Gravitational wave propagation in isotropic cosmologies

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We study the propagation of gravitational waves carrying arbitrary information through isotropic cosmologies. The waves are modeled as small perturbations of the background Robertson-Walker geometry. The perfect fluid matter distribution of the isotropic background is, in general, modified by small anisotropic stresses. For pure gravity waves, in which the perturbed Weyl tensor is radiative (i.e. type N in the Petrov classification), we construct explicit examples for which the presence of the anisotropic stress is shown to be essential and the histories of the wave fronts in the background Robertson-Walker geometry are shear-free null hypersurfaces. The examples derived in this case are analogous to the Bateman waves of electromagnetic theory.

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I. INTRODUCTION

This paper is primarily concerned with investigating the influence of gravitational waves on the matter content of the universe. The gravitational waves are modeled as small perturbations of Friedmann-Lemaître cosmological models with a spatially homogeneous and isotropic Robertson-Walker (RW) geometry. We use the gauge-invariant and covariant perturbation approach of Ellis and Bruni [1]. This technique has previously been used to study aspects of gravitational wave propagation in isotropic cosmologies which differ from those considered here (see, for example, [2,3]).

We look for gravitational waves which carry arbitrary information and this motivates us to require the Ellis-Bruni gauge-invariant variables to have an arbitrary dependence on a function. Thus by "arbitrary information" in the perturbations we mean that they depend upon an arbitrary function. This in turn means that the profile of the waves described by the perturbations is unspecified. The idea of introducing arbitrary functions into solutions of Einstein's equations describing gravitational waves goes back to pioneering work by Trautman [4]. This point of view was initiated in the present context by Hogan and Ellis [5], but whereas in that work the perturbed matter distribution was a perfect fluid, here we allow it to be completely general. The Ellis-Bruni approach does not involve working directly with Einstein's field equations but instead the basic equations used are the Ricci identities, Bianchi identities and the matter equations of motion and energy conservation equation, with Einstein's field equations incorporated in them. We work in the linear approximation in terms of the Ellis-Bruni variables and demonstrate that consistency of our assumptions with the basic equations necessarily leads to all perturbation variables vanishing except the perturbed shear of the matter world-lines and the anisotropic stress perturbation of the matter distribution. The consistency of the equations satisfied by these surviving variables is established. We then specialize our study to pure gravity wave perturbations for which the perturbed Weyl tensor is radiative (type N in the Petrov classification). We show that for such waves the histories of their wavefronts in the background RW space-time can be shear-free null hypersurfaces. The various open and closed RW geometries admit, in a natural way, families of shear-free null hypersurfaces. We derive explicit solutions of the perturbation equations describing gravitational waves propagating through these isotropic universes for which the histories of their wave fronts are these naturally occurring families of shear-free null hypersurfaces. These solutions are analogous to the Bateman waves [6] in electromagnetic theory.

The paper is organized as follows. In Sec. II we list the basic exact equations derived from the Ricci identities, Bianchi identities and the equations of motion and of energy conservation of the matter distribution. In Sec. III we introduce the Ellis-Bruni gauge-invariant variables and show how for us they have an arbitrary dependence on a function. The consistency of these assumptions with the basic equations in the linear approximation is then systematically studied, and the mathematical consistency of the surviving equations is established. This is followed in Sec. IV by specialization to pure gravity wave perturbations and the demonstration that their wave-front histories can be shear-free null hypersurfaces. In Sec. V some explicit families of pure gravity wave perturbations are derived. The paper ends with a discussion in which we comment on the solutions obtained in Sec. V.

II. THE BASIC EXACT EQUATIONS

To make the paper as self-contained as possible the basic equations required for our study are given in this section. We use the notation and sign conventions of [7] and all of the equations given here can be found in [7] with the exception that the Bianchi identities given in [7] apply to a perfect fluid matter distribution whereas we require the extension of these to a general matter distribution with energy-momentum-stress tensor given by Eq. (2.2) below. This covariant approach to cosmology began in a systematic way with the work of Schücking, Ehlers and Sachs (see [7,8] for example) while Hawking [9] gave the first description of cosmological perturbations in this context. We are concerned here with a four dimensional space-time manifold with metric tensor

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components g_{ab}, in a local coordinate system $\{x^a\}$, and a preferred congruence of world-lines tangent to the unit timelike vector field with components u^a (with $u^a u_a = -1$). With respect to this 4-velocity field the Weyl tensor, with components C_{abcd}, is decomposed into an "electric part" and a "magnetic part" given respectively by

$$E_{ab} = C_{apbq} u^p u^q, \qquad H_{ab} = {}^*C_{apbq} u^p u^q. \tag{2.1}$$

Here ${}^*C_{apbq} = \frac{1}{2} \eta_{ap}{}^{rs} C_{rsbq}$ is the dual of the Weyl tensor (the left and right duals being equal), $\eta_{abcd} = \sqrt{-g}\, \epsilon_{abcd}$ with $g = \det(g_{ab})$ and ϵ_{abcd} is the Levi-Civita permutation symbol. The expression for the Weyl tensor in terms of E_{ab} and H_{ab} is given in [7]. The symmetric energy-momentum-stress tensor T^{ab} can be decomposed with respect to the 4-velocity field u^a as

$$T^{ab} = \mu\, u^a u^b + p\, h^{ab} + q^a u^b + q^b u^a + \pi^{ab}, \tag{2.2}$$

where

$$h^{ab} = g^{ab} + u^a u^b, \tag{2.3}$$

is the projection tensor and

$$q^a u_a = 0, \qquad \pi^{ab} u_b = 0, \qquad \pi^a{}_a = 0, \tag{2.4}$$

with $\pi^{ab} = \pi^{ba}$. Here μ is interpreted as the matter energy density measured by the observer with 4-velocity u^a, p is the isotropic pressure, q^a is the energy flow (such as heat flow) measured by this observer and π^{ab} is the anisotropic stress (due, for example, to viscosity). We shall indicate covariant differentiation with a semicolon, partial differentiation by a comma, covariant differentiation in the direction of u^a by a dot and a definition by a colon followed by an equality sign. Thus the 4-acceleration of the timelike congruence is

$$\dot{u}^a := u^a{}_{;b} u^b, \tag{2.5}$$

and $u_{a;b}$ can be decomposed into

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta\, h_{ab} - \dot{u}_a u_b, \tag{2.6}$$

with

$$\omega_{ab} := u_{[a;b]} + \dot{u}_{[a} u_{b]}, \tag{2.7}$$

the vorticity tensor of the congruence tangent to u^a. The square brackets denote skew-symmetrization as usual. Also

$$\sigma_{ab} := u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{1}{3} \theta\, h_{ab}, \tag{2.8}$$

is the shear tensor of the congruence. The round brackets denote symmetrization and

$$\theta := u^a{}_{;a}, \tag{2.9}$$

is the expansion (or contraction) of the congruence.

The key equations we shall need are obtained by projections in the direction u^a and orthogonal to u^a (using the projection tensor h_{ab}) of the *Ricci identities*

$$u_{a;dc} - u_{a;cd} = R_{abcd} u^b, \tag{2.10}$$

where R_{abcd} is the Riemann curvature tensor, the *equations of motion* and the *energy conservation equation* contained in

$$T^{ab}{}_{;b} = 0, \tag{2.11}$$

and the *Bianchi identities* written conveniently in the form

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6} g^{c[a} R^{;b]}. \tag{2.12}$$

Here $R^{ca} := R^{cba}{}_b$ are the components of the Ricci tensor and $R := R^c{}_c$ is the Ricci scalar. Einstein's field equations, after absorbing the coupling constant into the energy-momentum-stress tensor, take the form

$$R_{ab} - \frac{1}{2} g_{ab} R = T_{ab}. \tag{2.13}$$

The Ricci identities yield *Raychaudhuri's equation*,

$$\dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^a{}_{;a} + 2(\sigma^2 - \omega^2) + \frac{1}{2}(\mu + 3p) = 0 \tag{2.14}$$

(here $\sigma^2 := \frac{1}{2}\sigma_{ab}\sigma^{ab}$, $\omega^2 := \frac{1}{2}\omega_{ab}\omega^{ab}$) the *vorticity propagation equation*,

$$h^a_b\, \dot{\omega}^b + \frac{2}{3}\theta\, \omega^a = \sigma^a{}_b\, \omega^b + \frac{1}{2}\eta^{abcd} u_b\, \dot{u}_{c;d} \tag{2.15}$$

