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Unified treatment of the electron propagator near the mass shell in three temperature regions

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The behavior of the electron propagator near the mass shell is quite different in three regimes: zero temperature; low temperature, $T \ll m$; and high temperature, $T \gg m$. A unified explanation of the behavior in all three regimes is given in terms of the space-time dependence of the photon propagator along the trajectory $\vec{r}=\vec{v}t$, as $t\to\infty$, where $\vec{v}$ is the appropriate group velocity of the electron.

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I. INTRODUCTION

In quantum field theories without massless particles, the renormalized self-energies are finite functions of momentum. In particular, the self-energies are analytic at the mass shell: the value of the self-energies are finite and their derivatives are finite. Consequently the mass shift and the wave-function renormalization are finite.

QED does not enjoy these properties because of the massless photon. At zero temperature the one-loop self-energy of the electron is finite on the mass shell, but the derivative is infinite. This behavior has long been understood [1–5]. At non-zero temperature the behavior is quite different and has not been explained. At low temperatures, $0 < T \ll m$, conventional finite-temperature perturbation theory applies and the electron self-energy is found to be analytic at the mass shell [6]. The analytic behavior does not hold at high temperatures. When $T \gg m$, it is necessary to use the Braaten-Pisarski hard-thermal-loop propagators [7,8]. In this regime the dispersion curve that defines the mass shell condition for the electron is a complicated function of momentum, $\omega(p)$. It turns out the one-loop electron self-energy is infinite at the mass shell $\omega(p)$ and has an infinite derivative at that energy. The behavior of the one-loop electron self-energy in these three temperature regimes can be summarized as follows:

$$T=0: \quad \Sigma(P)\sim e^2(p_0-E)\ln(p_0-E) \tag{1.1a}$$

$$0<T\ll m: \quad \Sigma(P)\sim \text{analytic} \tag{1.1b}$$

$$T\gg m: \quad {}^*\Sigma(P)\sim ie^2\ln(p_0-\omega). \tag{1.1c}$$

The result for $T \gg m$ emerged from calculations of fermion damping rates [9–12] in which the goal was to calculate $\mathrm{Im}\,{}^*\Sigma(P)$ evaluated at $p_0=\omega$. It was found that the quark damping rate in QCD has infrared divergences that arise from static, transverse gluons and that these are cut off by the putative magnetic mass. It was also found that the electron damping rate at high temperature has similar infrared divergences from static, transverse photons and there is no magnetic screening. Consequently the imaginary part of the electron self-energy diverges logarithmically at the mass shell.

The situation was greatly clarified by Blaizot and Iancu [13], who obtained Eq. (1.1c) and showed that it leads to a retarded electron propagator that decays more rapidly than an exponential: $S_R(t,\vec{p})=-ie^{-i\omega t}e^{-\alpha T t\ln(t/t_0)}$. The same result was found by Boyanovsky *et al.* [14] using completely different methods.

The purpose of this paper is to explain all three results in Eqs. (1.1a)–(1.1c) in a simple, physical fashion. The common feature of Eqs. (1.1a) and (1.1c) is that they are not analytic at the mass shell of the electron. It is curious that in the intermediate temperature range, $0<T\ll m$, the electron self-energy is analytic at the mass shell despite the massless photon [6].

Useful guidance as to the source of nonanalyticity can be gained from considering the zero-temperature self-energy, expressed in terms of the Feynman propagators:

$$\Sigma_F(P)=ie^2\int\frac{d^4K}{(2\pi)^4}\gamma_\mu S_F(P-K)\gamma_\nu D_F^{\mu\nu}(K). \tag{1.2}$$

If a fictitious photon mass λ is introduced into the photon propagator, the self-energy will have a branch point when $(P-K)^2=m^2$ and $K^2=\lambda^2$. The branch point will occur when $P^2=(m+\lambda)^2$ and is produced by two singularities in the integrand that pinch the integration contour at the value

$$K^\mu=\frac{\lambda}{m+\lambda}P^\mu.$$

As long as $\lambda\neq 0$, the branch point at $P^2=(m+\lambda)^2$ is separated from the electron pole at $P^2=m^2$. However, true QED requires that $\lambda\to 0$ and this shifts the branch point down to $P^2=m^2$. The region of integration which produces the branch point is then $K^\mu\approx 0$.

At non-zero temperature, both with and without HTL resummation, the region of integration that determines whether the self-energy has a branch cut at the electron mass shell will still be $K^\mu\approx 0$. Some simple analysis will show that the nonanalytic behavior of the one-loop electron self-energy is controlled entirely by the asymptotic behavior of the time-ordered photon propagator in space-time,

$$\mathcal{D}_{11}^{\mu\nu}(t,\vec{r})=-i\mathrm{Tr}\{\varrho\, T[A^\mu(x)A^\nu(0)]\}, \tag{1.3}$$

where ϱ is the density operator. The limit that contributes is $t\to\infty$, $\vec{r}=\vec{v}t\to\infty$, where $\vec{v}$ is the group velocity of the electron. At zero temperature, $\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t)$ falls like $1/t^2$. At any non-zero temperature below the electron mass, it falls exponentially with t [15]. At temperatures well above the electron mass, it is necessary to use Braaten-Pisarski resummed propagators. The resummed photon propagator $^*\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t)$ behaves asymptotically like $1/t$ [16]. The present paper will show that these three asymptotic behaviors directly determine the results displayed in Eqs. (1.1a)–(1.1c).

Section II computes the self-energy at zero temperature and at low temperature. Section III examines the high temperature behavior where the hard-thermal-loop propagators must be used. Section IV discusses the results in terms of the Bloch-Nordsieck approximation. Many of the details are contained in the Appendixes.

II. LOW TEMPERATURE: $0\leqslant T\ll m$

At zero or low temperature, $0\leqslant T\ll m$, there is no need for hard-thermal-loop resummation. Conventional finite-temperature perturbation theory is directly applicable. The results of such a perturbative calculation are most easily expressed in terms of the retarded self-energy for the electron, $\Sigma_R(P)$, which is related to the full retarded propagator by

$$S_R'^{-1}(P)=\gamma\cdot P-m-\Sigma_R(P). \tag{2.1}$$

Appendix A summarizes the relation between the retarded propagator and the path-ordered propagators; and the relation between the retarded self-energy and the path-ordered self-energies.

In the rest frame of the plasma, rotational invariance requires that Σ_R be a combination of the matrices $1,\gamma_0,\vec{\gamma}\cdot\vec{p}$, and $\gamma_0\vec{\gamma}\cdot\vec{p}$. With this decomposition it is straightforward to compute the inverse of Eq. (2.1) and obtain

$$S'(P)_R=\frac{\gamma_\mu P^\mu+m+\Sigma_R-\frac{1}{2}\mathrm{Tr}[\Sigma_R]}{P^2-m^2-\Pi_R(P)}. \tag{2.2}$$

The scalar self-energy in the denominator is

$$\Pi_R(P)=\frac{1}{2}\mathrm{Tr}[(\not{P}+m)\Sigma_R] -\frac{1}{4}\mathrm{Tr}[\Sigma_R^2]+\frac{1}{8}(\mathrm{Tr}[\Sigma_R])^2. \tag{2.3}$$

For one-loop calculations only the first term in $\Pi_R(P)$ should be retained.

