Timelike and null focusing singularities in spherical symmetry: A solution to the cosmological horizon problem and a challenge to the cosmic censorship hypothesis

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Extending the study of spherically symmetric metrics satisfying the dominant energy condition and exhibiting singularities of power-law type initiated by Szekeres and Iyer, we identify two classes of peculiar interest: focusing timelike singularity solutions with the stress-energy tensor of a radiative perfect fluid (equation of state: $p=\frac{1}{3}\rho$) and a set of null singularity classes verifying identical properties. We consider two important applications of these results: to cosmology, as regards the possibility of solving the horizon problem with no need to resort to any inflationary scenario, and to the strong cosmic censorship hypothesis to which we propose a class of physically consistent counterexamples.

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I. INTRODUCTION

In a recent work, devoted to some developments of the ''delayed big-bang'' (DBB) cosmological model [1], it was shown that any cosmological model exhibiting a null singularity surface is naturally and permanently free from any horizon problem. This property can easily be extended to models with a timelike singularity, as we shall see in Sec. II.

This important issue of the horizon problem in cosmology has induced us to investigate the constraints which can be imposed on the stress-energy tensor by the requirement of a nonspacelike singularity. Even though this requirement is only a sufficient, but not necessary, condition for the resolution of the problem (see discussion in Sec. II), it is an interesting cosmological question in its own right.

Part of the work has already been done by Szekeres and Iyer [2] (SI), who investigated, in the spherically symmetric case, the constraints imposed by the requirement of a timelike singularity, with the object of exploring the validity of the strong cosmic censorship hypothesis (SCCH). Albeit the situation considered by SI is a collapse, the results can be extended straightforwardly to the cosmological case by reversing the direction of time.

In this paper, we still concentrate our attention on spherically symmetric singularities. Despite this specialization, interesting information can undoubtedly be gained, and any results obtained should be usefully applied to both the cosmological issue and the SCCH.

Until now, our preliminary works aiming at solving the horizon problem without recourse to the inflationary paradigm [1,3,4] retain the simplifying approximation of a dust dominated universe, the DBB model. However, as stressed in [3], when going backward on the light cones issuing from the last-scattering surface towards the singularity, the energy density increases and one would expect on physical grounds that the pressure should do likewise. Thus, the radiation becomes the dominant component in the universe, and possibly a relativistic equation of state such as $P=\frac{1}{3}\rho$ should apply. In the present article we show that such an equation of state is compatible with a non-spacelike singularity, providing therefore new physically consistent models for a horizon problem free primordial universe.

Furthermore, the consideration of a special case which, for no good reason, has been omitted from the analysis given in SI yields an example of focusing timelike singularity compatible with the above radiative equation of state and at variance with the SCCH.

In Sec. II we review the way a non-spacelike singularity leads to the resolution of the standard cosmological horizon problem. In Secs. III to V, we identify the constraints imposed on the stress-energy tensor by the requirements of a timelike and a null singularity. Sections VI and VII are devoted to the application of the obtained results to the cosmological horizon problem and the SCCH issue, respectively. The conclusions are stated in Sec. VIII. A derivation of the timelike character of spherically symmetric shell-crossing surfaces is proposed in the Appendix.

II. SOLVING THE HORIZON PROBLEM

As shown in [1], the horizon problem develops sooner or later in any cosmological model exhibiting a spacelike singularity such as that occurring in standard Friedmann-Lemaître-Robertson-Walker (FLRW) universes. Simply stated, the horizon problem is this: In hot big-bang models the comoving region over which the cosmic microwave background radiation (CMBR) is observed to be homogeneous to better than one part in 10^5 at the last-scattering surface is much larger than the intersection of this surface with the future light cone from the ''big-bang.'' As this light cone provides the maximal distance over which causal processes could have propagated since a given point on the ''big bang,'' the observed isotropy of the CMBR remains unexplained.

Even inflation only postpones the occurrence of the horizon problem since it does not change the spacelike character of the singularity and is insufficient to solve it permanently. This is shown in Fig. 1, where thin lines represent light cones

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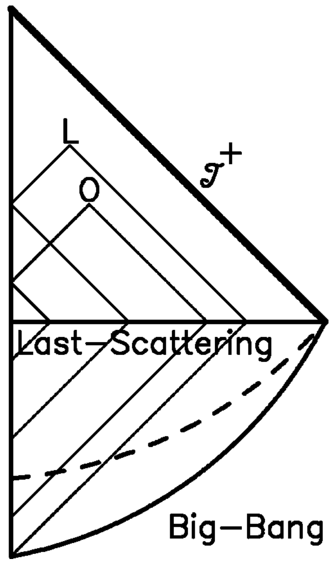

FIG. 1. Penrose-Carter diagram showing the horizon problem in a universe with spacelike singularity.

and the CMBR as seen by an observer O corresponds to the intersection of the observer's backward light cone with the last-scattering line. For a complete causal connection to occur between every pair of points in this intersection segment, backward light signals issuing from points therein must reach the vertical axis before they reach the spacelike ''big-bang'' curve. L is thus a limiting event beyond which any observer experiences the horizon problem. Adding an inflationary phase in the primordial history of the universe amounts to adding a slice of de Sitter space-time, indicated here by the region between the dashed line and the ''big bang.'' The effect of this region is merely to postpone the event L, allowing the current observer O to see a causally connected CMBR. At later times the observer reaches the region above L and the horizon problem reappears.

