarzschild spacetime near the matching hypersurface. These features taken into account, we set for the interior

$$(G_{00})_{\text{int}} = -(G_{11})_{\text{int}} = \frac{1}{(A + Br^3)^2}, \quad (38)$$

where A and B are two constants, to be determined. Using Eq. (18), we obtain

$$H_2(r) = \frac{r^2}{3A(A + Br^3)}, \quad (39)$$

where we have imposed Eq. (22). The matching conditions lead to

$$\frac{1}{A(A + Br^3)} = \frac{3}{R^2} H_1(R),$$

$$\frac{2A - BR^3}{A(A + BR^3)^2} = \frac{3}{R} H_1'(R), \quad (40)$$

where H_1 comes, as usual, from the external model.

In the exterior region, close to the matching hypersurface, the quantum contributions do not turn into a cosmological-like term. They are of the form $(G_{00})_{\text{ext}} \propto m^2/r^6$, as mentioned before. We thus have a quantum exterior which is different from the one encountered in the rest of the examples and sections before, which cannot be described by a member of the Kottler-Trefftz class. Fortunately, our preceding results are still useful. In fact, one realizes that it is possible to select a suitable exterior with a similar form as Eq. (38), just by setting $A_{\text{ext}} = 0$ and $B_{\text{ext}}^{-1} = \alpha m$, where $\alpha = \beta L_{\text{Pl}}$, being β of order unity, and L_{Pl} the Planck length (α is of order unity in Planckian units). β^2 is related with the number and type of the quantized fields [18,40]. This choice yields

$$H_1(r) = \frac{2m}{r} - \frac{1}{3} \left(\frac{\alpha m}{r^2} \right)^2, \quad (41)$$

where we have taken into account that the exterior region is dominated by the Schwarzschild geometry, with mass m , for large values of \tilde{r} .

Now, using Eqs. (40) and (41), A and B yield

$$A = \frac{\alpha}{6 - \frac{\alpha^2 m}{R^3}}, \quad B = \frac{2}{\alpha m} \left(1 - \frac{3}{6 - \frac{\alpha^2 m}{R^3}} \right). \quad (42)$$

Finally, using Eqs. (38), $(G_{22})_{\text{int}} = (2Br^3 - A)/(A + Br^3)^2$, where A, B have been given above. At the origin

$$G_{11}(0) = G_{22}(0) = -\frac{1}{B^2}. \quad (43)$$

To summarize, we have proven here that a spacetime model within our family satisfies all the required geometrical assumptions, and yields the particular form of G_{00} , both for the interior and the exterior of the body, as in the above mentioned references. A more throughout comparison of that model and ours will be given in Sec. VII B.

C. Dymnikova's model

Some time after the appearance of the previous cases a new model for a regular interior of a noncharged BH was proposed [20]. However, the approach was now quite different to that of the previous authors. Now Schwarzschild's

solution was only recovered in an asymptotical sense, for \tilde{r} approaching infinity only. However, if a sufficiently quick convergent matter model was obtained, then the quantity of mass outside the horizon of the collapsed body could become as negligible as desired with regard to the interior mass. Thus one would, at least, recover a trial model, interesting enough to support or reject the conclusions of the previous authors. In a later work [21], the model was extended to incorporate the observational fact in favor of a nonvanishing cosmological term in the exterior region. We will deal in this section with such a model, but considering a definite end to the collapsed body.

The imposition for the energy-matter content for the interior will be of the form

$$(G_{00})_{\text{int}} = -(G_{11})_{\text{int}} = A \exp(-\tilde{r}^3) + B, \quad (44)$$

where A and B are two constants to be determined and $\tilde{r} \equiv r/R$, where R is the matching radius.¹ We then integrate the expression of G_{00} , recall Eq. (18), in order to obtain H_2 , getting

$$H_2(\tilde{r}) = \frac{R^2}{3} \left[\frac{A}{\tilde{r}} (1 - e^{-\tilde{r}^3}) + B \tilde{r}^2 \right], \quad (45)$$

where we have already imposed the regularity conditions at the origin, Eq. (22). The matching conditions at the spatial hypersurface yield

$$A \frac{(e-1)}{e} + B = \frac{6m}{R^3} + \Lambda_1,$$

$$A \frac{(4-e)}{e} + 2B = -\frac{6m}{R^3} + 2\Lambda_1,$$

whence

$$A = \frac{6m}{R^3} \left(\frac{e}{e-2} \right), \quad B = \Lambda_1 - \frac{6m}{R^3} \frac{1}{(e-2)}. \quad (46)$$

Finally, using Eqs. (38),

$$(G_{22})_{\text{int}}(\tilde{r}) = A \left(\frac{3}{2} \tilde{r}^3 - 1 \right) e^{\tilde{r}^3} - B,$$

where A and B have been given above. At the origin

$$G_{11}(0) = G_{22}(0) = \Lambda_1 + \frac{6m}{R^3} \left(\frac{e-1}{e-2} \right). \quad (47)$$

In Sec. VII A we will compare, numerically, our results with those in the model of [20].

D. Ayón-Beato and García's models

In a series of papers, [23–26], some models of regular, electrically charged BHs with an energy-momentum tensor

of the form of a SSQV were presented, see also [24]. Their importance relied on the fact that the sources that give rise to those spacetimes could be linked with nonlinear electrodynamics (NED), which besides being a theory by itself, may be viewed as a low energy limit of string theory or M theory. Thus some plausible models of regular BHs, that took into account quantum effects in a clearer way than before, were put forward. The features of their models are analogous to the case of Dymnikova's model, though with a clear interpretation of the source origin. For the sake of brevity we will focus on the model in [26].