Appendix B performs some preliminary analyses of the one-loop retarded self-energy for the electron. The computation is organized into two parts:

$$\Sigma_R(P)=\Sigma^a(P)+\Sigma^b(P), \tag{2.4}$$

where

$$\Sigma^a(P)=ie^2\int\frac{d^4K}{(2\pi)^4}\gamma_\mu S_R(P-K)\gamma_\nu D_{11}^{\mu\nu}(K). \tag{2.5}$$

The other contribution, Σ^b, is defined in Eq. (B4b). Appendix B 2 shows that Σ^b is analytic at the mass shell and it will therefore be ignored. This section will analyze Σ^a to isolate the cause for a branch cut at the mass shell. The first step is to examine the free retarded propagator in Eq. (2.5):

$$S_R(P-K)=\frac{\gamma\cdot(P-K)+m}{(P-K)^2-m^2+i\eta(p_0-k_0)}.$$

The behavior of $\Sigma^a(P)$ at $P^2\approx m^2$ is controlled by the region in which K^μ is small. Since p_0 will always be positive and much larger than k_0, the imaginary part of the denominator can be replaced by $i\epsilon$. The p_0 derivative of $\Sigma^a(P)$ could only diverge at $p_0=(m^2+p^2)^{1/2}$ if the integration $\int d^4K$ diverges at small K^μ. One can safely omit the K^μ dependence of the numerator and omit the K^2 term in the denominator and use

$$S_R(P-K)\approx\frac{\gamma\cdot P+m}{P^2-m^2-2P\cdot K+i\epsilon}.$$

(Omitting the K^2 in the denominator spoils the ultraviolet convergence of the integration. The large K^μ behavior will have to be regulated later.) Since the mass shell is the region of interest, one can put $p_0=E+(p_0-E)$ and omit the terms of order $(p_0-E)^2$ from the denominator. This gives

$$S_R(P-K)\approx\frac{\gamma\cdot P+m}{2E(p_0-E+\vec{k}\cdot\vec{v}-k_0)+i\epsilon},$$

where the group velocity of the electron is

$$\vec{v}=\hat{p}\frac{\partial E}{\partial p}=\frac{\vec{p}}{E}. \tag{2.6}$$

It is convenient to write this approximate propagator as an integral over time:

$$S(P-K)\approx\frac{-i}{2E}(\gamma\cdot P+m)\int_0^\infty dt\, e^{it(p_0-E+\vec{k}\cdot\vec{v}-k_0+i\epsilon)}.$$

When this is substituted into Eq. (2.5) the integration over K becomes

$$\int\frac{d^4K}{(2\pi)^4}e^{i\vec{k}\cdot\vec{v}t-ik_0t}D_{11}^{\mu\nu}(K).$$

This integration produces the Fourier transform from $(k_0,\vec{k})$ to space-time $(t,\vec{r})$ on the trajectory $\vec{r}=\vec{v}t$. The Fourier transformed propagator is denoted by

$$\mathcal{D}_{11}^{\mu\nu}(t,\vec{r})=\int\frac{d^4K}{(2\pi)^4}e^{i\vec{k}\cdot\vec{r}-ik_0t}D_{11}^{\mu\nu}(K).$$

The part of the self-energy that could be nonanalytic at $p_0=E$ is

$$\Sigma^a(P)\approx\frac{e^2}{2E}\gamma_\mu(\gamma\cdot P+m)\gamma_\nu f^{\mu\nu}(P), \tag{2.7a}$$

where

$$f^{\mu\nu}(P)=\int_{t_0}^{\infty}dt e^{it(p_0-E+i\epsilon)}\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t). \quad (2.7b)$$

In $f^{\mu\nu}(P)$ the lower limit on the time integration has been set at t_0 in order to regulate the spurious short-distance divergence that was introduced by omitting the K^2 term from the denominator of $S_R(P-K)$.

Substituting the spinor self-energy Σ^a into Eq. (2.3) gives the scalar self-energy

$$\Pi^a(P)=\frac{2e^2}{E}P_\mu P_\nu f^{\mu\nu}(P)-\frac{e^2}{E}(P^2-m^2)g_{\mu\nu}f^{\mu\nu}(P).$$

Near the mass shell the first term is the more important and in $P_\mu P_\nu$ one can set $p_0=E$:

$$P^\mu|_{p_0=E}=E\,v^\mu; \qquad v^\mu=(1,\vec{v}). \quad (2.8)$$

Note that v^μ does not transform like a four-vector. The quantity v^μ determines both the polarization components of the virtual photon and the space-time trajectory of the photon. The part of the self-energy that is potentially nonanalytic at the mass shell is

$$\Pi^a(P)\approx 2Ee^2\int_{t_0}^{\infty}dt\, e^{it(p_0-E+i\epsilon)}v_\mu v_\nu \mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t). \quad (2.9)$$

This result shows that it is the large t behavior of the time-ordered photon propagator, $\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t)$, that determines whether $\Pi^a(P)$ is finite at $p_0=E$ and also whether $\partial\Pi^a(P)/\partial p_0$ is finite at $p_0=E$.

The remainder of this section evaluates Π^a in various cases. At zero temperature the asymptotic behavior $\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t)\rightarrow 1/t^2$ will produce a self-energy $\Pi^a(P)\sim(p_0-E)\ln(p_0-E)$. At non-zero temperature below the electron mass, the asymptotic behavior $\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t)\rightarrow \exp[-2\pi Tt(1-v)]$ will guarantee that $\Pi^a(P)$ is analytic at $p_0=E$.

It is worth emphasizing that one cannot set $m=0$ (or equivalently $v=1$) in these formulas. Proper treatment of the massless case requires hard thermal loop propagators and will be discussed in Sec. III.

A. Covariant gauges at $T=0$

The behavior of the electron self-energy near the mass shell has long been known at zero temperature and provides a useful check. In covariant gauges labeled by a parameter ξ the free photon propagator is

$$D^{\mu\nu}(K)=\frac{-g^{\mu\nu}}{K^2+i\epsilon}+(1-\xi)\frac{K^\mu K^\nu}{(K^2+i\epsilon)^2}.$$

The Fourier transform to space-time gives [15]

$$\mathcal{D}^{\mu\nu}(x)=-(\xi+1)\frac{ig^{\mu\nu}}{8\pi^2x^2}+(\xi-1)\frac{ix^\mu x^\nu}{4\pi^2(x^2)^2}. \quad (2.10)$$

The necessary projection is

$$v_\mu v_\nu \mathcal{D}^{\mu\nu}(x)|_{\vec{r}=\vec{v}t}=\frac{i(\xi-3)}{8\pi^2t^2}. \quad (2.11)$$

Note that from dimensional analysis alone the right-hand side must be proportional to $1/t^2$. The required time integration is

$$\int_{t_0}^{\infty}\frac{dt}{t^2}e^{it(p_0-E+i\epsilon)}=\frac{1}{t_0}-\frac{1}{t_0}\sum_{m=2}^{\infty}\frac{[i(p_0-E)t_0]^m}{(m-1)m!} -i(p_0-E)\{\ln[-i(p_0-E)t_0]-1+\gamma\}.$$

In the vicinity of the mass shell the non-analytic behavior is $(p_0-E)\ln[(p_0-E)t_0]$. The non-analytic part of the self-energy at $p_0\approx E$ is

$$\Pi^a(P)\approx(\xi-3)\frac{\alpha}{\pi}E(p_0-E)\ln[(p_0-E)t_0]. \quad (2.12)$$

This is the result indicated in Eq. (1.1a).

