Gauge and scheme dependence of mixing matrix renormalization

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We reexamine the issue of mixing matrix renormalization in theories that include Dirac or Majorana fermions. We show how a gauge-variant on-shell renormalized mixing matrix can be related to a manifestly gauge-independent one within a generalized $\overline{\text{MS}}$ scheme of renormalization. This scheme-dependent relation is a consequence of the fact that in any scheme of renormalization, the gauge-dependent part of the mixing-matrix counterterm is ultraviolet safe and has a pure dispersive form. Employing the unitarity properties of the theory, we can successfully utilize the aforementioned scheme-dependent relation to preserve basic global or local symmetries of the bare Lagrangian through the entire process of renormalization. As an immediate application of our study, we derive the gauge-independent renormalization-group equations of mixing matrices in a minimal extension of the standard model with isosinglet neutrinos.

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I. INTRODUCTION

One of the most fundamental properties of the well-established standard model (SM) [1] is its renormalizability [2]. Renormalizability endows the SM with enhanced predictive power that emanates from the fact that ultraviolet (UV) divergences due to high order quantum effects can always be successfully eliminated by a redefinition of a finite number of independent kinematic parameters of the theory, such as masses and couplings. The predictions of the SM have been tested and vindicated with a satisfactory accuracy at high-energy colliders, such as the Large Electron Positron (LEP) collider at CERN and the Tevatron collider at Fermilab, as well as in low-energy experiments, e.g. in the recent E821 experiment at BNL where the muon anomalous magnetic moment is measured [3].

In addition to masses and couplings of the SM particles, however, the quark-mixing matrix, the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix [4] V, needs be renormalized as well [5,6]. In this context, one of the renormalization schemes, most frequently adopted in the literature, is the on-shell (OS) scheme of renormalization [7–9], where the particle masses are renormalized so as to represent the physical masses at the poles of the propagators. It was shown in [6] that the complete UV structure of the counterterms (CTs) for the CKM matrix V can be entirely expressed in terms of quark wave-function renormalizations. Within this framework, a simple approach to renormalizing V in the OS scheme was also presented, consistent with the unitarity properties of the theory [10].

Even though radiative effects due to the renormalization of an off-diagonal CKM matrix were found to be undetectably small in the SM [6,11–13], this need not be the case for its minimal renormalizable extensions. In particular, in [14] the above formalism of mixing-matrix renormalization was extended to theories that include isosinglet neutrinos and so admit the presence of lepton-number-violating Majorana masses [15]. A minimal realization of such a theory is the SM with right-handed neutrinos [16,17]. As we will further discuss in Sec. II C, in this minimal model the charged and neutral current interactions of the W and Z bosons to leptons and neutrinos are described by two nonunitary mixing matrices B and C [17], respectively. Most importantly, the radiative effects on the light-heavy neutrino mixing angles contained in the B and C matrices were computed to be as large as 15% [14], close to present experimental sensitivities. The SM with right-handed neutrinos is an appealing scenario which may explain the smallness of the observed neutrino masses and adequately address the solar energy deficit problem [18] through neutrino oscillations [19]. Furthermore, this minimal extension of the SM may give rise to a number of observable phenomena, such as lepton-flavor and/or lepton-number violation in μ, τ [20–22] and Z-boson decays [23], or to possible lepton-number violating signals at high-energy colliders, e.g. at the CERN Large Hadron Collider (LHC) [24,25].

It has been noticed recently [26] that in the OS renormalization prescription presented in [6], the derived CTs for the CKM matrix naively depend on the choice of the gauge-fixing parameter ξ in the class of R_ξ gauges. This fact is not very desirable, as physical matrix elements will be gauge dependent after renormalization. To circumvent this problem of ξ dependence of the OS renormalized CKM matrix, several alternative schemes of renormalization have been suggested in the very recent literature [26,12,13,27,28]. As is expected, in all the proposed renormalization schemes, the UV-divergent parts of the CTs of the CKM matrix are identical to those derived in the modified minimal subtraction $\overline{\text{MS}}$ scheme [6]. Nevertheless, the UV-safe parts of the CTs differ from approach to approach by finite dispersive constants. Most interestingly, one may observe that even in the originally suggested OS scheme of [6], the gauge-dependent part of the CKM-matrix CTs is UV finite and also has a pure dispersive form, thus indicating the existence of a profound relation between gauge dependence and scheme dependence in mixing-matrix renormalization.

In this paper, we revisit the topic of mixing-matrix renormalization of the CKM matrix V and of the B and C matrices. In particular, we develop a generalized and manifestly gauge-invariant $\overline{\text{MS}}$ approach to mixing-matrix renormalization. The developed generalized $\overline{\text{MS}}$ approach provides a very convenient framework to address the problem of gauge and scheme dependences in the existing plethora of differently renormalized mixing matrices. Moreover, we show how our generalized $\overline{\rm MS}$ scheme can be successfully employed to maintain global or local symmetries of the bare Lagrangian after renormalization. Finally, with the help of our generalized $\overline{\rm MS}$ approach, we can derive the gauge-independent renormalization-group (RG) equations for mixing matrices. We explicitly demonstrate the theoretical advantages of this method by calculating the one-loop RG runnings of the B and C matrices in the SM with isosinglet neutrinos.

The paper is organized as follows: after briefly reviewing the basic formalism of mixing matrix renormalization in the OS scheme in Sec. II A, we present in Sec. II B our gauge-invariant generalized $\overline{\rm MS}$ approach to the renormalization of the CKM matrix V, and extend it in Sec. II C to the renormalization of the corresponding B and C mixing matrices in the SM with isosinglet neutrinos. In Sec. III, we show how our generalized $\overline{\rm MS}$ approach preserves additional global and local symmetries of the theory, which are manifested themselves as sum rules involving neutrino masses and the B and C matrices. As an immediate application of our considerations, we derive in Sec. IV the gauge-independent renormalization group equations RGEs of the B and C mixing matrices. Finally, our conclusions are summarized in Sec. V.