The choice there was $H(r) = (2m/r)[1 - \tanh(q^2/2mr)]$, for any $r \geq 0$. Ours will be

$$H(r) = \begin{cases} \frac{A}{r} \left[1 - \tanh \frac{B}{r} \right], & 0 \leq r \leq R, \\ \frac{2m}{r} - \frac{q^2}{r^2}, & R \leq r, \end{cases} \quad (48)$$

where A and B are constants to be determined. The matching conditions imply

$$A = \frac{q^2}{B} \cosh^2 \frac{B}{r}, \quad 1 + e^{-2B/R} = 2B \left(\frac{2m}{q^2} - \frac{1}{R} \right). \quad (49)$$

Defining $x \equiv A/A_0$, $y \equiv B/B_0$ with $A_0 = 2m$, $B_0 = q^2/2m$, that is the values of the model in [26], we get

$$x = \frac{1}{y} \cosh^2 \lambda y, \quad 1 + e^{-\lambda y} = 2(1 - \lambda)y, \quad (50)$$

where $\lambda \equiv B_0/R = q^2/2mR$. One has here to solve a transcendental equation in order to find the appropriate constants of the interior model. The parameter λ is the one controlling the set of solutions. In classical electrodynamics, $\lambda = 1$. We see that there is no solution in this case. In the context of general relativity, $\lambda = 1$ corresponds to the case where the exterior metric becomes flat at a spherical surface. But the choice of H_{int} cannot be zero for any positive value of r . Therefore the matching is impossible. The same happens for the other models in [23] and [24]. In the following section we will see which type of solutions arise for different values of λ .

E. Bronnikov's model

In [29] a model for static, regular, purely magnetically charged BHs with an energy-momentum tensor of the type of SSQVs was proposed. Its interest is twofold. Again the energy-momentum content of the objects was directly connected with NED. Second, it turns out that those BHs are the only ones based on a Lagrangian formulation of NED with a Maxwellian behavior in the weak field limit, regardless of the place the weak limit is taken. The example given there was a GNRSS metric with $H(r) = (|q_m|^{3/2}/ar)[1 - \tanh(a\sqrt{|q_m|}/r)]$ with m , the Arnowitt-Deser-Misner (ADM) mass, equal to $|q_m|^{3/2}/2a$, q_m being the magnetic charge. It is then obvious that the results of the previous section are valid now, just by changing q with q_m . The dif-

¹For other choices see, e.g., [39].

ference lies in the fact that now one has magnetic fields and also the theory describing NED is different to that of [23–26].

VII. NUMERICAL RESULTS

In order to study the approximate values of R for a given object, one needs to assume a particular behavior of the matter and energy inside the source. As of now, there is no agreement at this point. However, following several results, see, e.g., [15–23,41–43], the geometry of the core may be described by a dS solution. This has been the assumption used in most of the works dealing with regularized BHs. Here we will also include two examples with a different behavior and in Sec. X we will draw the main lines of a general behavior. In any case, the aim is to choose those physical models which are as consistent as possible, the dS model being one of them. In this case, at the core we will have $G_{00}(0) = -G_{11}(0) = -G_{22}(0) = -G_{33}(0) = \Lambda_2 = \text{const}$. Nonetheless, there is no present agreement about the scale at which regularization could act. A convenient way to handle and integrate this indeterminacy is to set $\Lambda_2 = 10^{3s}\Lambda_1$, s being the free parameter that governs the renormalization scale. For instance, if s is around 40, we are then considering that regularization takes place at Planck scales, and so on.

Finally, for the exterior region, in accordance with several recent observations [3,4], we will assume in what follows that $\rho_{\Lambda_1} \in [10^{-10}, 2 \times 10^{-8}] \text{ erg cm}^{-3}$. An analysis shows, however, that the fundamental contribution comes from the quantum gravitational model describing the core, and not from the type of quantum vacuum contribution that is assumed for the exterior region or near the surface of the body.

A. Two arbitrary powers

In this numerical analysis we will consider the uncharged case because there are no observed objects that can be associated with static, charged BHs. If there is a charge in the source, then the interesting situation involves rotation, which might be eventually connected with elementary particles (we refer the reader to [44]). The relation (34) is (now $q=0$)

$$\Lambda_1 = \Lambda_2 + \frac{6m}{R^3} \left(\frac{N+1}{N-2} \right), \quad \forall N \geq 3, \quad (51)$$

whence

$$R = R_\odot \sqrt[3]{M} \sqrt[3]{\frac{N+1}{4(N-2)}}, \quad (52)$$

where we have put $m = Mm_\odot$, m_\odot being the Sun mass, and $R_s \equiv \sqrt[3]{24m_\odot / (\Lambda_2 - \Lambda_1)}$. The last value only depends on the regularization scale and corresponds to the solution for a collapsed object of one solar mass in the case of the “lowest powers” model: $R_s \in [3 \times 10^{21-s}, 6 \times 10^{20-s}] \text{ cm}$. For $s = 40$ we get $R_s \in [3 \times 10^{-19}, 2 \times 10^{-20}] \text{ cm}$. Yet we see that the object has a quantum size very far from Planckian scales, even if s is bigger. In general $R_s / L_{\text{pl}} \geq 10^{13}$. Moreover, this

TABLE I. R in cm for various astrophysical and galactic objects and different scales of regularization ($s=30$ corresponds to a GUT’s regularization scale, $s=40$ to a Planckian one, etc.). In any case R/L_{Reg} is much bigger than 1 ($R/L_{\text{Reg}} \sim 10^{(-6+s/2)}$). Therefore all of them are quite far from their corresponding regularization scale.

| m | $s=30$ | $s=40$ | $s=50$ |
|----------------|-----------|------------|------------|
| M_\odot | 10^{-9} | 10^{-19} | 10^{-29} |
| $10^3 M_\odot$ | 10^{-8} | 10^{-18} | 10^{-28} |
| $10^6 M_\odot$ | 10^{-7} | 10^{-17} | 10^{-27} |
| $10^9 M_\odot$ | 10^{-6} | 10^{-16} | 10^{-26} |

result is valid for all N , since for any value of N we have that $R \in [0.6, 1] R_s \sqrt[3]{M}$. It is obvious that, for any astrophysical object, the final properties are very similar. Table I comprises different massive objects and regularization scales and their associated values of R within this model.