(here $\omega^a := \frac{1}{2}\eta^{abcd} u_b\, \omega_{cd}$ is the vorticity vector) the *shear propagation equation*,

$$h^f_a h^g_b (\dot{\sigma}_{fg} - \dot{u}_{(f;g)}) - \dot{u}_a \dot{u}_b + \omega_a \omega_b + \sigma_{af}\sigma^f{}_b + \frac{2}{3}\theta\,\sigma_{ab} + h_{ab}\left(-\frac{1}{3}\omega^2 - \frac{2}{3}\sigma^2 + \frac{1}{3}\dot{u}^c{}_{;c}\right) - \frac{1}{2}\pi_{ab} + E_{ab} = 0, \tag{2.16}$$

the $(0,\nu)$-*field equation* (see [7] for the explanation of this terminology),

$$\frac{2}{3} h^a_b\, \theta^{,b} - h^a_b\, \sigma^{bc}{}_{;d}\, h^d_c - \eta^{acdf} u_c (\omega_{d;f} + 2\,\omega_d\, \dot{u}_f) = q^a, \tag{2.17}$$

the *divergence of vorticity equation*,

$$\omega^a{}_{;b}\, h^b_a = \omega^a\, \dot{u}_a, \tag{2.18}$$

and the *magnetic part of the Weyl tensor*,

$$H_{ab} = 2\,\dot{u}_{(a}\,\omega_{b)} - h^t_a h^s_b (\omega_{(t}{}^{g;c} + \sigma_{(t}{}^{g;c})\, \eta_{s)fgc} u^f. \tag{2.19}$$

Next Eq. (2.11) projected orthogonal to u^a and along u^a respectively give the *equations of motion of matter*,

$$(\mu + p)\,\dot{u}^a + h^{ac}(p_{,c} + \pi^b{}_{c;b} + \dot{q}_c) + (\omega^{ab} + \sigma^{ab} + \frac{4}{3}\theta\, h^{ab})\, q_b = 0, \tag{2.20}$$

and the *energy conservation equation*,

$$\dot{\mu} + \theta(\mu + p) + \pi_{ab}\sigma^{ab} + q^a{}_{;a} + \dot{u}^a q_a = 0. \tag{2.21}$$

Finally the various projections of the Bianchi identities (2.12) along and orthogonal to u^a give analogous equations to Maxwell's equations for E_{ab} and H_{ab} in Eq. (2.1). These consist of the *div-E equation*,

$$h^b_g E^{gd}{}_{;f}\, h^f_d + 3\,\omega^s H^b_s - \eta^{bapq}\, u_a\, \sigma^d{}_p\, H_{qd} = \tfrac{1}{3} h^b_c\, \mu^{,c} + \tfrac{1}{2}\Big\{ -\pi^{bd}{}_{;d} + u^b\, \sigma_{cd}\, \pi^{cd} - 3\,\omega^{bd}\, q_d + \sigma^{bd}\, q_d - \tfrac{2}{3}\,\theta\, q^b + \pi^{bd}\, \dot{u}_d \Big\}, \tag{2.22}$$

the *div-H equation*,

$$h^b_g H^{gd}{}_{;f}\, h^f_d - 3\,\omega^s E^b_s + \eta^{bapq}\, u_a\, \sigma^d{}_p\, E_{qd} = (\mu+p)\,\omega^b + \tfrac{1}{2}\,\eta^b{}_{qac}\, u^q\, q^{a;c} + \tfrac{1}{2}\,\eta^b{}_{qac}\, u^q\,(\omega^{dc} + \sigma^{dc})\,\pi^a{}_d, \tag{2.23}$$

the $\dot{E}$-*equation*,

$$\begin{aligned} h^b_f\, \dot{E}^{fg}\, h^t_g &+ h^{(b}_a\, \eta^{t)rsd}\, u_r H^a_{s;d} - 2\, H^{(b}_s\, \eta^{t)drs}\, u_d \dot{u}_r - E^{(t}_s\, \omega^{b)s} - 3\, E^{(t}_s\, \sigma^{b)s} + h^{tb}\, E^{dp}\, \sigma_{dp} + \theta\, E^{bt} \\ &= -\tfrac{1}{2}(\mu+p)\,\sigma^{tb} - \tfrac{1}{6} h^{tb}\,\{\dot{\mu} + \theta\,(\mu+p)\} - q^{(b}\,\dot{u}^{t)} \\ &\quad - \tfrac{1}{2} u^{(b}\,\dot{q}^{t)} - \tfrac{1}{2} q^{(t;b)} + \tfrac{1}{2}\{\omega^{c(b} + \sigma^{c(b}\}\, u^{t)}\, q_c + \tfrac{1}{6}\,\theta\, u^{(t}\, q^{b)} \\ &\quad - \tfrac{1}{2}\,\dot{\pi}^{bt} + \pi^{c(b}\, u^{t)}\, \dot{u}_c - \tfrac{1}{2}\{\omega^{c(b} + \sigma^{c(b}\}\,\pi^{t)}{}_c - \tfrac{1}{6}\,\theta\,\pi^{bt}, \end{aligned} \tag{2.24}$$

and the $\dot{H}$-*equation*,

$$\begin{aligned} h^b_f\, \dot{H}^{fg}\, h^t_g &- h^{(b}_a\, \eta^{t)rsd}\, u_r E^a_{s;d} + 2\, E^{(b}_s\, \eta^{t)drs}\, u_d \dot{u}_r - H^{(t}_s\, \omega^{b)s} - 3\, H^{(t}_s\, \sigma^{b)s} + h^{tb}\, H^{dp}\, \sigma_{dp} + \theta\, H^{bt} \\ &= -q^{(t}\,\omega^{b)} - \tfrac{1}{2}\,\eta^{(t}{}_{rad}\,\{\omega^{b)d} + \sigma^{b)d}\}\, u^r\, q^a \\ &\quad - \tfrac{1}{2}\,\eta^{(b}{}_{rad}\,\pi^{t)a;d}\, u_r + \tfrac{1}{2}\,\eta^{(b}{}_{rad}\, u^{t)}\, u^r\,\{\omega^{cd} + \sigma^{cd}\}\,\pi^a{}_c. \end{aligned} \tag{2.25}$$

III. PERTURBATIONS OF ISOTROPIC COSMOLOGIES

We shall now assume that the space-time in Sec. II is a perturbed RW space-time. Thus the background metric tensor g_{ab} is the Robertson-Walker metric, the background energy-momentum-stress tensor is Eq. (2.2) specialized to a perfect-fluid (by putting $q^a = 0 = \pi^{ab}$) with fluid 4-velocity u^a and the background Weyl tensor vanishes. The Ellis-Bruni [1] approach to perturbations of this background is to work with gauge-invariant small quantities rather than small perturbations of the background metric. Such gauge-invariant quantities have the important property that they vanish in the background space-time. Thus for an isotropic background the Ellis-Bruni variables, which will henceforth be considered small of first order, are $E_{ab}, H_{ab}, \sigma_{ab}, \dot{u}^a, \omega_{ab}$ (or equivalently the vorticity vector ω^a), $X_a = h^b_a\,\mu_{,b}, Y_a = h^b_a\, p_{,b}$, $Z_a = h^b_a\,\theta_{,b}$, π_{ab} and q^a. The equations satisfied by these quantities are given by Eqs. (2.15)–(2.20) and by Eqs. (2.22)–(2.25) neglecting nonlinear terms in these variables. Raychaudhuri's equation (2.14) and the energy conservation equation (2.21) are not immediately useful because they are not expressed in terms of these gauge-invariant variables. We can however get useful equations from Eqs. (2.14) and (2.21) by calculating the spatial gradients of these equations and then retaining only linear terms in the gauge-invariant variables. This results in the two equations

$$\dot{Z}^a + \theta\, Z^a - \dot{\theta}\, \dot{u}^a - h^{ab}(\dot{u}^a{}_{;a})_{,b} + \tfrac{1}{2} X^a + \tfrac{3}{2} Y^a = 0, \tag{3.1}$$

and

$$\dot{X}^a + \tfrac{4}{3}\,\theta\, X^a + (\mu+p)\, Z^a + \theta\, Y^a - \dot{\mu}\,\dot{u}^a + h^{ab}\,(q^c{}_{;c})_{,b} = 0. \tag{3.2}$$

The background value of $\dot{\theta}$ to be substituted into Eq. (3.1) is given by

$$\dot{\theta} = -\tfrac{1}{3}\,\theta^2 - \tfrac{1}{2}(\mu + 3\,p), \tag{3.3}$$

which is obtained by specializing Raychaudhuri's equation (2.14) to the background. In Eq. (3.2) the background value of $\dot{\mu}$ is given by

$$\dot{\mu} = -\,\theta\,(\mu+p), \tag{3.4}$$

which follows from Eq. (2.21) specialized to the background. We shall assume an equation of state of the form $p = p(\mu)$ so that the Ellis-Bruni variables X_a and Y_a are related by

$$Y_a = \frac{dp}{d\mu} X_a. \tag{3.5}$$

Finally we shall assume that in the background $\mu + p \neq 0$ so that we do indeed have a cosmological model in which the Einstein tensor determines a unique timelike 4-velocity of the matter.