II. MIXING MATRIX RENORMALIZATION

In this section, we will first recall the basic analytic formulas for the wave-function and mass CTs in the OS renormalization scheme within the context of general fermionic theories, such as the SM and its natural extension with isosinglet neutrinos. Then, we will revisit the problem of gauge dependence of the OS-renormalized CKM matrix in the SM, and discuss its connection to scheme dependence within a generalized gauge-invariant $\overline{\rm MS}$ scheme of renormalization. Finally, our discussion will be extended to the renormalization of the mixing matrices B and C that parametrize the neutral- and charged-current interactions in the SM with singlet neutrinos.

A. OS renormalization scheme

In a theory with a number N_f of Dirac fermions, the bare kinetic Lagrangian has the following generic form:

$$\begin{aligned}\mathcal{L}_{\rm kin} &= i\overline{f^0_L}\partial\!\!\!/ f^0_L + i\overline{f^0_R}\partial\!\!\!/ f^0_R - \overline{f^0_L}M^0 f^0_R - \overline{f^0_R}M^0 f^0_L\\ &= i\overline{f_L}Z_L^{1/2\dagger}Z_L^{1/2}\partial\!\!\!/ f_L + i\overline{f_R}Z_R^{1/2\dagger}Z_R^{1/2}\partial\!\!\!/ f_R\\ &\quad -\overline{f_L}Z_L^{1/2\dagger}(M+\delta M)Z_R^{1/2}f_R\\ &\quad -\overline{f_R}Z_R^{1/2\dagger}(M+\delta M)Z_L^{1/2}f_L\,. \end{aligned}\tag{2.1}$$

In the above, we have employed a matrix notation in the space spanned by the N_f fermionic fields, i.e. $f^T=(f_1,f_2,\dots,f_{N_f})$. As usual, we adhere the superscript "0" to unrenormalized quantities. In Eq. (2.1), the $N_f\times N_f$ dimensional matrices $Z_L^{1/2}$ and $Z_R^{1/2}$ are the wave-function renormalizations for the left- and right-handed fermions, respectively. In addition, M^0, M and δM are diagonal $N_f\times N_f$ dimensional matrices that contain the bare masses, the renormalized masses and their respective counterterms (CTs).

The most general form of an unrenormalized $f_j\to f_i$ transition amplitude allowed by Hermiticity [8] reads

$$\begin{aligned}\Sigma_{ij}(p\!\!\!/) &= p\!\!\!/ P_L\Sigma^L_{ij}(p^2) + p\!\!\!/ P_R\Sigma^R_{ij}(p^2) + P_L\Sigma^D_{ij}(p^2)\\ &\quad + P_R\Sigma^{D*}_{ji}(p^2),\end{aligned}\tag{2.2}$$

supplemented by the constraints

$$\Sigma^L_{ij}(p^2)=\Sigma^{L*}_{ji}(p^2),\qquad \Sigma^R_{ij}(p^2)=\Sigma^{R*}_{ji}(p^2).\tag{2.3}$$

In the OS scheme of renormalization, the wave-function and mass CTs are given by [14][1]

$$\begin{aligned}\delta Z^L_{ij} &= \frac{2}{m_i^2-m_j^2}\big(m_j^2\Sigma^L_{ij}(m_j^2) + m_i m_j\Sigma^R_{ij}(m_j^2) + m_i\Sigma^D_{ij}(m_j^2)\\ &\quad + m_j\Sigma^{D*}_{ji}(m_j^2)\big),\end{aligned}\tag{2.4}$$

$$\begin{aligned}\delta Z^R_{ij} &= \frac{2}{m_i^2-m_j^2}\big(m_i m_j\Sigma^L_{ij}(m_j^2) + m_j^2\Sigma^R_{ij}(m_j^2) + m_j\Sigma^D_{ij}(m_j^2)\\ &\quad + m_i\Sigma^{D*}_{ji}(m_j^2)\big),\end{aligned}\tag{2.5}$$

$$\begin{aligned}\delta Z^L_{ii} &= -\Sigma^L_{ii}(m_i^2) + \frac{1}{2m_i}\big(\Sigma^D_{ii}(m_i^2) - \Sigma^{D*}_{ii}(m_i^2)\big)\\ &\quad - m_i^2\big(\Sigma^{L\prime}_{ii}(m_i^2) + \Sigma^{R\prime}_{ii}(m_i^2)\big)\\ &\quad - m_i\big(\Sigma^{D\prime}_{ii}(m_i^2) + \Sigma^{D*\prime}_{ii}(m_i^2)\big),\end{aligned}\tag{2.6}$$

$$\begin{aligned}\delta Z^R_{ii} &= -\Sigma^R_{ii}(m_i^2) - \frac{1}{2m_i}\big(\Sigma^D_{ii}(m_i^2) - \Sigma^{D*}_{ii}(m_i^2)\big)\\ &\quad - m_i^2\big(\Sigma^{L\prime}_{ii}(m_i^2) + \Sigma^{R\prime}_{ii}(m_i^2)\big)\\ &\quad - m_i\big(\Sigma^{D\prime}_{ii}(m_i^2) + \Sigma^{D*\prime}_{ii}(m_i^2)\big),\end{aligned}\tag{2.7}$$

$$\begin{aligned}\delta m_i &= \frac{1}{2}m_i\big(\Sigma^L_{ii}(m_i^2) + \Sigma^R_{ii}(m_i^2)\big)\\ &\quad + \frac{1}{2}\big(\Sigma^D_{ii}(m_i^2) + \Sigma^{D*}_{ii}(m_i^2)\big),\end{aligned}\tag{2.8}$$

where $\Sigma'(p^2)=d\Sigma(p^2)/dp^2$ and $\delta Z^{L,R}$ are the loop-induced wave-function renormalizations defined through the relation $Z^{1/2}_{L,R}=1+\frac{1}{2}\delta Z^{L,R}$. We should bear in mind that only the dis-

[1]Here, we have used the symmetry property of the Lagrangian (2.1) under the rephasings, $Z^{1/2}_{Lij}\to e^{i\theta_i}Z^{1/2}_{Lij}$ and $Z^{1/2}_{Rij}\to e^{i\theta_i}Z^{1/2}_{Rij}$, in order to cast δZ^L_{ii} and δZ^R_{ii} into a symmetric but fully equivalent form than the one presented in [14].

persive parts of the unrenormalized self-energies enter the renormalization such that the Hermiticity property of the local Lagrangian is maintained. In the SM, it is $\Sigma^D_{ij}(p^2) = m_i\Sigma^S_{ij}(p^2)$ and $\Sigma^S_{ij}(p^2) = \Sigma^{S*}_{ji}(p^2)$, and the formulas (2.4)–(2.8) reduce to those given in [6,9]. However, we should stress again that these relations are very specific to the SM and no longer apply to extended theories.

