

Nonsingular big bounces and the evolution of linear fluctuations

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We consider the evolutions of *linear* fluctuations as the background Friedmann world model goes from contracting to expanding phases through smooth and nonsingular bouncing phases. As long as gravity dominates over the pressure gradient in the perturbation equation, the growing mode in the expanding phase is characterized by a conserved amplitude; we call this a *C* mode. In spherical geometry with a pressureless medium, we show that there exists a special gauge-invariant combination Φ which stays constant throughout the evolution from the big bang to the big crunch, with the same value even after the bounce: it characterizes the coefficient of the *C* mode. We show this result by using a bounce model where the pressure gradient term is negligible during the bounce; this requires the additional presence of exotic matter. In such a bounce, even in more general situations for the equation of state before and after the bounce, the *C* mode in the expanding phase is affected only by the *C* mode in the contracting phase; thus the growing mode in the contracting phase decays away as the world model enters the expanding phase. When the background curvature plays a significant role during the bounce, the pressure gradient term becomes important and we cannot trace the *C* mode in the expanding phase to the one before the bounce. In such situations, perturbations in a fluid bounce model show exponential instability, whereas perturbations in a scalar field bounce model show oscillatory behavior.

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I. INTRODUCTION

The collapsing and bouncing phases of the FLRW (Friedmann-Lemaître-Robertson-Walker) world models we consider here are the possible ones in our past, before the big bang. The same physics, however, could work in the possible case in the future as well.

The reexpansion of a positive curvature Friedmann world model which is destined to collapse, or the cyclic repetition of the process with such a bounce, was proposed as early as the 1930s [1–3]. Specific realizations of the bounce and the conditions required to obtain the FLRW world model from a bounce were studied in [4,5]. Singularity-free cosmologies are possible if we give up the strong energy condition, which is often possible with quantum corrections [4]. Recently, the world model of a big bang preceded by a collapsing phase has attracted renewed attention in the context of brane cosmology [6].

In this paper we analyze the evolution of scalar-type curvature (often called adiabatic) fluctuations as the background world model goes through a smooth and nonsingular bounce which connects the contracting and expanding phases. We will *assume* that the classical general relativity is valid as the correct gravity theory throughout the evolution, and also consider scales where the linear approximation is valid.

In Sec. II we review the cosmological perturbation theory needed for our analyzes in later sections. In Sec. III we present the large-scale evolutions of various curvature perturbations near singularity. In Sec. IV we analyze the evolution using exact solutions in a pressureless situation. In Sec. V we show the evolution of perturbations through a bounce

using three different bounce models. Section VI presents a summary, implications of our work, and discussion of related work. We set $c \equiv 1$.

II. COSMOLOGICAL PERTURBATIONS

Our metric convention is [7,8]

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2(\beta_{,\alpha} + B_{\alpha}^{(v)})d\eta dx^{\alpha} + a^2[g_{\alpha\beta}^{(3)}(1+2\varphi) + 2\gamma_{,\alpha|\beta} + 2C_{(\alpha|\beta)}^{(v)} + 2C_{\alpha\beta}^{(t)}]dx^{\alpha}dx^{\beta}. \quad (1)$$

The perturbed order variables α , β , φ , and γ are scalar-type perturbations; the transverse $B_{\alpha}^{(v)}$ and $C_{\alpha}^{(v)}$ are vector-type perturbations (rotation); the transverse trace-free $C_{\alpha\beta}^{(t)}$ is a tensor-type perturbation (gravitational wave). The energy-momentum tensor is

$$\begin{aligned} T_0^0 &\equiv -(\bar{\mu} + \delta\mu), \\ T_{\alpha}^0 &\equiv (\mu + p)[-(1/k)v_{,\alpha} + v_{\alpha}^{(v)}], \\ T_{\beta}^{\alpha} &\equiv (\bar{p} + \delta p)\delta_{\beta}^{\alpha} + \pi_{\beta}^{\alpha}, \end{aligned} \quad (2)$$

where the trace-free π_{β}^{α} is the anisotropic stress.

The trace and trace-free parts of the extrinsic curvature (equivalently, the expansion $\hat{\theta}$ and the shear $\hat{\sigma}_{ab}$ of the normal frame vector field) and the intrinsic scalar curvature $R^{(h)}$ of the constant-time spacelike hypersurface are [see Eqs. (C3), (C14) of [8] and Eqs. (A6), (A7) of [9]],

$$\begin{aligned}\hat{\theta} &= 3H - \kappa, \\ \hat{\sigma}_{\alpha\beta} &= \chi_{,\alpha|\beta} - \frac{1}{3}g_{\alpha\beta}^{(3)}\Delta\chi + a\Psi_{(\alpha|\beta)}^{(v)} + a^2\dot{C}_{\alpha\beta}^{(t)}, \\ R^{(h)} &= \frac{1}{a^2}[6K - 4(\Delta + 3K)\varphi],\end{aligned}\quad (3)$$

where¹

$$\begin{aligned}\chi &\equiv a(\beta + a\dot{\gamma}), \quad \kappa \equiv 3(H\alpha - \dot{\phi}) - \frac{\Delta}{a^2}\chi, \\ \Psi_{\alpha}^{(v)} &\equiv B_{\alpha}^{(v)} + a\dot{C}_{\alpha}^{(v)}.\end{aligned}\quad (4)$$

We have $H \equiv \dot{a}/a$, K is the normalized background three-space curvature, and an overdot indicates a time derivative based on $t, dt \equiv a d\eta$. Thus, κ , χ , and φ are the perturbed expansion, the scalar-type shear, and the perturbed three-space scalar curvature of the normal hypersurface, respectively. $\Psi_{\alpha}^{(v)}$ and $\dot{C}_{\alpha\beta}^{(t)}$ give the vector- and tensor-type contributions to the shear tensor.

$C_{\alpha\beta}^{(t)}$, $\Psi_{\alpha}^{(v)}$, and $v_{\alpha}^{(v)}$ are gauge invariant. α , φ , χ , κ , v , $\delta\mu$, and δp are spatially gauge invariant but depend on the temporal gauge condition, i.e., they depend on the spatial hypersurface (time slicing) choice [9]. Setting any one of these temporally gauge dependent variables equal to zero corresponds to a fundamental gauge condition; except for the synchronous gauge ($\alpha \equiv 0$) each of the other conditions fixes the temporal gauge degree of freedom completely, and any variable in such a gauge condition uniquely corresponds to a gauge-invariant combination (of the variable and the variable used in the gauge condition) [7,8].

The equations describing the evolution of a spatially homogeneous and isotropic FLRW world model are

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \dot{\mu} = -3H(\mu + p). \quad (5)$$

The Λ term can also be considered as an ideal fluid with $\mu_{\Lambda} = -p_{\Lambda} = \Lambda/(8\pi G)$; in such cases, we have $\mu = \mu_m + \mu_{\Lambda}$ and $p = p_m + p_{\Lambda}$. In the case of a minimally coupled scalar field (ϕ) we have $\mu = \frac{1}{2}\dot{\phi}^2 + V$ and $p = \frac{1}{2}\dot{\phi}^2 - V$, with the equation of motion $\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$. Λ can be included as $V_{\Lambda} = \Lambda/(8\pi G)$.

We consider a general scalar-type perturbation. It is convenient to introduce [10–12]

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)}\varphi_{\chi}. \quad (6)$$

The scalar-type perturbation of a fluid with vanishing anisotropic stress in Einstein's gravity is described by [12]

$$\Phi = \frac{H^2}{4\pi G(\mu + p)a} \left(\frac{a}{H} \varphi_{\chi} \right), \quad (7)$$

$$\dot{\Phi} = -\frac{Hc_s^2}{4\pi G(\mu + p)} \frac{k^2}{a^2} \varphi_{\chi} - \frac{H}{\mu + p} e, \quad (8)$$

$$\frac{k^2 - 3K}{a^2} \varphi_{\chi} = 4\pi G\mu\delta_v, \quad (9)$$

where $w \equiv p/\mu$ and $c_s^2 \equiv \dot{p}/\dot{\mu}$. $\varphi_v \equiv \varphi - (aH/k)v$, $\varphi_{\chi} \equiv \varphi - H\chi$, and $\delta_v \equiv \delta + 3(aH/k)(1+w)v$ are gauge invariant combinations [7,8]; φ_v is the same as φ in the comoving gauge ($v \equiv 0$), and φ_{χ} is the same as φ in the zero-shear gauge ($\chi \equiv 0$), etc.

