

Aspects of higher order gravity and holography

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Some thermodynamical properties of Lovelock gravity in the vanishing cosmological constant limit are discussed in several space-time dimensions, the holographic principle being one of the ingredients of the discussion. As it turns out, the area law and the brickwall method, though correct for the Einstein-Hilbert theory, may fail to work in general.

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I. INTRODUCTION

Since the work of Bekenstein and Hawking [1,2] our knowledge about black hole physics has improved quite considerably. Moreover, black hole physics is also the main gate towards understanding gravity in extreme conditions, and as a consequence, quantum gravity. This led 't Hooft and Susskind [3,4] to generalize the area law relating entropy and the area of a black hole to any gravitational system by means of the introduction of the holographic principle, which in the past few years turned into a powerful means of understanding the possible ways towards the quantization of gravity.

Under such a motivation, the holographic principle was put forward, suggesting that microscopic degrees of freedom that build up the gravitational dynamics do not reside in the bulk space-time but on its boundary [3,4]. This principle is a large conceptual change in our thinking about gravity. Maldacena's conjecture on AdS conformal field theory (CFT) correspondence [5] is the first example realizing such a principle. Subsequently, Witten [5] convincingly argued that the entropy, energy, and temperature of CFT at high temperatures can be identified with the entropy, mass, and Hawking temperature of the AdS black hole [6], which further supports the holographic principle. In cosmological settings, testing the holographic principle is somewhat subtle. Fischler and Susskind (FS) [7] have shown that for flat and open Friedmann-Lemaître-Robertson-Walker (FLRW) universes the area of the particle horizon should bound the entropy on the backward-looking light cone. However violation of the FS bound was found for closed FLRW universes. Various different modifications of the FS version of the holographic principle have been raised subsequently [8]. In addition to the study of holography in homogeneous cosmologies, attempts to generalize the holographic principle to a generic realistic inhomogeneous cosmological setting were carried out in [9].

It is now natural to ask which premises should be forcefully satisfied in order to accommodate the holographic principle. In particular, what kind of dynamics requires holography as an outcome. This could provide a mechanism for selecting the correct gravity dynamics leading towards formulating quantum gravity.

The study of the thermodynamic properties of black holes has been extended to higher order gravity theories [10]. We study here however a special class of gravitational actions, of higher order in the curvature, known as Lovelock gravity

[11]. Lovelock gravity is exceptional in the sense that although containing higher powers of the curvature in the Lagrange density, the resulting equations of motion contain no more than second derivatives of the metric. It is also a covariant and ghost free theory as it happens in the case of Einstein's general relativity.

An important result that was found in the thermodynamic context is that the area law is a peculiarity of the Einstein-Hilbert theory [12]. This fact motivates us to perform a deeper study of the thermodynamics of the black hole solutions of such exotic theories. In [12] gravitational theories are considered with the dimension d and the degree k of the curvature in the respective Lagrangian as parameters. We shall first briefly review such a formulation and later consider holography in this context. Further discussions concerning higher order gravity can be found in [13].

II. HIGHER DIMENSIONAL GRAVITY

The Lanczos-Lovelock action is a polynomial of degree $[d/2]$ in the curvature, which can be expressed in the language of forms as [12]

$$I_G = \kappa \int \sum_{m=0}^{[d/2]} \alpha_m L^{(m)}, \quad (1)$$

where α_m are arbitrary constants, and $L^{(m)}$ is given by

$$L^{(m)} = \epsilon_{a_1 \dots a_d} R^{a_1 a_2} \dots R^{a_{2m-1} a_{2m}} e^{a_{2m+1}} \dots e^{a_d}. \quad (2)$$

R^{ab} are the Riemann curvature two-forms given by

$$R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}. \quad (3)$$

Here ω_{ab} are the spin connection one-forms and e^a the vielbein. A wedge product between forms is understood throughout.