One well-motivated extension of the SM is the one in which the SM field content is augmented by right-handed (isosinglet) neutrinos, thereby admitting the presence of Majorana masses in the Lagrangian [15–17]. In this case, the fermionic fields satisfy the Majorana constraints: $f^0_L = (f^0_R)^C$ and $f_L = (f_R)^C$, where the superscript C indicates charge conjugation. As a consequence of the Majorana constraints, we obtain the equalities:

$$Z^{1/2}_L = Z^{1/2*}_R, \quad \Sigma^L_{ij}(p^2) = \Sigma^{R*}_{ij}(p^2),$$
$$\Sigma^M_{ij}(p^2) = \Sigma^M_{ji}(p^2), \tag{2.9}$$

where we made the identification $\Sigma^D_{ij}(p^2) \equiv \Sigma^M_{ij}(p^2)$. Substituting Eq. (2.9) into Eqs. (2.4)–(2.8) yields the corresponding wave-function and mass CTs for Majorana fields [14].

The issue of mixing-matrix renormalization arises whenever one has to deal with the renormalization of a nontrivial rotation matrix that occurs in interactions relating flavor to mass eigenstates. To study this problem, we shall adopt a perturbative framework in which the classical tree-level Ward identities (WIs) are maintained after quantization. As such, one may consider the background field method (BFM) [29,30] or the pinch technique (PT) [31–34] or even possible diagrammatic generalizations of the latter, i.e. the generalized pinch technique (GPT) [35].

B. Renormalization of the CKM matrix in the SM

As a prototype example, let us consider the charged-current interaction in the quark sector of the SM. Specifically, we will revisit the renormalization of the CKM matrix elements V_{ud} that enter the vertex transition $W^+(p)d(p_d) \to u(p_u)$. Later on, we will generalize our results to the aforementioned $SU(2)_L \otimes U(1)_Y$ model with Majorana neutrinos. Within the perturbative approaches mentioned above, the following tree like WI is satisfied [33,11,26]:

$$p^\mu \Gamma^{W^+\bar{u}d,0}_\mu(p,p_u,p_d) + M^0_W \Gamma^{G^+\bar{u}d,0}(p,p_u,p_d)$$
$$= -\frac{g^0_w}{\sqrt{2}}\left(V^0_{u'd} S^{-1,0}_{uu'}(p_u) P_L - V^0_{ud'} P_R S^{-1,0}_{d'd}(p_d)\right), \tag{2.10}$$

where the summation convention over repeated quark-family indices is implied. In addition, in Eq. (2.10) we have defined

$$\Gamma^{W^+\bar{u}d,0}_\mu(p,p_u,p_d) = \Gamma^{W^+\bar{u}d,0}_{0\mu} + \Gamma^{W^+\bar{u}d}_{1\mu}(p,p_u,p_d), \tag{2.11}$$

$$\Gamma^{G^+\bar{u}d,0}(p,p_u,p_d) = \Gamma^{G^+\bar{u}d,0}_0 + \Gamma^{G^+\bar{u}d}_1(p,p_u,p_d), \tag{2.12}$$

$$S^{-1,0}_{uu'}(p_u) = \not{p}_u - m^0_u + \Sigma_{uu'}(\not{p}_u),$$

$$S^{-1,0}_{d'd}(p_d) = \not{p}_d - m^0_d + \Sigma_{d'd}(\not{p}_d), \tag{2.13}$$

where $\Gamma^{W^+\bar{u}d,0}_{0\mu}$ and $\Gamma^{G^+\bar{u}d,0}_0$ are the bare $W^+\bar{u}d$ and $G^+\bar{u}d$ couplings at the tree level, and $\Gamma^{W^+\bar{u}d}_{1\mu}(q,p_u,p_d)$ and $\Gamma^{G^+\bar{u}d}_1(q,p_u,p_d)$ are the corresponding higher-order unrenormalized one-particle irreducible vertices evaluated within e.g. the PT or the BFM. Similar identifications also apply for the unrenormalized two-point correlation functions $S^0_{uu'}(p_u)$ and $S^0_{d'd}(p_d)$.

Following the procedure outlined in [34], we require that the same tree-level WI (2.10), which involves unrenormalized quantities only, holds exactly true after renormalization. This condition can be successfully enforced within the gauge-independent $\overline{\text{MS}}$ scheme of renormalization. Nevertheless, in any other favorable scheme of renormalization, the renormalized parameters of the theory will differ from those in the $\overline{\text{MS}}$ scheme by UV-finite constants. The renormalized quantities may be determined in terms of the unrenormalized ones through the relations

$$g^0_w = Z_{g_w} g_w,$$

$$M^{02}_W = M^2_W + \delta M^2_W,$$

$$V^0 = V + \delta V, \tag{2.14}$$

$$\Gamma^{W^+\bar{u}d}_\mu(p,p_u,p_d) = Z^{1/2}_W Z^{1/2\dagger}_{Luu'} Z^{1/2}_{Ld'd} \Gamma^{W^+\bar{u}'d',0}_\mu \times (p,p_u,p_d),$$

$$\Gamma^{G^+\bar{u}d}(p,p_u,p_d) = Z^{1/2}_{G^+} Z^{1/2\dagger}_{uu'} Z^{1/2}_{d'd} \Gamma^{G^+\bar{u}'d',0} \times (p,p_u,p_d), \tag{2.15}$$

$$S^{-1}_{q_iq_j}(p_q) = Z^{1/2\dagger}_{q_iq_k} S^{-1,0}_{q_kq_l}(p_q) Z^{1/2}_{q_lq_j} \quad (\text{with } q = u,d), \tag{2.16}$$

where $Z^{1/2}_{q_iq_j} = Z^{1/2}_{Lq_iq_j} P_L + Z^{1/2}_{Rq_iq_j} P_R$ and δV stands for the mixing-matrix renormalization of the CKM matrix V. In Eq. (2.15), we required that $W^+\bar{u}d$ and $G^+\bar{u}d$ couplings be UV finite after the external wave-function CTs for the W boson, the would-be Goldstone boson G^+, and the u- and d-type quarks have been properly taken into account.