We emphasize that results in this section are valid considering general K , Λ , and time-varying equation of state $p(\mu)$. In the case of a minimally coupled scalar field, we have an additional nonvanishing entropic perturbation (the isotropic stress) $e = (1 - c_s^2)\delta\mu_v$. Its effect can be covered by changing c_s^2 in Eq. (8) to $c_A^2 \equiv 1 - 3(1 - c_s^2)K/k^2$ [13]; as Eqs. (7)–(9) are a complete set for a single component, this prescription always applies in the single scalar field case. It is convenient to have

$$\varphi_{\delta} \equiv \varphi + \frac{\delta}{3(1+w)} = \Phi + \frac{k^2}{a^2} \frac{1}{12\pi G(\mu + p)} \varphi_{\chi}. \quad (10)$$

We can show in general [13]

$$\varphi_{\kappa} = \varphi_{\delta} \left/ \left[1 + \frac{k^2 - 3K}{12\pi G(\mu + p)a^2} \right] \right., \quad (11)$$

where $\varphi_{\kappa} \equiv \varphi + H\kappa/(3\dot{H} - k^2/a^2)$. In the notation of Bardeen [9] we have

$$\delta_v = \epsilon_m, \quad \varphi_{\chi} = \Phi_H, \quad \varphi_v = \phi_m, \quad \varphi_{\kappa} = \phi_h. \quad (12)$$

We have $\varphi_{\delta} = \zeta$ in [7], and $\varphi_v = \mathcal{R}$ in [14]; φ_v was also originally introduced by Lukash as $-\frac{1}{3}q$ in [15]. From Eqs. (7),(8) we can derive equations in closed form

$$\ddot{\Phi} + (c_s^2 k^2/a^2 - \ddot{x}/x)\bar{\Phi} = 0, \quad (13)$$

$$\ddot{\bar{\varphi}}_{\chi} + (c_s^2 k^2/a^2 - \ddot{y}/y)\bar{\varphi}_{\chi} = 0, \quad (14)$$

where

$$\begin{aligned}\bar{\Phi} &\equiv x\Phi, \quad \bar{\varphi}_{\chi} \equiv (ay/H)\varphi_{\chi}, \\ y &\equiv H/\sqrt{(\mu + p)a} \equiv (a/c_s)x^{-1}.\end{aligned}\quad (15)$$

Equations using the conformal time were presented in [16].

In the large-scale limit (meaning the $c_s^2 k^2/a^2$ term is negligible; thus gravity dominates over pressure) we have the general solutions [11–13]

$$\Phi(k, t) = C(k) - d(k) \frac{k^2}{4\pi G} \int^t dt/x^2, \quad (16)$$

¹Indices of $\hat{\sigma}_{\alpha\beta}$ ($E_{\alpha\beta}$ and $H_{\alpha\beta}$, later) are based on the spacetime metric g_{ab} . All the other Greek indices are based on $g_{\alpha\beta}^{(3)}$.

$$\varphi_\chi(k,t) = 4\pi G C(k) \frac{H}{a} \int^t dt/y^2 + d(k) \frac{H}{a}, \quad (17)$$

where C and d are two spatially dependent integration constants: we call these the C mode and the d mode, respectively. Solutions for δ_v , φ_δ , and φ_κ follow from Eqs. (9)–(11). Notice that the d mode of Φ is higher order in the large-scale expansion compared with the d mode of φ_χ . In a pressureless medium, the above solutions are exact, and we have $\Phi = C$ [10]. In fact, for such a medium, instead of Eq. (13), Eq. (8) gives $\dot{\Phi} = 0$.

In order to use the large-scale solutions in Eqs. (16),(17) it is important to check whether we can ignore the $c_s^2 k^2/a^2$ term during the evolution. The large-scale condition implies that (pressure/gravity) $\ll 1$ where

$$\frac{\text{pressure}}{\text{gravity}} \sim \frac{c_s^2 k^2/a^2}{\ddot{x}/x}, \frac{c_s^2 k^2/a^2}{\ddot{y}/y}. \quad (18)$$

In a positive curvature (spherical) model the wave number varies as $k = \sqrt{(n^2 - 1)K}$ where $n = 1, 2, 3, \dots$; $n = 1, 2$ are known to be unphysical [17,9]. In a negative curvature (hyperbolic) model, $k > \sqrt{|K|}$ and $0 \leq k < \sqrt{|K|}$ correspond to the subcurvature and the supercurvature scales, respectively [18]. In a zero-curvature (flat) model we have $k \geq 0$.

The following two variables are continuous under a sudden jump of equation of state [19]:

$$\varphi_\chi \quad (\text{or } \delta_v), \quad \varphi_\delta. \quad (19)$$

These joining variables work for general K , Λ , and $p(\mu)$ in the general scale. This applies for perfect fluids, and for cases involving scalar fields; see [19]. For the background, a and \dot{a} should be continuous at the transition. Consider two phases I and II with different equations of state w_I and w_{II} , making a transition at t_1 . Assuming a flat background, in the large-scale limit, by matching φ_χ and φ_δ in Eqs. (16),(17),(10) we can see that to the leading order in the large-scale expansion we have

$$C_{II} = C_I,$$

$$d_{II} = d_I + 4\pi G C_I \left[\int^t \frac{a(\mu+p)}{H^2} dt \Big|_I - \int^t \frac{a(\mu+p)}{H^2} dt \Big|_{II} \right]. \quad (20)$$

Thus, to the leading order in the large-scale expansion the C mode of Φ remains the same, whereas the d mode of φ_χ is affected by the transition and also the previous history of the d and C modes [19]. Applications were made in [20].

Ignoring the anisotropic stress (π_β^α) and assuming $K = 0$, the equations for the rotation and the gravitational wave become [8]

$$[a^4(\mu+p)v_\alpha^{(v)}] = 0, \quad (21)$$

$$\frac{k^2}{a^2} \Psi_\alpha^{(v)} = 16\pi G(\mu+p)v_\alpha^{(v)},$$

$$\ddot{\bar{C}}_{\alpha\beta}^{(t)} + (k^2/a^2 - \ddot{z}/z)\bar{C}_{\alpha\beta}^{(t)} = 0, \quad (22)$$

where $\bar{C}_{\alpha\beta}^{(t)} \equiv z C_{\alpha\beta}^{(t)}$ and $z \equiv a^{3/2}$. The equation for $C_{\alpha\beta}^{(t)}$ satisfies the same equation as Φ in Eq. (13) with $x \propto a^{3/2}$. Thus, on the general scale for $\Psi_\alpha^{(v)}$ and in the large-scale limit for $C_{\alpha\beta}^{(t)}$ we have the general solutions

$$\Psi_\alpha^{(v)}(k,t) = d_\alpha(k) \frac{1}{a^2}, \quad (23)$$

$$C_{\alpha\beta}^{(t)}(k,t) = c_{\alpha\beta}(k) - d_{\alpha\beta}(k) \int^t dt/a^3. \quad (24)$$

As for the C mode of Φ , the amplitude of the $c_{\alpha\beta}$ mode simply stays constant.

III. LARGE-SCALE EVOLUTION OF CURVATURE PERTURBATIONS

The general large-scale solutions for scalar- and tensor-type perturbation, and the general solutions for vector-type perturbations are presented in Eqs. (16),(17),(23),(24). We call the solutions with C and $c_{\alpha\beta}$ the C modes, and the solutions with d , d_α , and $d_{\alpha\beta}$ the d modes. The vector-type perturbation has no C mode. In the expanding phase C modes are relatively growing solutions whereas d modes decay, and are thus transient in time. In a contracting phase, however, the opposite is the case, with the d modes often diverging as the background model approaches the singularity.