One can go a step further to make contact with conventional results for the full electron propagator. Appendix C summarizes how to obtain the full propagator in the Bloch-Nordsieck approximation from the one-loop self-energy and gives the final result, Eq. (C3).

B. Coulomb gauge at $T=0$

In the Coulomb gauge, $\mathcal{D}^{00}(x)=\delta(t)/4\pi r$ is instantaneous and thus does not contribute to Eq. (2.9). The transverse propagator

$$D^{ij}(K)=\left(\delta^{ij}-\frac{k^ik^j}{k^2}\right)\frac{1}{K^2+i\epsilon}$$

has a space-time transform [15]

$$\mathcal{D}^{ij}(x)=(\delta^{ij}-\hat{r}^i\hat{r}^j)\frac{i}{4\pi^2(t^2-r^2)} +(\delta^{ij}-3\hat{r}^i\hat{r}^j)\frac{i}{8\pi^2}\left(\frac{2}{r^2}-\frac{t}{r^3}\ln\left[\frac{t+r}{t-r}\right]\right). \quad (2.13)$$

The necessary projection is

$$v_iv_j\mathcal{D}^{ij}(x)|_{\vec{r}=\vec{v}t}=\frac{ib(v)}{8\pi^2t^2}. \quad (2.14)$$

From dimensional analysis thus must be proportional to $1/t^2$ as in Eq. (2.11). The function $b(v)$ is

$$b(v)=-2+\frac{1}{v}\ln\left[\frac{1+v}{1-v}\right]. \tag{2.15}$$

The required time integration is the same and the self-energy in the region $p_0 \approx E$ is

$$\Pi^a(P)\approx\frac{\alpha}{\pi}b(v)E(p_0-E)\ln[(p_0-E)t_0]. \tag{2.16}$$

The analytic structure is the same as in covariant gauges. Only the coefficient has changed. In the Bloch-Nordsieck approximation the one-loop correction exponentiates to give an electron propagator displayed in Eq. (C4).

C. Covariant gauges at $0<T\ll m$

At non-zero temperatures that are low, $0<T\ll m$, the time-ordered photon propagator in a general covariant gauge is

$$D_{11}^{\mu\nu}(K)=\left[-g^{\mu\nu}+(1-\xi)K^\mu K^\nu\frac{\partial}{\partial k^2}\right]\frac{1}{K^2+i\epsilon}$$
$$-\frac{2\pi i}{e^{\beta|k_0|}-1}\left[-g^{\mu\nu}+(1-\xi)K^\mu K^\nu\frac{\partial}{\partial k^2}\right]\delta(K^2).$$

The Fourier transform of this to space-time is computed in [15] with the result

$$\mathcal{D}_{11}^{\mu\nu}(x)=[-g^{\mu\nu}\Box+(1-\xi)\partial^\mu\partial^\nu]d_>(x),$$

where

$$d_>(x)=\frac{i}{16\pi^2}\ln\{\sinh[\pi T(r+t)]\sinh[\pi T(r-t)]\}.$$

One can check that at $T=0$ the propagator reduces to Eq. (2.10). The necessary projection is

$$v_\mu v_\nu\mathcal{D}_{11}^{\mu\nu}(x)|_{\vec r=\vec v t}=\frac{iT(1-v^2)}{8\pi v t}[\coth[\pi Tt(1+v)]$$
$$-\coth[\pi Tt(1-v)]]$$
$$+(\xi-1)\frac{iT^2}{16}\left[\frac{(1+v)^2}{\sinh^2[\pi Tt(1+v)]}\right.$$
$$\left.+\frac{(1-v)^2}{\sinh^2[\pi Tt(1-v)]}\right].$$

At zero temperature this coincides with Eq. (2.11). (Recall that $v\neq 1$ since $m\neq 0$.) The behavior of the electron self-energy in the region $p_0\approx E$ is determined by the large time behavior of the photon propagator. At large times, $t(1-v)\gg 1/\pi T$, the behavior is

$$v_\mu v_\nu\mathcal{D}_{11}^{\mu\nu}(x)|_{\vec r=\vec v t}\to\frac{iT(1-v^2)}{4\pi v t}[e^{-\pi Tt(1+v)}-e^{-\pi Tt(1-v)}].$$

Because of the exponential fall off with time, the function $f(p)$ in Eq. (2.9) is finite at $p_0=E$. Likewise all the derivatives $\partial^n f(P)/\partial p_0^n$ are finite at $p_0=E$. This implies that $f(P)$ is analytic at the mass shell as was found in [6].

D. Coulomb gauge at $0<T\ll m$

At low temperatures, $0<T\ll m$, the time-ordered propagator in the Coulomb gauge is

$$D_{11}^{ij}(K)=\left(\delta^{ij}-\frac{k^ik^j}{k^2}\right)\left[\frac{1}{K^2+i\epsilon}-\frac{2\pi i\,\delta(K^2)}{e^{\beta|k_0|}-1}\right].$$

The Fourier transform to space-time is performed in [15] with the result

$$\mathcal{D}_{11}^{ij}(x)=(\delta^{ij}-\hat r^i\hat r^j)\mathcal{D}_>(x)+(\delta^{ij}-3\hat r^i\hat r^j)E(x),$$

where

$$\mathcal{D}_>(x)=\frac{-iT}{4\pi r}\left[\frac{1}{e^{2\pi T(t+r)}-1}-\frac{1}{e^{2\pi T(t-r)}-1}\right]$$

$$E(x)=\frac{i}{8\pi^2r^2}\ln[1-e^{-2\pi T(t+r)}][1-e^{-2\pi T(t-r)}]$$
$$+\frac{i}{16\pi^3r^3T}\sum_{n=1}^{\infty}\frac{1}{n^2}[e^{-2\pi nT(t-r)}$$
$$-e^{-2\pi nT(t+r)}].$$

One can show [15] that at $T=0$ this propagator reduces to Eq. (2.13). The appropriate projection is

$$v_iv_j\mathcal{D}_{11}^{ij}(x)|_{\vec r=\vec v t}=-2v^2E(x)|_{\vec r=\vec v t}.$$

At large times, $t(1-v)\gg 1/\pi T$, the leading behavior is

$$v_iv_j\mathcal{D}_{11}^{ij}(x)|_{\vec r=\vec v t}\to\frac{-i}{8\pi^2v^2t^2}[e^{-2\pi Tt(1+v)}+e^{-2\pi Tt(1-v)}].$$

As before, the exponential fall off with time guarantees that $f(P)$ is analytic at $p_0=E$. This is a new result since the momentum space calculation in [6] did not include the Coulomb gauge.

