Experimental tests of curvature couplings of fermions in general relativity

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Spin-$\frac{1}{2}$ particles in geodesic trajectories experience no gravitational potential but they still have nonzero couplings to the curvature tensor. The effect of space-time curvature on fermions can be parametrized by a vector and a pseudovector potential. These apparent CPT-violating terms can be measured with satellite-based spin-polarized torsion balance and clock comparison experiments. The Earth's curvature effect is of the order of 10^{-37} GeV, which is not far from the present bounds of $\sim 10^{-29}$ GeV on such CPT-violating couplings.

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The effect of the gravitational potential on the quantum mechanics of elementary particles was observed in the classic Colella-Overhauser-Werner (COW) experiment [1] using neutron interferometry. In general relativity, it is possible to choose a local inertial (free fall) frame where the gravitational potential on an elementary particle is zero. The curvature tensor in such frames, however, cannot be made to vanish even locally. The experimental tests of this curvature coupling would be an important test of general relativity. The couplings of spin-$\frac{1}{2}$ particles to a curved space background have been studied by many authors [2–9]. Parker [2] studied the effect of the Riemann curvature of the Schwarschild metric on the energy levels of hydrogen atoms. It was found that the energy-level shift on the surface of the Sun is of the order of $GM_{\odot}/(R^3 m_e) \sim 10^{-51}$ GeV, which is too tiny to have any observational consequence. The effect of rotation in the curved space metric on spin-$\frac{1}{2}$ particles has been studied in [3–9].

In this paper, we show that the gravitational curvature couplings can be parametrized as an external vector $\bar{\psi}\gamma^{\mu}a_{\mu}\psi$ and a pseudovector $\bar{\psi}\gamma^5\gamma^{\mu}b_{\mu}\psi$ interaction term in the Lagrangian. In a rotating inertial frame (such as in the frame of a satellite orbiting the Earth), both a_{μ} and b_{μ} have some nonzero components, while in a nonrotating free fall frame only a_{μ} is nonzero. From the phenomenological point of view, a nonzero b_{μ} is of interest as it can be measured in various experiments [11–18] discussed below. The magnitude of $|\vec{b}|$ in Earth orbiting satellites is of the order of 6.5×10^{-37} GeV. This can be compared with the sensitivity of experiments [12–14] such as spin-polarized torsion balance and spin magnetization measurements, which can measure up to $|\vec{b}| \sim 10^{-29}$ GeV. The best prospects for measuring curvature effects on fermions is by using macroscopic spin-polarized substances in Earth orbit satellites.

The general invariant coupling of spin-$\frac{1}{2}$ particles to gravity is described by the Lagrangian [2–9]

$$\mathcal{L} = \sqrt{-g}(\bar{\psi} i \gamma^a D_a \psi - m\bar{\psi}\psi), \tag{1}$$

where

$$D_a = e_a^{\mu}\left(\partial_{\mu} - \frac{i}{4}\omega_{bc\mu}\sigma^{bc}\right), \tag{2a}$$

$$\sigma^{bc} = \frac{i}{2}[\gamma^b, \gamma^c], \tag{2b}$$

$$\omega_{bc\mu} = e_{b\lambda}(\partial_{\mu} e^{\lambda}{}_c + \Gamma^{\lambda}_{\gamma\mu} e^{\gamma}{}_c),$$

where a, b, c, etc., denote flat space indices, and α, β, γ, etc., are the curved space indices. We use coordinates which are locally inertial along the entire geodesic trajectory of the particles (called the Fermi normal coordinates [10]). The metric in these coordinates to second order takes the form

$$g_{00} = -1 - R_{0l0m} X^l X^m, \tag{3a}$$

$$g_{0i} = g^{0i} = \tfrac{2}{3} R_{0lim} X^l X^m, \tag{3b}$$

$$g_{ij} = \delta_{ij} - \tfrac{1}{3} R_{iljm} X^l X^m, \tag{3c}$$

where X^i are the spatial coordinates of an event occurring at time X^0. The corresponding vierbeins are given by

$$e^{\alpha}{}_0 = \delta^{\alpha}{}_0 - \tfrac{1}{2} R^{\alpha}{}_{l0m} X^l X^m, \tag{4a}$$

$$e^{\alpha}{}_i = \delta^{\alpha}{}_i - \tfrac{1}{6} R^{\alpha}{}_{lim} X^l X^m. \tag{4b}$$

In this coordinate system, Christoffel connections $\Gamma_{\mu}{}^{a}{}_{\nu} = 0$, on a geodesic, but their first derivatives are nonzero and are related to the Riemann tensor as given by

$$\Gamma_{\mu}{}^{\alpha}{}_{0,\nu} = R_{\mu}{}^{\alpha}{}_{\nu 0}, \tag{5a}$$

$$\Gamma_i{}^{\mu}{}_{j,k} = -\tfrac{1}{3}(R_i{}^{\mu}{}_{jk} + R_j{}^{\mu}{}_{ik}) \tag{5b}$$

on any point along the geodesic. Using the coordinates described in Eqs. (3)–(5), it can be shown, after some algebra, that the Dirac Lagrangian (1) can be written in the form

$$\mathcal{L} = i(\det e)\bar{\psi}[\gamma^0\partial_0 + \gamma^i\partial_i - a_0\gamma^0 - a_i\gamma^i - b_0\gamma_5\gamma^0 - b_i\gamma_5\gamma^i + im]\psi \tag{6}$$