In this section we assume a near *flat* background. With a constant w we have $c_s^2 = w$, and $a \propto |t|^{2/3(1+w)}$ for $-1 < w \leq 1$. In a medium with $w > -\frac{1}{3}$, as we approach the singularity $k/aH (\propto |t|^{(1+3w)/(3+3w)})$ becomes negligible for any given scaling with k ; thus the large-scale conditions are well satisfied. In such a case, during the dynamical time scale of the background evolution “light can travel only a small fraction of a wavelength” [9]; thus the scale becomes the super-horizon scale; Bardeen [9] called $k/aH \sim 1$ an “effective particle horizon.”² In general, we consider $w \geq 0$: “a single-component treatment of the matter is inappropriate when the net pressure is negative” [9].

In this case, from Eqs. (16),(17),(24) we have the C modes remaining constant in time:

²The effective particle horizon is the same as the “Hubble sphere” studied in [21], and closely resembles the “ z surface” introduced in [22]. Global concepts like particle and event horizons are not suitable to describe the local dynamically reachable ranges. Studies in [21,22] show that the Hubble sphere and the z surface are more suitable to describe concepts such as “scales becoming super-horizon size during the inflation era.” Similarly, these are suitable to describe the same physics during the contracting phase with $w > -1/3$. In this situation we can show that to an observer in the contracting phase an object separated by a given comoving distance appears more blueshifted as time goes on, i.e., the object is effectively receding from the observer.

$$\varphi_\chi = \frac{3+3w}{5+3w} C, \quad \varphi_v = C, \quad C_{\alpha\beta}^{(t)} = c_{\alpha\beta}. \quad (25)$$

Thus, in an expanding medium the perturbation evolutions in the super effective particle horizon are kinematic³ and are characterized by the conserved quantities; see [24].

Meanwhile, the d modes behave as

$$\begin{aligned} \varphi_\chi &\propto |t|^{-(5+3w)/(3+3w)}, \\ \varphi_v, C_{\alpha\beta}^{(t)} &\propto |t|^{-(1-w)/(1+w)}, \\ \Psi_\alpha^{(v)} &\propto |t|^{-4/3(1+w)}, \end{aligned} \quad (26)$$

for $-1 < w < 1$. For $w=1$, the above solution is valid for φ_χ ($\propto |t|^{-4/3}$) and $\Psi_\alpha^{(v)}$ ($\propto |t|^{-2/3}$), whereas we have $\varphi_v, C_{\alpha\beta}^{(t)} \propto \ln|t|$ instead.

In the case of constant w , a complete set of solutions for a scalar-type perturbation in six different fundamental gauge conditions is presented in Tables 2–5 of [25]. Although the solutions in [25] are presented in the context of the expanding phase, by changing the time variables to their absolute values with the singularity at $|t|=0$, the same solutions apply in the contracting phase as well.

A. Intrinsic curvature

φ is a dimensionless measure of the (intrinsic) curvature perturbation of the hypersurface (temporal gauge condition) we choose. Thus, its value depends on the chosen hypersurface (temporal gauge condition). From Eq. (A6) of [9] we notice that $C_{\alpha\beta}^{(t)}$ gives a dimensionless contribution to the tensor-type intrinsic curvature perturbation, and the vector-type perturbation does not contribute to the curvature perturbation; see also Eq. (C14) in [8]. Table 2 of [25] shows that for the d mode⁴

$$\begin{aligned} \varphi_\kappa, \varphi_\alpha, \varphi_\delta, \varphi_v, C_{\alpha\beta}^{(t)} &\propto |t|^{-(1-w)/(1+w)}, \\ \varphi_\chi &\propto |t|^{-(5+3w)/(3+3w)}, \end{aligned} \quad (27)$$

for $-1 < w < 1$. For $w=1$ we have,⁵ from Table 4 of [25],

$$\varphi_\kappa, \varphi_\alpha, \varphi_\delta, \varphi_v, C_{\alpha\beta}^{(t)} \propto \ln|t|, \quad \varphi_\chi \propto |t|^{-4/3}. \quad (28)$$

Thus, even for $w=1$, φ diverges in all gauge conditions considered. φ in the zero-shear gauge diverges more strongly

³This kinematic nature of the evolution is reflected in the conserved behavior of linear perturbations in the expanding phase [9,7]. What is conserved is the amplitude of the C mode of the curvature perturbation in diverse gauge conditions. In the expanding phase the situation is also well represented by the separated Friedmann picture of the perturbed Friedmann world model pioneered (to our knowledge) by Harrison in [23].

⁴Although φ_α , the φ in the synchronous gauge ($\alpha \equiv 0$), is not gauge invariant, we are considering the physical solutions.

⁵One other special case occurs for $w=0$ where we have no d modes for φ_v and φ_α ; see Table 5 of [25].

compared with the ones in the other gauge conditions. The strong divergence in the zero-shear gauge is known to be due to the strong curvature of the hypersurface (temporal gauge condition) [9]. φ is set to zero in the uniform curvature gauge. For the C modes we have Eq. (25) and $\varphi_\kappa = \varphi_\alpha = \varphi_\delta = \varphi_v = C$.

B. Extrinsic curvature

Although we have no shear in the background model, the perturbed scalar-type shear is still a gauge dependent quantity. The dimensionless measure of the shear variable (the shear divided by the background expansion rate) becomes

$$\frac{\hat{\sigma}}{H} \sim \left(\frac{k^2}{a^2 H^2} H\chi, \frac{k}{aH} \Psi_\alpha^{(v)}, \frac{1}{H} \dot{C}_{\alpha\beta}^{(t)} \right) \quad (29)$$

for the three perturbation types; $\hat{\sigma} \equiv \sqrt{\hat{\sigma}_{\alpha\beta} \hat{\sigma}^{\alpha\beta}}/2$. The d modes of all types of perturbation show the *same* temporal behavior

$$\frac{\hat{\sigma}}{H} \propto |t|^{-(1-w)/(1+w)}, \quad (30)$$

for $-1 < w \leq 1$; thus with no logarithmic divergences for the $w=1$ case. The results for vector- and tensor-type perturbations follow from Eqs. (23),(24), and that for a scalar-type perturbation follows from Tables 2 and 4 of [25]; thus, the solutions apply to the gauge conditions considered (we set $\chi=0$ in the zero-shear gauge). We can check that the C mode contributions to $\hat{\sigma}/H$ are all regular near the singularity for $-1 < w \leq 1$. The behavior of κ/H , a dimensionless measure of the perturbed trace part of the extrinsic curvature, varies widely depending on the gauge condition; see Tables 2 and 4 of [25].

C. Weyl curvature

Durrer has informed us of another useful measure of the spacetime fluctuation which behaves regularly for $w=1$, the Weyl curvature C_{abcd} . The Weyl curvature (the conformal tensor) vanishes in the FLRW background geometry, and is naturally gauge invariant. The Weyl tensor can be covariantly decomposed into the electric E_{ab} and magnetic H_{ab} parts [26]. Using Eq. (C9) in [8] [see also Eqs. (2.26),(2.27) of [27]], we can show that

$$\frac{E}{R} \sim \left(\frac{k^2}{a^2 H^2} \varphi_\chi, \frac{k}{aH^2} \dot{\Psi}_\alpha^{(v)}, \frac{1}{H} \dot{C}_{\alpha\beta}^{(t)} \right), \quad (31)$$

where $E \equiv \sqrt{E^{\alpha\beta} E_{\alpha\beta}}/2$ and R is the scalar curvature ($\sim H^2$). Thus, the d modes of all types of perturbation behave exactly like $\hat{\sigma}/H$ ($\propto |t|^{-(1-w)/(1+w)}$), and thus behave regularly for $w=1$. H_{ab} contributes only to the vector- and tensor-type perturbations, and we can also show that the d modes behave as $\sqrt{H^{\alpha\beta} H_{\alpha\beta}}/2/R \propto |t|^{-(2/3)(1-3w)/(1+w)}$, and thus behave more regularly for $w=1$. This regular behavior of the Weyl curvature at the singularity for $w=1$ was used to argue the validity of perturbation theory in such a situation [see around Eq. (5.20) of [28]; however, see our discussion below Eq. (68) here].

IV. EXACT EVOLUTION IN A PRESSURELESS CASE

For $K > 0$, $\Lambda = 0$, and $p = 0$, Eq. (5) gives a cycloid [29,2]

$$a = c_m(1 - \cos \eta), \quad t = c_m(\eta - \sin \eta), \quad (32)$$

where $c_m \equiv (4\pi G/3)\mu a^3$ and $d\eta \equiv dt/a$. K is normalized to unity; thus $0 \leq \eta \leq 2\pi$.