III. HIGH TEMPERATURE: $T\gg m$

This section deals with the extreme high temperature regime, where it is necessary to use hard-thermal-loop propagators [7,8]. Some simple analysis leads to Eq. (3.11), which links the nonanalytic behavior of the electron self-energy to the asymptotic time dependence of the HTL photon propagator.

The starting point is the retarded HTL electron propagator

$${}^*S_R(P)=\frac{\frac{1}{2}(\gamma_0-\vec\gamma\cdot\hat p)}{D_+(P)}+\frac{\frac{1}{2}(\gamma_0+\vec\gamma\cdot\hat p)}{D_-(P)}, \tag{3.1}$$

where $D_{\pm}(P)$ are known functions of p_0 and p. Perturbative corrections beyond the HTL approximation give a self-energy $^*\Sigma_R(P)$ and thus an inverse propagator

$$^*S_R^{\prime -1}(P)=[\,^*S_R(P)]^{-1}-\,^*\Sigma_R(P). \qquad (3.2)$$

Invariance under chirality and parity limits the self-energy to be a linear combination of γ_0 and $\vec{\gamma}\cdot\vec{p}$. This makes it easy to invert Eq. (3.2) and obtain

$$^*S_R(P)=\frac{\frac{1}{2}(\gamma_0-\vec{\gamma}\cdot\hat{p})}{D_+(P)-\Pi_+(P)}+\frac{\frac{1}{2}(\gamma_0+\vec{\gamma}\cdot\hat{p})}{D_-(P)-\Pi_-(P)}, \qquad (3.3)$$

where the scalar functions $\Pi_{\pm}$ are given by

$$\Pi_{\pm}(P)=\frac{1}{4}\mathrm{Tr}[(\gamma_0\mp\vec{\gamma}\cdot\hat{p})\Sigma_R(P)]. \qquad (3.4)$$

The question under investigation is how $\Pi_{\pm}(P)$ behave near the zeros of $D_{\pm}(P)$.

The HTL propagator (3.1) has two positive-energy poles: $D_+(P)$ has a simple zero at $p_0=\omega_+(p)$ and $D_-(P)$ has a simple zero at $p_0=\omega_-(p)$. Both dispersion curves start at $\omega_{\pm}(0)=eT/\sqrt{8}$ and have the limiting behavior $\omega_{\pm}(p)\to p$ as $p\to\infty$. The residues of the HTL propagator poles are defined by

$$\frac{1}{Z_\sigma}=\left.\frac{\partial D_\sigma(P)}{\partial p_0}\right|_{p_0=\omega_\sigma},$$

so that

$$p_0\to\omega_\sigma: \quad D_\sigma(P)\to\frac{p_0+i\epsilon-\omega_\sigma}{Z_\sigma}. \qquad (3.5)$$

At zero momentum, $Z_+=Z_-=1/2$; at infinite momentum, $Z_+\to 1$ and $Z_-\to 0$.

The group velocity along each dispersion relation is

$$\vec{v}_\sigma=\hat{p}\frac{d\omega_\sigma(p)}{dp}. \qquad (3.6)$$

It will be necessary to express the group velocity differently. When the defining relation $D_\sigma(\omega_\sigma(p),p)=0$ is differentiated with respect to p the result is

$$\frac{d\omega_\sigma}{dp}\left.\frac{\partial D_\sigma}{\partial p_0}\right|_{p_0=\omega}+\left.\frac{\partial D_\sigma}{\partial p}\right|_{p_0=\omega}=0.$$

This allows the group velocity to be expressed as

$$\vec{v}_\sigma=-\hat{p}\left.\frac{\partial D_\sigma(P)/\partial p}{\partial D_\sigma(P)/\partial p_0}\right|_{p_0=\omega_\sigma}. \qquad (3.7)$$

A. One-loop self-energy

The modified Feynman rules that result from HTL expansion include a momentum-dependent vertex function for the photon and a new two-photon vertex [7,8]. These must be included in computing the one-loop self-energy of the electron. However, if the electron momentum is large, $p\gg T$, then only the bare vertex γ^μ is required. The following analysis will deal only with a hard electron.

Appendix B 1 shows that the one-loop retarded self-energy for the HTL propagator can be organized into two parts,

$$^*\Sigma_R(P)=\,^*\Sigma^a(P)+\,^*\Sigma^b(P), \qquad (3.8)$$

as defined in Eqs. (B6a) and (B6b). Appendix B 3 shows that $^*\Sigma^b$ is analytic at $p_0\approx\omega_\sigma(p)$. Therefore we need only examine $^*\Sigma^a$:

$$^*\Sigma^a(P)=ie^2\int\frac{d^4K}{(2\pi)^4}\gamma_\mu\,^*S_R(P-K)\gamma_\nu\,^*D_{11}^{\mu\nu}(K). \qquad (3.9)$$

As before, a divergence in the p_0 derivative of this at $p_0=\omega_{\pm}(p)$ can only come from K^μ small. Thus in the electron propagator one can omit K^μ in the numerator and can linearize the denominators with respect to K:

$$D_\sigma(P-K)\approx D_\sigma(P)-K^\mu\frac{\partial D_\sigma(P)}{\partial P^\mu}.$$

In the region $p_0\approx\omega_{\pm}(p)$ one can linearize in the small difference $p_0-\omega_{\pm}(p)$ using Eqs. (3.5) and (3.7) to get

$$D_\sigma(P-K)\approx\frac{1}{Z_\sigma}(p_0-\omega_\sigma+i\epsilon+\vec{k}\cdot\vec{v}_\sigma-k_0).$$

The HTL electron propagator that occurs in the loop integration can be approximated as

$$^*S_R(P-K)\approx\frac{Z_\sigma\frac{1}{2}(\gamma_0-\sigma\vec{\gamma}\cdot\vec{p})}{p_0-\omega_\sigma+i\epsilon+\vec{k}\cdot\vec{v}_\sigma-k_0}.$$

As before it is convenient to express the denominator as an integral over time:

$$-iZ_\sigma\frac{1}{2}(\gamma_0-\sigma\vec{\gamma}\cdot\vec{p})\int_0^\infty dt\,e^{i(p_0-\omega_\sigma+i\epsilon+\vec{k}\cdot\vec{v}_\sigma-k_0)t}.$$