The corresponding field equations can be obtained varying with respect to e^a and ω^{ab} . In [12] the expression for the coefficients α_m was found requiring the existence of a unique cosmological constant. In such a case these theories are described by the action

$$I_k = \kappa \int \sum_{p=0}^k c_p^k L^{(p)}, \quad (4)$$

which corresponds to Eq. (1) with the choice

$$\alpha_p := c_p^k = \begin{cases} \frac{l^{2(p-k)}}{(d-2p)} \binom{k}{p}, & p \leq k, \\ 0, & p > k, \end{cases} \quad (5)$$

for the parameters, where $1 \leq k \leq [(d-1)/2]$. For a given dimension d , the coefficients c_m^k give rise to a family of inequivalent theories, labeled by k which represent the highest power of curvature in the Lagrangian. This set of theories possesses only two fundamental constants, κ and l , related respectively to the gravitational constant G_k and the cosmological constant Λ through

$$\kappa = \frac{1}{2(d-2)! \Omega_{d-2} G_k}, \quad (6)$$

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2}. \quad (7)$$

Since we are interested in the black hole solutions that are asymptotically flat we consider the vanishing cosmological constant limit case. When $l \rightarrow \infty$ the only nonvanishing term in Eq. (4) is the k th one; therefore the action is obtained from Eq. (1) with the choice of coefficients

$$\alpha_p := \tilde{c}_p^k = \frac{1}{(d-2k)} \delta_p^k, \quad (8)$$

in which case the action reads

$$\tilde{I}_k = \frac{\kappa}{(d-2k)} \int \epsilon_{a_1 \dots a_d} R^{a_1 a_2} \dots R^{a_{2k-1} a_{2k}} e^{a_{2k+1} \dots a_d}. \quad (9)$$

Note that for $k=1$ the Einstein action without cosmological constant is recovered, while for $k=2$ we obtained the Gauss-Bonnet action

$$I_2 = \frac{(d-2)! \kappa}{(d-4)} \int d^d x \sqrt{-g} (-R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + 4R_{\mu\nu} R^{\mu\nu} - R^2). \quad (10)$$

The existence of physical black hole solutions is used as a criterion to assess the validity of those theories. In the vanishing cosmological constant limit the black hole solution is [12]

$$ds^2 = -\left(1 - \left(\frac{r_h}{r}\right)^\gamma\right) dt^2 + \frac{dr^2}{1 - \left(\frac{r_h}{r}\right)^\gamma} + r^2 d\Omega_{d-2}^2, \quad (11)$$

where¹ $r_h = (2G_k M)^{1/(d-2k-1)}$ is the radius of the event horizon and

¹The case $d=2k+1$ is excluded because the solution does not describe a black hole. See [12].

$$\gamma = \frac{d-2k-1}{k}. \quad (12)$$

The thermodynamic properties of the black holes in higher order gravity have been studied in various works [10]. In the case of the black hole solution (11) the Hawking temperature is given by

$$T = \frac{\gamma}{4\pi r_h}. \quad (13)$$

Furthermore, using the partition function, obtained from the Euclidean path integral, the entropy can be calculated leading to the result

$$S_k = \frac{2\pi k}{G_k} \frac{r_h^{d-2k}}{d-2k} \quad (14)$$

that is an increasing function of r_h which is consistent with the second law of thermodynamics.

III. BOUNDS IN HIGHER ORDER GRAVITY

Some time ago Bekenstein [14] proposed that there exists a universal upper bound to the entropy-to-energy ratio of any system of total energy E and effective proper radius R given by the inequality

$$S/E \leq 2\pi R. \quad (15)$$

This bound has been checked in many physical situations, either for systems with maximal gravitational effects (i.e., strong gravity, such as black holes) or systems with negligible self-gravity [15].

In this section we want to consider how this bound behaves with the Lovelock gravity. First we will obtain the bound for the black hole solutions (11). Using the entropy relation (14) and the horizon radius expression we get the bound in an obvious way:

$$S/E = \frac{2\pi k}{G_k} \frac{r_h^{d-2k}}{d-2k} = \frac{4\pi k r_h}{d-2k} = \frac{2k}{d-2k} (S/E)_{Bek}. \quad (16)$$

We thus obtain that the bound for S/E is $2k/(d-2k)$ times the bound found by Bekenstein for the Schwarzschild case ($d=4, k=1$). A real upper bound of S/E for these black hole solutions is achieved for the maximal value of the function $2k/(d-2k)$, namely $(d-3)/3$ for d odd and $(d-2)/2$ for d even. In the case of weak self-gravity systems finding the bound requires more steps. We consider a neutral body of rest mass m , and proper radius R , that is dropped into the Lovelock type black hole. We also demand that this process satisfies the generalized second law (GSL).