B. Israel-Poisson model

We have found that the corresponding model within our family must satisfy

$$A = \frac{\alpha}{6 - \frac{\alpha^2 m}{R^3}}. \quad (53)$$

In this case, $A^{-2} = \lim_{r \rightarrow 0} G_{00} = \Lambda_2$, so that

$$R^3 = \frac{\alpha^2 m}{6 - \alpha \sqrt{\Lambda_2}} = \frac{\beta^2}{6 - \beta \sqrt{\Lambda_2 L_{\text{pl}}^2}} m L_{\text{pl}}^2. \quad (54)$$

This model clearly depends on the coefficient β . For instance, in order to obtain a solution, we must have $\beta^2 < 36 / (\Lambda_2 L_{\text{pl}}^2)$. The natural scale of regularization in this model is the Planckian one since from the beginning the coefficient α was related to the Planck length. Obviously other regularization scales would simply change L_{pl} by the corresponding scale. Using standard values for Λ_2 that use a Planckian regularization scale, and the fact that β^2 should be at most of order unity [18,40], we get $R \sim \sqrt[3]{M} \times 10^{-20} \text{ cm}$. This result is in complete agreement with the foregoing values, even though the models possess very different functions $H(r)$.

C. Dymnikova’s model

From Sec. VI C and the assumption of a dS core, we have

$$\Lambda_1 + \frac{6m}{R^3} \left(\frac{e-1}{e-2} \right) = \Lambda_2, \quad (55)$$

whence

$$R = \sqrt[3]{\frac{6m}{\Lambda_2 - \Lambda_1} \left(\frac{e-1}{e-2} \right)}. \quad (56)$$

Comparing this result with the one in Eq. (52), we get $R = \lambda R_{\text{Two powers}}$, with $0.84 < \lambda < 1.34$, for any N . Therefore R is again of the same order of magnitude, despite the differences in the choice of the profile of the energy density and of the tangential pressures.

Comparing now the model proposed here with the original one in [20], we see that both yield similar conclusions (in the instances they can be compared). For example, in the mentioned work, a characteristic radius was found for the collapsed body. Its expression is $R_c = \sqrt[3]{6m/(\Lambda_2 - \Lambda_1)}$, that yields $\sqrt[3]{(e-2)/(e-1)}R \sim 1.34R$. Besides having a different description for this in our model, the values of the coefficients A and B are also quite different (numerically).

D. Ayón–Beato and García’s, and Bronnikov’s models

In the papers dealing with those models, there is no analysis of the orders of magnitude of an eventual characteristic radius. The only such condition on these models is to have an event horizon. We can now compute which are the ranges of R corresponding to different cases of λ .

First of all, Eq. (50) only has a solution for $0 < \lambda < 1$. Therefore extremely charged objects (those with $|q|/m \gg 1$) cannot be described within the present framework. This would require $R \gg m$, so that the regularized object would not be a BH but a “visible” object, such as an electron (its size, though, being bigger than the classical radius, or Compton size, $q^2/2m$).

For strongly charged objects, i.e., $|q|/m \sim 1$, we get that in order to have a BH $1/2 < R/m < 2$. Thus the regularized object is of a similar size as that of the event horizon. Much bigger than in the uncharged case.

The solution given by [26], i.e., $A = 2m$, $B = q^2/2m$, can only be valid now for very weakly charged objects, $|q|/m \ll 1$, and satisfying $R/m < q^2/m^2$. They showed that their model was acceptable for $|q|/m \leq 1.05$. Now, we see that the values of A and B in our model change for most of these cases.

The same is valid for Bronnikov’s model, changing the electric field for a magnetic one. Nevertheless, rotation should be introduced in such a case, at least when $|q|/m$ is not very small.

VIII. HORIZONS AND AN INTERPRETATION OF THE REGULARIZED BH

Looking at Eq. (30) in [21] and comparing it with our result,

$$g_{00} = -1 + H_2, \quad (57)$$

we realize that, substituting here our corresponding H_2 for that model, these expressions turn out to be very similar, except for a possible overall sign difference, due to the different signatures [e.g., $(+, -, -, -)$ instead of our $(-, +, +, +)$]. We conclude that the same structure for the horizons and Cauchy hypersurfaces is obtained. In [21], the solutions are obtained by approximations of the exact solu-

tion, so that these results and ours are really coincident (the relative error with respect to both exact solutions being completely negligible).

In general, the horizons result from the cancellation of g_{00} . Thus we are left with a general set of horizons. A global study for all the candidates encountered has not yet been carried out. We could focus on examples, and try then to extract some general features from them, but we do not find this of primary importance.² The main point is here, in fact, that the matching occurs at a radius which is substantially smaller than the Schwarzschild radius of the object. Therefore we will always have a typical exterior, a vacuum transition region extending until the matching with the object happens, and a quantum-dominated interior, which finally converges to a de Sitter core. In the vacuum interior region and in some part of the quantum object, the role of t and r are not interpreted as usual (∂_t changes its character). This is the reason for adequately treating the horizons: to see where exactly such changes appear. But, we can still perfectly agree in ordinary physical terms without requiring a general determination of the precise radii at which horizons occur.