When this is substituted into Eq. (3.9) the integration over d^4K performs the Fourier transform of the photon propagator to space-time on the trajectory $\vec{r}=\vec{v}_\sigma t$:

$$^*\Sigma_R(P)\approx e^2Z_\sigma\frac{1}{2}\gamma_\mu(\gamma_0-\sigma\vec{\gamma}\cdot\hat{p})\gamma_\nu\times\int_0^\infty dt\,e^{it(p_0-\omega_\sigma+i\epsilon)}\,^*\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}_\sigma t). \qquad (3.10)$$

As before, the lower limit of the time integration gives spurious ultraviolet divergences that were introduced by keeping only the terms in the electron propagator that are important for small K^μ. The ultraviolet divergences are regularized by changing the lower limit on the time integration to some non-zero value t_0. The trace in Eq. (3.4) then gives for the part of the self-energy that is potentially non-analytic at $p_0 \approx \omega_\sigma(p)$ is

$$\Pi_\sigma(P) \approx e^2 Z_\sigma \int_{t_0}^{\infty} dt\, e^{i(p_0-\omega_\sigma+i\epsilon)t} \phi_\mu \phi_\nu {}^*\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}_\sigma t), \tag{3.11}$$

where ϕ^μ is a light-like vector

$$\phi^\mu = (1,\sigma\hat{p}). \tag{3.12}$$

B. Coulomb gauge results

The asymptotic behavior of ${}^*\mathcal{D}_{11}^{\mu\nu}(t,\vec{v}t)$ for $t\to\infty$ was calculated in [16] in the Coulomb gauge. The momentum space propagator ${}^*D_{11}^{\mu\nu}(K)$ contains quasiparticle poles in the region of time-like K and an electron-positron cut in the region of space-like K. The contribution to ${}^*\mathcal{D}_{11}^{00}(t,\vec{v}t)$ from the poles falls like $1/t^{3/2}$ and the contribution from the cut falls like $1/t^3$. The contribution to ${}^*\mathcal{D}_{11}^{ij}(t,\vec{v}t)$ from the pole falls like $1/t^{3/2}$ and the contribution from the cut falls like $1/t$. The latter is thus the dominant contribution. The specific behavior is

$${}^*\mathcal{D}_{11}^{ij\,\mathrm{cut}}(t,\vec{r}) \to \frac{-iT}{8\pi r}(\delta^{ij}+\hat{r}^i\hat{r}^j), \tag{3.13}$$

in the limit $t \gg 1/m_\gamma$ with a fixed ratio $r/t<1$. Here $m_\gamma = eT/3$ is the photon effective thermal mass that occurs in the HTL photon propagator. This behavior is quite unusual: It comes from the momentum space region in which $k_0 \sim k^3/m_\gamma^2$ [16].

The non-analytic contribution to the electron self-energy comes from ${}^*\mathcal{D}^{ij\,\mathrm{cut}}(t,\vec{r}t)$ with $\vec{r}=\vec{v}t$. On this trajectory, $\hat{r}=\hat{v}=\hat{p}$. The necessary projection is

$$\phi_\mu\phi_\nu {}^*\mathcal{D}^{\mu\nu}(t,\vec{v}_\sigma t) \to \frac{-iT}{4\pi v_\sigma t}. \tag{3.14}$$

The only part of the electron self-energy that can be non-analytic at $p_0 \approx \omega_\sigma$ is

$$\Pi_\sigma(P) \approx -\frac{ie^2T}{4\pi v_\sigma} Z_\sigma \int_{t_0}^{\infty} \frac{dt}{t} e^{i(p_0-\omega_\sigma+i\epsilon)t}. \tag{3.15}$$

For simplicity t_0 has been chosen larger than $1/m_\gamma$. The time integration gives a logarithmic singularity:

$$\int_{t_0}^{\infty}\frac{dt}{t}e^{i(p_0-\omega_\sigma+i\epsilon)t} = -\gamma + i\frac{\pi}{2} - \ln[(p_0-\omega_\sigma)t_0] - \sum_{n=1}^{\infty}\frac{[i(p_0-\omega_\sigma)t_0]^n}{n!\,n}.$$

Since the logarithm is the only non-analytic term, the non-analytic part of the self-energy is

$$\Pi_\sigma(P) \approx \frac{ie^2T}{4\pi v_\sigma} Z_\sigma \ln[(p_0-\omega_\sigma)t_0]. \tag{3.16}$$

The self-energy and its derivative are both infinite in value at $p_0=\omega_\sigma$. For the $\sigma=+$ mode at high momentum $v_+\to 1$ and $Z_+\to 1$ and the result agrees with Blaizot and Iancu [13] and with Boyanovsky *et al.* [14]. For the $\sigma=-1$ mode the result is new. Of course, at large momentum $v_-\to 1$ but the residue is exponentially small: $Z_-\to R\exp(-R-1)$ where $R = 16p^2/(eT)^2$. Appendix C summarizes the arguments that lead from this result to the propagator displayed in Eq. (C8).

IV. DISCUSSION

The known results displayed in Eqs. (1.1a)–(1.1c) have been obtained by isolating that part of the electron self-energy integral that could diverge at the mass shell or whose derivative could diverge at the mass shell. The emphasis was on isolating the nonanalytic contributions.

At zero temperature and low temperature the only possible nonanalytic contribution was Eq. (2.5). In coordinate space this reads

$$\Sigma^a(x) = ie^2\gamma_\mu S_R(x)\gamma_\nu \mathcal{D}_{11}^{\mu\nu}(x).$$

The approximations in momentum space that led to Eq. (2.7) are equivalent to employing the approximate propagator

$$S_R(x) \to \frac{-i}{2E}(\gamma\cdot P + m)\, e^{-iEt}\, \delta^3(\vec{r}-\vec{v}t).$$

This is the standard Bloch-Nordsieck approximation for the electron propagator [2–4].

Similarly, in the high temperature regime the only possible non-analytic contribution to the electron self-energy for a hard electron ($p \geq T$) was isolated in Eq. (3.9). In coordinate space that contribution is

$${}^*\Sigma(x) = ie^2\gamma_\mu {}^*S_R(x)\gamma_\nu {}^*\mathcal{D}_{11}^{\mu\nu}(x).$$

The approximations in momentum space that led to Eq. (3.10) can be summarized by the replacement

$${}^*S_R(x) \to \sum_{\sigma=\pm}\frac{-iZ_\sigma}{2}(\gamma_0 - \sigma\vec{\gamma}\cdot\hat{p})\, e^{-i\omega_\sigma t}\, \delta^3(\vec{r}-\vec{v}_\sigma t),$$

where $\sigma=\pm$ accounts for the two positive energy modes, one with the helicity equal to the chirality and the other with the helicity opposite to the chirality. This approximation is equivalent to the high-temperature Bloch-Nordsieck approximation used by Blaizot and Iancu [13].

The nonanalytic part of the electron self-energy comes from soft photons and in retrospect could be obtained by using the approximate Bloch-Nordsieck propagators shown above. However, for the analysis to succeed it was essential to first split the fermion self-energy into two parts, Eq. (B3) at low temperature and Eq. (B5) at high temperature, and then to eliminate Σ^b and $^*\Sigma^b$ because they could not spoil the analyticity at the mass shell.