For $p = 0$, Eqs. (9),(14) give

$$[a^2 H^2 (\delta_v / H)'] / (a^2 H) = \ddot{\delta}_v + 2H\dot{\delta}_v + 4\pi G\mu\delta_v = 0, \quad (33)$$

which coincides with the density perturbation equation in the synchronous gauge [17] or in the Newtonian context [30]; δ_v is the energy density perturbation in the comoving gauge [31]. Assuming $\Lambda = 0$, the two independent *exact* solutions for $\varphi_\chi \propto \delta_v/a$ in Eq. (17) are [32]

$$\begin{aligned} c_m \frac{H}{a} \int^t \frac{dt}{a^2 H^2} &= -\frac{3\eta \sin \eta}{(1 - \cos \eta)^3} + \frac{5 + \cos \eta}{(1 - \cos \eta)^2} \\ &\equiv \varphi_+(\eta)/3, \\ c_m^2 \frac{H}{a} &= \frac{\sin \eta}{(1 - \cos \eta)^3} \equiv \varphi_d(\eta). \end{aligned} \quad (34)$$

In asymptotes we have

$$\varphi_+(\eta) \approx 3/5, \quad \varphi_d(\eta) \approx 8/\eta^5 \quad (\eta \ll 1),$$

$$\varphi_+(\eta)/(18\pi) \approx (8/\bar{\eta}^5) \approx -\varphi_d(\eta) \quad (\bar{\eta} \ll 1), \quad (35)$$

where $\bar{\eta} \equiv 2\pi - \eta$. We have

$$\varphi_+(2\pi - \eta) = \varphi_+(\eta) + 18\pi\varphi_d(\eta) \equiv \varphi_-(\eta), \quad (36)$$

where φ_- shows the time inverted evolution of φ_+ [33]. Equations (16),(17),(9)–(11) give the exact solutions

$$\Phi = C, \quad (37)$$

$$\varphi_\chi = \varphi_+ C + \varphi_d \bar{d} = \frac{3}{k^2 - 3} \frac{\delta_v}{1 - \cos \eta}, \quad (38)$$

$$\varphi_\delta = C + (k^2/9)(1 - \cos \eta)\varphi_\chi, \quad (39)$$

and similarly for φ_κ ; $\bar{d} \equiv d/c_m^2$ is dimensionless. For $\eta \ll 1$ we have $\varphi_\chi = \frac{3}{5}C$ and $\varphi_\delta = \varphi_\kappa = C$ for the C modes. From Eq. (8) we have $\dot{\Phi} = 0$ exactly for a pressureless fluid considering general K and Λ . Φ has only the C mode (it is identified as C), and no d mode. The evolution of φ_χ and Φ is shown in Fig. 1. Notice that both φ_+ and φ_d diverge as the model approaches the big crunch singularity.

Let us consider a scenario where the big crunch is succeeded by an expanding phase: thus we have two phases $\eta \leq \eta_1$ (phase I) and $\eta \geq \eta_1$ (phase II) with $\eta_1 = 0$. For φ_χ we can take two of three forms of solutions (φ_+ , φ_- , φ_d) as the general solutions in either phase. Although \dot{a} could be discontinuous at the transition reaching the singularity let us try

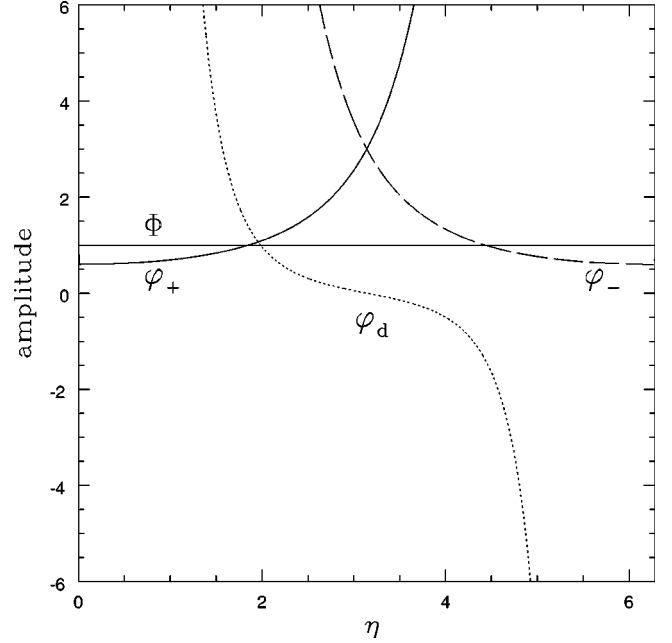


FIG. 1. Evolutions of φ_+ (solid line), φ_- (dashed line), φ_d (dotted line), and Φ (horizontal line).

matching φ_χ and φ_δ directly at η_1 ; after all, we assume a nonsingular bounce near η_1 (see later). Using Eqs. (38),(39),(35) we can show that

$$C_{\text{II}} = C_{\text{I}}, \quad \bar{d}_{\text{II}} = \bar{d}_{\text{I}} - 18\pi C_{\text{I}}. \quad (40)$$

This implies that the value of the Φ variable is conserved even through the bounce. In terms of the general solutions we notice the following. Using Eq. (36) we can decompose φ_+ in Eq. (38) into φ_- and φ_d . Near the bounce, although $\varphi_-^{\text{I}} \approx \frac{3}{5}$ is negligible compared with $\varphi_+^{\text{I}} \approx 18\pi(8/|\eta|^5)$ and $\varphi_d^{\text{I}} \approx -8/|\eta|^5$, we can show that it is this constant mode of φ_-^{I} that feeds the C mode after the bounce. Thus, in the collapsing phase it is appropriate to write Eq. (38) as

$$\varphi_\chi = \varphi_- C + \varphi_d(\bar{d} - 18\pi C). \quad (41)$$

Therefore, if such a bounce is allowed, we have shown that φ_-^{I} feeds the growing mode φ_+^{II} in the expanding phase. The apparent growing (diverging) solutions φ_+^{I} or φ_d^{I} feed only the φ_d^{II} or φ_-^{II} modes, which are the decaying modes in the expanding phase. The time scale of a cycle is encoded in c_m of Eq. (32) and can affect only the decaying solution in the expanding phase. The value of Φ , which is C , is not affected by the different duration of each cycle.

Notice, however, that if we strictly consider the singular and cuspy bounce at η_1 implied by Eq. (32) we have \dot{a} discontinuous, which forbids us from relying on the matching conditions. We have assumed that such a singular bounce can be regarded as a limiting case of a smooth and nonsingular bounce; a concrete example will be considered in the next section. We note that the curvature term has a negligible role near the big crunch/bang. We have also assumed the linearity of the fluctuations involved.

V. THROUGH THE BOUNCE

Assume two expanding phases I and II with equations of state w_I and w_{II} . In the near flat situation we have

$$a(t) = a_{0i}(t - t_i)^{2/3(1+w_i)}, \quad (42)$$

where $i=I,II$. The coefficients should be determined by matching a and \dot{a} at the transitions t_1 . In phases I and II the large-scale solutions in Eqs. (16),(17),(10) give for $-1 < w_i < 1$

$$\Phi = C + k^2 \frac{4}{9} \frac{w_i}{1-w_i^2} \frac{1}{a^3 H} d, \quad (43)$$

$$\varphi_\chi = \frac{3+3w_i}{5+3w_i} C + \frac{H}{a} d, \quad (44)$$

$$\varphi_\delta = C + k^2 \frac{2}{9} \frac{1}{1-w_i} \frac{1}{a^3 H} d, \quad (45)$$

where C and d in the phase I should be regarded as C_I and d_I , and similarly for the phase II. For $w_i=1$, the d mode of Φ (and part of φ_δ) contains a $\ln(t-t_i)$ term instead of $(1-w_i)^{-1}$. Equations (6),(11) give φ_v and φ_κ . As long as we have a and \dot{a} continuous through the transition from phase I to phase II at t_1 we can use our joining variables in Eq. (19). Examples are the radiation-matter transition and the inflation-radiation transition. Using Eqs. (44),(45), to the leading order in the large-scale expansion we have

$$C_{II} = C_I,$$

$$d_{II} = d_I + \frac{6(w_I - w_{II})}{(5+3w_I)(5+3w_{II})} \frac{a_I}{H_I} C_I. \quad (46)$$

This is consistent with the result derived in Eq. (17) of [19]. Similar results hold for two contracting phases as well.