Since our main interest is to compute the UV divergent part of the $W^+\bar{u}d$ vertex in the presence of flavor mixing and so determine the UV-divergent structure of the mixing-matrix CT δV, we shall therefore focus our attention only on the chirally-projected WI (2.10) related to the expression $P_R \Gamma^{W^+\bar{u}d,0}_\mu P_L$. In particular, we have

$$Z_W^{1/2} Z_L^{u,1/2\dagger} P_R (p^\mu \Gamma_\mu^{W^+\bar{u}d,0} + M_W^{02} \Gamma^{G^+\bar{u}d,0}) P_L Z_L^{d,1/2}$$
$$= - Z_W^{1/2} Z_{g_w} \frac{g_w}{\sqrt{2}} Z_L^{u,1/2\dagger} P_R (S^{-1,0}(p_u) V^0 - V^0 S^{-1,0}(p_d)) P_L Z_L^{d,1/2}. \qquad (2.17)$$

To simplify notation in Eq. (2.17), we have employed the matrix representation for the quark wave-functions and their inverse propagators, i.e. $Z_L^{q,1/2} = Z_{Lq_iq_j}^{1/2}$ and $S^{-1,0}(p_q) = S_{q_iq_j}^{-1,0}(p_q)$, with $q = u,d$. Substituting Eqs. (2.15) and (2.16) into Eq. (2.17) gives

$$P_R (p^\mu \Gamma_\mu^{W^+\bar{u}d} + Z_W^{1/2} Z_{G^+}^{-1/2} M_W^{02} \Gamma^{G^+\bar{u}d}) P_L$$
$$= - Z_W^{1/2} Z_{g_w} \frac{g_w}{\sqrt{2}} \times P_R (S^{-1}(p_u) Z_L^{u,-1/2} V^0 Z_L^{d,1/2} - Z_L^{u,1/2\dagger} V^0 Z_L^{d,-1/2\dagger} S^{-1}(p_d)) P_L. \qquad (2.18)$$

The requirement now that the WI (2.18) retains its original form (2.10) where all quantities are replaced by their renormalized ones gives rise to the following consistency conditions:

$$Z_W^{1/2} = Z_{g_w}^{-1}, \quad Z_{G^+} = Z_W \left(1 + \frac{\delta M_W^2}{M_W^2} \right), \qquad (2.19)$$

$$V = Z_L^{u,-1/2} V^0 Z_L^{d,1/2} = Z_L^{u,1/2\dagger} V^0 Z_L^{d,-1/2\dagger}. \qquad (2.20)$$

The two equalities in Eq. (2.19) are exactly satisfied within the PT and BFM frameworks [30,34,26]. The double equality in Eq. (2.20) assures the unitarity property of the renormalized mixing matrix V, i.e. $V^{-1} = V^+$. Most importantly, Eq. (2.20) determines the analytic structure of the CT δV. Employing the usual decomposition for the wave-function renormalizations, i.e. $Z_L^{u,1/2} = 1 + \frac{1}{2}\delta Z^{u,L}$ and $Z_L^{d,1/2} = 1 + \frac{1}{2}\delta Z^{d,L}$, we arrive at the perturbative and more familiar form for δV [6]:[2]

$$\delta V = \frac{1}{4}(\delta Z^{u,L} - \delta Z^{u,L\dagger}) V - \frac{1}{4} V (\delta Z^{d,L} - \delta Z^{d,L\dagger}), \qquad (2.21)$$

where the u- and d-quark wave functions have to satisfy the constraining relation [11,26]

$$\frac{1}{2}(\delta Z^{u,L} + \delta Z^{u,L\dagger}) V = \frac{1}{2} V (\delta Z^{d,L} + \delta Z^{d,L\dagger}). \qquad (2.22)$$

In the absence of flavor mixing, i.e. for $V = 1$, this last relation simplifies to the known one $Z^{u,L} = Z^{d,L}$ [30,34].

Several important remarks and observations regarding mixing-matrix renormalization are now in order:

(i) The UV poles of δV are entirely specified by the wave-function CTs of the left-handed u and d quarks to all orders in perturbation theory. Moreover, with the definition of δV in Eq. (2.21), V is automatically unitary through the order considered.

(ii) The left-hand side (LHS) of the WI (2.18) is gauge-independent, when the chirally-projected amplitudes $P_R \Gamma_\mu^{W^+\bar{u}d}(p,p_u,p_d) P_L$ and $P_R \Gamma^{G^+\bar{u}d}(p,p_u,p_d) P_L$ are evaluated by setting the external particles on their mass shells. Consequently, the RHS of Eq. (2.18) must be gauge-independent as well. This can only happen, if V and hence δV are gauge-independent [11,26]. For example, unlike in the $\overline{\mathrm{MS}}$ scheme [6], δV is ξ dependent [12,13,27,28] in the OS scheme of renormalization. As we will see below in (iii), however, because the gauge-dependent part of δV is UV finite and has a pure dispersive form, the ξ-dependent terms of an OS renormalized CKM matrix V can always be related to finite gauge-independent constants in a generalized and manifestly gauge-invariant $\overline{\mathrm{MS}}$ scheme of renormalization.