Following Carter [16] and using the constants of motion (we consider the metric form $ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 d\Omega_{d-2}^2$)

$$E = -\pi_t = -g_{tt}\dot{t}, \quad (17)$$

$$m = (-g_{\alpha\beta}P^\alpha P^\beta)^{1/2}, \quad (18)$$

we get the equation of motion of the body on the background (11)

$$E = m\sqrt{-g_{tt}}. \quad (19)$$

The energy at $r = r_h + \epsilon$ is given by

$$E = m\gamma^{1/2}\left(\frac{\epsilon}{r_h}\right)^{1/2}. \quad (20)$$

In order to find the change in the black hole entropy caused by the assimilation of the body, one should evaluate E at the point of capture, a proper distance R outside the horizon

$$R = \int_{r_h}^{r_h + \epsilon(R)} \frac{dr}{\sqrt{1 - \left(\frac{r_h}{r}\right)^\gamma}}. \quad (21)$$

Integrating we get

$$R = 2\sqrt{\frac{r_h\epsilon}{\gamma}}. \quad (22)$$

Therefore we can rewrite the energy as

$$E = \frac{m\gamma R}{2r_h}. \quad (23)$$

The assimilation of the body results in a change $dM = E$ in the black hole mass. Using the first law of thermodynamics

$$dM = T dS \quad (24)$$

and the temperature relation (13) we get that the black hole entropy increases as

$$(dS)_{bh} = 2\pi mR. \quad (25)$$

However, we know from GSL that the relation $(\Delta S)_T \equiv (dS)_{bh} - S_{bo} \geq 0$ must be satisfied. This implies an upper limit for the entropy of the body

$$S_{bo} \leq 2\pi ER. \quad (26)$$

Once more it is found that the bound (26) is universal for negligible self-gravity systems because it depends only on the system parameters not on the black hole parameters.

IV. BRICK WALL METHOD

Another interesting point is to check the method of brick wall [17] for this kind of black hole. As an example, we perform the calculations for black holes in $d=8$ and $k=2$. Therefore, we have

$$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2d\Omega_4^2, \quad (27)$$

where the $h(r)$ function which describes the event horizon is given by

$$h = 1 - \left(\frac{r_h}{r}\right)^{3/2}. \quad (28)$$

In this background, we consider a minimally coupled scalar field which satisfies the Klein-Gordon equation [18]

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - m^2\Phi = 0. \quad (29)$$

The 't Hooft method consists of introducing a brick wall cutoff near the event horizon, such that the boundary condition

$$\Phi = 0 \quad \text{for } r \leq r_h + \epsilon \quad (30)$$

is satisfied. In order to eliminate infrared divergencies, another cutoff is introduced at a large distance from the horizon, $L \gg r_h$, where we have

$$\Phi = 0 \quad \text{for } r \geq L. \quad (31)$$

In the spherically symmetric space, the scalar field can be decomposed as

$$\Phi(t, r, \theta) = e^{-iEt}R(r)Y(\theta), \quad (32)$$

where θ represents all the angular variables. Substituting this expression back into Eq. (29) and using the eigenvalue equation for the generalized spherical function $Y(\theta)$,

$$\Delta Y(\theta) = -l(l+5)Y(\theta), \quad (33)$$

we obtain, after some manipulations, the radial equation

$$\begin{aligned} h^{-1}E^2R(r) + \frac{1}{r^6}\partial_r[r^6h\partial_rR(r)] - \frac{l(l+5)}{r^2}R(r) - m^2R(r) \\ = 0. \end{aligned} \quad (34)$$

Using the WKB approximation, we substitute $R(r) = \rho(r)e^{iS(r)}$, the function $\rho(r)$ being a slowly varying amplitude and $S(r)$ is a rapidly varying phase. To leading order, only the contribution from the first derivatives of S are important. Then from Eq. (34) we get for the radial wave number $K \equiv \partial_r S$ the expression

$$\begin{aligned} K = \left(1 - \left(\frac{r_h}{r}\right)^{3/2}\right)^{-1} \\ \times \sqrt{E^2 - \left(1 - \left(\frac{r_h}{r}\right)^{3/2}\right)\left(\frac{l(l+5)}{r^2} + m^2\right)}. \end{aligned} \quad (35)$$