In [1] a permanent solution to this problem was proposed, using the DBB class of models, valid for all observers regardless of their location in the universe. These models have a nonspacelike singularity which can arise, for example, as shell crossings (see [5] for a detailed characterization of a shell crossing singularity). In [1] shell-crossings were mistakenly claimed to be null surfaces, whereas they are in fact timelike. A derivation of this property, valid for general spherically symmetric models, is given in the Appendix. Figure 2 shows that a nonspacelike singularity always gives rise to an everywhere causally connected model of the universe. Every pair of points in the CMBR seen by the current observer O are causally connected since a past light signal from any point in the segment of the last-scattering surface seen by O reaches the vertical axis before arriving at the null (straight line) or timelike (curved line) singularity. The same holds for any event O' in the observer's past or future.

Therefore, any cosmological model exhibiting the equation of state of radiation near a nonspacelike (i.e., timelike or null) singularity, whatever its type (shell cross or focus), could be considered as a physically consistent candidate to represent the primordial universe.

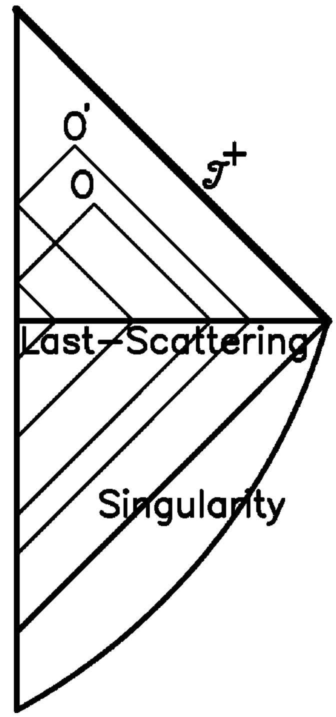

FIG. 2. Penrose-Carter diagram showing causal connectedness of a universe with nonspacelike singularity.

III. DOUBLE NULL COORDINATES AND SINGULARITIES OF POWER-LAW TYPE

In SI spherically symmetric metrics were studied, having the form

$$ds^2 = -dt^2 + [t-\tau(r)]^{2a} f^2(r,t) dr^2 + [t-\tau(r)]^{2b} g^2(r,t) d\Omega^2, \tag{3.1}$$

where f and g are functions of r and t which are regular and nonvanishing at the singularity surface $t=\tau(r)$. This type of singularity was said to be ''of power-law type'' and is present in spherically symmetric dust solutions [Lemaître-Tolman-Bondi (LTB) metrics [7–9]], all FLRW perfect fluid solutions, and cosmologies with singularities of the Lifshitz-Kalatnikov type [10]. We will be proposing a slightly different and more general definition of this concept. While the only purpose of SI was to discuss the validity of the cosmic censorship hypothesis by considering the behavior of collapsing space-times near the singularity, most of the results of that paper can readily be generalized to a cosmological setting by reversing the direction of time. As the present paper is mostly directed at cosmological issues, we are mainly interested in the region $t>\tau(r)$, and have made an appropriate sign change in Eq. (3.1) to that used in SI. The newly obtained results will again be applied to the SCCH issue, by reversing the direction of time.

The method adopted in SI is to pass to double null coordinates $u(r,t)$ and $v(r,t)$ such that

$$ds^2 = -2e^U du dv + e^V d\Omega^2. \tag{3.2}$$

Such coordinates have a certain rigidity, in that the only available coordinate freedoms are of the form

$$u' = \mu(u), \quad v' = \nu(v), \tag{3.3}$$

where μ and ν are arbitrary differentiable functions. Radial null lines (θ=const, ϕ=const) are exactly the curves u =const or v=const. We assume the future pointing directions on these null lines are those with increasing values of v and u, respectively.

In SI a coordinate transformation converting the metric of Eq. (3.1) into the double null coordinates of Eq. (3.2) is performed in a neighborhood of $r=r_0$, $t=\tau(r_0)$ by carrying out a series expansion of the form

$$r = r_0 + u + f_1(u)x^{a_1} + f_2(u)x^{a_2} + \cdots \tag{3.4}$$

$$t = \tau(r_0+u) + g_1(u)x^{b_1} + g_2(u)x^{b_2} + \cdots \tag{3.5}$$

$$= \tau(r_0) - \tau'(r_0+u)f_1(u)x^{a_1} + g_1(u)x^{b_1} + \cdots \tag{3.6}$$

where $0<a_1<a_2<\cdots$, and $0<b_1<b_2<\cdots$. By Eq. (3.6) the singularity at $t=\tau(r)$ occurs at $x=0$, and the freedom of Eqs. (3.3) can be used to express the function $x(u,v)$ in the form

$$x = lu + kv \quad \text{where} \quad l,k = \pm 1 \quad \text{or} \quad 0. \tag{3.7}$$

The signs of l and k should be chosen such that $x>0$ for $t>\tau(r)$ in the neighborhood of the singularity. It is easy to verify that the singularity $x=0$ has spacelike character if $lk=1$, timelike if $lk=-1$, and null if $lk=0$. For example, if $lk=1$, then in the limit as $x\to 0$, the surface $x=\text{const}>0$ intersects both null lines u=const and v=const in positive (future) values if $l=k=1$, while it intersects them in the past if $l=k=-1$. The first case $x=u+v$ therefore corresponds to a spacelike singularity in the past (cosmological), while $x=-u-v$ represents a spacelike singularity in the future (collapse). A similar analysis for the case $lk=-1$ results in one ingoing null ray and the other outgoing. It therefore is timelike, which can be thought of as having both a cosmological and collapse character. When $lk=0$ one of the rays $u=0$ or $v=0$ is tangential to the singularity surface, which can be thought of as null.