In the case of a transition from a contracting to an expanding phase, however, \dot{a} can be discontinuous at the transition. In order to handle this case properly, we need an intermediate bouncing phase B which smoothly connects the two phases I and II. We consider the collapsing (I) and expanding (II) phases smoothly connected by a nonsingular bouncing phase (B). Assuming that the curvature is not important in phases I and II near the bounce, and assuming w_I and w_{II} for the two phases, we have

$$a_I(t) = a_{I0}[-(t - t_1)]^{2/3(1+w_I)}, \quad (47)$$

$$a_{II}(t) = a_{II0}(t - t_2)^{2/3(1+w_{II})}. \quad (48)$$

The coefficients should be determined by matching a and \dot{a} at the transitions t_1 and t_2 .

In the expanding phase II, the d modes of Φ , φ_χ , φ_δ , φ_v , and φ_κ in Eqs. (43)–(45) decay away whereas the C modes remain constant and have the role of the relatively

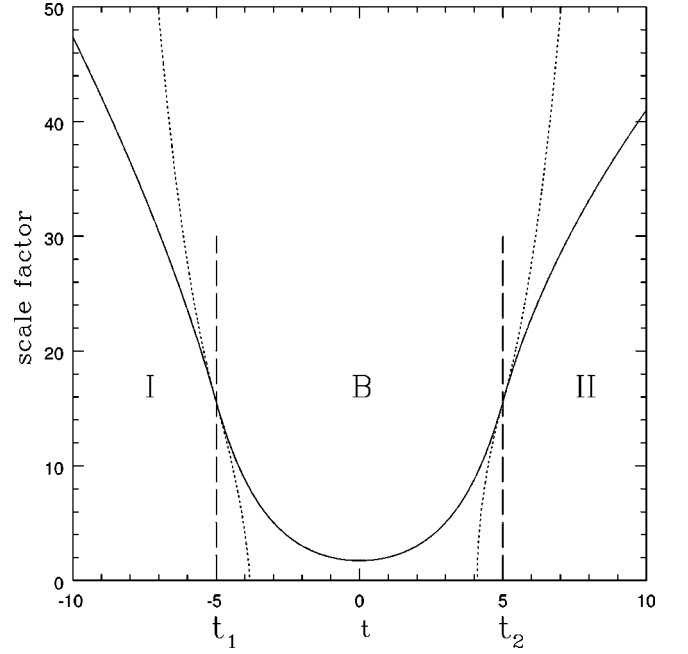


FIG. 2. Evolution of the scale factor (solid line). The dotted lines indicate the extensions of the collapsing, bouncing, and expanding phases without matching. The bouncing phase lasts for $|t_B| < 5$ with $\Lambda = 1$ and $K = 1$. As an example, we take $w_I = 0$ and $w_{II} = 1/3$.

growing modes; whereas, as $t \rightarrow t_1$ in the contracting phase, although the C modes remain constant, the d modes diverge (for $-1 < w_I < 1$) as

$$\begin{aligned} \Phi &\propto \varphi_v \propto \varphi_\delta \propto \varphi_\kappa \propto \frac{1}{a^3 H} \propto |t - t_1|^{-(1-w_I)/(1+w_I)}, \\ \varphi_\chi &\propto \frac{H}{a} \propto |t - t_1|^{-(5+3w_I)/(3+3w_I)}. \end{aligned} \quad (49)$$

For $w_I = 0$, the d mode of Φ vanishes exactly, and the d mode of φ_v vanishes in a near flat situation. For $w_I = 1$, the d modes of Φ , φ_v , φ_δ , and φ_κ show $\ln|t-t_1|$ divergence, instead. A complete set of solutions in several different gauge conditions is presented in Tables 2–5 of [25]; although the solutions were derived in the expanding phase, the same solutions remain valid in the collapsing phase with the time replaced by its absolute value.

A simple example of the bounce is the case with $K > 0$ and a positive Λ [34]:

$$a_B(t) = \sqrt{3K/\Lambda} \cosh(\sqrt{\Lambda/3}t). \quad (50)$$

Evolution of the scale factor is plotted in Fig. 2. Either for a vanishing fluid with pure Λ or for a Λ -type fluid, we have $\mu + p = 0$. If we have $\mu + p = 0$ strictly, the basic set of perturbation equations becomes trivial and we cannot determine the perturbations properly, i.e., we do not have meaningful perturbations.

In the following we consider the perturbation evolution in three other examples of the bouncing phase. The first two

models rely on the fluid and field, which give effectively a $w < -\frac{1}{3}$ equation of state during the bounce. To have a bounce in such models the positive curvature should play a significant role during the bounce; thus these are not suitable for the bouncing model assumed in Sec. IV. In addition, as the background curvature becomes important, all the perturbation scales go through the small-scale regime where the pressure gradient term becomes important. The third model relies on the presence of exotic matter which gives a negative contribution to the total energy density. In this case we have a bounce without resorting to positive curvature; thus the scales remain large during the bounce and the model suits the requirements for the bounce assumed in Sec. IV.

A. Bounce with a $w = -2/3$ fluid

For an ideal fluid with $w = \text{const}$, $K > 0$, and $\Lambda = 0$, from the Friedmann equation we have $H = 0$ at $a(t_*) = (2c_0/K)^{1/(1+3w)}$ where $c_0 \equiv (4\pi G/3)\mu a^{3(1+w)}$. We can show that $a(t_*)$ is a maximum for $w > -\frac{1}{3}$ and a minimum for $w < -\frac{1}{3}$. As a simple example which gives a bounce we consider the $w = -\frac{2}{3}$ case [35]. Although it is uncertain whether it is appropriate to consider an ideal fluid for the $w < 0$ case, we will take the ideal fluid assumption (see a cautionary remark in Sec. VII of [9]). We will find a fundamentally different result in a more realistic (in the sense that we have a concrete action and equation of motion) case based on a scalar field (see Sec. V B). Equation (5) gives the exact solution

$$a = (c_0/2)(t^2 + K/c_0^2). \quad (51)$$

We have $c_s^2 = -\frac{2}{3}$ and

$$\frac{\ddot{x}}{x} = \frac{c_0^2}{2a^2} \frac{3t^4 + (K/c_0^2)^2}{t^2}, \quad \frac{\ddot{y}}{y} = \frac{c_0^2}{2a^2} (t^2 - 3K/c_0^2). \quad (52)$$

The pressure terms become important compared with the gravity near the bounce. Thus, in order to follow the evolution we need to handle perturbations based on the full equations. Equation (14) gives

$$\ddot{\varphi}_\chi + \frac{4t}{t^2 + K/c_0^2} \dot{\varphi}_\chi - \frac{8}{3c_0^2} \frac{k^2 - 3K}{(t^2 + K/c_0^2)^2} \varphi_\chi = 0. \quad (53)$$

Ignoring the k^2 term, the exact solutions in Eqs. (16),(17) can be integrated, and for φ_χ we have

$$\varphi_\chi = C \frac{1}{3} \frac{t^4 + 6(K/c_0^2)t^2 - 3(K/c_0^2)^2}{(t^2 + K/c_0^2)^2} + d \frac{4t/c_0}{(t^2 + K/c_0^2)^2}. \quad (54)$$

Since the k^2 terms become negligible away from the bounce we can use these solutions as the proper initial conditions for the C and d modes. A typical evolution is presented in Fig. 3.

As we have $c_s^2 = -\frac{2}{3}$ we anticipate an exponential growth and/or decay of the perturbation while the pressure gradient term becomes important. In Fig. 3 both the C and d modes in the contracting phase become the (relatively growing) C mode in the expanding phase. As the large-scale conditions

are violated both the C and d modes will be dominated by the exponentially growing mode in the small-scale limit, and eventually we cannot trace the C and d modes in the expanding phase to the ones in the contracting phase.