Moreover, in [40], the authors studied the stability of the model. The same considerations there hold for our whole family of solutions, as can be easily seen after a careful analysis.

Finally, there is still the issue of the topology of the solutions, which is connected with the possibility of a “universe reborn” in the extended spacetime. Its general structure can be found in [45] for the case where the sources satisfy weak energy conditions (see the next section). There, it was shown that the topology of any regular BH, satisfying the weak energy conditions, should be similar to that of a singularity-free Reissner-Nordström spacetime. However, there are relevant solutions in our family that violate the weak energy conditions (WEC). It would be worth studying what happens in those cases.

IX. ENERGY CONDITIONS

A common point when dealing with the avoidance of singularities is to show that the energy conditions required in the singularity theorems (see, e.g., [7]) fail to be valid.

Here we will study the strong energy conditions (SEC), the weak energy conditions (WEC), the null energy conditions (NEC), and the dominant energy conditions (DEC), within the GNRSS family (see [46] for the case of a general spherically symmetric spacetime). The SEC are related with the formation of singularities in the collapse of an object. The WEC are directly related with the energy density measured by an observer. The NEC are useful in order to include some spacetimes which violate the first two, but are predicted by some quantum models, e.g., AdS. Finally, the DEC are in fact related with the causal structure of the energy-

²With respect to the other models encountered here, we have found that the results are rather similar to those in Dymnikova and Soltysek’s model [21].

matter content of a spacetime [47].³ Even though an analysis of energy conditions helps to understand the physics of a model, one has to be cautious on ascribing to them more relevance than they actually have. In several systems, mainly when quantum effects play a fundamental role, they all may be violated with less difficulty (see, e.g., the review in [48]).

Let $\{\vec{e}_a\}$, $a=0,1,2,3$, be a dual vector basis of the cobasis used in Sec. III B, defined by $\Theta^b \vec{e}_a \equiv \delta_a^b$, $b=0,1,2,3$. Any timelike vector field, \vec{V} , in the manifold can be represented by

$$\vec{V} = A^b \vec{e}_b, \quad (A^0)^2 = 1 + \sum_{i=1}^3 (A^i)^2, \quad (58)$$

where A^b are some functions.

On the other hand, from the results of Sec. II, the Ricci tensor is

$$\begin{aligned} \text{Ricci} = & R_{00}(\Theta^0 \otimes \Theta^0 - \Theta^1 \otimes \Theta^1) \\ & + R_{22}(\Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3), \end{aligned} \quad (59)$$

where \otimes denotes tensor product. A similar expression holds for the Einstein tensor.

A. Strong energy conditions

SEC require $R_{VV} \equiv R_{ab} V^a V^b \geq 0$, for all \vec{V} . From the expressions above, we obtain $R_{VV} = R_{00} + (R_{00} + R_{22})[(A^2)^2 + (A^3)^2]$. Taking into account the expressions of the Ricci and Einstein's tensor given in Sec. III B we get $R_{VV} = G_{22} + (G_{00} + G_{22})[(A^2)^2 + (A^3)^2]$. Finally, using Einstein's equations, and the fact that A^2, A^3 are free, we get

$$\text{SEC} \leftrightarrow \rho + p_2 \geq 0, \quad p_2 \geq 0, \quad (60)$$

where ρ is the energy density measured by \vec{e}_0 , $8\pi\rho = G_{00}$, and p_2 is the tangential pressure (or stress) of the source, $8\pi p_2 = G_{22}$. This is the usual representation of SEC. However, the GNRSS family allows for a new, and more useful, expression. Indeed, as mentioned elsewhere, $p_2 = -(\rho + r\rho'/2)$, where $(\)' \equiv d(\)/dr$. Therefore we can write

$$\text{SEC} \leftrightarrow p_2 \geq 0, \quad \rho' \leq 0 \quad (61)$$

in all the examples given before SEC are violated. This is natural since they are regular. Particularly, SEC are violated for $r \leq R_{\text{SEC}}$, with

$$R_{\text{SEC}} = \frac{N-2}{N} \sqrt{\frac{2}{N}} R, \quad R_{\text{SEC}} = \left(\sqrt[3]{\frac{\alpha^2 m}{4(6R^3 - \alpha^2 m)}} \right) R,$$

$$R_{\text{SEC}} = 0.68R, \quad R_{\text{SEC}} = 0.83B,$$

where all the quantities have been defined in Sec. VI and the solutions correspond to the two-power model, the Israel-

Poisson's model, Dymnikova's model, and Ayón-García's [26] and Bronnikov's model, respectively. Indeed, SEC are violated in a main portion of the object, i.e., $R_{\text{SEC}} \leq R$. For the evaluation of the Israel-Poisson's model, we have used the same numerical values as in Sec. VII A. In the case of Dymnikova's model the value displayed corresponds to the case $\Lambda_2 \gg \Lambda_1$. For any other case with $\Lambda_2 > \Lambda_1 > 0$ or $\Lambda_2 < 0 < \Lambda_1$, as expected, SEC are violated from bigger values of R_{SEC} . Finally, in the latter case, one should evaluate B for different possibilities (see Sec. VI D and the next case).

B. Weak energy conditions

Following analogous steps, one finds, for WEC ($G_{VV} \geq 0$, for all \vec{V})

$$\text{WEC} \leftrightarrow \rho \geq 0, \quad \rho' \leq 0. \quad (62)$$

It turns out that WEC are satisfied in the models of Secs. VI A–VI C, very easily for any value of r (e.g., for de Sitter core, $\rho' = 0$). One only needs to impose $\Lambda_1 < \Lambda_2$.