In such a case, the number of radial modes n_r is given by

$$\pi n_r = \int_{r_h + \epsilon}^L dr K(r, l, E). \quad (36)$$

In order to find the entropy of the system we calculate the free energy of a thermal bath of scalar particles with an inverse temperature β , that is

$$e^{-\beta F} = \sum e^{-\beta E_{N_\tau}}, \quad (37)$$

where E_{N_τ} is the total energy corresponding to the quantum state τ . Since the sum also includes the degeneracies of the quantum state, we have

$$e^{-\beta F} = \prod_{n_\tau} \frac{1}{1 - \exp(-\beta E)}, \quad (38)$$

where (n_τ) represents the set of quantum numbers associated with this problem. The product \prod take into account the contribution from all the modes. The factor $(1 - e^{-\beta E})^{-1}$ is due to the fact that we are dealing with bosons where the occupation number can take on the value of all positive integers as well as zero, so that

$$\sum_{n=0}^{\infty} e^{-\beta n E} = \frac{1}{1 - \exp(-\beta E)}. \quad (39)$$

From the previous equation we can write the free energy as

$$\begin{aligned} F &= \frac{1}{\beta} \sum \log(1 - e^{-\beta E}) \\ &= \frac{1}{\beta} \int dD_l \int dn_r \log(1 - e^{-\beta E}), \end{aligned} \quad (40)$$

where

$$D_l = \frac{(2l+5)(l+4)!}{5!l!} = \frac{(2l+5)(l+1)(l+2)(l+3)(l+4)}{5!} \quad (41)$$

is the degeneracy of the spherical modes [19].

Integrating by parts and using Eq. (36) we get

$$\begin{aligned} F &= - \int dD_l \int dE \frac{1}{\exp(\beta E) - 1} n_r \\ &= - \frac{1}{\pi} \int dD_l \int dE \frac{1}{\exp(\beta E) - 1} \int_{r_h + \epsilon}^L dr \\ &\quad \times \left(1 - \left(\frac{r_h}{r}\right)^{3/2}\right)^{-1} \\ &\quad \times \sqrt{E^2 - \left(1 - \left(\frac{r_h}{r}\right)^{3/2}\right) \left(\frac{l(l+5)}{r^2} + m^2\right)}. \end{aligned} \quad (42)$$

The l integration can be performed explicitly and it is taken only over those values for which the square roots exist,

$$\begin{aligned} &\int dD_l \sqrt{E^2 - \left(1 - \left(\frac{r_h}{r}\right)^{3/2}\right) \left(\frac{l(l+5)}{r^2} + m^2\right)} \\ &= \frac{16r^6(E^2 - hm^2)^{7/2}}{5!105h^3} + \frac{8r^4(E^2 - hm^2)^{5/2}}{5!3h^2} \\ &\quad + \frac{16r^2(E^2 - hm^2)^{3/2}}{5!h}. \end{aligned} \quad (43)$$

We are interested in the leading contribution to the free energy near the horizon. Then we just take the first term from the previous equation, that is,

$$\begin{aligned} F &= - \frac{16}{5!105\pi} \int dE \frac{1}{\exp(\beta E) - 1} \\ &\quad \times \int_{r_h + \epsilon}^L dr r^6 h^{-4} [E^2 - hm^2]^{7/2}. \end{aligned} \quad (44)$$

Introducing the change of variable $y = (r/r_h)^{3/2}$ and substituting it back into Eq. (44) we find

$$\begin{aligned} F &= - \frac{32r_h^7}{5!315\pi} \int dE \frac{1}{\exp(\beta E) - 1} \\ &\quad \times \int_{(1+\bar{\epsilon})^{3/2}}^{\bar{L}^{3/2}} dy y^{11/3} \left(1 - \frac{1}{y}\right)^{-4} \\ &\quad \times \left[E^2 - \left(1 - \frac{1}{y}\right)m^2\right]^{7/2}, \end{aligned} \quad (45)$$

where $\bar{\epsilon} = \epsilon/r_h$ and $\bar{L} = L/r_h$.

Near the horizon, that is for y near 1, we find the expression [20]

$$\begin{aligned} F &= - \frac{32r_h^7}{5!315\pi} \int_0^\infty dE \frac{E^7}{\exp(\beta E) - 1} \\ &\quad \times \int_{(1+\bar{\epsilon})^{3/2}}^{\bar{L}^{3/2}} dy (y-1)^{-4}. \end{aligned} \quad (46)$$

We next use the formula

$$\int_0^\infty dE \frac{E^7}{\exp(\beta E) - 1} = \frac{7! \zeta(8)}{\beta^8} \quad (47)$$

and integrate over y . The expression for F reduces to

$$F = - \frac{2^9 \zeta(8)}{45\pi 3^3} \frac{r_h^{10}}{\epsilon^3 \beta^8}, \quad (48)$$

allowing us to compute the entropy from

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{2^{12} \zeta(8)}{45\pi 3^3} \frac{r_h^{10}}{\epsilon^3 \beta^7}. \quad (49)$$