After some analysis, the functions U and V appearing in Eq. (3.2) can be shown to have the form

$$e^U = 2t_u t_v = x^p e^{\alpha}, \quad e^V = (t-\tau)^{2b} g^2 = x^q e^{\beta} \tag{3.8}$$

where

$$\alpha = \alpha_0(u) + \alpha_1(u)x^{p_1} + \cdots, \tag{3.9}$$

$$\beta = \beta_0(u) + \beta_1(u)x^{q_1} + \cdots, \tag{3.10}$$

and the exponents p and q depend on a and b in a variety of ways detailed in SI. Exponents $p_1, p_2, \ldots, q_1, \ldots$, appearing in the expansions of α and β can also be evaluated in principle from the exponents $a_1, a_2, \ldots \; b_1, b_2, \ldots$ occurring in Eqs. (3.4) to (3.6), though it may be difficult to give general expressions for them.

The following two examples should give an idea of the kind of results expected

A. Einstein–de Sitter dust solution

These are only spatially flat dust solutions of the form given in Eq. (3.1) having $\tau(r)=\text{const}=0$. The metric can be written

$$ds^2 = -dt^2 + t^{4/3}(dr^2 + r^2 d\Omega^2) \tag{3.11}$$

and the singularity is well known to be spacelike (since future pointing null geodesics emanate from $t=0$ in all directions). It is for this reason that the horizon problem occurs in cosmological models of this type.

The singularity has $a=b=\frac{2}{3}$, and the transformation to double null coordinates is straightforward to perform. The series expansion has $a_1=1$ and $b_1=3$ and gives rise to the following exponents in Eqs. (3.8), (3.9), and (3.10):

$$p=q=4, \quad p_1=q_1=1, \quad \ldots .$$

B. Zero energy LTB solutions

The zero energy LTB solutions are

$$ds^2 = -dt^2 + (t-t_0(r))^{-2/3}(t-t_1(r))^2 dr^2 + r^2(t-t_0(r))^{4/3} d\Omega^2, \tag{3.12}$$

where $t=t_0(r)$ is the focusing singularity and $t=t_1(r) = t_0(r) + \frac{2}{3} r t_0'(r)$ is a shell-crossing singularity.

A t_0 singularity [with $t_0'(r_0)\neq 0$] has $a=-\frac{1}{3}$, $b=\frac{2}{3}$, and the exponents in the power series of Eqs. (3.4) and (3.5) turn out to be

$$a_1=1, \quad a_2=\frac{5}{4}, \quad \ldots \quad b_1=\frac{3}{4}, \quad b_2=1, \ldots$$

and give rise to the following exponents in the expansion of U and V:

$$p=\frac{2a}{1-a}=-\frac{1}{2}, \quad q=\frac{2b}{1-a}=1, \quad p_1=q_1=\frac{1}{2}, \ldots .$$

In this case it turns out that one must have $l=k=1$ as in the Einstein–de Sitter case and the singularity is spacelike.

A t_1 singularity [with $t_1'(r_0)\neq 0$] has $a=1$ and $b=0$, with transformation exponents

$$a_1=\frac{1}{2}, \quad \ldots \quad b_1=1, \quad \ldots,$$

and

$$p=q=0, \quad p_1=q_1=1, \quad p_2=q_2=\frac{3}{2}, \quad \ldots .$$

The singularity in this case has $lk=-1$, is timelike (see the Appendix) and has no horizon problem. Accordingly, the dust DBB models have been constructed in such a way that the first singularity encountered when going backward on timelike curves is of type t_1.

Although the singularity surface $x=0$ of a metric of the type described by Eq. (3.1) is generally non-null, there is no reason to impose this restriction when starting from the double null form of the metric of Eq. (3.2). We will therefore define the singularity surface of a spherically symmetric metric as being *of power-law type* if it can be expressed in the form of Eq. (3.2) with

$$U=p\ln x+\alpha_0(u)+\alpha_1(u)x^{p_1}+\alpha_2(u)x^{p_2}+\cdots, \tag{3.13}$$

$$V=q\ln x+\beta_0(u)+\beta_1(u)x^{q_1}+\beta_2(u)x^{q_2}+\cdots, \tag{3.14}$$

where $0<p_1<p_2\ldots,\ 0<q_1<q_2<\ldots$ and x is given by Eq. (3.7). As there are essentially no further coordinate freedoms, this definition is invariant. In case the reader wonders why the functions $\alpha_1,\ \beta_1$ are not postulated to be regular functions of both variables u and v, it is simple to express v as either $u\pm x$ or $-(u\pm x)$, substitute in the functions and expand as a power series. The result would then be that given in Eqs. (3.13) and (3.14).

IV. STRESS-ENERGY TENSOR NEAR TIMELIKE SINGULARITIES OF POWER-LAW TYPE

The Einstein tensor for the metric of Eq. (3.2) has the following nonvanishing components (setting $x^0=u,\ x^1=v,\ x^2=\theta,\ x^3=\phi$):

$$G_0^0=G_1^1=-e^{-V}-e^{-U}(V_{01}+V_0V_1)$$

$$=-x^{-q}e^{-\beta}-e^{-\alpha}x^{-p}\left(\frac{kl(q^2-q)}{x^2}+\frac{k\beta_u+l\beta_v}{x}+\beta_{uv}+\beta_u\beta_v\right) \tag{4.1}$$