(where $i = 1,2,3$), the vector $(a_0, \vec{a})$ is given by

$$a_0 = \tfrac{1}{4} R_0{}^i{}_{im} X^m, \tag{7a}$$

$$a_i = \tfrac{1}{2} R_{i00m} X^m - \tfrac{1}{4} R_i{}^j{}_{jm} X^m, \tag{7b}$$

and the pseudovector $(b_0, \vec{b}_i)$ is given by

$$b_0 = \tfrac{1}{8}\epsilon_{0ijk}R^{jki}{}_{m}X^m, \tag{8a}$$

$$b_i = \tfrac{1}{4}\epsilon_{0kji}R^{kj}{}_{0m}X^m + \tfrac{1}{4}\epsilon_{0kji}R^{0jk}{}_{m}X^m. \tag{8b}$$

From Eq. (6), we see that the effect of space-time curvature appears as an external vector coupling $\gamma^\mu a_\mu$ and pseudovector coupling $\gamma^5\gamma^\mu b_\mu$ and is formally similar to the effective CPT- and Lorentz-violating Lagrangian [11–18].

Although the terms arising from gravitational curvature couplings in Eq. (6) are the same as in the explicit CPT-violating interactions studied in [11–18], there is a fundamental difference between the two, in that, for the case of gravitational couplings, there is no CPT of Lorentz symmetry violation. If one treats the vectors a_μ and b_μ as fixed external vectors which do not transform under CPT (as is done in Refs. [11–18]), then since both $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma_5\gamma_\mu\psi$ are odd under CPT [19], the interaction terms $a_\mu\bar{\psi}\gamma^\mu\psi$ and $b_\mu\bar{\psi}\gamma_5\gamma^\mu\psi$ explicitly violate CPT. Similarly, the existence of preferred external four-vectors a_μ and b_μ explicitly violates the Lorentz invariance of vacuum. In the case of gravitational couplings (6), however, it can be checked explicitly that if one transforms the source currents which generate a_μ and b_μ correctly under CPT, then the interaction terms in Eq. (6) do not violate CPT. One can check this explicitly from the expressions for a_μ and b_μ given in Eqs. (10) and (12), respectively, for the case of a satellite orbiting a central body with angular velocity ω. Under time reversal operation $T{:}\,\omega \to -\omega$ and under parity P, the spatial three-vector $(X,Y,Z) \to (-X,-Y,-Z)$. One can see explicitly from the expressions for a_μ [Eq. (10)] and b_μ [Eq. (12)] that under CPT, both a_μ and b_μ pick up negative signs which compensate for the negative signs picked up by the fermion bilinears $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma_5\gamma_\mu\psi$ under CPT transformation, and the gravitational curvature terms (6) do not violate CPT. Similarly, since both a_μ and b_μ transform as four-vectors under Lorentz transformation, the interaction terms in Eq. (6) are Lorentz-invariant. If one considers the dynamics of fermions in a background gravitational field and neglects the backreaction of the fermions to the gravitating sources, then the phenomenological effects of gravitational curvature couplings in Eq. (6) will be the same as that of an external fixed vector field as studied in Refs. [11–18], and the experiments which can be used for testing CPT violation using fermions as test particles can also be used in principle for looking for gravitational curvature couplings.

We consider a coordinate system in a free fall orbit around a gravitating body. Such a system of coordinates would ideally describe satellite-based experiments. For Earth-based experiments, the gravitational source would be the Sun. Choosing without loss of generality orbital angular momentum around the Z axis, $\vec{\omega} = (0,0,\omega)$, we can write the nonzero components of the Riemann tensor in the orbital free fall coordinates as follows:

$$
\begin{aligned}
R_{1010} &= \frac{2GM}{R^3},\\
R_{0202} &= R_{0303} = -\frac{GM}{R^3},\\
R_{1212} &= \frac{GM}{R^3}[1+\omega^2(-Y^2+2X^2)],\\
R_{1313} &= \frac{GM}{R^3}(1-\omega^2Y^2),\\
R_{2323} &= -\frac{GM}{R^3}(2+\omega^2X^2),\\
R_{1202} &= R_{1303} = -\frac{GM}{R^3}\omega Y,\\
R_{2101} &= -\frac{2GM}{R^3}\omega X,\\
R_{2303} &= \frac{GM}{R^3}\omega X,\\
R_{2313} &= \frac{GM}{R^3}\omega^2 XY,
\end{aligned}
\tag{9}
$$