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APPENDIX A: RETARDED PROPAGATORS

The Feynman rules for finite-temperature calculations in real time are conventionally formulated by doubling the number of degrees of freedom [8,17,18]. All propagators become 2×2 matrices in an auxiliary space. It is also possible to formulate real-time Feynman rules directly in terms of retarded and advanced propagators [19]. This appendix gives the standard relations between the propagators in the different bases.

1. Photons

The retarded and advanced propagators for photons depend upon the thermal average of the commutator:

$$\mathcal{D}_R^{\mu\nu}(x)=-i\,\theta(t)\mathrm{Tr}\{\varrho\,[A^\mu(x),A^\nu(0)]\} \tag{A1a}$$

$$\mathcal{D}_A^{\mu\nu}(x)=i\,\theta(-t)\mathrm{Tr}\{\varrho\,[A^\mu(x),A^\nu(0)]\} \tag{A1b}$$

where $\varrho=\exp(-\beta H)/\mathrm{Tr}[\exp(-\beta H)]$ is the density operator at temperature $T=1/\beta$. In the following formulas $n(k_0)$ is the Bose-Einstein function with no absolute value bars on the energy:

$$n(k_0)=1/[e^{\beta k_0}-1].$$

In momentum space the path-ordered photon propagators can be expressed as [19]

$$D_{11}^{\mu\nu}(K)=[1+n(k_0)]D_R^{\mu\nu}(K)-n(k_0)D_A^{\mu\nu}(K) \tag{A2a}$$

$$D_{12}^{\mu\nu}(K)=e^{\sigma k_0}n(k_0)[D_R^{\mu\nu}(K)-D_A^{\mu\nu}(K)] \tag{A2b}$$

$$D_{21}^{\mu\nu}(K)=e^{(\beta-\sigma)k_0}n(k_0)[D_R^{\mu\nu}(K)-D_A^{\mu\nu}(K)] \tag{A2c}$$

$$D_{22}^{\mu\nu}(K)=n(k_0)D_R^{\mu\nu}(K)-[1+n(k_0)]D_A^{\mu\nu}(K). \tag{A2d}$$

The parameter σ lies in the range $0\leqslant\sigma\leqslant\beta$. The real-time self-energies are related to the inverse full propagator by

$$[D'(K)^{-1}]_{ab}^{\mu\nu}=[D(K)^{-1}]_{ab}^{\mu\nu}-\Pi_{ab}^{\mu\nu}(K). \tag{A3}$$

In terms of the retarded and advanced self-energies this implies

$$\Pi_{11}^{\mu\nu}(K)=[1+n(k_0)]\Pi_R(K)-n(k_0)\Pi_A(K) \tag{A4a}$$

$$\Pi_{12}(K)=e^{\sigma k_0}n(k_0)[-\Pi_R(K)+\Pi_A(K)] \tag{A4b}$$

$$\Pi_{21}(K)=e^{(\beta-\sigma)k_0}n(k_0)[-\Pi_R(K)+\Pi_A(K)] \tag{A4c}$$

$$\Pi_{22}(K)=n(k_0)\Pi_R(K)-[1+n(k_0)]\Pi_A(K). \tag{A4d}$$

2. Electrons

For electrons the retarded and advanced propagators are defined in terms of the anticommutator:

$$[S_R(x)]_{\alpha\beta}=-i\,\theta(t)\mathrm{Tr}[\varrho\{\psi_\alpha(x),\overline{\psi}_\beta(0)\}] \tag{A5a}$$

$$[S_A(x)]_{\alpha\beta}=i\,\theta(-t)\mathrm{Tr}[\varrho\{\psi_\alpha(x),\overline{\psi}_\beta(0)\}], \tag{A5b}$$

where ϱ is again the density operator. The Fermi-Dirac function (without absolute value bars) is denoted

$$f(p_0)=1/[e^{\beta p_0}+1].$$

The path-ordered propagators in momentum space are

$$S'_{11}(P)=[1-f(p_0)]S'_R(P)+f(p_0)S'_A(P) \tag{A6a}$$

$$S'_{12}(P)=e^{\sigma p_0}f(p_0)[-S'_R(P)+S'_A(P)] \tag{A6b}$$

$$S'_{21}(P)=e^{(\beta-\sigma)p_0}f(p_0)[S'_R(P)-S'_A(P)] \tag{A6c}$$

$$S'_{22}(k)=-f(p_0)S'_R(P)-[1-f(p_0)]S'_A(P), \tag{A6d}$$

where $0\leqslant\sigma\leqslant\beta$. The real-time self-energies are related to the inverse full propagator by

$$[S'(P)^{-1}]_{ab}=[S(P)^{-1}]_{ab}-\Sigma_{ab}(P). \tag{A7}$$

In terms of the retarded and advanced self-energies this implies

$$\Sigma_{11}(P)=[1-f(p_0)]\Sigma_R(P)+f(p_0)\Sigma_A(P) \tag{A8a}$$

$$\Sigma_{12}(P)=e^{\sigma p_0}f(p_0)[\Sigma_R(P)-\Sigma_A(P)] \tag{A8b}$$

$$\Sigma_{21}(P)=e^{(\beta-\sigma)p_0}f(p_0)[-\Sigma_R(P)+\Sigma_A(P)] \tag{A8c}$$

$$\Sigma_{22}(P)=-f(p_0)\Sigma_R(P)-[1-f(p_0)]\Sigma_A(P). \tag{A8d}$$

APPENDIX B: ELECTRON SELF-ENERGY

This appendix will derive the one-loop contribution to the retarded electron self-energy and perform some preliminary analysis leading to Eqs. (2.5) and (3.9), which isolate the possible nonanalytic behavior.

1. The retarded self-energy

The unperturbed electron propagator in the path-ordered basis is the matrix S_{ij}. The magnitude of the temperature determines what the appropriate unperturbed propagator is—at low temperature S_{ij} denotes the free thermal propagator; at high temperature S_{ij} will be replaced by the HTL resummed propagator $^*S_{ij}$.

