Wormholes supported by pure exotic radiation

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Traversible wormhole space-times are found as static, spherically symmetric solutions to the Einstein equations with ingoing and outgoing pure radiation of negative energy density. Switching off the radiation causes the wormhole to collapse to a Schwarzschild black hole.

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The idea of space-time wormholes originated with Wheeler [1], with traversible wormholes becoming popular after the article of Morris and Thorne [2]. More recently, renewed interest has focused on outstanding questions such as how to construct wormholes [3], their dynamical behavior [4], their stability [5,6] and the nature of the negative-energy matter needed to support them [7]. This paper proposes a very simple model of negative-energy radiation and exhibits resulting wormhole solutions, which can be used to study the other questions.

The simplest radiation model is pure radiation, also known as null dust, for which the energy tensor is $\rho u\otimes u$, where u is a null vector and ρ is the energy density (which, if positive, can be absorbed in the normalization of u). This represents incoherent radiation and occurs in the geometric-optics limit of massless particles such as photons. The matter model proposed here is simply pure radiation with negative energy density. One might expect this pure exotic radiation to be a similarly valid model of negative-energy radiation. More realistic models, such as semiclassical quantum fields such as those produced by Hawking radiation [8], are often not analytically tractable, so a simple model has theoretical merit.

With pure radiation of the usual positive energy density, the static, spherically symmetric Einstein system has been studied by Date [9], with solutions found by Kramer [10] and, more generally, by Gergely [11]. Reversing appropriate signs leads to the wormhole solutions derived below. However, the geometry is quite different, in particular being nonsingular. It is curious that benign wormhole solutions naturally appear under such situations.

Liberally adopting the method and notation of Gergely, the line element of a static, spherically symmetric space-time may be written as

$$ds^2=r^2d\Omega^2+f(r)^{-1}dr^2-h(r)dt^2 \tag{1}$$

where r is the areal radius, t is the time, $d\Omega^2$ refers to the unit sphere and the metric functions f and h are positive. The active gravitational mass-energy [12,13] of such a space-time is

$$m(r)=[1-f(r)]r/2. \tag{2}$$

The energy tensor of pure exotic radiation is $-\tau u\otimes u$, where u is a null vector and $\tau\geqslant 0$ is the negative energy density. With both ingoing and outgoing radiation, the energy tensor is

$$T=-\tau_+u_+\otimes u_+-\tau_-u_-\otimes u_- \tag{3}$$

where $u_\pm$ are null vectors. For the above metric, one may take

$$\sqrt{2}u_\pm=\frac{1}{\sqrt{h}}\frac{\partial}{\partial t}\pm\sqrt{f}\frac{\partial}{\partial r} \tag{4}$$

so that the null vectors are relatively normalized. Static solutions require the same negative energy density $\tau=\tau_\pm$ for both ingoing and outgoing radiation. This specialized matter model can also be regarded as an anisotropic fluid with density $-\tau$, radial pressure $-\tau$ and vanishing tangential pressure. Here and throughout, units are such that the speed of light and Newton's constant are unity.

The energy-momentum conservation equations $\nabla\cdot T=0$ reduce to

$$(hr^2\tau)'=0 \tag{5}$$

where the prime denotes d/dr. Thus the (gravitationally redshifted) radial tension or negative linear mass density

$$\lambda=4\pi r^2h\tau \tag{6}$$

is a positive constant. It is possible to absorb the magnitude of λ in the static Killing vector, but it will be retained here. In terms of the function

$$\beta=-2\lambda/h \tag{7}$$

the Einstein equations $G=8\pi T$ reduce to

$$rf'=1-\beta-f \tag{8}$$

$$rfh'=h(1+\beta-f) \tag{9}$$

$$rf\beta'=-\beta(1+\beta-f) \tag{10}$$

where one equation is redundant, having already solved the conservation equation. Transforming to $\rho=\ln r$ and using a dot to denote $d/d\rho$, the reduced system is

$$\dot{f}=1-\beta-f \tag{11}$$

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$$f\dot{\beta} = -\beta(1+\beta-f). \tag{12}$$

The second equation yields

$$f = \frac{1+\beta}{\beta-\dot{\beta}}\beta \tag{13}$$

which can be eliminated from the first equation to yield

$$(1+\beta)\beta\ddot{\beta} - (\beta+2)\dot{\beta}^2 + \beta(1-2\beta)\dot{\beta} + 2\beta^3 = 0. \tag{14}$$

Dividing by β^3, this becomes an exact differential, integrating to

$$\frac{1+\beta}{\beta^2}\dot{\beta} - \frac{1}{\beta} - 2\ln\frac{\beta}{r} = D \tag{15}$$

where D is the integration constant. This leads to the algebraic relation

$$\frac{(1+\beta)^2}{2f\beta} = \ln\left(-\frac{r}{a\beta}\right) \tag{16}$$

where $a = \exp[-(1+D)/2]$ is a positive constant. The branch of the logarithm corresponds to negative β, reflecting the negative energy density. Returning to the reduced system, one may eliminate ρ or r to give

$$\frac{df}{d\beta} = \frac{f(\beta+f-1)}{\beta(\beta-f+1)}. \tag{17}$$

Since solutions with negative β are being sought, the method of Gergely is now modified by defining