$$G_1^0=e^{-U}\left(V_{11}-U_1V_1+\frac{1}{2}V_1^2\right)$$

$$=-e^{-\alpha}x^{-p}\left(\frac{k^2\left(q+pq-\frac{1}{2}q^2\right)}{x^2}+\frac{k((p-q)\beta_v+q\alpha_v)}{x}-\beta_{vv}+\alpha_v\beta_v-\frac{1}{2}\beta_v^2\right) \tag{4.2}$$

$$G_0^1=e^{-U}\left(V_{00}-U_0V_0+\frac{1}{2}V_0^2\right)$$

$$=-e^{-\alpha}x^{-p}\left(\frac{l^2\left(q+pq-\frac{1}{2}q^2\right)}{x^2}+\frac{l((p-q)\beta_u+q\alpha_u)}{x}-\beta_{uu}+\alpha_u\beta_u-\frac{1}{2}\beta_u^2\right) \tag{4.3}$$

$$G_2^2=G_3^3=-e^{-U}\left(U_{01}+V_{01}+\frac{1}{2}V_0V_1\right)$$

$$=-e^{-\alpha}x^{-p}\left(\frac{lk\left(\frac{1}{2}q^2-p-q\right)}{x^2}+\frac{q(k\beta_u+l\beta_v)}{2x}+\alpha_{uv}+\beta_{uv}+\frac{1}{2}\beta_u\beta_v\right). \tag{4.4}$$

The stress-energy tensor arising from Einstein's equations

$$T_\nu^\mu=G_\nu^\mu$$

has the form

$$T_\nu^\mu=\rho u^\mu u_\nu+P_r f^\mu f_\nu+P_\perp h_\nu^\mu,$$

where u^μ is the unit timelike eigenvector and f^μ the unit spacelike radial eigenvector,

$$T_\nu^\mu u^\nu=-\rho u^\mu,\quad T_\nu^\mu f^\nu=P_r f^\mu$$

having components

$$u^\mu=\left(u^0,u^1=u^0\sqrt{\frac{G_0^1}{G_1^0}},0,0\right),$$

$$g_{\mu\nu}u^\mu u^\nu=-2e^U u^0u^1=-1,$$

$$f^\mu=\left(f^0,f^1=-f^0\sqrt{\frac{G_0^1}{G_1^0}},0,0\right),$$

$$g_{\mu\nu}f^\mu f^\nu=-2e^U f^0f^1=1,$$

and h_ν^μ is the projection tensor into the space orthogonal to u^μ and f^μ,

$$h_\nu^\mu=\delta_\nu^\mu+u^\mu u^\nu-f^\mu f^\nu,$$

whose only nonvanishing components are $h_2^2=h_3^3=1$. In order for the eigenvectors u^μ and f^μ to be real it is necessary that G_0^1 and G_1^0 have the same sign. The density and radial pressure are given by

$$\rho=-G_0^0-G_1^0\sqrt{\frac{G_0^1}{G_1^0}},\quad P_r=G_0^0-G_1^0\sqrt{\frac{G_0^1}{G_1^0}},$$

while tangential pressure is given by

$$P_\perp=G_2^2=G_3^3. \tag{4.5}$$

It is common to impose the dominant energy condition $\rho>|P|$ as a physical requirement on the system, where P is the pressure in any direction (and discounting the "extreme" case $\rho=\pm P$). Using $\rho+P_r>0$ we obtain

$$G_1^0<0,\quad G_0^1<0 \tag{4.6}$$

and the density and radial pressure are given by

$$\rho=-G_0^0+\sqrt{G_1^0G_0^1},\quad P_r=G_0^0+\sqrt{G_1^0G_0^1}. \tag{4.7}$$

The dominant energy conditions $\rho>|P_r|$, $\rho>|P_\perp|$ imply the further inequalities

$$G_0^0<0,\quad |G_2^2|<-G_0^0+\sqrt{G_1^0G_0^1}, \tag{4.8}$$

which also guarantee positive density, $\rho>0$.

For a perfect fluid with a baryotropic equation of state, we have $P=P_r=P_\perp$ with

$$P=\gamma\rho,\quad -1<\gamma<1$$

and substitution in Eqs. (4.5) and (4.7) results in

$$G_2^2=G_3^3=\frac{2\gamma}{\gamma-1}G_0^0,\quad \sqrt{G_1^0G_0^1}=\frac{\gamma+1}{\gamma-1}G_0^0\quad (G_0^0<0). \tag{4.9}$$

For radiation, $\rho=\frac{1}{3}P$, we have

$$G_2^2=-G_0^0,\quad \sqrt{G_1^0G_0^1}=-2G_0^0.$$

From Eqs. (4.1)–(4.3) and (4.7) we see that if $\rho\to\infty$ as $x\to0$, then $q>0$ or $p>-2$. If the pressure is nonextreme in this limit, $P_r\gtrsim-\rho$ as $x\to0$ then $q\leqslant p+2$. The only way these conditions are consistent is if

$$p>-2,\quad q\leqslant p+2.$$

A detailed discussion for the case $kl\neq0$ (timelike or spacelike singularity) and $q<p+2$ results in the following conclusion of SI:

A timelike singularity of power-law type, in whose neighborhood the energy-stress tensor satisfies the dominant energy condition, must either

(1) *be a (dustlike, $P=0$) shell-cross singularity, or*

(2) *have an asymptotically extreme equation of state ($|P_r|\approx\rho$ or $|P_\perp|\approx\rho$), or*

(3) *possess a negative pressure ($P_r<0$ or $P_\perp<0$) in its neighborhood.*

However the case $q=p+2$ has, for no good reason, been omitted in the analysis given in SI. We now give details of this case.