We look for solutions σ_{ab}, $\dot{u}^a, \omega_{ab}$ $(\Leftrightarrow \omega^a), X_a$ [and thus Y_a by Eq. (3.5)], Z^a, π^{ab} and q^a of the linearized versions of Eqs. (2.15)–(2.20) and (2.22)–(2.25) along with Eqs. (3.1) and (3.2) for which these variables depend upon an arbitrary function. This is because we expect that this dependence of perturbations will describe gravitational waves carrying arbitrary information. We note that E_{ab}, H_{ab} are derived variables. Specifically we assume that

$$\begin{aligned} \sigma_{ab} &= s_{ab}\, F(\phi), \dot{u}^b = a^b\, F(\phi), \\ \omega^{ab} &= w^{ab}\, F(\phi) [\Leftrightarrow \;\; \omega^a = w^a\, F(\phi)], \\ X^a &= x^a\, F(\phi), Z^a = z^a\, F(\phi), \\ \pi^{ab} &= \Pi^{ab}\, F(\phi), q^a = Q^a\, F(\phi), \end{aligned} \tag{3.6}$$

where F is an arbitrary real-valued function of its argument $\phi(x^a)$. This form for the gauge-invariant variables was first introduced by Hogan and Ellis [5] where the perturbed matter distribution was taken to be a perfect fluid. As we mentioned in the Introduction, the idea of introducing arbitrary functions into solutions of Einstein's equations describing gravitational waves goes back to pioneering work by Trautman [4]. We note that all of the quantities in Eqs. (3.6) are orthogonal to u^a and that s_{ab}, Π_{ab} are trace-free with respect to the background metric g_{ab} (i.e. $s^a{}_a = 0 = \Pi^a{}_a$).

When Eqs. (3.6) are substituted into the linearized versions of Eqs. (2.15)–(2.20), (2.22)–(2.25) and (3.1), (3.2) the following extensive but surveyable list of equations emerges as indicated.

From the shear propagation equation,

$$E_{ab} = \left(\tfrac{1}{2}\Pi_{ab} + p_{ab}\right)F + m_{ab}F', \tag{3.7}$$

with $F' = dF/d\phi$ and

$$p_{ab} = a_{(a;b)} + u_{(a}\dot{a}_{b)} - \tfrac{1}{3}\theta u_{(a}a_{b)} - \tfrac{1}{3}a^f{}_{;f}h_{ab} - \dot{s}_{ab} - \tfrac{2}{3}\theta s_{ab}, \tag{3.8}$$

$$m_{ab} = a_{(a}\lambda_{b)} - \tfrac{1}{3}\chi h_{ab} - \dot{\phi} s_{ab}. \tag{3.9}$$

Here and throughout $\lambda_a = h_a^b \phi_{,b}$, $\chi = \phi_{,f}a^f$ and $\dot{\phi} = \phi_{,a}u^a$.

From the magnetic part of the Weyl tensor,

$$H_{ab} = q_{ab}F + l_{ab}F', \tag{3.10}$$

with

$$q_{ab} = w_{(a;b)} + \dot{w}_{(a}u_{b)} - \tfrac{1}{3}\theta w_{(a}u_{b)} - s_{(a}{}^{p;c}\eta_{b)fpc}u^f - h_{ab}w^c{}_{;c}, \tag{3.11}$$

$$l_{ab} = w_{(a}\lambda_{b)} - w_c\phi^{,c}h_{ab} - s_{(a}{}^p\eta_{b)fpc}u^f\phi^{,c}. \tag{3.12}$$

From the div-E equation,

$$m^{ab}\phi_{,b} = 0, \tag{3.13}$$

$$\Pi^{ab}\phi_{,b} + p^{ab}\phi_{,b} + m^{ab}{}_{;b} = 0, \tag{3.14}$$

$$\Pi^{ab}{}_{;b} + p^{ab}{}_{;b} = \tfrac{1}{3}x^a - \tfrac{1}{3}\theta Q^a. \tag{3.15}$$

From the div-H equation,

$$l^{ab}\phi_{,b} = 0, \tag{3.16}$$

$$q^{ab}\phi_{,b} + l^{ab}{}_{;b} = \tfrac{1}{2}\eta^a{}_{qbc}u^qQ^b\phi^{,c}, \tag{3.17}$$

$$q^{ab}{}_{;b} - (\mu+p)w^a = \tfrac{1}{2}\eta^a{}_{qbc}u^qQ^{b;c}. \tag{3.18}$$

From the $\dot{E}$-equation,

$$\dot{\Pi}^{bt} + \tfrac{2}{3}\theta\Pi^{bt} - \tfrac{1}{6}h^{bt}Q^a{}_{;a} + \tfrac{1}{2}u^{(t}\dot{Q}^{b)} + \tfrac{1}{2}Q^{(t;b)} - \tfrac{1}{6}\theta u^{(t}Q^{b)} = -\dot{p}^{bt} - u_r q^{(b}{}_{s;d}\eta^{t)rsd} - \theta p^{bt} - \tfrac{1}{2}(\mu+p)s^{bt}, \tag{3.19}$$

$$\dot{\phi}\Pi^{bt} - \tfrac{1}{6}h^{bt}Q^a\phi_{,a} + \tfrac{1}{2}Q^{(t}\lambda^{b)} = -\dot{\phi}p^{bt} - \dot{m}^{bt} - \theta m^{bt} - u_r(q^{(b}{}_s\phi_{,d} + l^{(b}{}_{s;d})\eta^{t)rsd}, \tag{3.20}$$

$$\dot{\phi}m^{bt} + l^{(b}{}_s\eta^{t)rsd}u_r\phi_{,d} = 0. \tag{3.21}$$

From the $\dot{H}$-equation,

$$\dot{q}^{bt} - u_r p^{(b}{}_{s;d}\eta^{t)rsd} + \theta q^{bt} = 0, \tag{3.22}$$

$$\dot{\phi}q^{bt} + \dot{l}^{bt} + \theta l^{bt} - u_r(p^{(b}{}_s\phi_{,d} + m^{(b}{}_{s;d})\eta^{t)rsd} = 0, \tag{3.23}$$

$$\dot{\phi}l^{bt} - m^{(b}{}_s\eta^{t)rsd}u_r\phi_{,d} = 0. \tag{3.24}$$

From the $(0,\nu)$-field equation,

$$s^{ab}\phi_{,b} + \eta^{acdf}u_cw_d\phi_{,f} = 0, \tag{3.25}$$

$$Q^a = \tfrac{2}{3}z^a - s^{ab}{}_{;b} - \eta^{acdf}u_cw_{d;f}. \tag{3.26}$$

From the vorticity propagation equation,

$$\dot{w}^a + \frac{2}{3}\theta w^a = \tfrac{1}{2}\eta^{acbd}u_ca_{b;d}, \tag{3.27}$$

$$\dot{\phi}w^a = \tfrac{1}{2}\eta^{acbd}u_ca_b\phi_{,d}. \tag{3.28}$$

From the divergence of vorticity equation,

$$w^a{}_{;a} = 0, \tag{3.29}$$

$$w^a\phi_{,a} = 0. \tag{3.30}$$

From the spatial gradient of Raychaudhuri's equation,

$$\dot{z}^c + \theta z^c + \tfrac{1}{2}x^c + \tfrac{3}{2}y^c = h^{cb}(a^d{}_{;d})_{,b} + \dot{\theta}a^c, \tag{3.31}$$

$$\dot{\phi}z^c = a^d{}_{;d}\lambda^c + h^{cb}(a^d\phi_{,d})_{,b}, \tag{3.32}$$

$$(a^d\phi_{,d})\lambda_c = 0. \tag{3.33}$$

From the equations of motion of matter,

$$y^a + \dot{Q}^a = -(\mu+p)a^a - \Pi^{ab}{}_{;b}, \tag{3.34}$$

$$\dot{\phi}Q^a = -\Pi^{ab}\phi_{,b}. \tag{3.35}$$

From the spatial gradient of the energy conservation equation,

$$\dot{x}^c + \tfrac{4}{3}\theta x^c + \theta y^c + h^{cb}(Q^a{}_{;a})_{,b} = -(\mu+p)\theta a^c - (\mu+p)z^c, \tag{3.36}$$

$$\dot{\phi}x_c + Q^a{}_{;a}\lambda_c + h_c^b(Q^a\phi_{,a})_{,b} = 0, \tag{3.37}$$

$$(Q^a\,\phi_{,a})\,\lambda_b=0. \qquad (3.38)$$

On account of Eq. (3.5) the variable y^a appearing in Eqs. (3.31), (3.34) and (3.36) is related to x^a by

$$y^a=\frac{dp}{d\mu}x^a. \qquad (3.39)$$

We must now examine the internal consistency of these equations.

Our first enquiry parallels the work in [5]. Putting

$$V^{bt}=m^{bt}+i\,l^{bt}, \qquad (3.40)$$

we can write Eqs. (3.21) and (3.24) as a single complex equation

$$2\,\dot{\phi}\,V^{bt}=i\,\eta^{trsd}\,u_r\,V^b{}_s\,\phi_{,d}+i\,\eta^{brsd}\,u_r\,V^t{}_s\,\phi_{,d}\,. \qquad (3.41)$$

From this we calculate that

$$\dot{\phi}\,\eta_{bpql}\,V^{bt}\,u^p=2\,i\,V^t{}_{[q}\,\lambda_{l]}\,. \qquad (3.42)$$

When this is substituted into each term on the right-hand side of Eq. (3.41) we obtain

$$2\,\dot{\phi}\,V^{bt}=2\,\dot{\phi}\,V^{bt}+2\,\dot{\phi}^{-1}\phi_{,d}\,\phi^{,d}\,V^{bt}. \qquad (3.43)$$

Hence with $\dot{\phi}\neq0$ and $V^{bt}\neq0$ we must have

$$\phi_{,d}\,\phi^{,d}=0. \qquad (3.44)$$

The hypersurfaces $\phi(x^a)=\text{const}$ in the background isotropic cosmological model must be null [see Sec. IV below where the physical implications of Eq. (3.44) are discussed]. Thus $\lambda_a=h^b_a\,\phi_{,b}=\phi_{,a}+\dot{\phi}\,u_a\neq0$ and so we now have, in addition to Eq. (3.44), the following vanishing scalar products:

$$w^a\,\phi_{,a}=0, \qquad a^a\phi_{,a}=0, \qquad Q^a\,\phi_{,a}=0, \qquad (3.45)$$

on account of Eqs. (3.30), (3.33) and (3.38). The latter two simplify Eqs. (3.32) and (3.37) respectively.