B. Bounce with a massive scalar field

We consider a bouncing model based on a massive minimally coupled scalar field with a positive curvature [36]. The results up to Eq. (18) in Sec. II remain valid for a field with the prescription mentioned below Eq. (9). The background equations are presented in Eq. (5) and below it with $\mu_\phi = \frac{1}{2}(\dot{\phi}^2 + m^2\phi^2)$, etc. Equation (14) gives

$$\ddot{\varphi}_\chi + \left(7H + 2\frac{m^2\phi}{\dot{\phi}}\right) \dot{\varphi}_\chi + \left(\frac{m^2\phi^2}{M_{pl}^2} + 2H\frac{m^2\phi}{\dot{\phi}} + \frac{k^2 - 8K}{a^2}\right) \varphi_\chi = 0, \quad (55)$$

where $M_{pl}^2 \equiv 1/(8\pi G)$. Once we have φ_χ , the rest of the perturbations Φ , φ_v , φ_δ , and φ_κ follow from Eqs. (6),(7), (10),(11). We have

$$c_A^2 \frac{k^2}{a^2} = \frac{2m^2\phi}{H\dot{\phi}} \frac{K}{a^2} + \frac{k^2}{a^2}, \quad (56)$$

$$\frac{\ddot{y}}{y} = \frac{2m^2\phi}{H\dot{\phi}} \frac{K}{a^2} + m^2 - \frac{3}{4} \frac{K}{a^2} + \frac{7\phi^2 + 25m^2\phi^2}{24M_{pl}^2} + 2m^2 \frac{\phi}{\dot{\phi}} \left(m^2 \frac{\phi}{\dot{\phi}} + 4H\right). \quad (57)$$

We can show that near the bouncing era the pressure term in Eq. (56) dominates over the gravity term in Eq. (57); this is true even for $k^2 = 0$. Thus, near the bounce the large-scale condition is violated, and with the positive sign in front of the k^2 term in Eq. (56) we can show that perturbations show oscillatory behavior while on the small scale; see Fig. 4. Although the first term in the right-hand side (RHS) of Eq. (56) diverges near the bounce, the same term appears in the gravity part in Eq. (57) as well. Due to the positive sign in the second term we expect oscillatory instability as the pressure term dominates the gravity part. Near the bounce we have $H \approx 0$; thus $\mu_\phi \approx \text{const}$, and

$$a \approx \sqrt{3M_{pl}^2 K / \mu_\phi} \cosh[\sqrt{\mu_\phi / (3M_{pl}^2)} t]. \quad (58)$$

In this model, during the bounce all scales reach the small scale where we cannot apply our large-scale solutions. In Fig. 4 we used an arbitrary initial condition at the minimum of the bounce ($t=0$), and as the scale becomes large we have only the C mode because the d mode in the expanding phase is decaying (and thus transient) and yields to the relatively growing C mode within a few expansion times. Thus, as in the previous example based on the $w = -\frac{2}{3}$ fluid the C and d modes during the collapsing phase are mixed up while on the small scale, and we cannot trace the C and d modes in the expanding phase to the ones in the collapsing phase. One important difference of the scalar field compared with the

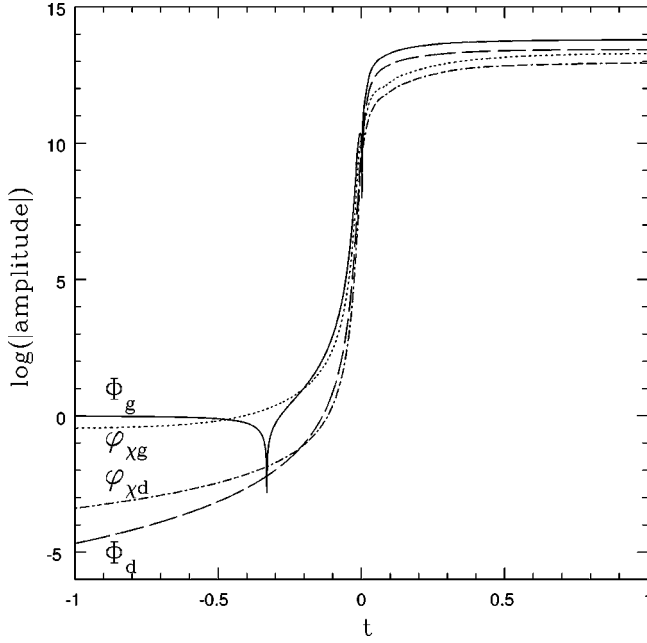


FIG. 3. Evolutions of Φ for the C mode (solid line) and d mode (long-dashed line) initial conditions, and of φ_χ for the C mode (dotted line) and d mode (dot-short-dashed line) initial conditions. We take $n=10$. The pressure terms become important at $|t|<0.4$ for φ_χ and at $|t|<0.2$ for Φ . Notice that the C mode of Φ changes sign twice, whereas the d mode changes once. The sign changes in the expanding phase occur at the same time.

$w = -\frac{2}{3}$ fluid is that, while fluctuation of the fluid shows exponential instability due to the negative c_s^2 term, the field fluctuation shows oscillatory behavior [37]. This difference comes from the presence of a nonvanishing entropic perturbation term e in the case of a scalar field; see below Eq. (9).

In the later expanding phase as ϕ starts oscillating near the potential minimum the background model enters an era with effectively $w_\phi=0$ (dust) as the equation of state. As ϕ starts oscillating we cannot solve Eq. (55) directly. Instead, we can handle the situation analytically using proper time averaging over the coherent oscillations of the background and the perturbed field [38]. In [39] it was shown that, while the background enters the dust era, the perturbations also behave like cold dark matter even in the large scale limit.

C. Bounce model with $\mu = \mu_m - \mu_X$

The positively curved FLRW world model with only radiation and matter does not allow bouncing after the big crunch. If the physical state near the big crunch allows the presence of additional matter X with its effective energy density behaving as $-\mu_X(t) = -\mu_{X0}a^{-3(1+w_X)}$ and $w_X > \frac{1}{3}$, we can have a smooth and nonsingular bouncing phase; this is the generalized case of the “desperate” example mentioned in [3], p. 368. Thus, for the bounce only, we do not even need positive curvature in the background world model.

As a toy model allowing such a smooth and nonsingular transition with the relevant scale satisfying the large scale condition, we consider the case with pressureless matter and exotic matter with $w_X = \frac{1}{3}$. Thus, we consider a model with

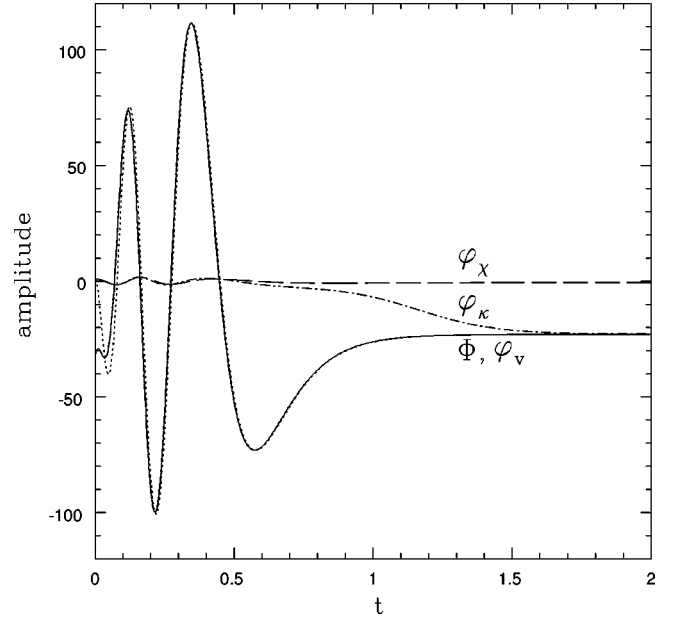


FIG. 4. Evolution of φ_χ (long-dashed line), Φ (solid line), φ_v (dotted line), and φ_κ (dot-short-dashed line). We omit φ_δ which has a larger amplitude and approaches Φ on the large scale. We take $n=10$. $t=0$ corresponds to the minimum of the bounce, and the pressure dominates over the gravity until $t \sim 1$. We can show that Φ , φ_v , φ_κ , and φ_δ stay constant on the large scale, whereas φ_χ is still adjusting its value even in the large scale as the background equation of state effectively changes from $w_\phi \approx -1$ while ϕ is rolling to $w_\phi = 0$ as ϕ starts oscillating, which occurs for $t > 11$.

dust and radiation with a negative sign in the radiation component:

$$H^2 = \frac{8\pi G}{3}(\mu_m - \mu_X) - \frac{K}{a^2}, \quad (59)$$

where $\mu_m \propto a^{-3}$ and $\mu_X \propto a^{-4}$. Certainly, this is not a realistic model for the bounce because we need to assume that there is no conventional radiation component present. Later we will show, however, that this toy model captures the basic physics of more realistic situations.