(iii) There exists an underlying symmetry in the renormalization of V^0, reflecting the presence of a general intrinsic freedom in redefining mixing matrices at higher orders.[3] The presence of this higher-order scheme arbitrariness in the renormalization of V^0 may be described as follows. We know that the CKM matrix is the product of two unitary matrices $U_L^{u,0}$ and $U_L^{d,0}$ relating the weak to mass eigenstates of the left-handed u- and d-type quark fields, respectively, i.e. $V^0 = U_L^{u,0} U_L^{d,0\dagger}$. If we now perform the following perturbative shifts in the left-handed quark wave functions and their mixing-matrix CTs:

$$\delta Z^{u,L} \to \delta Z^{u,L} + c^u, \quad \delta Z^{d,L} \to \delta Z^{d,L} + c^d, \qquad (2.23)$$

$$\delta U_L^u \to \delta U_L^u - \frac{1}{2} c^u U_L^u, \quad \delta U_L^{d\dagger} \to \delta U_L^{d\dagger} + \frac{1}{2} U_L^{d\dagger} c^d, \qquad (2.24)$$

where c^u and c^d are anti-Hermitian, gauge-independent UV-finite constant matrices, i.e. $c^{u,d} = -c^{u,d\dagger}$ and $c^{u,d} = \mathcal{O}(\delta Z^{u,L}, \delta Z^{d,L})$, then none of the important key equalities

[2]Here, we should remark that the analytic results for the wave-function CTs and δV obtained within the (G)PT or BFM are identical to those derived within the conventional framework of R_ξ gauges [13,12], with the additional restriction that the gauge-fixing parameters related to the photon and the Z boson are equal, i.e. $\xi_\gamma = \xi_Z$.

[3]The existence of such a scheme dependence in the renormalization of V^0 was first pointed out in [14].

(2.20), (2.21) and (2.22) will change through the order considered. On the basis of the above unitarity symmetry, we may generally define the manifestly gauge-independent CKM matrix CT:

$$\delta V = \delta V^{\overline{\rm MS}} - \frac{1}{2} c^u V + \frac{1}{2} V c^d, \tag{2.25}$$

where $\delta V^{\overline{\rm MS}}$ is the corresponding CT in the gauge-invariant $\overline{\rm MS}$ scheme evaluated in Eq. (2.21). Evidently, δV does not give rise to gauge dependences in the computation of physical transition amplitudes, e.g. in the top-decay amplitude $t \to W + b$. However, the UV-finite constants c^u and c^d introduce a scheme dependence into the renormalization of the CKM matrix V^0. Thus, all the various schemes of renormalization proposed in the literature [6,14,26,12,13,27,28], including the OS scheme [6,14], may be represented by the gauge-invariant expression (2.25) with appropriate choices for the UV-finite matrices c^u and c^d.

Finally, it is interesting to notice that the number of independent parameters contained in the anti-Hermitian matrices c^u and c^d is exactly equal to the number of group parameters, the so-called mixing angles, that generate the unitary rotations in the left-handed u- and d-quark flavor space. To elucidate the above point, let us consider the renormalized unitary matrix in the left-handed u-quark flavor space, U^u_L. In particular, U^u_L can be written as

$$U^u_L(\theta^a_u) = \exp(i\,\theta^a_u T^a), \tag{2.26}$$

where θ^a_u are the group parameters and T^a are the associate generators of the U(n) flavor group, satisfying the usual Lie-algebra relations: $[T^a, T^b] = i f^{abc} T^c$, with $T^a = T^{a\dagger}$, $T^0 = \mathbf{1}_n$ and $f^{0ab} = 0$. If we now shift θ^a_u by a finite higher-order amount $\delta\theta^a_u$, then the unitary matrix (2.26) exhibits the following variation:

$$U^u_L(\theta^a_u + \delta\theta^a_u) = \left[\mathbf{1}_n + i \left(\delta^{bc} - \frac{1}{2} f^{bcd} \theta^d_u - \frac{1}{6} f^{bdx} \times f^{cex} \theta^d_u \theta^e_u + \cdots \right) \delta\theta^b_u T^c \right] \times U^u_L(\theta^a_u) + \mathcal{O}(\delta\theta^{a2}_u). \tag{2.27}$$

In writing the RHS of Eq. (2.27), we have employed the Baker-Hausdorff formula for infinitesimal non-Abelian rotations. Comparing Eq. (2.27) with Eq. (2.24), we immediately recognize that the anti-Hermitian matrix c^u is given by

$$c^u = -2i \left(\delta^{ab} - \frac{1}{2} f^{abc} \theta^c_u - \frac{1}{6} f^{acx} \times f^{bdx} \theta^c_u \theta^d_u + \cdots \right) \delta\theta^a_u T^b, \tag{2.28}$$

where the $\delta\theta^a_u$'s parametrize the scheme-dependent shifts in the mixing angles θ^a_u. Notice that the anti-Hermitian matrix c^u in Eq. (2.28), determined by means of Eq. (2.27), preserves the unitarity properties of U^u_L by construction. Analogous determinations may also be found for c^d and hence for the scheme-dependent part of V in Eq. (2.25).

C. Mixing renormalization in the SM with singlet neutrinos

We will now discuss the problem of mixing-matrix renormalization in an $SU(2) \otimes U(1)_Y$ theory with a number N_G of fermionic doublets and a number N_R of right-handed (isosinglet) neutrinos. The interaction Lagrangians of this model which describe the couplings of the $W^\pm$, Z, and Higgs (H) bosons to the N_G charged leptons, l_i, and $(N_G + N_R)$ Majorana neutrinos, n_i, are given by

$$\mathcal{L}_W = -\frac{g_w}{\sqrt{2}} W^-_\mu \bar{l}_i B_{ij} \gamma^\mu P_L n_j + \text{H.c.}, \tag{2.29}$$

$$\mathcal{L}_Z = -\frac{g_w}{4 c_w} Z_\mu \bar{n}_i \gamma^\mu (C_{ij} P_L - C^*_{ij} P_R) n_j \tag{2.30}$$

$$\mathcal{L}_H = -\frac{g_w}{4 M_W} H \bar{n}_i [(m_i C_{ij} + m_j C^*_{ij}) P_L + (m_i C^*_{ij} + m_j C_{ij}) P_R] n_j. \tag{2.31}$$