We will next show that for consistency of our equations we must have $Q^a=0$. Let us, for convenience, write Eq. (3.8) as

$$p^{ab}=A^{ab}-\dot{s}^{ab}-\tfrac{2}{3}\,\theta\,s^{ab}, \qquad (3.46)$$

with

$$A^{ab}=a^{(a;b)}+u^{(a}\,\dot{a}^{b)}-\tfrac{1}{3}\,\theta\,u^{(a}\,a^{b)}-\tfrac{1}{3}a^f{}_{;f}\,h^{ab}. \qquad (3.47)$$

Using the fact that in the background $\theta_{,a}=-\dot{\theta}\,u_a$ we have from Eq. (3.46),

$$p^{ab}{}_{;b}=A^{ab}{}_{;b}-\dot{s}^{ab}{}_{;b}-\tfrac{2}{3}\,\theta\,s^{ab}{}_{;b}\,. \qquad (3.48)$$

In the background $u_{a;b}=\frac{1}{3}\,\theta\,h_{ab}$ and so this equation can be written

$$p^{ab}{}_{;b}=A^{ab}{}_{;b}-s^{ab}{}_{;cb}\,u^c-\theta\,s^{ab}{}_{;b}\,. \qquad (3.49)$$

The Ricci identities satisfied by s^{ab} give

$$s^{ab}{}_{;cb}\,u^c=(s^{ab}{}_{;b})^{\cdot}, \qquad (3.50)$$

and so Eq. (3.49) becomes

$$p^{ab}{}_{;b}=A^{ab}{}_{;b}-(s^{ab}{}_{;b})^{\cdot}-\theta\,s^{ab}{}_{;b}\,. \qquad (3.51)$$

Now Eq. (3.26) is

$$s^{ab}{}_{;b}=-Q^a+\tfrac{2}{3}z^a+\mathcal{A}^a, \qquad (3.52)$$

with

$$\mathcal{A}^a=-\,\eta^{acdf}\,u_c\,w_{d;f}\,. \qquad (3.53)$$

Putting Eq. (3.52) into Eq. (3.51) yields

$$p^{ab}{}_{;b}=A^{ab}{}_{;b}+\dot{Q}^a+\theta\,Q^a-\tfrac{2}{3}\dot{z}^a-\tfrac{2}{3}\,\theta\,z^a-\dot{\mathcal{A}}^a-\theta\,\mathcal{A}^a. \qquad (3.54)$$

Alternatively from Eqs. (3.15) and (3.34) we have

$$p^{ab}{}_{;b}=\tfrac{1}{3}x^a-\tfrac{1}{3}\,\theta\,Q^a+y^a+\dot{Q}^a+(\mu+p)\,a^a. \qquad (3.55)$$

Thus Eqs. (3.54) and (3.55) are consistent provided

$$\tfrac{4}{3}\,\theta\,Q^a=\tfrac{2}{3}(\dot{z}^a+\theta\,z^a+\tfrac{1}{2}\,x^a+\tfrac{3}{2}y^a)+(\mu+p)\,a^a+\dot{\mathcal{A}}^a+\theta\,\mathcal{A}-A^{ab}{}_{;b}\,. \qquad (3.56)$$

Making use of Eq. (3.28) to write w^a in terms of a^a and using the propagation equation (3.27) for w^a along u^a it follows that

$$\dot{\mathcal{A}}^a+\theta\,\mathcal{A}^a=\tfrac{1}{2}\ddot{a}^a+\tfrac{1}{2}\,\theta\,\dot{a}^a-\{\tfrac{1}{4}(\mu-p)-\tfrac{1}{6}\,\dot{\theta}-\tfrac{1}{9}\,\theta^2\}\times a^a-\tfrac{1}{2}h^{ac}\,(a^b{}_{;b})_{,c}-\tfrac{1}{3}\,\theta\,a^b{}_{;b}\,u^a+\tfrac{1}{2}a^{a;b}{}_{;b}\,. \qquad (3.57)$$

Direct calculation from Eq. (3.47) yields

$$A^{ab}{}_{;b}=\tfrac{1}{2}\ddot{a}^a+\tfrac{1}{2}\,\theta\,\dot{a}^a+\{\tfrac{1}{4}(\mu-p)-\tfrac{1}{6}\,\dot{\theta}-\tfrac{2}{9}\,\theta^2\}\,a^a+\tfrac{1}{6}h^{ac}\,(a^b{}_{;b})_{,c}-\tfrac{1}{3}\,\theta\,a^b{}_{;b}\,u^a+\tfrac{1}{2}a^{a;b}{}_{;b}\,. \qquad (3.58)$$

Putting Eqs. (3.57) and (3.58) into Eq. (3.56) and using the background Raychaudhuri equation (3.3) gives

$$\tfrac{4}{3}\,\theta\,Q^a=\tfrac{2}{3}\{\dot{z}^a+\theta\,z^a+\tfrac{1}{2}x^a+\tfrac{3}{2}y^a-h^{ac}\,(a^b{}_{;b})_{,c}-\dot{\theta}\,a^a\}. \qquad (3.59)$$

It thus follows from Eq. (3.31) that

$$\theta\,Q^a=0. \qquad (3.60)$$

Since $\theta>0$ we must have

$$Q^a=0. \qquad (3.61)$$

It now follows from Eqs. (3.37) and (3.38) that

$$x^a=0, \qquad (3.62)$$

and so Eq. (3.39) yields

$$y^a=0. \tag{3.63}$$

Now Eq. (3.32) with Eq. (3.45) becomes

$$\dot\phi\, z^c=a^d{}_{;d}\,\lambda^c, \tag{3.64}$$

while Eq. (3.36) reduces to

$$(\mu+p)\,(z^c+\theta\, a^c)=0. \tag{3.65}$$

Contracting Eq. (3.64) with $\phi_{,c}$ and using Eq. (3.44) results in

$$z^c\,\phi_{,c}=\dot\phi\, a^d{}_{;d}\,, \tag{3.66}$$

and, noting Eq. (3.45) again, contracting Eq. (3.65) with $\phi_{,c}$ gives

$$\dot\phi\,(\mu+p)\,a^d{}_{;d}=0. \tag{3.67}$$

With $\dot\phi\neq 0,\mu+p\neq 0$ we must have $a^d{}_{;d}=0$ and so Eq. (3.64) becomes

$$z^c=0. \tag{3.68}$$

Now Eq. (3.65) with $\theta>0$ and $\mu+p\neq 0$ yields

$$a^c=0. \tag{3.69}$$

It now follows from Eq. (3.28) that

$$w^a=0. \tag{3.70}$$

At this stage the only surviving gauge-invariant small quantities from the list (3.6) are σ^{ab} and π^{ab} or equivalently s^{ab},Π^{ab}. We see that now Eqs. (3.25) and (3.35) require

$$s^{ab}\,\phi_{,b}=0,\qquad \Pi^{ab}\,\phi_{,b}=0. \tag{3.71}$$

Also Eqs. (3.26) and (3.34) require s^{ab}, Π^{ab} to satisfy

$$s^{ab}{}_{;b}=0,\qquad \Pi^{ab}{}_{;b}=0. \tag{3.72}$$

We will write the remaining equations from Eqs. (3.13)–(3.38), which have not reduced to $0=0$, in terms of s^{ab},Π^{ab}. We will then check that the equations we obtain for s^{ab},Π^{ab} [including Eqs. (3.71) and (3.72)] are consistent. We begin with Eq. (3.20) with $Q^a=0$ and substitute into it p^{ab} from Eq. (3.8) with $a^a=0$ and m^{ab} from Eq. (3.9) with $a^a=0$ to obtain

$$\dot\phi\,\Pi^{bt}=2\,\dot\phi\,\dot s^{bt}+\tfrac{5}{3}\theta\,\dot\phi\, s^{bt}+\ddot\phi\, s^{bt} -u_r\,(q^{(b}{}_s\,\phi_{,d}+l^{(b}{}_{s;d})\,\eta^{t)rsd}. \tag{3.73}$$

With q^{ab} given by Eq. (3.11) with $w^a=0$ we calculate

$$\eta^{brsd}\,q^t{}_s\,u_r\,\phi_{,d}=-\dot s^{tp}\,\phi_{,p}\,u^b+s^{tb;p}\,\phi_{,p}+\dot\phi\,\dot s^{tb} -s^{tp;b}\,\phi_{,p}+\tfrac{1}{3}\theta\,\dot\phi\, s^{tb}. \tag{3.74}$$