For a positive curvature $K > 0$, Eq. (59) gives the exact solution

$$a = \frac{1}{K} c_m [1 - \sqrt{1 - 2(c_X/c_m^2)K} \cos(\sqrt{K}\eta)], \quad (60)$$

where $c_X \equiv (4\pi G/3)\mu_X a^4$; c_X/c_m^2 is dimensionless. We have normalized the time axis so that a has minimum values at $\eta = 2\bar{n}\pi/\sqrt{K}$ with \bar{n} an integer number. For vanishing X component, $c_X = 0$, we recover the solution in Eq. (32). With the X component the model shows cyclic behavior. The basic picture of cyclic bounces remains valid in more realistic situations with $w_X > \frac{1}{3}$.

The K term becomes important near a_{\max} , and near a_{\min} we have $\mu_X \approx \mu_m$. The curvature term is negligible near the bounce, thus allowing the existence of the large scale where we can ignore the Laplacian term coming from the pressure

gradient. If we do not need cyclic behavior we can actually take a flat model with $K=0$. Assuming a negligible K term we have

$$a \simeq \frac{c_m}{2} (\eta^2 + 2c_X/c_m^2), \quad (61)$$

which is an exact solution of Eq. (59) for $K=0$.

It is not entirely clear how to handle the perturbation of the exotic component X which is introduced to have a bounce in the background without resorting to positive curvature. If such an exotic state of matter can be modeled by using a field or modifying terms in gravity theory (coming from quantum corrections or higher dimensions, for example), we need the correct form of the equation of motion or the modified action to handle the perturbations properly. In the present situation, lacking a concrete model for the exotic matter, we will take a phenomenological approach based on a fluid approximation.

Let us *assume*, except for its negative contribution to the density, that the fluid X behaves as an ordinary ideal fluid with an equation of state w_X . In such a case, near the bounce we need to consider a two-component system with m and X . [The other possibility is to derive solutions in the m - and X -dominated eras separately and connect them by using the matching conditions. As the X fluid cannot dominate the total fluid (we have $\mu_m \geq \mu_X$ with the equality holding at the bounce) such a situation is forbidden.]

We use the conventional decomposition of the system into the adiabatic (curvature) perturbation, characterized by φ_X (or Φ), and the relative (often called the isocurvature) perturbation defined as $S \equiv S_{12} \equiv \delta_1/(1+w_1) - \delta_2/(1+w_2)$ [40]. The basic equations in the two-component system become [see Eqs. (23),(35),(46),(57) in [13]]

$$\begin{aligned} & \frac{\mu+p}{H} \left[\frac{H^2}{a(\mu+p)} \left(\frac{a}{H} \varphi_X \right) \right]' + c_s^2 \frac{k^2}{a^2} \varphi_X \\ &= -4\pi G e = -4\pi G \frac{(\mu_1+p_1)(\mu_2+p_2)}{\mu+p} (c_{(1)}^2 - c_{(2)}^2) S, \end{aligned} \quad (62)$$

$$\begin{aligned} & \ddot{S} + H(2-3c_z^2)\dot{S} + c_z^2 \frac{k^2}{a^2} S \\ &= -\frac{k^2(k^2-3K)}{a^4} \frac{c_{(1)}^2 - c_{(2)}^2}{4\pi G(\mu+p)} \varphi_X, \end{aligned} \quad (63)$$

where $c_{(i)}^2 \equiv \dot{p}_i/\dot{\mu}_i$ and

$$c_z^2 \equiv \frac{c_{(2)}^2(\mu_1+p_1) + c_{(1)}^2(\mu_2+p_2)}{\mu+p}. \quad (64)$$

In our case $1=m$ and $2=X$; thus, $w_1=c_{(1)}^2=0$, $w_2=c_{(2)}^2=1/3$, $\mu+p=\mu_m - \frac{4}{3}\mu_X$, etc. In the large scale limit the curvature mode can be sourced by the isocurvature one [see Eq. (62)], whereas the isocurvature mode decouples from the curvature mode in general [41] [see Eq. (63)]. [The situation

is different in the case of multiple numbers of scalar fields: in such a case, effectively, the RHS of Eq. (63) has a k^2/a^2 factor instead of k^4/a^4 ; thus the isocurvature modes are less decoupled from the adiabatic one (see [13,42]).]

Let us *assume* an adiabatic initial condition, thus setting $S=0$ at an early era in the contracting phase, for simplicity. More precisely, we are assuming $S \ll \varphi_X$ in the initial epoch, which is natural because it means we are assuming no significant fluctuations in the X component on the early matter dominated era. Since the curvature mode does not source the isocurvature mode in the large scale, the isocurvature mode will remain small. In such a case the RHS of Eq. (62) vanishes, and the curvature equations in the single component situation, Eq. (14), remain valid without any change. Thus, for scales satisfying the large scale limit we have the same solutions as for Eqs. (16),(17) remaining valid. The solutions become

$$\begin{aligned} \Phi &= C + \tilde{d} k^2 \frac{64}{27} \frac{c_X}{c_m^2} \int^\eta \left(\eta^4 + \frac{4c_X}{3c_m^2} \eta^2 - \frac{4c_X^2}{3c_m^4} \right)^{-2} \eta^2 d\eta, \\ \varphi_X &= \left[\left(1 + \frac{c_m^2}{2c_X} \eta^2 + \frac{5c_m^4}{12c_X^2} \eta^4 + \frac{3c_m^6}{40c_X^3} \eta^6 \right) C + \frac{c_m^6}{c_X^3} \eta \tilde{d} \right] / \\ & \quad \times \left(1 + \frac{c_m^2}{2c_X} \eta^2 \right)^3. \end{aligned} \quad (65)$$

For φ_X the contribution from the lower bound of integration of the C mode is absorbed into the \tilde{d} mode. In the matter dominated era $|\eta| \gg 2c_X/c_m^2$, we have

$$\Phi = C - k^2 \frac{64}{135} \frac{c_X}{c_m^2} \eta^{-5} \tilde{d}, \quad \varphi_X = \frac{3}{5} C + 8 \eta^{-5} \tilde{d}. \quad (66)$$

Near the big bang/crunch, the solution for φ_X coincides with the one for a pressureless medium considered in Eq. (38). For $c_X/c_m^2 \rightarrow 0$, Φ also coincides with the one known in the pressureless medium. Near the bounce, $|\eta| \ll 2c_X/c_m^2$, we have

$$\Phi = C + k^2 \frac{4}{9} \frac{c_m^6}{c_X^3} \eta^3 \tilde{d}, \quad \varphi_X = C + \frac{c_m^6}{c_X^3} \eta \tilde{d}, \quad (67)$$

which are regular and finite.

Since the present bounce model allows the scales to stay in the large-scale limit during the transition, it can be considered as a concrete model of the smooth and nonsingular bounce assumed in Sec. IV. Indeed, the curvature term is negligible near the bounce as was the case near the big crunch/bang in Sec. IV. In the $c_X/c_m^2 \rightarrow 0$ limit, Eq. (65) reduces to Eq. (66), which also coincides with the known solution considered in Sec. IV.

Clearly, we can also make a more realistic model with radiation, matter, and X where $w_X > 1/3$. We note that Eqs. (62),(63) remain valid for any two-component system of matter perturbations. Even in such a case the X fluid can

cause a smooth and nonsingular bounce and the curvature term has a negligible role near the bounce. Thus, essentially the same conclusions (e.g., the C mode feeding the C mode) remain valid. We have considered a simple toy model only because it allows analytic handling of the background and the perturbations, thus showing the situation explicitly.