The case kl=±1, q=p+2, p>−2

Since we must have $q>0$ in this case, the dominant behavior of the various components of G_ν^μ as $x\to0$ is, on setting $\varepsilon=kl=\pm1$,

$$G_0^0\approx-x^{-q}(e^{-\beta_0}+\varepsilon e^{-\alpha_0}(q^2-q)) \tag{4.10}$$

$$G_0^1\approx G_1^0\approx-x^{-q}e^{-\alpha_0}\frac{1}{2}q(q-2) \tag{4.11}$$

$$G_2^2\approx-x^{-q}e^{-\alpha_0}\varepsilon\frac{1}{2}(q-2)^2. \tag{4.12}$$

By Eq. (4.6) and $q>0$ it follows that $q>2$, and using Eqs. (4.5) and (4.7), we have

$$\rho\approx x^{-q}\left(e^{-\beta_0}+e^{-\alpha_0}\frac{q}{2}(2\varepsilon(q-1)+q-2)\right) \tag{4.13}$$

$$P_r\approx x^{-q}\left(-e^{-\beta_0}+e^{-\alpha_0}\frac{q}{2}(-2\varepsilon(q-1)+q-2)\right) \tag{4.14}$$

$$P_\perp\approx-x^{-q}e^{-\alpha_0}\varepsilon\frac{(q-2)^2}{2}. \tag{4.15}$$

For the case of a spacelike singularity $\varepsilon=1$,

$$\rho\approx x^{-q}\left(e^{-\beta_0}+e^{-\alpha_0}\frac{q(3q-4)}{2}\right)>0\quad \text{since}\ \ q>2.$$

However, pressures in both radial and tangential directions are negative,

$$P_r\approx x^{-q}\left(-e^{-\beta_0}-e^{-\alpha_0}\frac{q^2}{2}\right)<0,$$

$$P_\perp\approx-x^{-q}e^{-\alpha_0}\frac{(q-2)^2}{2}<0.$$

Thus, while the dominant energy conditions

$$\rho+P_r=x^{-q}e^{-\alpha_0}q(q-2)>0$$

and

$$\rho+P_\perp=x^{-q}(e^{-\beta_0}+e^{\alpha_0}(q^2-2))>0$$

clearly hold for $q>2$, the negative pressures do not allow for a radiation limit.

In the case of a timelike singularity, $\varepsilon=-1$, we have

$$\rho\approx x^{-q}\left(e^{-\beta_0}-e^{-\alpha_0}\frac{q^2}{2}\right),$$

$$P_r\approx x^{-q}\left(-e^{-\beta_0}+e^{-\alpha_0}\frac{q(3q-4)}{2}\right),$$

$$P_\perp \approx x^{-q} e^{-\alpha_0} \frac{(q-2)^2}{2} > 0.$$

The positive density condition, $\rho > 0$, gives

$$e^{-\beta_0} > e^{-\alpha_0} \frac{q^2}{2}$$

from which, using $q > 2$, it is possible to verify the radial dominant energy inequality $G_0^0 < 0$. Setting

$$e^{-\beta_0} = e^{-\alpha_0}\left(\frac{q^2}{2} + C_0(u)\right) \quad \text{where} \quad C_0(u) > 0,$$

we have

$$\rho \approx x^{-q} e^{-\alpha_0} C_0, \tag{4.16}$$

$$P_r \approx x^{-q} e^{-\alpha_0}(-C_0 + q^2 - 2q), \tag{4.17}$$

$$P_\perp \approx x^{-q} e^{-\alpha_0} \frac{(q-2)^2}{2}. \tag{4.18}$$

Essentially any sensible equation of state can be obtained in the vicinity of the singularity $x=0$ by a judicious choice of the function $C_0(u)$. For example, in the case of a perfect fluid (isotropic pressure),

$$P_r = P_\perp \Rightarrow C_0(u) = \frac{q^2}{2} - 2 > 0$$

and

$$\gamma = \frac{P}{\rho} = \frac{q-2}{q+2} > 0.$$

A radiative equation of state $\gamma = \frac{1}{3}$ is achieved if

$$q = 4, \quad C_0(u) = 6.$$

V. NULL SINGULARITIES

The case where $x=0$ is a null singularity, $kl=0$ is not considered in SI, and should not be discarded without further investigation. In this case the dominant energy condition with $\rho \not\to -P_r$ as $x \to 0$, together with $\rho \to \infty$ implies that

$$q \leqslant p+1, \quad p > -1. \tag{5.1}$$

There are three essential cases to consider.

(i) l=1, k=0

Since $x=u$ in this case the particular choice of series expansion in Eqs. (3.13) and (3.14) means that both functions U and V appearing in the double null coordinate form of the metric are functions of u alone: $U=U(u)$ and $V=V(u)$. Hence $\alpha_v = \beta_v = \beta_{vv} = 0$ in Eqs. (4.1)–(4.4), and consequently $G_1^0 = 0$. By using Eq. (4.7) we arrive at the physically unacceptable condition $\rho = -P_r$ in the neighborhood of the singularity.

(ii) l=0, k=1, p>−1, q<p+1

In this case $x=v$ and the principal terms in Eqs. (4.1)–(4.4) are

$$G_0^0 \approx -v^{-p-1} e^{-\alpha_0} \beta_0'(u),$$

$$G_1^0 \approx -v^{-p-2} e^{-\alpha_0} q(1+p-\tfrac{1}{2}q),$$

$$G_0^1 \approx -v^{-p} e^{-\alpha_0}(-\beta_0'' + \alpha_0'\beta_0' - \tfrac{1}{2}(\beta_0')^2),$$

$$G_2^2 \approx -v^{-p-1} e^{-\alpha_0} \frac{q}{2} \beta_0'.$$

The inequality $G_1^0 < 0$ implies $q(1+p-\frac{1}{2}q) > 0$. Hence, if $q<0$ then $p < \frac{1}{2}q - 1 < -1$, contradicting the stated condition $p > -1$. Thus $q > 0$. On the other hand, the inequality $G_0^0 < 0$ gives $\beta_0' > 0$, and the tangential pressure must be negative, $P_\perp = G_2^2 < 0$. This certainly does not permit an isotropic radiative fluid to be present near a null singularity of this type.