Let us now consider the series of models in Secs. VI D and VI E. We have already seen that $0 < \lambda < 1$. This implies that $y, x > 0$, and hence that $A, B > 0$. In general, we have $H(r) = (A/r)[1 - \tanh(B/r)]$. We then get $8\pi\rho = (1/r^2)(Hr)' = (AB/r^4)\cosh^{-2}(B/r)$ and $8\pi\rho' = (2AB/r^5)\cosh^{-2}(B/r) \times [-2 + (B/r)\tanh(B/r)]$. The energy density ρ is positive for any r , although ρ' may become positive. To see this, we first solve $\rho' = 0$. Its solution is $r \approx 0.48B$. Therefore we have for $r < 0.48B \cap r < R$ (outside the body, WEC are satisfied), WEC are violated.

In the model of Refs. [26] and [30], one has $B = q^2/2m$ and $|q| < 1.05m$. This gives that WEC are violated for $r < 0.27m$, already far away from the core.

In our revisited model, we have basically two different possibilities. First, for weakly charged sources, i.e., those with $|q|/m \ll 1$, the conclusions are the same as for the model in [26] and [30]. Second, for sources with $|q|/m \sim 1$, we have, recall Sec. VII D, $m/2 < R < m$, for a BH. Two limiting alternatives appear.

The first one is that $R \rightarrow 2m$. In this case $y \sim 1$ and, therefore, WEC are violated for $r < 0.27m$. The other one is that $R \rightarrow (m/2) + \epsilon$, with $\epsilon \ll 1$. Now, $y \sim 1/4\epsilon$. WEC are violated for $r < 0.06m/\epsilon \cap r < 2m$, that is everywhere inside the source. In conclusion, WEC are again violated almost everywhere.

Finally, if one lets $R > 2m$ (one does not have now a BH, but a “visible” object) large enough to have $\lambda < 1$ for any $|q|, m$, we find that WEC are violated everywhere in the object.

This adds a new (elementary) example to the violation of WEC when quantum effects play an important role (see [48] for a recent review) and shows clearly that, although energy conditions do help to understand the models, they should not necessarily restrict the search for new solutions (Fig. 1).

C. Null energy conditions

In the case of NEC, \vec{V} is a null vector field, $\vec{V} \cdot \vec{V} = 0$, and requires the evaluation of $R_{ab} V^a V^b = G_{ab} V^a V^b \geq 0$, $\forall \vec{V}$. One obtains

³Let us note that their Eq. (2.25), expressing the DEC, are wrong. For our case, the correct ones are given in Eq. (64).

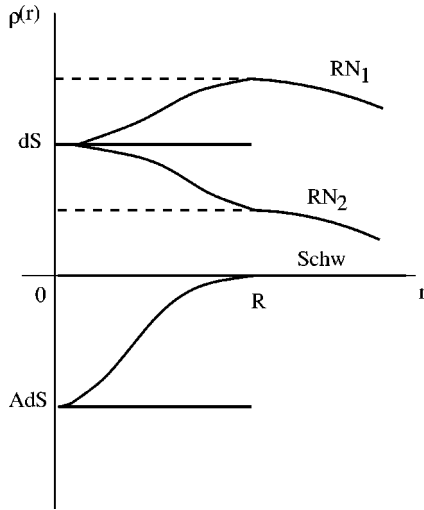


FIG. 1. Plot of the density ρ , in arbitrary units, in terms of the coordinate r . RN means Reissner-Nordström, dS means de Sitter, Schw means Schwarzschild, and AdS means anti-de Sitter. $\rho_{\text{RN}} = e^2/r^4$, $\rho_{\text{dS}} = \Lambda_2$, $\rho_{\text{Schw}} = 0$, and $\rho_{\text{AdS}} = -|\Lambda_2|$. In the region $r \in [0, R]$, ρ can be any (smooth) function matching continuously with ρ at the center and at the surface. Regions where ρ is increasing violate SEC, WEC, and NEC. These are clearly most but not all of the possibilities. From the plot, $e^2/R^4 < 8\pi\Lambda_2$, if one wants that the model fulfills WEC or NEC. The addition of an external Λ simply shifts the horizontal axis a quantity Λ_1 .

$$\text{NEC} \leftrightarrow \rho' \leq 0. \quad (63)$$

Thus one sees that a necessary condition common to SEC, WEC, and NEC is that the energy profile of the sources be a nonincreasing function. NEC are satisfied in the models of Secs. VI A–VI C, for $\Lambda_2 > \Lambda_1$, regardless of the signs of Λ_2 or Λ_1 . In the models of Secs. VI D and VI E, NEC are violated in the same regions as WEC are, contrary to the belief expressed in [30].

D. Dominant energy conditions

DEC are satisfied if and only if $|T_0^0| \geq |T_j^j|$, $i, j = 1, 2, 3$. For the GNRSS family one gets

$$\text{DEC} \leftrightarrow \text{sign } \rho = \text{sign}(-\rho') = \text{sign}(\rho' + 4\rho/r). \quad (64)$$

Two immediate consequences are that if ρ changes its sign, DEC are violated, and if WEC are violated in a region with $\rho \geq 0$, then DEC are also violated.

Let us turn now to the models considered here. In the case of the two-powers model we will assume $\Lambda_1 \geq 0$. In this case, ρ is positive everywhere. WEC were satisfied in these models. However, DEC may be violated. A study of the sign of $\rho' + 4\rho/r$ tells us that SEC are satisfied for $\tilde{r}^{N-2} \leq 4\Lambda_2/(\Lambda_2 - \Lambda_1) \times 1/(N+2)$. Now the question is whether \tilde{r} is less than 1 or not.