VI. DISCUSSION

In Sec. IV we showed that the perturbation in a positively curved FLRW model filled with a pressureless matter is described by the conservation of Φ . Assuming a transition of a big crunch followed by a big bang in such a model, by using the known matching conditions we showed that Φ maintains the same value even after the transition. Using the matching conditions we also showed that the diverging solution in the contracting phase is matched to the decaying solution in the subsequent expanding phase, whereas the other solution, which stays constant during the contracting phase, is matched into the same constant solution in the expanding phase. That constant mode is characterized by Φ ; the other solution of Φ which decays in the expanding phase is higher-order in k^2 compared with the one for φ_χ and vanishes for vanishing background pressure.

In order to confirm these results based on matching at a singular bounce, in Sec. V we considered three different nonsingular and smooth bounce models. For the bounce models based on a fluid (Sec. V A) and a massive scalar field (Sec. V B) in a positively curved background, the role of background curvature is important to make the bounce. In such cases, all the perturbation scales come inside the sound horizon near the bounce, and the large-scale conditions are violated. As the pressure gradient terms become important, perturbations in the fluid model show exponential instability, whereas the ones in the massive field model show oscillatory behavior. For both situations the two independent perturbation modes in the large-scale limit during the collapsing phase get mixed up with the two independent modes on the small scale during the bouncing phase. Thus, we cannot trace the two independent solutions (C and d modes) in the expanding phase to the ones in the contracting phase.

In Sec. V B we considered a bouncing model based on exotic matter with a negative contribution to the total energy density. In such a case the positive curvature is not important during the bounce. Thus, we could have the relevant scales remaining in the large-scale limit, and could apply the general large-scale solutions. In this case, however, we have to handle the perturbation of the exotic matter in addition to the ordinary one simultaneously. By considering the adiabatic initial condition we have shown that the same curvature perturbation equation known in the single-component situation remains valid; thus the known large-scale solutions are valid as well throughout the bounce. Therefore, this third type of bounce model can be regarded as an example of the smooth and nonsingular bouncing assumed in Sec. IV. As an analytically manageable concrete example, we considered a simple toy model with dust and exotic matter with a radiationlike equation of state. Even in more general situations for the equations of state before and after the bounce, similar analy-

ses can be made which show that the C mode in the expanding phase is affected only by the C mode in the contracting phase; thus the growing mode in the contracting phase decays away as the world model enters the expanding phase.

Our analyses are based on two assumptions: (i) the contracting phase is converted into the expanding one by a smooth and nonsingular bounce, and (ii) linear perturbation theory holds during the evolution. The large-scale evolution can be characterized by the conservation of Φ . We have shown that the C mode of Φ , which is the proper growing mode in the expanding phase, is simply conserved during the evolution and through the bounces. The results are true as long as the two assumptions made above are valid, and, in addition, if the large scale condition is met during the transition as considered in Sec. V C.

In Sec. III we showed that the three dimensionless measures, the intrinsic curvatures (φ and $C_{\alpha\beta}^{(r)}$), the trace-free part of the extrinsic curvature ($\hat{\sigma}/H$), and the Weyl curvature (E/R), *diverge* at singularity for $-1 < w < 1$. Thus, for $-1 < w < 1$, the spacetime perturbations become singular as the background approaches the singularity. An ambiguity remains for the $w=1$ case because, although φ and $C_{\alpha\beta}^{(r)}$ *diverge* logarithmically, $\hat{\sigma}/H$ and E/R remain finite. These results apply to all perturbation types and for all gauge conditions we have considered. The behaviors of the other variables (the perturbed lapse function α , the dimensionless measure of the perturbed expansion κ/H , the relative density perturbation δ , etc.) depend more strongly on the gauge conditions (see Tables 2 and 4 of [25]). Thus, these variables apparently have less physical significance in characterizing the spacetime fluctuations compared with the other three measures, whose behaviors are gauge independent at least in the pool of gauge conditions we have investigated. Do the above results imply diverging spacetime fluctuations for $-1 < w < 1$, and regular ones for $w=1$? In Table 4 of [25] we find that in no gauge condition do *all* the perturbations remain finite for $-1 < w \leq 1$.

The authors of [43] argued that as the model goes through a singular bounce the perturbation becomes nonlinear. We have shown that, *if* the fluctuations survive the bounce as linear ones, the diverging mode in the contracting phase should be matched to the decaying one in the expanding phase. Lyth in [44] made the following simple and powerful argument. As we have under the gauge transformation that ($\tilde{x}^a = x^a + \xi^a$ with $\xi^t \equiv a \xi^\eta$)

$$\tilde{\varphi} = \varphi - H \xi^t, \quad \tilde{\delta} = \delta + 3H(1+w)\xi^t, \quad (68)$$

if φ diverges while δ remains finite, or vice versa, in any single gauge condition [this is the case for $-1 < w \leq 1$; see Eqs. (9),(49) for φ_v and δ_v] no temporal gauge transformation ξ^t can be found that makes both φ and δ finite. Therefore, for $-1 < w \leq 1$ we find that the d mode perturbations of the Friedmann world model become *singular* near the big crunch in one form or another in *all* gauge conditions.

We note that Φ , which becomes φ_v for $K=0$, simply stays constant in a pressureless medium; thus its magnitude *cannot* characterize the breakdown of linearity of the per-

turbation. As we have from Eqs. (6),(37),(38) $\varphi_v = C + (1/3)(1 - \cos \eta)\varphi_\chi$ where we set $K=1$, φ_v itself could diverge near the singularity. From Eq. (11), near the big crunch in the pressureless medium we have $\varphi_\kappa \simeq \varphi_\delta$ where φ_δ , given in Eq. (39), has a diverging part. Thus, near the big crunch the diverging modes behave as

$$\varphi_v \propto \varphi_\delta \propto \varphi_\kappa \propto |\eta|^{-3}, \quad \varphi_\chi \propto |\eta|^{-5}, \quad (69)$$

whereas Φ has no diverging mode in the pressureless case. For the situation with general w , see Eq. (49). Bardeen has argued that the behavior of φ_χ “overstates the physical strength of the singularity;” see below Eq. (5.12) in [9].

At the singular big crunch, we certainly have d modes of many perturbation variables unambiguously becoming singular for $-1 < w \leq 1$ (see Tables 2–4 in [25]). Do large amplitudes of some dimensionless measures of perturbations imply the breakdown of linear theory? Due to the gauge dependence of relativistic perturbations, the large (larger than unity, say) amplitudes of some gauge-invariant perturbation variables do not guarantee the breakdown of linear theory. However, what Lyth [44] has shown is that in the collapsing phase we could encounter situations where the amplitudes of perturbation variables become large in one form or the other in *all* gauge conditions. Lyth has argued this as a violation of the *necessary condition* of the linear perturbation theory.

In our models, which avoid the singularity by a smooth and nonsingular bounce, it is likely that certain scales can safely go through the bounce and retain their linear nature. As we have assumed the linearity of perturbations, our analyses and results are applicable to such scales only. In Secs. IV and V C we showed that the diverging solutions in the contracting phase in Eq. (69) affect only the decaying (thus transient) solutions in the subsequent expanding phase. In such a scenario, however, one could anticipate a large (compared with the C mode) amount of the decaying (d) mode present in the early big-bang phase for a while, as a remnant from the preceding phase before the big bang.

In a recently proposed ekpyrotic scenario it was argued that the final scalar-type perturbation is scale invariant [45]. In [46] it was shown that the scale-invariant spectrum generated in the zero-shear gauge during the collapsing phase should be identified as the d mode; thus after the bounce we have a different power spectrum [47]. Our results in this paper confirm that, while the large scale condition is met during the (smooth and nonsingular) transition, the d mode in the contracting phase does not affect the (properly growing) C mode in the expanding phase. The background curvature is flat in the ekpyrotic scenario and the scale remains large during the bounce. However, since the bounce of the ekpyrotic scenario goes through a singularity the author of [44] has argued that one cannot rely on linear analyses as the model approaches the singularity. Thus, either the final spectrum is not scale invariant (which is the case if the linear perturbation survives) or the issue should be handled in the future in the string theory context with a concrete mechanism for the bounce.

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