We are left to consider one final case.

(iii) l=0, k=1, p>−1, q=p+1

The components of the Einstein tensor are asymptotically dominated by the following terms:

$$G_0^0 \approx -v^{-p-1}(e^{-\beta_0} + e^{-\alpha_0}\beta_0'(u)),$$

$$G_1^0 \approx -v^{-p-2} e^{-\alpha_0} \frac{q^2}{2},$$

$$G_0^1 \approx -v^{-p} e^{-\alpha_0}(-\beta_0'' + \alpha_0'\beta_0' - \tfrac{1}{2}(\beta_0')^2),$$

$$G_2^2 \approx -v^{-p-1} e^{-\alpha_0} \frac{q}{2} \beta_0'.$$

By the inequalities (4.6) and (4.8) we may set

$$e^{-\beta_0} = -e^{-\alpha_0}\beta_0'(u) + e^{-\alpha_0} A_0(u)$$

where

$$A_0(u) > 0,$$

and

$$-\beta_0'' + \alpha_0'\beta_0' - \frac{1}{2}(\beta_0')^2 = 2B_0^2(u)$$

where

$$B_0(u) > 0.$$

Density and pressure components are found from Eqs. (4.5) and (4.7),

$$\rho \approx v^{-p-1} e^{-\alpha_0}(A_0(u) + qB_0(u))$$

$$P_r \approx v^{-p-1} e^{-\alpha_0}(-A_0(u) + qB_0(u))$$

$$P_\perp \approx -v^{-p-1} e^{-\alpha_0} \frac{q}{2} \beta_0'.$$

If we require space-time to be radiation dominated in the neighborhood of $v=0$, then

$$-\frac{q}{2}\beta_0'(u)=-A_0(u)+qB_0(u)=\frac{1}{3}(A_0(u)+qB_0(u))$$

which gives the readily satisfied conditions

$$\beta_0'(u)<0,\quad A_0(u)=-\frac{q}{2}\beta_0'(u),$$

$$B_0(u)=-\beta_0'(u). \tag{5.2}$$

The conclusion is that for a spherically symmetrical solution, a power-law type singularity surface can occur for an isotropic radiative equation of state $P=\frac{1}{3}\rho$, which is either timelike or null, provided there is the simple relation $q=p+2$, with $q=4$ and $p=2$, and $q=p+1$ with $p>-1$ respectively in the leading exponents of U and V in Eq. (3.8).

An interesting property of these singularities is linked to the area whose magnitude is 4π times the coefficient of $d\Omega^2$ in the expression for the metric. If the singularity has zero area, it can be considered as a central focus. If its area is finite, the singularity is usually regarded as being a shell cross. In Eq. (3.2), the coefficient of $d\Omega^2$ is e^V, which, from the definition retained in Eq. (3.8), is equal to $x^q e^\beta$. Therefore, every singularity $x=0$ such that $q>0$ is a central focus. This is the case for both the timelike and null singularity solutions identified above.

VI. COSMOLOGICAL APPLICATIONS

We now turn our attention to the cosmological consequences we can derive from the above stated results. If we consider the physically consistent picture of a universe which is first radiation dominated and after a period of cooling, becomes dust dominated, we are now provided with two different ways of giving a final solution to the horizon problem, using the scheme of Fig. 2. The first is to assume, as in [1,3,4], that a consistent approximation of the dust dominated region of the universe can be a model pertaining to the DBB class. We shall discuss below some salient features of this class of models. The second is to take advantage of the new results to suggest that in a radiation dominated primordial universe timelike or null singularities can occur therefore getting rid of any horizon problem, whatever the properties of the dust dominated era to come.

One feature of the DBB model worth taking into account is the nature of the constant energy density surfaces. We have seen indeed, in Sec. II, that causality is restored between every pair of points on the last-scattering surface, provided the backward light cone issued from these points reconnects at the "center" of the model before reaching the singularity. This can be achieved by the virtue of a nonspacelike constant energy density surface interposed between the last-scattering surface and the singularity. In a pure dust DBB model, the shell-crossing singularity, which can be viewed as a surface of infinite "constant" energy density, is timelike. We show, in the following, that this timelike property is shared, in this model, by a set of constant high-energy density surfaces, but that less such energetic surfaces are spacelike. We are therefore induced to look for peculiar subclasses of DBB models for which the nonspacelike nature would be shared by a sufficiently broad set of constant high-energy density surfaces such as to include surfaces with energy densities smaller than the limit where the dust and radiative energy densities are of the same order of magnitude. Such models would be free of any horizon problem, as one can convince oneself by replacing, in Fig. 2, the nonspacelike singularity by a nonspacelike (timelike) constant density surface.