Here, we follow the conventions of [17,20]. In Eqs. (2.29)–(2.31), B and C are $N_G \times (N_G + N_R)$ and $(N_G + N_R) \times (N_G + N_R)$-dimensional mixing matrices, defined as

$$B_{lj} = \sum_{k=1}^{N_G} V^l_{lk} U^{n*}_{kj}, \tag{2.32}$$

$$C_{ij} = \sum_{k=1}^{N_G} U^n_{ki} U^{n*}_{kj}. \tag{2.33}$$

In Eq. (2.32), the $N_G \times N_G$ unitary matrix V^l occurs in the diagonalization of the charged-lepton mass matrix and relates the weak to the mass eigenstates of the left-handed charged leptons. Correspondingly, the $(N_G + N_R) \times (N_G + N_R)$ unitary matrix U^n in Eq. (2.33) diagonalizes the symmetric neutrino-mass matrix

$$M^n = \begin{pmatrix} 0 & m_D \\ m^T_D & m_M \end{pmatrix}, \tag{2.34}$$

through the unitary transformation

$$U^{nT} M^n U^n = \hat{M}^n = \text{diag}(m_1, m_2, \ldots, m_{N_G+N_R}). \tag{2.35}$$

At this point, we should recall again [17] that the mixing matrices B and C obey a number of basic identities which ensure the renormalizability of the theory:

$$\sum_{k=1}^{N_G+N_R} B_{lk} B^*_{l'k} = \delta_{ll'}, \tag{2.36}$$

$$\sum_{k=1}^{N_G+N_R} C_{ik}C_{kj}=C_{ij}\,, \tag{2.37}$$

$$\sum_{k=1}^{N_G+N_R} B_{lk}C_{ki}=B_{li}\,, \tag{2.38}$$

$$\sum_{l=1}^{N_G} B^*_{li}B_{lj}=C_{ij}\,, \tag{2.39}$$

$$\sum_{k=1}^{N_G+N_R} m_k B_{lk}B_{l'k}=0, \tag{2.40}$$

$$\sum_{k=1}^{N_G+N_R} m_k B_{lk}C_{ik}=0, \tag{2.41}$$

$$\sum_{k=1}^{N_G+N_R} m_k C_{ik}C_{jk}=0. \tag{2.42}$$

The last three relations (2.40)–(2.42) are manifestations of the presence of lepton-number violation in the neutrino sector. Instead, if theory conserves lepton number, these three identities are not necessary. In this case, Majorana neutrinos are either degenerate in pairs forming massive Dirac neutrinos or unpaired giving rise to massless Majorana-Weyl two-component spinors. As a consequence of lepton-number conservation, the mixing matrix elements C^*_{ij} are absent from the Zn_in_j and Hn_in_j couplings in Eqs. (2.30) and (2.31), so the Zn_in_j coupling becomes purely chiral, proportional to the operator $\bar{n}_i\gamma_\mu P_L n_j$.

The renormalization of the mixing matrices B and C can be carried out in a way very similar to the SM case. Following analogous steps for the one-particle irreducible vertex functions $\Gamma^{W^-\bar{l}n_j}$ and $\Gamma^{Z\bar{n}_in_j}$ in the BFM or PT, we find

$$B=Z_L^{l,-1/2}B^0Z_L^{n,1/2}=Z_L^{l,1/2\dagger}B^0Z_L^{n,-1/2\dagger}, \tag{2.43}$$

$$C=Z_L^{n,-1/2}C^0Z_L^{n,1/2}=Z_L^{n,1/2\dagger}C^0Z_L^{n,-1/2\dagger}, \tag{2.44}$$

where $Z_L^{l,1/2}$ and $Z_L^{n,1/2}$ are wave-function renormalization matrices for the left-handed charged leptons and Majorana neutrinos, respectively. Equations (2.43) and (2.44) lead perturbatively to the mixing-matrix CTs [14]

$$\delta B=\frac{1}{4}(\delta Z^{l,L}-\delta Z^{l,L\dagger})B-\frac{1}{4}B(\delta Z^{n,L}-\delta Z^{n,L\dagger}), \tag{2.45}$$

$$\delta C=\frac{1}{4}(\delta Z^{n,L}-\delta Z^{n,L\dagger})C-\frac{1}{4}C(\delta Z^{n,L}-\delta Z^{n,L\dagger}). \tag{2.46}$$

In addition, the following constraining relations are satisfied:

$$\frac{1}{2}(\delta Z^{l,L}+\delta Z^{l,L\dagger})B=\frac{1}{2}B(\delta Z^{n,L}+\delta Z^{n,L\dagger}), \tag{2.47}$$

$$\frac{1}{2}(\delta Z^{n,L}+\delta Z^{n,L\dagger})C=\frac{1}{2}C(\delta Z^{n,L}+\delta Z^{n,L\dagger}). \tag{2.48}$$

It is important to observe that the renormalized mixing matrices B and C given in Eqs. (2.43) and (2.44) as well as the perturbative definitions of the mixing-matrix CTs δB and δC by means of Eqs. (2.45) and (2.46) fully satisfy the identities (2.36)–(2.39). However, the compatibility of δB and δC with the remaining identities (2.40)–(2.42) proves more subtle and will be discussed in Sec. III.

In the $\overline{\text{MS}}$ scheme, the mixing-matrix CTs δB and δC become gauge independent, only after the corresponding gauge-dependent part of the tadpole graphs are included [31,32,30,27]. To further illuminate our procedure, we calculate the Higgs-boson tadpole graph Γ^H induced by the W^+ boson, and the associate would-be Goldstone boson G^+ and ghost fields $c^+,\bar{c}^+$, i.e.

$$\Gamma^H_{(W)}(0)=\frac{g_w}{32\pi^2}\frac{M_H^2}{M_W}[\xi_W M_W^2(1+B_0(0,\xi_W M_W^2,\xi_W M_W^2))]+\frac{g_w}{16\pi^2}(D-1)M_W[M_W^2(1+B_0(0,M_W^2,M_W^2))], \tag{2.49}$$

where the one-loop function $B_0(p^2,m_1^2,m_2^2)$ is defined in the Appendix A. From Eq. (2.49) it is easy to see that only the M_H^2-dependent part of the tadpole depends on ξ_W and should be included in the scalar part of the self-energy transitions Σ^D_{ij} [cf. Eq. (2.2)]. More precisely, the M_H^2-dependent part of the tadpole graph effectively induces a gauge-dependent shift to the H-boson VEV v:

$$\left(\frac{\delta v}{v}\right)^{\xi_W}=\frac{g_w}{2M_W}\frac{(\Gamma^H)^{\xi_W}}{M_H^2}=\frac{\alpha_w}{16\pi}\xi_W(1+B_0(0,\xi_W M_W^2,\xi_W M_W^2)). \tag{2.50}$$