Next using l^{ab} in Eq. (3.12) with $w^a=0$ we find

$$\eta^{brsd}\,u_r\,l_{st}=-\dot\phi\, s^d{}_t\,u^b+s^b{}_t\,\phi^{,d}+\dot\phi\, s^b{}_t\,u^d-s^d{}_t\,\phi^{,b}. \tag{3.75}$$

The first of Eqs. (3.72) and the background expression $u^d{}_{;d}=\theta$ help us to deduce from Eq. (3.75) that

$$\eta^{brsd}\,u_r\,l^t{}_{s;d}=-\dot\phi_{,d}\,s^{td}\,u^b-\tfrac{1}{3}\theta\dot\phi\, s^{tb}+s^{tb}{}_{;d}\,\phi^{,d}+s^{tb}\,\phi^{,d}{}_{;d} +\ddot\phi\, s^{tb}+\dot\phi\dot s^{tb}+\theta\,\dot\phi\, s^{tb}-\phi^{,b}{}_{;d}\,s^{td}. \tag{3.76}$$

Putting Eqs. (3.74) and (3.76) together we get

$$\eta^{brsd}\,u_r\,(q^t{}_s\,\phi_{,d}+l^t{}_{s;d})=2\,s^{tb;d}\,\phi_{,d}+\phi^{,d}{}_{;d}\,s^{tb}+2\,\dot\phi\,\dot s^{tb} +\ddot\phi\, s^{tb}+\theta\,\dot\phi\, s^{tb}, \tag{3.77}$$

which is symmetric in (t,b). Substituting this into Eq. (3.73) gives the equation

$$s'_{tb}+(\tfrac{1}{2}\phi^{,d}{}_{;d}-\tfrac{1}{3}\theta\,\dot\phi)s_{tb}=-\tfrac{1}{2}\dot\phi\,\Pi_{tb}, \tag{3.78}$$

with $s'_{tb}:=s_{tb;d}\,\phi^{,d}$. *This is a propagation equation for s_{tb} along the null geodesics tangent to $\phi^{,d}$.*

We now turn our attention to Eq. (3.23). Substituting for p^{ab},m^{ab} from Eqs. (3.8) and (3.9) with $a^a=0$ we see that

$$p^b{}_s\,\phi_{,d}+m^b{}_{s;d}=-(\phi_{,d}\,s^b{}_s)^{\cdot}-\theta\,\phi_{,d}\,s^b{}_s -\tfrac{1}{3}\theta\,\dot\phi\, u_d\,s^b{}_s-\dot\phi\, s^b{}_{s;d}\,. \tag{3.79}$$

We notice in passing from this that now Eq. (3.14), which reads

$$p^{ab}\,\phi_{,b}+m^{ab}{}_{;b}=0, \tag{3.80}$$

on account of the second of Eqs. (3.71), is automatically satisfied because of the first equation in Eqs. (3.71) and in Eq. (3.72). Writing out Eq. (3.23) with q^{ab} substituted from Eq. (3.11) with $w^a=0$ we have

$$\dot l^{bt}+\theta\, l^{bt}-u_r\,(\dot\phi\, s^{(b}{}_{s;d}+p^{(b}{}_s\,\phi_{,d}+m^{(b}{}_{s;d})\,\eta^{t)rsd}=0. \tag{3.81}$$

Now using Eq. (3.79) we can write this as

$$\dot l^{bt}+\theta\, l^{bt}+u_r\{(\phi_{,d}\,s^{(b}{}_s)^{\cdot}+\theta\,\phi_{,d}\,s^{(b}{}_s\}\,\eta^{t)rsd}=0. \tag{3.82}$$

This can be rearranged as

$$(l^{bt}+u_r\,\phi_{,d}\,s^{(b}{}_s\,\eta^{t)rsd})^{\cdot}+\theta\,(l^{bt}+u_r\,\phi_{,d}\,s^{(b}{}_s\,\eta^{t)rsd})=0. \tag{3.83}$$

With l^{bt} given in Eq. (3.12) with $w^a=0$ we see that Eq. (3.83) is identically satisfied.

We next examine Eq. (3.22). It is identically satisfied and this can be seen as follows: with p^{ab} in Eq. (3.8) and $a^a=0$, we find

$$p^b{}_{s;d}=-\dot s^b{}_{s;d}+\tfrac{2}{3}\dot\theta\, u_d\,s^b{}_s-\tfrac{2}{3}\theta\, s^b{}_{s;d}\,. \tag{3.84}$$

From the Ricci identities satisfied by s^{ab} we have

$$\dot{s}^b{}_{s;d} = (s^b{}_{s;d})^{\cdot} + (\tfrac{1}{6}\mu + \tfrac{1}{2}p)(u^b s_{ds} + s^b{}_d u_s) + \tfrac{1}{3}\theta(s^b{}_{s;d} + \dot{s}^b{}_s u_d). \quad (3.85)$$

This allows us to write

$$\eta^{trsd} u_r p^b{}_{s;d} = -(\eta^{trsd} u_r s^b{}_{s;d})^{\cdot} - \theta\, \eta^{trsd} u_r s^b{}_{s;d}. \quad (3.86)$$

When this is entered into Eq. (3.22) the equation can be rearranged as

$$(q^{bt} + u_r s^{(b}{}_{s;d}\, \eta^{t)rsd})^{\cdot} + \theta(q^{bt} + u_r s^{(b}{}_{s;d}\, \eta^{t)rsd}) = 0, \quad (3.87)$$

which is an identity on account of (3.11) with $w^a = 0$.

We now consider Eq. (3.19) with $Q^a = 0$. If we first substitute for p^{ab} into it from Eq. (3.8) with $a^a = 0$ we obtain

$$-\ddot{s}^{bt} - \tfrac{5}{3}\theta\dot{s}^{bt} - \tfrac{2}{3}\dot{\theta}s^{bt} - \tfrac{2}{3}\theta^2 s^{bt} + \tfrac{1}{2}(\mu + p)s^{bt} + u_r q^{(b}{}_{s;d}\, \eta^{t)rsd} = -\dot{\Pi}^{bt} - \tfrac{2}{3}\theta\Pi^{bt}. \quad (3.88)$$

However using q^{ab} in Eq. (3.11) with $w^a = 0$ we find that

$$\eta^{brsd} u_r q^t{}_s = -\tfrac{1}{3}\theta s^{td} u^b + \tfrac{1}{3}\theta s^{tb} u^d - \dot{s}^{td} u^b + \dot{s}^{bt} u^d - s^{td;b} + s^{tb;d}, \quad (3.89)$$

and thus, using the first of Eqs. (3.72),

$$\eta^{brsd} u_r q^t{}_{s;d} = \tfrac{2}{9}\theta^2 s^{tb} + \tfrac{1}{3}\dot{\theta}s^{tb} + \ddot{s}^{tb} + \theta\dot{s}^{tb} - s^{td;b}{}_{;d} + s^{tb;d}{}_{;d}. \quad (3.90)$$

The second to last term here can be simplified using the Ricci identities satisfied by s^{ab} and the first of Eqs. (3.72) to read

$$s^{td;b}{}_{;d} = (\tfrac{5}{6}\mu - \tfrac{1}{2}p)s^{tb}. \quad (3.91)$$

We see that now Eq. (3.90) is symmetric in (b,t) and on substitution into Eq. (3.88) we arrive at *a wave equation for* s^{ab}, namely,

$$s^{ab;d}{}_{;d} - \tfrac{2}{3}\theta\dot{s}^{ab} - (\tfrac{1}{3}\dot{\theta} + \tfrac{4}{9}\theta^2)s^{ab} + (p - \tfrac{1}{3}\mu)s^{ab} = -\dot{\Pi}^{ab} - \tfrac{2}{3}\theta\Pi^{ab}. \quad (3.92)$$

With $w^a = 0 = Q^a$ we have from Eq. (3.18) that

$$q^{ab}{}_{;b} = 0. \quad (3.93)$$

This equation is satisfied by q^{ab} given by Eq. (3.11) with $w^a = 0$. Substituting the latter expression for q^{ab} into the left-hand side of Eq. (3.93) and using Eq. (3.91) we have

$$q^{ab}{}_{;b} = -\tfrac{1}{4}(s^a{}_{p;cb} - s^a{}_{p;bc})\,\eta^{bcfp} u_f, \\ = -\tfrac{1}{4}(R^a{}_{gcb}s^g{}_p + R_{pgcb}s^{ag})\,\eta^{bcfp} u_f, \quad (3.94)$$

using the Bianchi identities. Here $R^a{}_{gcb}$ are the components of the Riemann tensor of the isotropic background. These components can be easily written in terms of the perfect-fluid energy-momentum-stress tensor of the background because the background, being isotropic, is conformally flat. When this is done it is found that each of the Riemann tensor terms in Eq. (3.94) separately vanishes and so Eq. (3.93) is satisfied.

With the second of Eq. (3.72) holding we have from Eq. (3.15)

$$p^{ab}{}_{;b} = 0. \quad (3.95)$$

That this is satisfied by p^{ab} given by Eq. (3.8) with $a^a = 0$ is straightforward when one notes that $\dot{s}^b{}_{s;b} = (s^b{}_{;b})^{\cdot} = 0$, which follows from Eq. (3.85) after summation over (b,d) and the properties of s^{ab}.