Obviously, for any N exceeding $N^* \equiv 4\Lambda_2/(\Lambda_2 - \Lambda_1) - 2 = 2(\Lambda_2 + \Lambda_1)/(\Lambda_2 - \Lambda_1)$, we have that DEC are violated. One may ask whether this is too odd or easy. Since one expects $\Lambda_2 \gg \Lambda_1$, we readily get $N^* \approx 2$. This, together with

the fact that N must be bigger than 2 (in order to be singularity-free), implies that, in practice, DEC are violated in these models, recall WEC are satisfied throughout.

For the model of Sec. VI B, we can assume B, C to be positive (for, if $BC < 0$ one gets a negative Schwarzschild's mass outside the body and if $B, C < 0$, B, C can be substituted by $|B|, |C|$). Following a similar analysis as with the previous models, one gets that DEC are satisfied for $r \leq r^*$, where $r^* \equiv (2B/C)^{1/3}$. Therefore if r^* is less than the matching R , there is a region where DEC are violated. This is the case of our corrected model. Incidentally, in the original model, DEC are violated for $r > r^*$. This conclusion is against physics, since thus far one expects Schwarzschild's solution to be valid, and it is a vacuum's solution with no problems in its causal structure. Therefore the corrected version not only describes a more realistic picture but also solves this undesired property.

Turning back to our corrected model we have to answer whether r^* may be smaller than R , see the expressions given in Sec. VI B. We get that for $\beta > \beta_+$ or $\beta < \beta_-$, where $\beta_+ \equiv 6(\sqrt{3}-1) \times \sqrt{(L_{\text{pl}}^2 \Lambda_2)}$ and $\beta_- \equiv -6(\sqrt{3}+1) \times \sqrt{(L_{\text{pl}}^2 \Lambda_2)}$. On the other hand, it is expected, [49], that $L_{\text{pl}}^2 \Lambda_2 \sim \mathcal{O}(1)$ and, consequently, β_+, β_- are of order unity. Therefore even though there are several parameters for which DEC may hold, there are also many others for which DEC will fail. A more definite answer can only be provided after a particular field model is chosen which will yield a particular value of β . What is this plausible field model remains, as of now, unknown.

For the next model (the one in Sec. VI C), it is easy to show that DEC are satisfied throughout the source if $\Lambda_2 > \Lambda_1 > 0$, as expected. This is contrary to the other models since this one departs from them through the causal connection in its stress-energy content. It is to be noticed that DEC give a new input to understand the models. (The case $\Lambda_2 < \Lambda_1 < 0$ also satisfies DEC, whereas the rest of the possibilities violates them.)

Finally, for the models in Secs. VI D and VI E, as ρ is positive and ρ' is positive near the core, DEC are violated together with WEC.

Some concluding remarks are in order. First, although DEC are known to be different from WEC, here we see more: it turns out that in cases with $\rho \geq 0$, DEC are more restrictive than WEC. Another consequence is that DEC violation and the spacetime region where it occurs are not related. That is, DEC may be violated in regions where $H(r)$ is larger or smaller than 1. It happens, however, that after substituting expected numerical values for the physical parameters involved, the values of $H(r)$ where DEC is violated belong mainly in the region where $H(r) \geq 1$ and a “signature change in spacetime has occurred.” The region with $H(r) \leq 1$ is then at Planckian (regularization) scales and can thus be forgotten. On the other hand, when WEC are violated, one usually accepts that the energy-matter content of the model can no longer be described by a classical matter source model. However, DEC deserve some especial attention.

These remarks impel us to further interpret the violation of DEC from the causal interpretation of DEC [47]. A pos-

sibility is that the breakdown of causality in matter interaction may be interpreted in similar, though properly adapted, terms as is the Einstein-Podolski-Rosen paradox interpreted in quantum mechanics.

If this is so, or something similar can be proven, DEC may be a more natural sign of quantum effects in matter than WEC, for the case of positive densities. This point deserves further investigation.

We will now analyze the main features arising when one replaces the de Sitter core by a different spherically symmetric solution.

X. THE MATCHING OF STATIC SPHERICALLY SYMMETRIC SPACETIMES

In previous sections we have worked with the assumption that the energy-momentum tensor satisfies $T_0^0 = T_1^1$. Now we would like to make the first steps towards the general case where T_0^0 and T_1^1 may be independent of each other. Therefore our aim here is to match two spacetimes that share the existence of an integrable Killing field and spherical symmetry. In order to get the most natural junction, we need to take profit of both symmetries exhaustively. The metric can always be written, for any of them, as

$$ds^2 = g_{AB}(R)dx^A dx^B + G^2(R)d\Omega^2, \quad (65)$$

where $A, B = T, R$. ∂_T has been chosen to be the integrable Killing vector and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Moreover, if $G'(R) = 0$, we already know that they belong to the generalized Nariai family, in which case they only match with another member of its own family. Thus we will only deal with the situation $G'(R) \neq 0$. In this case, a direct redefinition of the R coordinate allows us to write

$$ds^2 = g_{AB}(r)dx^A dx^B + r^2 d\Omega^2, \quad (66)$$

where $A, B = T, r$.

Spherical symmetry has thus been completely used. We now extract consequences from the presence of ∂_T . The natural thing to do is to identify both vector fields, i.e., $\partial_{T_1} \equiv \Sigma \partial_{T_2}$. However, this is not a right choice, in general, because if a Killing vector is multiplied by a constant factor, the resulting vector field is obviously a Killing vector field. Therefore normalizing each Killing vector, when possible, gives the natural way to identify them. This is indeed implemented in the junction process if the hypersurfaces are spacelike or timelike everywhere. On the contrary, in the general case, we cannot rely on such normalization.