The one-loop contribution to the electron self-energy in the path-ordered basis is given by

$$\Sigma_{ij}(P)=ie^2(-1)^{i+j}\int\frac{d^4K}{(2\pi)^4}\gamma_\mu S_{ij}(P-K)\gamma_\nu D^{\mu\nu}_{ij}(K).$$

From Eqs. (A8a) and (A8b) the retarded self-energy for the electron is

$$\Sigma_R(P)=\Sigma_{11}(P)+e^{-\sigma p_0}\Sigma_{12}(P). \tag{B1}$$

Direct substitution gives

$$\Sigma_R(P)=ie^2\int\frac{d^4K}{(2\pi)^4}[\gamma_\mu S_{11}(P-K)\gamma_\nu D^{\mu\nu}_{11}(K) - e^{-\sigma p_0}\gamma_\mu S_{12}(P-K)\gamma_\nu D^{\mu\nu}_{12}(K)].$$

This can be reorganized into a form that will be more convenient for the subsequent analysis. First use Eqs. (A6a) and (A6b) to express S_{11} and S_{12} in terms of S_R and S_A. The difference between S_R and S_A defines the fermion spectral function:

$$i\,\rho_f(P)=S_R(P)-S_A(P), \tag{B2}$$

which, of course, involves Dirac matrices. The retarded self-energy becomes

$$\Sigma_R(P)=ie^2\int\frac{d^4K}{(2\pi)^4}\{\gamma_\mu S_R(P-K)\gamma_\nu D^{\mu\nu}_{11}(K) - if(p_0-k_0)\gamma_\mu\rho_f(P-K)\gamma_\nu \times[D^{\mu\nu}_{11}(K)-e^{-\sigma k_0}D^{\mu\nu}_{12}(K)]\}.$$

The combination of photon propagators in square brackets is $D_R(K)$ because of Eqs. (A2a) and (A2b). The retarded self-energy is therefore

$$\Sigma_R(P)=\Sigma^a(P)+\Sigma^b(P). \tag{B3}$$

The two integrals are

$$\Sigma^a(P)=ie^2\int\frac{d^4K}{(2\pi)^4}\gamma_\mu S_R(P-K)\gamma_\nu D^{\mu\nu}_{11}(K) \tag{B4a}$$

$$\Sigma^b(P)=e^2\int\frac{d^4K}{(2\pi)^4}\frac{\gamma_\mu\rho_f(P-K)\gamma_\nu}{e^{\beta(p_0-k_0)}+1}D^{\mu\nu}_R(K). \tag{B4b}$$

The first of these is displayed in Eq. (2.5).

As already discussed, at high temperature the unperturbed propagators are the HTL resummed propagators. The self-energy correction to the resummed electron propagator is computed in terms of loop integrals containing resummed propagators. For electron momenta $p \geqslant T$ the bare vertices may be used [7,8]. The retarded self-energy then has the decomposition:

$$^*\Sigma_R(P)={}^*\Sigma^a(P)+{}^*\Sigma^b(P). \tag{B5}$$

Only the bare vertices appear in the loop integrations:

$$^*\Sigma^a(P)=ie^2\int\frac{d^4K}{(2\pi)^4}\gamma_\mu\,{}^*S_R(P-K)\gamma_\nu\,{}^*D^{\mu\nu}_{11}(K) \tag{B6a}$$

$$^*\Sigma^b(P)=e^2\int\frac{d^4K}{(2\pi)^4}\frac{\gamma_\mu\,{}^*\rho_f(P-K)\gamma_\nu}{e^{\beta(p_0-k_0)}+1}{}^*D^{\mu\nu}_R(K). \tag{B6b}$$

The first of these is displayed in Eq. (3.9).

2. Low T: Analysis of $\boldsymbol{\Sigma^b}$

It is straightforward to show that the contribution in Eq. (B4b) cannot produce a logarithmic singularity. Specifically the derivative $\partial\Sigma^b(P)/\partial p_0$ will be finite when p_0 is on the mass shell:

$$\frac{\partial\Sigma^b(P)}{\partial p_0}=e^2\int\frac{d^4K}{(2\pi)^4}\frac{\gamma_\mu\rho_f(P-K)\gamma_\nu}{e^{\beta(p_0-k_0)}+1}\frac{\partial D^{\mu\nu}_R(K)}{\partial k_0}. \tag{B7}$$

The photon propagator is the free thermal propagator. For example, in Feynman gauge the contribution is

$$\frac{\partial}{\partial k_0}\left(\frac{-g^{\mu\nu}}{K^2}\right)=\frac{g^{\mu\nu}2k_0}{(K^2)^2}. \tag{B8}$$

The spectral function for the free electron is

$$\rho_f(P-K)=2\pi\epsilon(p_0-k_0)\,\delta((P-K)^2-m^2) \times(\gamma_\mu(P-K)^\mu+m).$$

When the self-energy is evaluated at $p_0=E(p)$, the support of the spectral function is at $k_0=E(p)\pm E(\vec{p}-\vec{k})$. For the upper sign, k_0 is large and Eq. (B8) is never large. For the lower sign, at small $\vec{k}$, $k_0\approx\vec{v}\cdot\vec{k}$, where $\vec{v}=\vec{p}/E$. The potentially divergent contribution to Eq. (B7) is

$$\int d^3k \frac{\vec{v}\cdot\vec{k}+\mathcal{O}(k^2)}{[(\vec{v}\cdot\vec{k})^2-k^2]^2}.$$

The leading term in the integrand is odd in $\vec{k}$ and integrates to zero. Therefore the surviving terms are $d^3k\, k^2/k^4$ and give finite integrals. This shows that Eq. (B7) is finite at the electron mass shell.

3. High T: Analysis of $^*\Sigma^b$

A similar argument applies to Eq. (B6b). The HTL resummed propagator $^*S_R(P)$ has poles at two positive energies: $p_0=\omega_+(p)$ and $p_0=\omega_-(p)$. We are interested in whether $^*\Sigma_R(p)$ contains either a term of the form $(p_0-\omega_\sigma)\ln(p_0-\omega_\sigma)$ or a term $\ln(p_0-\omega_\sigma)$. In either case the p_0 derivative of the self-energy would diverge at $p_0=\omega_\sigma$:

$$\frac{\partial\, {}^*\Sigma^b(P)}{\partial p_0}=e^2\int \frac{d^4K}{(2\pi)^4}\frac{\gamma_\mu\, {}^*\rho_f(P-K)\gamma_\nu}{e^{\beta(p_0-k_0)}+1}\frac{\partial\, {}^*D_R^{\mu\nu}(K)}{\partial k_0}. \tag{B9}$$

The HTL spectral function for the electron has support in the space-like region $|p_0-k_0|<|\vec{p}-\vec{k}|$, but when the self-energy is evaluated at $p_0=\omega_\sigma(p)>p$ the space-like condition cannot be satisfied for vanishingly small components of K^μ. The spectral function also has support on the dispersion curves, i.e. at $p_0-k_0=\pm\omega_{\sigma'}(\vec{p}-\vec{k})$ where $\sigma'=\pm$. When the self-energy is evaluated on one of the dispersion curves, $p_0=\omega_\sigma(p)$, the support is at $k_0=\omega_\sigma(p)\mp\omega_{\sigma'}(\vec{p}-\vec{k})$. Of the eight possible cases, the only ones in which this k_0 can be small is when $\sigma'=\sigma$ and the minus sign is chosen. Then at small $\vec{k}$, $k_0\approx\vec{v}_\sigma\cdot\vec{k}$, where $\vec{v}_\sigma=d\omega_\sigma(p)/dp$ is the group velocity. Using this gives for the only contribution to Eq. (B7) that is potentially divergent

$$\int d^3k \frac{\partial\, {}^*D_R^{\mu\nu}(K)}{\partial k_0}\bigg|_{k_0=\vec{v}_\sigma\cdot\vec{k}}$$

However, since the HTL photon propagator is a even function of k_0, after setting $k_0=\vec{v}_\sigma\cdot\vec{k}$, the above integrand is odd in $\vec{k}$. It therefore integrates to zero, which shows that Eq. (B9) is finite at $p_0=\omega_\sigma(p)$.