Finally we must check Eq. (3.17) with $Q^a = 0$. This reads

$$q^d{}_b\,\phi_{,d} + l^d{}_{b;d} = 0. \quad (3.96)$$

To see that q^{ab} and l^{ab} satisfy this equation we start with Eq. (3.77) and multiply it by η_{bpqt}. Since the right-hand side of Eq. (3.77) is symmetric in (t,b) we obtain

$$(q^m{}_p\,\phi_{,m} + l^m{}_{p;m})u_q - (q^m{}_q\,\phi_{,m} + l^m{}_{q;m})u_p = 0. \quad (3.97)$$

If this is multiplied by u^p then Eq. (3.96) results.

At this point all of the equations (3.13)–(3.38) are satisfied provided s^{ab}, Π^{ab} satisfy the algebraic relations with $\phi_{,a}$ in Eq. (3.71), are divergence-free as indicated in Eq. (3.72) and satisfy the propagation equation (3.78) for s^{ab} along the integral curves of $\phi^{,a}$ in the background space-time and satisfy the wave equation (3.92). These equations reduce to those obtained in [5] when $\Pi^{ab} = 0$. We need to check that Eqs. (3.71), (3.72), (3.78) and (3.92) are consistent.

Using the Bianchi identities we can show that

$$(s^{ab;d}{}_{;d})_{;b} = (s^{ab}{}_{;b})^{;d}{}_{;d} + (\tfrac{7}{6}\mu - \tfrac{1}{2}p)s^{ab}{}_{;b}, \quad (3.98)$$

$$\dot{s}^{ab}{}_{;b} = (s^{ab}{}_{;b})^{\cdot} + \tfrac{1}{3}\theta s^{ab}{}_{;b}, \quad (3.99)$$

$$\dot{\Pi}^{ab}{}_{;b} = (\Pi^{ab}{}_{;b})^{\cdot} + \tfrac{1}{3}\theta\Pi^{ab}{}_{;b}. \quad (3.100)$$

With the help of these equations it is straightforward to see that the wave equation (3.92) is consistent with Eq. (3.72). Also using

$$s^{ab;d}{}_{;d}u_b = (s^{ab}u_b)^{;d}{}_{;d} - \tfrac{2}{3}\theta s^{ab}{}_{;b}, \quad (3.101)$$

one can easily see that the wave equation (3.92) is consistent with $s^{ab}u_b = 0 = \Pi^{ab}u_b$. The wave equation (3.92) is also consistent with $s^{ab}\phi_{,b} = 0 = \Pi^{ab}\phi_{,b}$. This follows from

$$s^{ab;d}{}_{;d}\,\phi_{,b} = -2(s^{ab;d}\phi_{,d})_{;b} - s^{ab}(\phi^{,d}{}_{;d})_{,b}, \quad (3.102)$$

which is obtained using the Bianchi identities satisfied by s^{ab} and $\phi_{,a}$. The propagation equation (3.78) allows us to write Eq. (3.102) as

$$s^{ab;d}{}_{;d}\,\phi_{,b}=-\tfrac{2}{3}\theta\,\dot{\phi}_{,b}\,s^{ab}+\dot{\phi}_{,b}\Pi^{ab}. \tag{3.103}$$

One can also derive this by multiplying the wave equation (3.92) by $\phi_{,b}$ and using the fact that $(\phi_{,b})^{\cdot}=\dot{\phi}_{,b}-\frac{1}{3}\,\theta\,\lambda_b$ with, as always, $\lambda_b=h_b^c\,\phi_{,c}$.

The propagation equation (3.78) is clearly consistent with Eq. (3.71) because the integral curves of $\phi^{,a}$ are geodesics. That Eq. (3.78) is also consistent with $s^{ab}\,u_b=0=\Pi^{ab}\,u_b$ follows from $u'_b=u_{b;c}\,\phi^{,c}=\frac{1}{3}\,\theta\,\lambda_b$. The consistency of Eq. (3.78) with Eq. (3.72) requires the wave equation (3.92). This is because on taking the divergence of Eq. (3.78) and using the Ricci identities one arrives at

$$-\tfrac{1}{2}\dot{\phi}\,\Pi^{ab}{}_{;b}+\tfrac{1}{2}\phi_{,b}\,\dot{\Pi}^{ab}=-\tfrac{1}{2}\phi_{,b}\,s^{ab;d}{}_{;d}+\tfrac{1}{3}\theta\,\dot{s}^{ab}\,\phi_{,b}. \tag{3.104}$$

Substituting for $s^{ab;d}{}_{;d}$ here from the wave equation (3.92), this equation reduces to $\Pi^{ab}{}_{;b}=0$.

IV. PURE GRAVITY WAVE PERTURBATIONS

As a result of the calculations outlined in Sec. III the perturbations of the Weyl tensor now have "electric" and "magnetic" parts given by

$$E_{ab}=\left(\tfrac{1}{2}\Pi_{ab}+p_{ab}\right)F+m_{ab}\,F', \tag{4.1}$$

$$H_{ab}=q_{ab}\,F+l_{ab}\,F', \tag{4.2}$$

with

$$p_{ab}=-\dot{s}_{ab}-\tfrac{2}{3}\theta\,s_{ab},\qquad m_{ab}=-\dot{\phi}\,s_{ab}, \tag{4.3}$$

$$q_{ab}=-s_{(a}{}^{p;c}\,\eta_{b)fpc}\,u^f,\qquad l_{ab}=-s_{(a}{}^{p}\,\eta_{b)fpc}\,u^f\,\phi^{,c}. \tag{4.4}$$

Also s^{ab},Π^{ab} satisfy the consistent equations (3.71), (3.72), (3.78) and (3.92) and $\phi^{,a}$ is a null vector field in the background RW space-time. From the first of Eqs. (3.71) we see from Eqs. (4.3) and (4.4) that

$$m^{ab}\,\phi_{,b}=0,\qquad l^{ab}\,\phi_{,b}=0, \tag{4.5}$$

verifying that Eqs. (3.13) and (3.16) are satisfied. Thus the F'-parts of E_{ab},H_{ab} above are type N in the Petrov classification with degenerate principal null direction $\phi^{,a}$. We therefore consider the F'-part of this perturbed field as describing gravitational waves having propagation direction $\phi^{,a}$ in the RW background and the histories of the wave fronts are the null hypersurfaces $\phi(x^a)=\text{const}$. This interpretation is based on the well-known analogy with electromagnetic radiation [10]. The F-parts of E_{ab},H_{ab} are not in general type N and so do not necessarily describe gravitational waves.

For the remainder of this paper we will consider pure type N perturbations (i.e. pure gravity wave perturbations) of the RW background. We could do this by requiring the F-parts of E_{ab},H_{ab} in Eqs. (4.1) and (4.2) to vanish. It is possible to exhibit solutions of our basic equations (3.71), (3.72), (3.78) and (3.92) having this property (see Sec. VI below) but it is less restrictive to require that the F-parts of E_{ab},H_{ab} also be type N in the Petrov classification with $\phi^{,a}$ as degenerate principal null direction. This means that, in the light of the second of Eqs. (3.71), we should require

$$p^{ab}\,\phi_{,b}=0,\qquad q^{ab}\,\phi_{,b}=0, \tag{4.6}$$

with p^{ab},q^{ab} given above in Eqs. (4.3) and (4.4). The first of these can be written

$$s^{ab}\,\phi_{,b;c}\,u^c=0, \tag{4.7}$$

while the second gives us

$$s^{ab}\,\phi_{,a}{}^{;c}-s^{ac}\,\phi_{,a}{}^{;b}=0. \tag{4.8}$$

To elucidate the meaning of Eqs. (4.7) and (4.8) it is convenient to make use of a null tetrad in the background RW space-time. First we note that $k_a=-\dot{\phi}^{-1}\phi_{,a}$ and $l_a=u_a-\frac{1}{2}k_a$ are two real covariant vector fields satisfying $k_a\,k^a=0$, $l_a\,l^a=0$ and $k_a\,l^a=-1$. Let $m_a,\bar{m}_a$ be a complex covariant vector field and its complex conjugate (indicated by a bar) chosen so that they are null ($m_a\,m^a=0=\bar{m}_a\,\bar{m}^a$), are orthogonal to k^a and l^a and satisfy $m_a\,\bar{m}^a=1$. Now k^a, l^a, m^a, $\bar{m}^a$ constitute a null tetrad with respect to which s^{ab} can be written (because $s^a{}_a=0$, $s^{ab}\,u_b=0$, $s^{ab}\,k_b=0$ and so $s^{ab}\,l_b=0$)

$$s^{ab}=\bar{s}\,m^a\,m^b+s\,\bar{m}^a\,\bar{m}^b. \tag{4.9}$$

Thus $|s|^2=\frac{1}{2}\,s^{ab}\,s_{ab}$. Substituting Eq. (4.9) into Eq. (4.7) we easily see that Eq. (4.7) is equivalent to

$$s\,\phi_{,b;c}\,\bar{m}^b\,l^c=0, \tag{4.10}$$

from which we conclude that provided $s\neq 0$ ($\Leftrightarrow s^{ab}\neq 0$) we must have

$$\phi_{,b;c}\,\bar{m}^b\,l^c=0. \tag{4.11}$$

On using Eq. (4.9) in Eq. (4.8) we find that, in addition to Eq. (4.11),

$$\bar{s}\,\phi_{,a;b}\,m^a\,m^b=s\,\phi_{,a;b}\,\bar{m}^a\,\bar{m}^b. \tag{4.12}$$

A simple way to satisfy this with $s\neq 0$ is to require the null hypersurfaces $\phi(x^a)=\text{const}$ to satisfy

$$\phi_{,a;b}\,m^a\,m^b=0. \tag{4.13}$$

This means that *the complex shear of the null geodesic congruence tangent to $\phi^{,a}$ in the background RW space-time vanishes*. If we can find a family of null hypersurfaces $\phi(x^a)=\text{const}$ in the background space-time satisfying Eqs. (4.11) and (4.13) then Eq. (4.6) will be satisfied. The solutions we then obtain of Eqs. (3.71), (3.72), (3.78) and (3.92) will be analogous to the Bateman waves [6] of electromagnetic theory.