In the dust dominated region of a DBB model, corresponding to a zero energy (spatially flat) Lemaître-Tolman-Bondi solution [7–9], the line element in comoving coordinates (r,θ,φ) and proper time t is

$$ds^2=-c^2dt^2+R'^2(r,t)dr^2+R^2(r,t)(d\theta^2+\sin^2\theta d\varphi^2). \tag{6.1}$$

With the radial coordinate r defined as in [3], we obtain an expression for the metric component R,

$$R(r,t)=\left(\frac{9GM_0}{2}\right)^{1/3}r[t-t_0(r)]^{2/3}, \tag{6.2}$$

and for the energy density

$$\rho(r,t)=\frac{1}{2\pi G[3t-3t_0(r)-2rt_0'(r)][t-t_0(r)]}, \tag{6.3}$$

where $t_0(r)$ is an arbitrary function of r, such that $t=t_0(r)$ is the focusing "big bang" singularity surface for which $R(r,t)=0$.

We see from Eq. (6.3) that the equation for the surfaces with constant energy density can be written

$$D(r,t)=[3t-3t_0(r)-2rt_0'(r)][t-t_0(r)]=\text{const.} \tag{6.4}$$

The normal, n_β, to this surface is

$$n_\beta\propto(\dot{D},D',0,0), \tag{6.5}$$

where a dot denotes the derivative with respect to t, and a prime the derivative with respect to r.

From Eq. (6.4), we get

$$\dot{D}=2(3t-3t_0-rt_0'), \tag{6.6}$$

$$D'=2[rt_0'^2-(4t_0'+rt_0'')(t-t_0)], \tag{6.7}$$

and substitution into Eq. (6.5), after simplifying by the constant factor 2, results in

$$n_\beta\propto(3t-3t_0-rt_0',rt_0'^2-(4t_0'+rt_0'')(t-t_0),0,0). \tag{6.8}$$

Using the metric tensor components as they appear in Eq. (6.1), we can write

$$n_\beta n^\beta=-c^2(3t-3t_0-rt_0')^2+R'^2[rt_0'^2-(4t_0'+rt_0'')(t-t_0)]^2. \tag{6.9}$$

It can now be verified that on the shell-crossing surface $R'=0$, corresponding to $3t-3t_0-2rt_0'=0$ [3], the vector magnitude $n_\beta n^\beta$ is negative for every value of r and t, confirming the timelike nature of this surface (see the Appendix).

We now consider a constant energy density surface located in the $R'>0$ region, with ρ sufficiently large to be allowed to write

$$3t-3t_0-2rt_0'=\frac{1}{2\pi G\rho(t-t_0)}=\epsilon(r,t), \qquad (6.10)$$

with $0<\epsilon(r,t)\ll rt_0'(r)$ for every r and t. In this case,

$$n_\beta n^\beta=-c^2(\epsilon+rt_0')^2+\left(\frac{GM_0}{2}\right)^{2/3}\frac{\epsilon^2}{(\epsilon+2rt_0')^{2/3}} \times\left[rt_0'^2-(4t_0'+rt_0'')\left(\frac{\epsilon+2rt_0'}{3}\right)\right]^2, \qquad (6.11)$$

where the right-hand side is dominated by the negative term of zero order in ϵ, namely $-c^2r^2t_0'^2$. The corresponding surface of constant ρ is therefore timelike.

On the other hand, a constant low energy density surface, satisfying

$$3t-3t_0-2rt_0'=\frac{1}{2\pi G\rho(t-t_0)}=\frac{1}{\epsilon(r,t)}, \qquad (6.12)$$

where $0<\epsilon(r,t)rt_0'(r)\ll 1$ for every r and t, gives, after an expansion in powers of ϵ^{-1},

$$n_\beta n^\beta=-\frac{c^2}{\epsilon^2}(1+\epsilon rt_0')^2+\left(\frac{GM_0}{2}\right)^{2/3}\frac{(4t_0'+rt_0'')^2}{9\epsilon^{10/3}} \times[1-\mathcal{O}(\epsilon rt_0')][1-\mathcal{O}(\epsilon rt_0')]^2. \qquad (6.13)$$

The positive term of $\frac{10}{3}$th order in ϵ^{-1} dominates in this equation, and the corresponding surface of constant ρ is therefore spacelike.

In [1,3] the physical assumption is made that the surface of last-scattering is a spacelike constant temperature (i.e., constant low energy density) surface, located in the dust dominated era. When traveling backward on an incoming light cone emitted from any point on this surface, we therefore cross the spacelike $\rho=$const surfaces exhibiting growing energy densities until we reach either the region where these surfaces become timelike or the radiation dominated era.

If the first timelike $\rho=$const surface is still located in the dust dominated region, the light cone is bound to reconnect at the center before reaching this surface and the horizon problem naturally disappears for any observer looking at any point on the last-scattering surface. If the radiation dominated domain is reached first, we can contemplate two possibilities.

First, we may be brought back to an inflationlike configuration (see Fig. 1), where the horizon problem can be (temporarily) solved for a given observer, provided the backward incoming light cone issued from the "point" (i.e., two-surface) she sees on the last scattering spends enough space-time in the dust dominated era. This implies some fine tuning of the parameters of the model, i.e., the observer location r_0 and the expression for the $t_0(r)$ function, of which the essential variable is the slope. Although the allowed values for these parameters can be chosen from related infinite sets, such a solution provides less intellectual satisfaction than our alternative proposal. We need a means to discriminate between DBB models which do or do not exhibit timelike ρ_d surfaces such that $\rho_d<\rho_{eq}$, where ρ_{eq} is the value of the energy density for which the dust ρ_d and radiation components are equivalent. While this cannot be done analytically, the problem can easily be solved numerically for any given profile of the "bang" function $t_0(r)$ (see, e.g., [3] where two examples have been numerically solved).