The inverse of the Hawking temperature is

$$\beta = \frac{8\pi}{3} r_h \quad (50)$$

and we can subsequently find the entropy, that is,

$$S = \frac{3^4 \zeta(8)}{45 \pi^8 2^9} \frac{r_h^3}{\epsilon^3}. \quad (51)$$

This expression can be transformed making use of the invariant distance

$$\int ds = \int_{r_h}^{r_h + \epsilon} dr \frac{1}{\sqrt{1 - (r_h/r)^{3/2}}} = \sqrt{\frac{8r_h \epsilon}{3}}, \quad (52)$$

in terms of which we can rewrite the entropy as a function of invariants only,

$$S = \frac{\zeta(8)}{15 \pi^8} \frac{r_h^6}{(r_h \epsilon)^3} = \frac{A}{D_{(8)}^{(2)} \left(\int ds \right)^6}, \quad (53)$$

where $A = \frac{16}{15} \pi^3 r_h^6$ is the horizon area and $D_{(8)}^{(2)} = 2^4 \pi^{11} / \zeta(8)$.

Therefore the entropy of the scalar field is proportional to the area and diverges cubically with the cutoff ϵ .

In Ref. [21] it was shown that the question of finiteness of the entropy can be solved by the renormalization of Newton's gravitational constant. Here that is not possible because the bare entropy (14) does not have the same power of the horizon radius as the divergent term (53).

Repeating the same procedure we can find the general expression for the free energy, for given values of d and k , which are

$$F_\epsilon^{(d)} = -C_{(d)}^{(k)} \frac{r_h^{\kappa_d}}{\epsilon^{(d-2)/2} \beta^d}, \quad (54)$$

where $\kappa_d = d + (d-4)/2$ and the constants $C_{(d)}^{(k)}$ are given in the Appendix.

From the previous equation the entropy can also be obtained also in an easy way,

$$S^{(d)} = \frac{d C_{(d)}^{(k)} \gamma^{d/2} 2^{d-2}}{(4\pi)^{d-1}} \frac{r^{d-2}}{\left(\int ds \right)^{d-2}} = \frac{A}{D_{(d)}^{(k)} \left(\int ds \right)^{d-2}}, \quad (55)$$

where $A = [2\pi^{(d-1)/2} / \Gamma((d-1)/2)] r^{d-2}$ and $D_{(d)}^{(k)} = 2^{d+1} \pi^{3/2(d-1)} / d C_{(d)}^{(k)} \gamma^{d/2} \Gamma((d-1)/2)$.

This result implies that the brick wall method works just for linear gravity.

V. CONCLUSIONS

In this paper we have studied some properties of the black hole solutions in the Lovelock gravity in the zero cosmological constant limit, with the coefficients shown in Sec. II. One of the main conclusions from this study is that we cannot infer the holographic bound from the generalized second law (GSL). In other words, the area law is not respected despite the fact that the second law of thermodynamics is satisfied. Another interesting outcome is that the brick wall method works well only for the Einstein-Hilbert theory ($k=1$). A possible explanation is that this method, by construction, computes the modes living in a shell and therefore at the end of the calculations always reflects this geometrical setup.

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APPENDIX

Here we show the value of the constants $C_{(d)}^{(k)}$ in the free energy expression (54) for the different values of the dimension d and the degree in curvature k :

$$C_{(6)}^{(2)} = \frac{64 \zeta(6)}{3 \pi},$$

$$C_{(7)}^{(2)} = \frac{3 \zeta(7)}{4},$$

$$C_{(8)}^{(3)} = \frac{243^3 \zeta(8)}{5 \pi},$$

$$C_{(9)}^{(2)} = \frac{5 \zeta(9)}{2^{9/2} 16}, \quad C_{(9)}^{(3)} = \frac{5 \zeta(9)}{16},$$

$$C_{(10)}^{(2)} = \frac{2^{11} \zeta(10)}{7 \pi 5^6}, \quad C_{(10)}^{(3)} = \frac{8^2 \zeta(10)}{35 \pi}, \quad C_{(10)}^{(4)} = \frac{2^{16} \zeta(10)}{35 \pi}.$$

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