V. EXPLICIT EXAMPLES

To exhibit explicit examples of pure type N perturbations of the RW space-times we first select in such space-times some naturally occurring shear-free null hypersurfaces $\phi(x^a)=\text{const}$. We begin with the general Robertson-Walker line-element in standard form,

$$ds^2=R^2(t)\,\frac{[(dx^1)^2+(dx^2)^2+(dx^3)^2]}{\left(1+\frac{k}{4}r^2\right)^2}-dt^2, \quad (5.1)$$

where $R(t)$ is the scale factor, $r^2=(x^1)^2+(x^2)^2+(x^3)^2$ and $k=0,\pm 1$ is the Gaussian curvature of the spacelike hypersurfaces $t=\text{const}$. We can put these line-elements in the following interesting forms for our purposes [11]:

$$ds^2=R^2(t)\,\{dx^2+p_0^{-2}f^2(dy^2+dz^2)\}-dt^2, \quad (5.2)$$

with $p_0=1+(K/4)(y^2+z^2), K=\text{const}, f=f(x)$. The following cases arise: (i) if $k=+1$ then $K=+1$ and $f(x)=\sin x$; (ii) if $k=0$ then $K=0,+1$ with $f(x)=1$ when $K=0$ and $f(x)=x$ when $K=+1$; (iii) if $k=-1$ then $K=0,\pm 1$ with $f(x)=\frac{1}{2}e^x$ when $K=0$, $f(x)=\sinh x$ when $K=+1$ and $f(x)=\cosh x$ when $K=-1$.

Case (i) above arises because when $k=+1$ the closed model universe with line-element (5.1) has $t=\text{const}$ hypersurfaces with line-element which can be put in the form $dl^2=R^2(t)\,ds_0^2$ with

$$ds_0^2=dx^2+\sin^2 x\,(d\vartheta^2+\sin^2\vartheta\, d\varphi^2), \quad (5.3)$$

and we then use stereographic coordinates y,z such that $y+iz=2\, e^{i\varphi}\cot(\vartheta/2)$ in place of the polar angles ϑ,φ.

Case (ii) arises because when $k=0$ the open, spatially flat universe with line-element (5.1) has $t=\text{const}$ hypersurfaces with line-element $dl^2=R^2(t)\,ds_0^2$ where

$$ds_0^2=dx^2+dy^2+dz^2, \quad (5.4)$$

or

$$ds_0^2=dx^2+x^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2), \quad (5.5)$$

and in the latter we introduce the stereographic coordinates y,z again in place of ϑ,φ.

Case (iii) is due to the fact that in Eq. (5.1) when $k=-1$ the $t=\text{const}$ hypersurfaces can, modulo the factor $R^2(t)$, each be viewed as the future sheet of a unit timelike hypersphere $\mathcal{H}_3$ in four dimensional Minkowskian space-time $\mathcal{M}_4$. Thus if the line-element of $\mathcal{M}_4$ is written

$$ds_0^2=(dz^1)^2+(dz^2)^2+(dz^3)^2-(dz^4)^2, \quad (5.6)$$

then $\mathcal{H}_3$ is given by

$$(z^1)^2+(z^2)^2+(z^3)^2-(z^4)^2=-1, \qquad z^4>0. \quad (5.7)$$

The different parts of case (iii) above are due to the different ways one can parametrize Eq. (5.7). One possibility is with

$$z^1+iz^2=(y+iz)\,p_0^{-1}\sinh x, \quad (5.8)$$

$$z^3=\left(\tfrac{1}{4}(y^2+z^2)-1\right)p_0^{-1}\sinh x, \quad (5.9)$$

$$z^4=\cosh x, \quad (5.10)$$

with $p_0=1+\frac{1}{4}(y^2+z^2)$. Substitution into Eq. (5.6) gives

$$ds_0^2=dx^2+p_0^{-2}\sinh^2 x\,(dy^2+dz^2). \quad (5.11)$$

The next possibility is

$$z^1+iz^2=\tfrac{1}{2}e^x\,(y+iz), \quad (5.12)$$

$$z^3=\tfrac{1}{4}e^x\,(y^2+z^2-1)+e^{-x}, \quad (5.13)$$

$$z^4=\tfrac{1}{4}e^x\,(y^2+z^2+1)+e^{-x}, \quad (5.14)$$

and now Eq. (5.6) reads

$$ds_0^2=dx^2+\tfrac{1}{4}\,e^{2x}\,(dy^2+dz^2). \quad (5.15)$$

Finally we can take

$$z^1+iz^2=(y+iz)\,p_0^{-1}\cosh x, \quad (5.16)$$

$$z^3=\sinh x, \quad (5.17)$$

$$z^4=\left(\tfrac{1}{4}(y^2+z^2)+1\right)p_0^{-1}\cosh x, \quad (5.18)$$

with $p_0=1-\frac{1}{4}(y^2+z^2)$. With this Eq. (5.6) takes the form

$$ds_0^2=dx^2+p_0^{-2}\cosh^2 x\,(dy^2+dz^2). \quad (5.19)$$

In the space-times with line-elements (5.2), with the special cases outlined following Eq. (5.2), the hypersurfaces

$$\phi(x^a):=x-T(t)=\text{const}, \quad (5.20)$$

with $dT/dt=R^{-1}$ are *null hypersurfaces*. They are generated by null geodesics having expansion

$$\tfrac{1}{2}\,\phi^{,a}{}_{;a}=\frac{f'}{R^2 f}+\frac{\dot R}{R^2}. \quad (5.21)$$

Here $f'=df/dx, \dot R=dR/dt$. The integral curves of the vector field $\partial/\partial t$ are the world lines of the fluid particles. The components of this vector field are denoted by u^a and using Eq. (5.20) we can show that

$$2\,\phi_{,a;b}=\xi_a\,\phi_{,b}+\xi_b\,\phi_{,a}+\phi_{,d}{}^{;d}\,g_{ab}, \quad (5.22)$$

with

$$\xi_a=-\frac{f'}{f}\,\phi_{,a}+R\,\phi_{,d}{}^{;d}\,u_a. \quad (5.23)$$

With $s^{ab}\,\phi_{,b}=0=s^{ab}\,u_b$ we see on substituting Eq. (5.22) that Eqs. (4.7) and (4.8) are now satisfied. On account of Eq. (5.22) it follows that $\phi_{,a}$ *is shear-free* [12]. Alternatively we can easily verify Eqs. (4.11) and (4.13) using the null tetrad described following Eq. (4.8) which is given via the 1-forms

$$k_a\,dx^a = R\,dx - dt, \qquad l_a\,dx^a = -\tfrac{1}{2}(R\,dx + dt),$$

$$m_a\,dx^a = \frac{1}{\sqrt{2}} R\,p_0^{-1} f\,(dy + i\,dz). \tag{5.24}$$

For convenience we have used the same coordinate labels $\{x,y,z,t\}$ for all of the special cases included in Eq. (5.2). Of course the ranges of some of these coordinates will be different in the different cases [for example, in case (ii) $x \in (-\infty, \ +\infty)$ if $K=0$ whereas $x \in [0,+\infty)$ if $K=+1$]. Similarly the shear-free null hypersurfaces (5.20) differ from case to case, and within cases (ii) and (iii), as can be seen by noting the intersections of these null hypersurfaces with the spacelike hypersurfaces $t=$const. In case (i) the intersection is a 2-sphere. In case (ii) it is a 2-sphere if $K=+1$ and a 2-plane if $K=0$. Thus Eq. (5.20) describes two quite different families of shear-free null hypersurfaces that can arise in an open, spatially flat universe. In case (iii) the intersection of Eq. (5.20) with $t=$const can be a 2-space of positive ($K=+1$), negative ($K=-1$) or zero ($K=0$) curvature giving three different families of shear-free null hypersurfaces in a $k=-1$ open universe. A geometrical explanation for these subcases is given in [11].