Alternatively, the radiation dominated region may be smoothly connected to the dust dominated region by using a model among the solutions of Einstein's equations exhibiting a singularity of the timelike or null type identified in Secs. IV and V. It remains a problem, however, that if such solutions exist, we do not have sufficient knowledge of their other properties to provide a matching of these solutions with the DBB ones.

Notwithstanding these difficulties, it seems to us that the simplest way of resolving the horizon problem is to take advantage of the new results stated in the present article and only consider the primordial region of the universe, i.e., the neighborhood of the singularity which we can physically assimilate to an era of energy density approaching the Planck scale. If this region can be represented by one of the radiative models identified in Secs. IV and V as exhibiting a timelike or null singularity, then the problem is definitely solved.

VII. APPLICATION TO THE SCCH

The strong cosmic censorship hypothesis (SCCH) was proposed in 1979 by Penrose [11] after the debate which had followed his first proposal, in 1969, of the cosmic censorship hypothesis [12]. The SCCH runs as follows:

No physically realistic collapse leads to a locally naked (i.e., timelike) singularity.

These singularities are visible from regular points of space-time, but possibly not at infinity. However, from the point of view of infalling particles, such singularities must be as worrying as those visible at infinity, since they are likely to upset the physical conditions in their space-time neighborhood. (See, e.g., SI for a further discussion of this issue.)

In Sec. IV, we have identified a class of power-law type focusing timelike singularities, with spherical symmetry, exhibiting in their vicinity the stress-energy tensor of a radiative perfect fluid. Reversing the sign of time, we obtain a corresponding class of solutions to Einstein's equations which represent, to a good approximation, a spherical cloud of collapsing gas near its focusing point. It therefore constitutes an interesting physically consistent counterexample to the SCCH.

VIII. CONCLUSION

In this paper, we have extended the study, initiated in SI, of spherically symmetric metrics satisfying the dominant energy condition and of which the singularities are of power-law type. We have identified two classes of peculiar interest:

(1) A *timelike* class exhibiting in the neighborhood of its *focusing* singularity the stress-energy tensor of a perfect fluid, with the equation of state of *radiation*: $p=\frac{1}{3}\rho$.

(2) A set of *null* classes verifying identical properties.

We have considered two important applications of these results:

(1) *In cosmology, the possibility of solving the horizon problem.* We have reviewed in Sec. II how a timelike or null singularity is a sufficient condition for a cosmological model to be rid of this cumbersome problem. Therefore, if we consider the physically consistent picture of a universe which is first radiation dominated and, after a period of cooling, becomes dust dominated, we can take advantage of our new results to state that Einstein's equations permit the existence of solutions exhibiting nonspacelike singularities having physical conditions in their neighborhood consistent with the primordial region of the universe. These can be assimilated to a region of energy density approaching the Planck scale (beyond which general relativity is generally believed to break down). Choosing to describe this region with one of the radiative models corresponding to the timelike or null singularities identified in Secs. IV and V allows us to solve *permanently* the horizon problem, as has been stressed in Sec. II.

Together with the DBB model [3], first proposed to solve the horizon problem in a geometrical way, these results provide us with new candidates to achieve this without need to resort to an inflationary scenario. It is worth noting that, contrary to the DBB solutions which exhibit a shell-crossing singularity, those proposed here arise from a *focus*.

(2) *In gravitational collapse, a counterexample to the SCCH.* If we limit ourselves to the consideration of the focusing *timelike* singularities identified in Sec. IV, the corresponding class of solutions to Einstein's equations represents, to a good approximation, a spherical cloud of collapsing gas near its focusing point, contradicting the commonly believed strong cosmic censorship hypothesis.

Some further cosmological conclusions can incidentally be derived from the results first obtained in SI. We recall that, in that work, it was shown that in the neighborhood of a timelike singularity of power-law type which is not a shell cross, the dominant energy condition can be satisfied if there is an asymptotically extreme energy-stress tensor ($|P_r|\sim\rho$ or $|P_\perp|\sim\rho$) or one of the pressures is negative. As a negative pressure is characteristic of a cosmological constant dominated universe, such a model is not likely to exhibit any horizon problem.

We finish by noting that the timelike nature of any spherically symmetric shell-crossing singularity, stated in the Appendix, would allow us to extend from the DBB cosmological models to *nonflat* predominantly dust cosmological models the property of solving the horizon problem. We leave the discussion of such models to future works, stressing once more the nice geometrical properties possessed by the geodesics in some peculiar classes of *inhomogeneous* models of the universe.

APPENDIX

In this appendix, we give a derivation of the timelike character of the shell-crossing singularity in a spherically symmetric model, generalizing a line of reasoning first proposed by Hellaby and Lake in [6].

The general spherically symmetric line element can be written

$$ds^2=-B^2(r,t)dt^2+A^2(r,t)dr^2+R^2(r,t)(d\theta^2+\sin^2\theta d\varphi^2). \tag{A1}$$

A typical shell-crossing surface $t=b(r)$ is such that

$$A=[t-b(r)]^a f(r,t)=0, \quad B\neq 0, \quad R\neq 0. \tag{A2}$$

The normal, n_α, to the surface $A=\text{const}$ (here $A=0$), is

$$n_\alpha \propto (\dot{A},A',0,0). \tag{A3}$$

With the metric of Eq. (A1), the squared norm of this normal vector is

$$n_\alpha n^\alpha=-\dot{A}^2B^2+A'^2A^2. \tag{A4}$$

According to Eq. (A2), $A^2=0$, and the above expression for $n_\alpha n^\alpha$ is always negative, implying, with our choice of the metric signature, a timelike shell-crossing surface.

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