Similarly, the Z-boson loop causes an analogous gauge-dependent shift to v, i.e.

$$\left(\frac{\delta v}{v}\right)^{\xi_Z}=\frac{g_w}{2M_W}\frac{(\Gamma^H)^{\xi_Z}}{M_H^2}=\frac{\alpha_w}{32\pi c_w^2}\xi_Z(1+B_0(0,\xi_Z M_Z^2,\xi_W M_Z^2)). \tag{2.51}$$

Then, the ξ-dependent VEV shifts (2.50) and (2.51) contribute the following term to the scalar part of the Majorana-neutrino self-energy transitions:

$$\Sigma_{ij}^{M,\mathrm{tad}}P_L=-\left(\frac{\delta v}{v}\right)^{\xi_{W,Z}}(m_iC_{ij}+m_jC_{ij}^*)P_L. \quad (2.52)$$

Note that if neutrinos are Dirac particles, one has just to drop the term proportional to C_{ij}^* on the RHS of Eq. (2.52). As we will see more explicitly in the next sections, the tadpole contribution (2.52) plays an instrumental role in rendering the mixing-matrix CTs δB and δC gauge independent.

III. NEUTRINO MASS-MIXING SUM RULES

As was already mentioned in Sec. II, the neutrino mass-mixing sum rules (2.40)–(2.42) are very essential to ensure the renormalizability of the theory. These sum rules are obtained by projecting out the zero texture in the Majorana-neutrino mass matrix (2.34) as follows:

$$\sum_{k=1}^{N_G+N_R} m_k U_{lk}^n U_{l'k}^n=(U^n\hat{M}^n U^{nT})_{ll'}=M_{ll'}^{n*}=0, \quad (3.1)$$

for $l,l'=1,2,\ldots,N_G$. The zero texture is protected by the gauge symmetry of the theory, since the contributing 5-dimensional gauge-invariant operator $\overline{L}^T\Phi^T\Phi L^C$ is absent from the local renormalizable Lagrangian, where L and Φ are the lepton and Higgs doublets, respectively. This operator is radiatively generated at the one- [36,17] and two- [37] loop levels and is UV finite. The neutrino mass-mixing sum rule (3.1) is no longer valid, if the theory is extended by one Higgs triplet Δ_L, since the aforementioned lepton-number-violating operator can now appear in the tree level Lagrangian through the term $\overline{L}^T\Delta_L L^C$.

In the following, we will show that renormalization of U^n does not spoil the basic identity (3.1) in the $\overline{\mathrm{MS}}$ scheme. Within the scheme of renormalization outlined in Sec. II, the CT matrix δU^n of U^n is given by

$$\delta U^n=\frac{1}{4}U^n(\delta Z^{n,LT}-\delta Z^{n,L*}). \quad (3.2)$$

Observe that Eq. (3.2) may also be derived by setting $V^l=1$ in Eq. (2.45). In order that the bare and renormalized mixing matrices $U^{n,0}$ and U^n obey Eq. (3.1), one has to show that

$$(\delta U^n\hat{M}^nU^{nT}+U^n\hat{M}^n\delta U^{nT}+U^n\delta\hat{M}^nU^{nT})_{ll'}=0, \quad (3.3)$$

namely the corresponding mixing and mass CTs obey also Eq. (3.1) up to higher orders of perturbation proportional to $(\delta U^n)^2$.

Before offering a proof of Eq. (3.3) for the most general case, let us first gain some insight from considering a one-generation model with one right-handed neutrino only. The mass spectrum of this simple model, which essentially resembles the known seesaw scenario [15], consists of two Majorana neutrinos: a light neutrino ν observed in experiment and a yet-undetected superheavy one N. Most interestingly, in this model the elements of the bare mixing matrix $U^{n,0}$ are entirely determined by the two bare mass eigenvalues, m_ν^0 and m_N^0, of $M^{n,0}$ in Eq. (2.34):

$$U_{\nu\nu}^0=\sqrt{\frac{m_N^0}{m_\nu^0+m_N^0}},\quad U_{\nu N}^0=i\sqrt{\frac{m_\nu^0}{m_N^0}}U_{\nu\nu}^0,$$
$$U_{N\nu}^0=U_{\nu N}^0,\quad U_{NN}^0=U_{\nu\nu}^0, \quad (3.4)$$

where we have dropped the superscript "n" from U^n and have chosen the phase convention in which the elements $U_{\nu\nu}^0$ and U_{NN}^0 are positive. In the $\overline{\mathrm{MS}}$ scheme, the mass, wave-function and mixing CTs are found to be

$$\delta m_\nu=-\frac{\alpha_w}{16\pi}\frac{m_\nu m_N}{m_\nu+m_N}\left(\frac{3m_l^2}{M_W^2}-\frac{m_\nu^2}{M_W^2}-\frac{m_\nu m_N}{M_W^2}\right)C_{\mathrm{UV}},$$
$$\delta m_N=-\frac{\alpha_w}{16\pi}\frac{m_\nu m_N}{m_\nu+m_N}\left(\frac{3m_l^2}{M_W^2}-\frac{m_N^2}{M_W^2}-\frac{m_\nu m_N}{M_W^2}\right)C_{\mathrm{UV}}, \quad (3.5)$$