4. Alternate decomposition of Σ_R

There are various other ways to organize Σ_R. In a previous paper [6] on the low-temperature behavior, which performed using momentum-space integration, it was convenient to use the decomposition

$$\Sigma_R(P)=\Sigma^\gamma(P)+\Sigma^e(P), \tag{B10}$$

in which the two contributions were defined as follows:

$$\Sigma^\gamma(P)=ie^2\int \frac{d^4K}{(2\pi)^4}\gamma_\mu S_R(P-K)\gamma_\nu \times\frac{1}{2}\coth\left(\frac{k_0}{2T}\right)[D_R^{\mu\nu}(K)-D_A^{\mu\nu}(K)] \tag{B11}$$

$$\Sigma^e(P)=ie^2\int \frac{d^4K}{(2\pi)^4}\gamma_\mu[S_R(P-K)-S_A(P-K)]\gamma_\nu \times\frac{1}{2}\tanh\left(\frac{p_0-k_0}{2T}\right)D_R^{\mu\nu}(K). \tag{B12}$$

This was convenient because in Σ^γ the photon is on shell; in Σ^e the electron was on shell. The difference between the on-shell decomposition and that in Eq. (B3) is as follows. The difference I is defined as

$$\Sigma^a(P)-\Sigma^\gamma(P)=I(P), \tag{B13}$$

where

$$I(P)=\frac{ie^2}{2}\int \frac{d^4K}{(2\pi)^4}\gamma_\mu S_R(P-K) \times\gamma_\nu[D_R^{\mu\nu}(K)+D_A^{\mu\nu}(K)].$$

Similarly, the difference J is defined as

$$\Sigma^b(P)-\Sigma^e(P)=J(P), \tag{B14}$$

where

$$J(P)=\frac{-ie^2}{2}\int \frac{d^4K}{(2\pi)^4}\gamma_\mu[S_R(P-K) -S_A(P-K)]\gamma_\nu D_R^{\mu\nu}(K)$$

The sum of I and J is

$$I(P)+J(P)=\frac{ie^2}{2}\int \frac{d^4K}{(2\pi)^4}\gamma_\mu S_R(P-K)\gamma_\nu D_A^{\mu\nu}(K) +\frac{ie^2}{2}\int \frac{d^4K}{(2\pi)^4}\gamma_\mu S_A(P-K)\gamma_\nu D_R^{\mu\nu}(K).$$

Both these integrands vanish: the first, because the integrand has no singularities in the lower-half of the complex k_0 plane; the second, because the integrand has no singularities in the upper-half of the complex k_0 plane. This confirms that

$$\Sigma^a+\Sigma^b=\Sigma^\gamma+\Sigma^e, \tag{B15}$$

and thus the two decompositions of the retarded self-energy are equivalent.

APPENDIX C: BLOCH-NORDSIECK PROPAGATOR

The Bloch-Nordsieck approximation allows one to take a one-loop electron self-energy and immediately compute an approximation to the full electron propagator. The proof of this method is best done using functional integrals [2,4,13] and will not be repeated here. This appendix only displays the Bloch-Nordsieck prescription for passing from the one-loop self-energy to the full propagator.

1. Low temperature: $0 \leqslant T \ll m$

At zero temperature or low temperature, the electron self-energy approximation given in Eq. (2.9) can be expressed as a function of time as

$$\Pi(t,\vec{p}) = 2Ee^2\, e^{-iEt}\, v_\mu v_\nu \mathcal{D}^{\mu\nu}_{11}(t,\vec{v}t).$$

From this self-energy compute the phase

$$\psi(t,\vec{p}) = e^2 \int_{t_0}^{\infty} dt'\,(t-t')\, v_\mu v_\nu \mathcal{D}^{\mu\nu}_{11}(t',\vec{v}t'). \tag{C1}$$

Then the Bloch-Nordsieck approximation to the electron propagator is

$$S_{BN}(t,\vec{p}) = -i e^{-iEt - i\psi(t)}. \tag{C2}$$

At zero temperature in covariant gauges Eq. (2.11) implies that

$$\psi(t,\vec{p}) = \frac{i(\xi-3)\alpha}{2\pi}\int_{t_0}^{t} dt' \frac{t-t'}{t'^2}$$
$$= \frac{i(\xi-3)\alpha}{2\pi}\left[-1+\frac{t}{t_0}-\ln\frac{t}{t_0}\right].$$

The logarithm is the nontrivial part and gives

$$S_{BN}(t,\vec{p}) = -i\,\exp\left[-iEt-(\xi-3)\frac{\alpha}{2\pi}\ln\frac{t}{t_0}\right].$$

The Fourier transform of this is

$$S_{BN}(P) \approx \frac{1}{(p_0-E)}[(p_0-E)t_0]^{(\xi-3)\alpha/2\pi}. \tag{C3}$$

This is the standard result for the electron propagator in covariant gauges [2–4]. At zero temperature the electron propagator in the Coulomb gauge follows directly from Eq. (2.14):

$$S_{BN}(P) \approx \frac{1}{(p_0-E)}[(p_0-E)t_0]^{b(v)}. \tag{C4}$$

In both cases the Dirac matrices $(\gamma\cdot P+m)/2E$ have been omitted.

2. High temperature: $T \gg m$

In the high temperature regime, where hard thermal loop propagators are required, the justification of the Bloch =Nordsieck scheme is non-trivial. The detailed arguments are presented by Blaizot and Iancu [13]. The approximate propagator is

$$S_{BN}(t,\vec{p}) = -iZ_\sigma e^{-i\omega_\sigma t - iZ_\sigma \psi(t)}. \tag{C5}$$

where the ψ function

$$\psi(t,\vec{p}) = e^2 \int_{t_0}^{\infty} dt'\,(t-t')\,\phi_\mu\phi_\nu{}^{*}\mathcal{D}^{\mu\nu}_{11}(t',\vec{v}_\sigma t'). \tag{C6}$$

Using Eq. (3.14) this gives

$$\psi(t) = -i\frac{\alpha T Z_\sigma}{v_\sigma}\left\{t\ln\left[\frac{t}{t_0}\right]-t+t_0\right\}. \tag{C7}$$

The propagator is

$$S_{BN}(t,\vec{p}) = -iZ_\sigma \exp\left\{-i\omega_\sigma t - \frac{\alpha T Z_\sigma^2}{v_\sigma} t\ln\left[\frac{t}{t_0}\right]\right\}. \tag{C8}$$

For the $\sigma=+$ mode this coincides with the well-known result [13,14]. The Dirac matrices $(\gamma_0-\sigma\vec{\gamma}\cdot\hat{p})/2$ have been omitted.

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