We begin with $\phi(x^a)$ given by Eq. (5.20) and $u^a\partial/\partial x^a = \partial/\partial t$. Since s^{ab} and Π^{ab} are orthogonal to u^a and $\phi^{,a}$ and trace-free, with respect to the metric tensor given via the line-element (5.2), each have only two independent components. If the coordinates are labeled $x^1=x, x^2=y, x^3=z, x^4=t$ then the surviving components are $s^{33}=-s^{22}=\alpha(x,y,z,t), s^{23}=s^{32}=\beta(x,y,z,t)$ and $\Pi^{33}=-\Pi^{22}=A(x,y,z,t), \Pi^{23}=\Pi^{32}=B(x,y,z,t)$. We can conveniently express these on the null tetrad (5.24). We have s^{ab} given by Eq. (4.9) with

$$\bar{s} = -R^2 p_0^{-2} f^2(\alpha + i\,\beta), \tag{5.25}$$

and

$$\Pi^{ab} = \bar{\Pi}\,m^a\,m^b + \Pi\,\bar{m}^a\,\bar{m}^b, \tag{5.26}$$

with

$$\bar{\Pi} = -R^2 p_0^{-2} f^2(A + i\,B). \tag{5.27}$$

Calculation of the first of Eqs. (3.72) to be satisfied by s^{ab} shows that α,β must satisfy the Cauchy-Riemann equations

$$\frac{\partial}{\partial y}(p_0^{-4}\alpha) - \frac{\partial}{\partial z}(p_0^{-4}\beta) = 0, \tag{5.28}$$

$$\frac{\partial}{\partial y}(p_0^{-4}\beta) + \frac{\partial}{\partial z}(p_0^{-4}\alpha) = 0. \tag{5.29}$$

In addition Π^{ab} and thus A,B must satisfy the same equations,

$$\frac{\partial}{\partial y}(p_0^{-4}A) - \frac{\partial}{\partial z}(p_0^{-4}B) = 0, \tag{5.30}$$

$$\frac{\partial}{\partial y}(p_0^{-4}B) + \frac{\partial}{\partial z}(p_0^{-4}A) = 0. \tag{5.31}$$

We shall find it convenient to work with α_0,β_0 rather than α,β where

$$\alpha_0 = \alpha\,f^3R^3, \qquad \beta_0 = \beta\,f^3R^3. \tag{5.32}$$

Since $f=f(x),\ R=R(t)$ we have Eqs. (5.28) and (5.29) satisfied by α_0 and β_0 and these equations can be written economically as

$$\frac{\partial}{\partial \bar{\zeta}}\{p_0^{-4}(\alpha_0 + i\,\beta_0)\} = 0, \tag{5.33}$$

with $\zeta = y + i\,z$, giving

$$\alpha_0 + i\,\beta_0 = p_0^4\,\mathcal{G}(\zeta,x,t), \tag{5.34}$$

where $\mathcal{G}$ is an analytic function of ζ. Now Eq. (5.25) reads

$$\bar{s} = -R^{-1}p_0^2 f^{-1}\mathcal{G}(\zeta,x,t). \tag{5.35}$$

The propagation equation (3.78) for s^{ab} along the integral curves of $\phi^{,a}$ gives A,B in terms of α_0,β_0. Writing this in terms of Π given by Eq. (5.27) we find that

$$\bar{\Pi} = -2\,R^{-2}p_0^2 f^{-1}(D\mathcal{G} + \dot{R}\,\mathcal{G}). \tag{5.36}$$

Here the operator D is given by $D=\partial/\partial x + R\,\partial/\partial t = \partial/\partial x + \partial/\partial T$ with $T(t)$ introduced in Eq. (5.20). Also the dot indicates differentiation of $R(t)$. It follows from this and Eq. (5.27) that $A + i\,B$ is analytic in ζ and so Eqs. (5.30) and (5.31) are automatically satisfied. The only remaining equation to satisfy is the wave equation (3.92). With s^{ab} given by Eqs. (4.9) and (5.35) and with Π^{ab} given by Eqs. (5.26) and (5.36) we find, after a lengthy calculation, that Eq. (3.92) reduces to the remarkably simple wave equation

$$D^2\mathcal{G} + k\,\mathcal{G} = 0, \tag{5.37}$$

with $k=0,\pm 1$ labeling the RW backgrounds with line-elements of the form (5.2). Thus we have for $k=0$,

$$\mathcal{G}(\zeta,x,t) = a(\zeta,x-T)\,(x+T) + b(\zeta,x-T), \tag{5.38}$$

for $k=+1$,

$$\mathcal{G}(\zeta,x,t) = a(\zeta,x-T)\sin\left(\frac{x+T}{2}\right) + b(\zeta,x-T)\cos\left(\frac{x+T}{2}\right), \tag{5.39}$$

and for $k=-1$,

$$\mathcal{G}(\zeta,x,t)=a(\zeta,x-T)\sinh\left(\frac{x+T}{2}\right)+b(\zeta,x-T)\cosh\left(\frac{x+T}{2}\right), \qquad (5.40)$$

where in each case $a(\zeta,x-T)$, $b(\zeta,x-T)$ are arbitrary functions. In deriving Eq. (5.37) we have made use of the equations

$$f''=-k\,f, \qquad (f')^2+k\,f^2=K, \qquad (5.41)$$

which are satisfied in the cases (i)–(iii) described following Eq. (5.2) above.

With s^{ab} and Π^{ab} known we can calculate m^{ab}, l^{ab}, p^{ab} and q^{ab} in order to form the electric and magnetic parts of the perturbed Weyl tensor as indicated in Eqs. (4.1) and (4.2). We can write the result compactly as

$$E^{ab}+i\,H^{ab}=-2\,R^{-2}p_0^2f^{-1}\frac{\partial}{\partial x}(\mathcal{G}F)\,m^a\,m^b. \qquad (5.42)$$

We emphasize that $\mathcal{G}$ is given by Eqs. (5.38)–(5.40) in the various cases and now $F=F(x-T)$ so that $F'=\partial F/\partial x$. Also $p_0=1+(K/4)(y^2+z^2), f=f(x)$ described following Eq. (5.2), and $R(t)$ is the scale factor. It is immediately clear from Eq. (5.42) that the perturbations of the RW background which we have constructed here are pure gravity wave perturbations. We will discuss some of their properties in Sec. VI.

VI. DISCUSSION

The propagation equation (3.78) for s^{ab} along the null geodesics tangent to $\phi^{,a}$ shows that if $s^{ab}=0$ then $\Pi^{ab}=0$. An important converse property of the pure type N perturbations described in Sec. V is that *if* $\Pi^{ab}=0$ *then* $s^{ab}=0$ *provided* $\mu+p\neq 0$. To see this we have from Eq. (5.36) that $\Pi^{ab}=0$ is equivalent to

$$D\mathcal{G}+\dot{R}\,\mathcal{G}=0. \qquad (6.1)$$

Substituting this into the wave equation (5.37) results in

$$(\dot{R}^2-R\,\ddot{R}+k)\,\mathcal{G}=0. \qquad (6.2)$$

For the background RW space-time the fluid proper density μ and isotropic pressure p satisfy

$$\frac{2}{R^2}(\dot{R}^2-R\,\ddot{R}+k)=\mu+p, \qquad (6.3)$$

as a consequence of Einstein's field equations. Hence we can rewrite Eq. (6.2) simply as

$$(\mu+p)\,\mathcal{G}=0. \qquad (6.4)$$

From this and Eq. (5.35) it follows that $s^{ab}=0$ provided $\mu+p\neq 0$.

The perturbed Weyl tensor given via Eq. (5.42) for the pure type N perturbations can be infinite where $p_0(y,z)$ is infinite (when $y^2+z^2\rightarrow+\infty$ if $K\neq 0$) and where $f(x)$ vanishes [when $k=+1$ at $x=0$, when $k=-1$ at $x\rightarrow-\infty$ ($K=0$) or at $x=0$ ($K=0$)]. *There is one nonsingular case corresponding to* $k=0$, $K=0$ for which $p_0=1$ and $f=1$. In this case the expansion of the history of the wave fronts (5.21) is entirely due to the expansion of the universe. This case is as close as one can get to plane waves in the present context and is analogous to plane Bateman waves in electromagnetic theory.

Had we wished to construct examples of type N perturbations for which the F-parts of Eqs. (4.1) and (4.2) vanish we see from Eq. (5.42) that these would be given by Eq. (5.42) with $\partial\mathcal{G}/\partial x=0$. This condition would then be incorporated into the wave equation (5.37) and the appropriate solutions $\mathcal{G}(\zeta,t)$ replacing Eqs. (5.38)–(5.40) could easily be obtained.

We note that we have used the assumption $\theta>0$ in the background cosmological models to conclude from Eq. (3.60) that $Q^a=0$. If the background were an Einstein static universe then $\theta=0$ and we would have $Q^a\neq 0$. It is well-known (see, for example [13]) that the Einstein universe is unstable and it might be interesting to investigate how this instability manifests itself in the formalism we use in this paper.

There are exact cosmological solutions of Einstein's field equations known which contain gravitational waves (see [14,15] and references therein). These solutions describe universes with a stiff equation of state so that the speed of sound is equal to the speed of light. Our perturbations describing gravitational waves propagating through isotropic cosmologies place no restriction on the equation of state of the isotropic background and thus we would not in general expect them to approximate to these known exact solutions.

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