where we have chosen X along the radial direction and Y along the tangential direction of the orbit. Using the curvature components (9), we can compute the vector a_μ and pseudovector b_μ given in Eqs. (7) and (8) for a free fall coordinate rotating with angular velocity ω around the Z axis as follows. The vector couplings $\bar{\psi}\gamma^\mu a_\mu\psi$ of fermions is given by a_μ:

$$
\begin{aligned}
a_0 &= \frac{3}{4}\frac{GM\omega}{R^3}XY,\\
a_1 &= -\frac{GMX}{2R^3} + \frac{GM\omega^2X}{4R^3}(2X^2-Y^2),\\
a_2 &= \frac{GMY}{4R^3}[1+\omega^2(2X^2-Y^2)],\\
a_3 &= \frac{GMZ}{4R^3}[1-\omega^2(X^2+Y^2)].
\end{aligned}
\tag{10}
$$

The vector field a_μ can arise in a Schwarzschild metric [2]. In a solar mass star it can split the $2P_{3/2}$ levels of a hydrogen atom by the amounts [2]

$$\Delta E(2P) \simeq \frac{GM}{R^3}\left(\frac{1}{m_e}\right). \tag{11}$$

On the surface of the Sun, this level split is 3.8×10^{-51} GeV, and on the surface of the Earth, the hydrogen $3P_{3/2}$ levels will split by 1.2×10^{-51} GeV. This energy is unobservably small and at present there are no known experiments which can hope to measure the gravity-induced vector potential a_μ.

The effective pseudovector couplings $\bar{\psi}\gamma^5\gamma^\mu b_\mu \psi$ are described by components of b_μ:

$$(b_0, b_1, b_2, b_3) = \left(0,0,0,\frac{GM\omega}{R^3}X^2 \right). \tag{12}$$

Much more stringent bounds can be put on the pseudovector b_μ from CPT- and Lorentz-violation tests [11–18]. The pseudovector term in the nonrelativistic limit is equivalent to the interaction energy

$$H_I = -\vec{s}\cdot\vec{b} \tag{13}$$

due to the interaction of the fermion spin $\vec{s}$ with the external field $\vec{b}$. In experiments where a macroscopic number of fermions can be polarized in the same direction, this interaction energy may be measurable. In the Eot-Wash II experiment [12], the spin-polarized torsion balance has $N = 8\times 10^{22}$ aligned spins (with negligible net magnetic moment). There will be a torque on such a torsion balance of the magnitude $\tau = (N/\pi)\Delta E$, where $\Delta E = |\vec{b}|$ is the energy difference between the fermion spins polarized parallel and antiparallel to the external $\vec{b}$ field. From the results of this experiment [12], it is possible to measure up to $|\vec{b}| \sim 10^{-28}$ GeV [13].

Another method of probing $\vec{b}$ is to measure the net magnetization in a paramagnetic material using a squid [14]. An external $\vec{b}$ field appears as an effective magnetic field of strength $\vec{B}_{\rm eff} = (\vec{b}/\mu_B)$. The magnitude of the effective magnetic field which can be probed in this experiment is $B_{\rm eff} = 10^{-12}$ G, which translates to a measurement of the $\vec{b}$ field at the level of 10^{-29} GeV [13,14].

Bounds on spatial components of b_μ can also be put from muon properties [15], tests of QED in Penning traps [16], spectroscopy of hydrogen and antihydrogen [17], and in future clock-comparison tests with satellite-based atom clocks [18].

We have shown above that a nonzero b_μ arises due to curvature couplings of fermions in a rotating frame. On a satellite orbiting the Earth with a typical velocity of 7.5 km/sec, the value of $b_3 = (GM_\oplus / R_\oplus)\omega = 6.5\times 10^{-37}$ GeV. This is a factor of 10^{-6} smaller than the best available bounds at present, but it may be possible to measure curvature effects in future satellite-based experiments. If one considers the Earth's motion around the Sun, $b_3 = (GM_\oplus / R_{ES})\omega = 1.2 \times 10^{-40}$ GeV is much smaller. On the Mercury orbit around the Sun, taking the Mercury-Sun distance $\sim 55\times 10^6$ km and the period of Mercury orbit ~ 0.24 years, $b_3 = 2.1 \times 10^{-39}$ GeV. These estimates show that the best prospect for observing the curvature effects on fermions is probably by spin-polarized torsion balance experiments in low orbit Earth satellites.

There can be some interesting cosmological applications of gravitational curvature couplings in the early universe where curvature effects of the expanding universe are likely to be large. It has been shown by Bertolami *et al.* [20] that explicit CPT-violating terms in the Lagrangian can give rise to a net chemical potential for baryon number and generate a net baryon asymmetry in the presence of baryon-number-violating processes. In a separate paper [21], we show that T violation induced by the expanding universe R-W metric gives rise to a CP-violating interaction term in the Lagrangian which can give rise to net baryon asymmetry in the present universe.

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