$$p = \sqrt{-2f\beta}, \quad l = \frac{1+\beta}{\sqrt{-2f\beta}}. \tag{18}$$

Then the above equation becomes

$$\frac{dp}{dl} = 2(pl-1). \tag{19}$$

Integrating the corresponding homogeneous equation yields

$$p = -2e^{l^2}\phi \tag{20}$$

and the solution is completed by

$$\phi(l) = \int_0^{\ell} e^{-\ell^2} dl + b = \frac{\sqrt{\pi}}{2}\operatorname{erf}(l) + b \tag{21}$$

where b is constant and erf denotes the error function, the symmetric integral of the standard normal distribution. The solution is mathematically familiar compared with the case of positive energy density. Then

$$\beta = pl - 1 = -1 - 2le^{l^2}\phi. \tag{22}$$

Collecting results, the class of solutions depends on the parameters (a,b) and the normalizable λ, and may be summarized by

$$r = -a\beta e^{-l^2} = a(e^{-l^2} + 2l\phi) \tag{23}$$

$$h = -\frac{2\lambda}{\beta} = \frac{2\lambda}{1+2le^{l^2}\phi} \tag{24}$$

$$f = -\frac{p^2}{2\beta} = \frac{2e^{2l^2}\phi^2}{1+2le^{l^2}\phi} \tag{25}$$

as substituted into the line element (1). By construction, this is the unique class of static, spherically symmetric solutions to the Einstein equations with pure exotic radiation.

It is straightforward to calculate

$$\frac{dr}{dl} = 2a\phi \tag{26}$$

$$m = (e^{-l^2} + 2l\phi - 2e^{l^2}\phi^2)a/2 \tag{27}$$

$$\frac{dm}{dl} = -(1+2le^{l^2}\phi)a\phi. \tag{28}$$

The line element may then be written explicitly in (t,l) coordinates as

$$ds^2 = a^2(e^{-l^2}+2l\phi)^2 d\Omega^2 + 2a^2 e^{-l^2}(e^{-l^2}+2l\phi)dl^2 - \frac{2\lambda dt^2}{1+2le^{l^2}\phi}. \tag{29}$$

In the case $b=0$, $\phi(l)$ is an odd function and the metric is even in the spatial coordinate l. This describes a symmetric wormhole with spatial topology $R\times S^2$ and minimal surfaces at the wormhole throat $l=0$, with radius $r=a$. The spacetime is not asymptotically flat, but otherwise constitutes a Morris-Thorne wormhole. There are no singularities, unlike in the corresponding positive-energy solutions. The $b\neq 0$ cases include asymmetric wormholes which are analogous to the asymmetric Ellis wormholes [14] for an exotic Klein-Gordon field.

The solutions may be written in dual-null form

$$ds^2 = r^2 d\Omega^2 - h\,dx^+ dx^- \tag{30}$$

in terms of null coordinates $x^\pm$ defined by

$$dx^\pm = dt \pm \frac{a}{\sqrt{\lambda}}(e^{-l^2} + 2l\phi)dl. \tag{31}$$

Integration by parts yields an analytic solution:

$$x^\pm = t \pm \frac{a}{2\sqrt{\lambda}}[le^{-l^2} + (1+2l^2)\phi]. \tag{32}$$

Thus the metric functions are implicitly known as functions of $x^+ - x^-$. Using the relations

$$\partial_\pm l = \pm \frac{\sqrt{\lambda}}{2a(e^{-l^2} + 2l\phi)} \tag{33}$$

the solutions have been checked by substitution in the dual-null form of the Einstein equations [12,13] for the general spherically symmetric line element (30) with non-zero energy components $T_{\pm\pm} = -h\tau/2$.

Returning to more general issues, the simplicity of pure radiation means that it is easier to study how the wormholes react to changes in the radiation level. This has not been analytically tractable previously except in a two-dimensional model [3]. In particular, the ingoing and outgoing radiation simply pass through one another without interaction, following null geodesics with propagation equations $\partial_\pm(hr^2\tau_\pm) = 0$. For instance, it is not difficult to see what happens if the radiation supporting the wormhole is turned off, as follows.

Consider a static wormhole for $\{x^+ < 0, x^- < 0\}$, with the radiation then switched off from both sides of the wormhole, so that $\tau_\mp = 0$ for $x^\pm > 0$, respectively. The solutions in the regions $\{x^\pm > 0, x^\mp < 0\}$ will be similar to Vaidya solutions [15], while the solution in the future region $\{x^+ > 0, x^- > 0\}$ will be vacuum and therefore, by a version of Birkhoff's theorem valid through the horizons of a black hole [12], Schwarzschild. Further, this region can be seen to be the interior of a Schwarzschild black hole with mass $a/2$, by differentiability of the metric at $x^+ = x^- = 0$. Specifically, continuity of the trapping horizons $\partial_\pm r = 0$ [12,13], which initially form the throat $\{x^+ = x^-, x^\pm < 0\}$ of the wormhole, means that they can be joined to the event horizons $\{x^\pm = 0, x^\mp > 0\}$ of a black hole. In summary, if the supporting radiation is switched off, the wormhole collapses to a Schwarzschild black hole.

This generally unexpected connection between wormholes and black holes was predicted by a general theory of both [4]. By close analogy to the results obtained in the two-dimensional model [3], it should be possible to demonstrate the dynamic stability of the wormholes, and how to construct them by irradiating a Schwarzschild black hole.

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