Any metric of interest (to our purposes) can be written as [recall the coordinate change to obtain Eq. (6), setting now $F = 1 - H$]

$$ds^2 = -(1-H)dT^2 + \frac{2H}{g}dT dr + \frac{1+H}{g^2}dr^2 + r^2 d\Omega^2, \quad (67)$$

where H , $g \neq 0$ are functions of r only. Looking back to expressions (12) and the coordinate changes mentioned there, we will now put $dT = g_0 dt + (g_0 - g_0^{-1})dr$, where g_0

is a constant, that will be related with the function g , as we shall see in a moment. With this coordinate change the metric takes the form

$$ds^2 = -g_0^2(1-H)dt^2 + 2\tilde{G}dt dr + \tilde{F}dr^2 + r^2 d\Omega^2, \quad (68)$$

where $\tilde{G} = g_0^2(H-1) + 1 + H(g_0 - g)/g$, and $\tilde{F} = 2 + g_0^2(H-1) + (g_0 - g)[2H/g + 2/g_0^2 g + (g_0 - g)(H+1)/g_0^2 g]$.

The junction conditions (for any type of hypersurface, see [50]) are then

$$[r] = 0, \quad [\dot{r}] = 0 \quad (69)$$

$$[\tilde{H}]i^2 + 2[\tilde{G}]i\dot{r} + \tilde{F}\dot{r}^2 = 0, \quad (70)$$

$$[\tilde{F}]i\ddot{r} - [\tilde{G}](\ddot{t} - \ddot{r}\dot{r}) + [\tilde{H}]\dot{r}\ddot{t} + [\tilde{G}']r^3 + [\tilde{H}' - (\tilde{F}'/2)]r^2\dot{t} - [\tilde{H}']i^3/2 = 0, \quad (71)$$

where $[f] = \Sigma f_2 - f_1$, and where we have put $\tilde{H} \equiv g_0^2(H-1) + 1$. In Eqs. (70) and (71) \dot{t} and \dot{r} are either \dot{t}_1, \dot{r}_1 or \dot{t}_2, \dot{r}_2 , and $A' \equiv dA(r)/dr|_{r=r(\lambda)}$. The same conditions lead, in general, to a second order ordinary differential equation for r . In principle there is the possibility for asymptotic stopping solutions, i.e., solutions for which $r \rightarrow \text{const}$ as $t \rightarrow \infty$, and also for null ones. The special case $r_1 = r_2 = R = \text{const}$ is of great interest, since it constitutes the solution towards which any transitory solution should converge. Under this restriction, the conditions become, simply,

$$[\dot{t}] = 0, \quad [\tilde{H}] = 0, \quad 2[\tilde{G}]\dot{t} - [\tilde{H}']i^2 = 0, \quad (72)$$

where t is either t_1 or t_2 . Choosing g_0 as g_Σ one gets $[\tilde{G}] = 0$ (the same result comes out directly in the case when the normal vector of Σ is non-null). The last conditions become then $[\tilde{H}'] = 0$. Thus the conditions emerging from the matching of two spherically symmetric spacetimes with an integrable Killing vector field are, for the case $r = R = \text{const}$ and taking the maximum identification between them,

$$[\tilde{H}] = 0, \quad [\tilde{H}'] = 0, \quad (73)$$

where $\tilde{H} \equiv g_\Sigma^2(H-1) + 1$. An intrinsic characterization, valid for any representation of the form (65) or (66) (the ones most often dealt with in the literature) is $\tilde{H} \equiv -g_\Sigma^2(\tilde{\xi} \cdot \tilde{\xi}) + 1$, $g_\Sigma \equiv [G'/|\det(g_{AB})|^{1/2}]_{r=R}$, where $\tilde{\xi}$ is the Killing vector associated with the staticity of the solution (in some regions) of (65) or (66). Finally, notice that the first condition on \tilde{H} is nothing but the requirement of the mass function to be continuous across the hypersurface, while the second one is related with the continuity of the radial stress or pressure [see, e.g., Eq. (3)]. Needless to say, if one restricts oneself to the family of metrics in Eq. (12), one gets the conditions of Sec. IV B.

XI. AN APPLICATION TO SUPERSYMMETRIC STRINGY BLACK HOLES

The semiclassical expressions for supersymmetric stringy black holes are well-established (see, e.g., [51,52] and references therein). There are also other objects of interest, such as black strings, higher dimensional black holes, etc. In all cases, one looks for a correspondence principle with general relativistic black holes. This transition is usually reflected in the strength of the coupling constant, or the entropy (see, e.g., [51–54] and references therein). Here we take a complementary viewpoint.

The most interesting case to our aims is that of a self-gravitating string (see, e.g., [53,54]). However, the necessary ingredients, specially the corresponding spacetime metric, in order to tackle this problem are still under study. Here we will consider the most simple (and widely considered) case, that of a supersymmetric black hole.

A family of such black holes, related with electrically charged black holes, is given by (see [51,52] for details)

$$ds^2 = -f^{-1/2}(r) \left(1 - \frac{r_0}{r} \right) dt^2 + f^{1/2}(r) \left[\left(1 - \frac{r_0}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \right], \quad (74)$$

where $f(r) = \prod_{i=1}^4 [1 + (r_0 \sinh^2 \alpha_i / r)]$, and where the α_i are related with the integer charges of the D-branes being used. If the correspondence occurs at a constant value of r , we get

$$r_1 + r_0 \sinh^2 \alpha = r_2 f_2^{1/4}(r_2) \equiv R = \text{const}, \quad (75)$$

$$\frac{2m}{R} - \frac{Q^2}{R^2} = 1 + \left[\left(\frac{r_0}{r} - 1 \right) \left(1 + \frac{rf'}{4f} \right)^2 \right]_{\Sigma_2}, \quad (76)$$

$$-\frac{2m}{R^2} + \frac{2Q^2}{R^3} = \left\{ \left(1 + \frac{rf'}{4f} \right)^2 \left[\frac{f'}{2f} \left(1 - \frac{r_0}{r} \right) - 2 \frac{r_0}{r^2} \right] \right\}_{\Sigma_2}, \quad (77)$$

where we have used $g_{\Sigma} = G'|_{r=G^{-1}(R)}$, $G(r) = rf^{1/4}(r)$, and $\vec{\xi} \cdot \vec{\xi} = -f^{-1/2}(1 - r/r_0)$. The subscript Σ_2 means that all these quantities refer to the interior region, to be evaluated at $r = r_2$. For the exterior metric we have put $\alpha_i = \alpha_j \equiv \alpha$ for all i, j , because the exterior metric is that of a Reissner-Nordström black hole, for which

$$2m = r_0 \cosh 2\alpha, \quad Q^2 = r_0^2 \sinh^2 \alpha \cosh^2 \alpha, \quad (78)$$

$$r_0 = 2\sqrt{m^2 - Q^2}, \quad 2 \sinh^2 \alpha = -1 + m/\sqrt{m^2 - Q^2},$$

where m is the (ADM) mass of the black hole and Q is its electric charge. Since $f_2(r_2) = \prod_{i=1}^4 [1 + (r_0 \sinh^2 \alpha_i / r)]_{\Sigma_2}$, the above conditions yield R as a function of six of the seven parameters, $M, Q, (r_0)_2, \alpha_i$. Detailed analysis shows that these conditions are easily fulfilled when $r_0 \rightarrow 0$, $\alpha_i \rightarrow \pm \infty$, with $r_0 \sinh^2 \alpha_i$ fixed. The resulting R is very close to R_0

$\equiv m + \sqrt{m^2 - Q^2}$, i.e., the event horizon of the black hole. We remark that $rf^{1/4}(r)$ is the radial coordinate which has a direct interpretation in terms of the “size” of the object, and not r alone. All this being in complete agreement with the expected transitions for extreme, and nearly extreme, supersymmetric black holes. The same idea should be extended to self-gravitating strings when their (four-dimensional) spacetime metric is obtained. For instance, the expected order of magnitude of R found in [53] should be recovered. This issue will be the matter of subsequent research.

XII. FINAL REMARKS

The first thing to be noticed is the intrinsic freedom present in our model, which is as large as the measure of the set of analytic functions of one variable. This is a very rewarding feature, since it allows one to impose further restrictions coming from new, more accurate proposals. In particular, it will be a helpful tool when trying to find explicitly a quantum field responsible for the G_{11} and G_{22} in the fundamental uncharged case. For comparison, in all previous works, based on individual models, the prospect of finding a quantum field related with their energy-matter content was hopeless. To that end, we would like to draw attention to [55], where a useful framework to deal with the interior region is given.

In the charged case, let us notice that any GNRSS spacetime can be linked with a solution to NED (see [56]). Of course, the case of the Schwarzschild solution is a solution with zero charge, and the Reissner-Nordström one is the only one which is linear, i.e., Maxwellian. Therefore the whole family of GNRSS metrics has indeed an immediate interpretation in terms of a field theory which is well established when the object is electrically or magnetically charged. This is another useful result. A careful study of this fact will be reported elsewhere.

Finally, one can free the requirement that there must be an event horizon. The objects would then become “visible” and the study of the entropy of the solutions as well as their associated Hawking temperature would bring some clues on the time evolution of (classically) static black holes.

XIII. CONCLUSIONS

In this work we have investigated, under quite general conditions, the question of using Einstein’s theory of gravitation, extended to include semiclassical effects, with the purpose of constraining the physical structure of the emerging spacetime solutions that might be suitable for the description of the interiors of nonrotating black holes.⁴

In the first part of the work we have exploited the idea that vacuum polarization may indeed play an essential role in the interior region. We have obtained the result that only two

⁴The rotating case, which is of major astrophysical interest, and the rotating and electrically charged one, which may be associated with spinning particles, seem to yield results very similar to the ones presented here, see [41,44].

families fulfill the imposed requirement and, moreover, we have shown that only one of them is suitable for representing static black hole interiors, what is certainly a most remarkable result.

Then we have turned our attention to other sources for the core. Given that a promising alternative, self-gravitating strings, needs still to be studied in more detail, we have started this program by first giving the general conditions to be fulfilled by any spacetime with spherical symmetry and having some static region. Finally, we have applied the results obtained to a supersymmetric black hole as a preliminary case. We have seen that, in such a situation, the matching is generically compatible, including the case of extreme black holes. This last setting is precisely the same for which the correspondence between semiclassical black holes and stringy ones has been recently confirmed in the literature (see, e.g., [53,54]).

Briefly, our overall conclusion is the following. First, the results in the first sections have opened a new window for

the search of a compatible quantum field that, once regularized, may yield the same result for, at least, a particular energy-momentum tensor inside the general family of models considered (for instance, within NED, see also [56]). Second, once a corresponding Einsteinian metric associated with a quantum model is known, the scheme developed here has been proven to be well suited to check the consistency of the involved physical parameters and even, in some cases, to assign explicit values to them.

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