$$\delta Z_{\nu\nu}^L=-\frac{\alpha_w}{16\pi}C_{\nu\nu}\left(2\xi_W+\frac{\xi_Z}{c_w^2}+\frac{m_l^2}{M_W^2}+\frac{m_\nu^2}{M_W^2}+\frac{m_\nu m_N}{M_W^2}\right)C_{\mathrm{UV}},$$
$$\delta Z_{\nu N}^L=-\frac{\alpha_w}{8\pi}C_{\nu N}\left(\xi_W+\frac{\xi_Z}{2c_w^2}-\frac{m_l^2}{M_W^2}+\frac{3m_l^2}{M_W^2}\frac{m_\nu}{m_\nu+m_N}\right)C_{\mathrm{UV}},$$
$$\delta Z_{N\nu}^L=-\frac{\alpha_w}{8\pi}C_{N\nu}\left(\xi_W+\frac{\xi_Z}{2c_w^2}-\frac{m_l^2}{M_W^2}+\frac{3m_l^2}{M_W^2}\frac{m_N}{m_\nu+m_N}\right)C_{\mathrm{UV}},$$
$$\delta Z_{NN}^L=-\frac{\alpha_w}{16\pi}C_{NN}\left(2\xi_W+\frac{\xi_Z}{c_w^2}+\frac{m_l^2}{M_W^2}+\frac{m_N^2}{M_W^2}+\frac{m_\nu m_N}{M_W^2}\right)C_{\mathrm{UV}}, \quad (3.6)$$

$$\delta U_{\nu\nu}=\delta U_{NN}=U_{\nu\nu}\frac{3\alpha_w}{32\pi}\frac{m_\nu(m_N-m_\nu)}{(m_\nu+m_N)^2}\frac{m_l^2}{M_W^2}C_{\mathrm{UV}},$$

$$\delta U_{\nu N}=\delta U_{N\nu}=-U_{\nu N}\frac{3\alpha_w}{32\pi}\frac{m_N(m_N-m_\nu)}{(m_\nu+m_N)^2}\frac{m_l^2}{M_W^2}C_{\mathrm{UV}}. \quad (3.7)$$

In the above, $C_{ij}=U_{\nu i}U_{\nu j}^*$ according to Eq. (2.33) (with $i,j=\nu,N$), m_l is the charged-lepton mass, and C_{UV} is an UV constant defined in the Appendix. As was discussed in Sec. II, we find that the mass CTs δm_ν and δm_N, and the CT matrix δU^n computed by Eq. (3.2) are gauge independent only after the tadpole contributions are included. Moreover, these CTs satisfy the basic identity (3.3), i.e.

$$\delta m_\nu U^2_{\nu\nu} + \delta m_N U^2_{\nu N} + 2 m_\nu U_{\nu\nu} \delta U_{\nu\nu} + 2 m_N U_{\nu N} \delta U_{\nu N} = 0. \tag{3.8}$$

In the above simple Majorana-neutrino model, it is still possible to follow an alternative approach. Specifically, the same results would have been obtained if we had considered the elements of $U^{n,0}$ as functions of m^0_ν and m^0_N, i.e. $U^{n,0} = U^{n,0}(m^0_\nu, m^0_N)$. In this case, the mixing-matrix CT δU^n is calculated as [28]

$$\delta U^n = \delta m_\nu \frac{\partial U^n(m_\nu, m_N)}{\partial m_\nu} + \delta m_N \frac{\partial U^n(m_\nu, m_N)}{\partial m_N}, \tag{3.9}$$

and the basic relation (3.8) of the CTs will be satisfied by construction, even within the OS scheme of renormalization. In addition, δU^n defined in terms of δm_ν and δm_N is gauge-independent [28]. Instead, if we had employed Eq. (3.2) to compute δU^n in the OS scheme, the resulting expression would have naively been gauge-dependent and have violated the CT relation (3.8) by UV finite terms.[4] Nevertheless, even if the mass CTs are evaluated in the OS scheme, we can always restore the validity of the sum rule (3.1), along with the constraining relation (3.3), by redefining δU^n in a gauge-invariant manner. To be specific, exactly as we did in Sec. II, we add a gauge-independent and UV-finite anti-Hermitian matrix c^n to the $\overline{\rm MS}$ CTs of U^n:

$$\delta U^n = \delta U^{n,\overline{\rm MS}} + \frac{1}{2} c^n U^n, \tag{3.10}$$

with $c^n = -c^{n\dagger}$. In agreement with our phase conventions for the matrix elements of U^n in Eq. (3.4), it is sufficient to assume that the matrix c^n takes on the form

$$c^n = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \tag{3.11}$$

where c is a real constant. Then, the parameter c can be uniquely determined by requiring that the constraining relation (3.8), with the mass CTs δm_ν and δm_N computed in the OS scheme, holds exactly true. In this way, we were able to verify that the so-derived mixing-matrix CTs $\delta U_{\nu\nu}$ and $\delta U_{\nu N}$ are identical with those obtained by virtue of Eq. (3.9).

For the more realistic case, for which the SM contains more than one right-handed neutrino, the unitary matrix U^n cannot be entirely expressed in terms of neutrino masses, and the alternative approach based on Eq. (3.9) turns out to be not very practical. Instead, one may utilize the more general approach described above, in which anti-Hermitian constants c^n are added to the $\overline{\rm MS}$ CT δU^n. Within this generalized framework of the $\overline{\rm MS}$ scheme, the constraining relations (3.3) reduce the number of independent constants c^n. The remaining freedom should be fixed by comparing the theoretical predictions for observables involving the undetermined matrix elements of U^n with experiment. Here, we should stress again that the anti-Hermitian constants c^n are only required if the neutrino-mass renormalizations δm_i are computed by Eq. (2.8) in the OS scheme.

Unlike in the OS scheme, in the $\overline{\rm MS}$ scheme one has to pay the price that the $\overline{\rm MS}$-renormalized neutrino masses are not the physical ones, namely the poles of the neutrino propagators. However, as we will now show, all the basic symmetries of the theory, including the one reflected in the sum rule (3.1), are preserved and the addition of anti-Hermitian constants c^n is no longer needed. In particular, we will provide a general proof of the validity of the constraining relation (3.3) governing the neutrino mass and mixing CTs in the $\overline{\rm MS}$ scheme. To this end, we first substitute Eq. (3.2) into Eq. (3.3)

$$\begin{aligned}
\delta U^n \hat{M}^n U^{nT} + U^n \hat{M}^n \delta U^{nT} + U^n \delta\hat{M}^n U^{nT}\Big|_{ll'} &= U^n\left[\frac{1}{4}(\delta Z^{n,LT} - \delta Z^{n,L*})\hat{M}^n + \frac{1}{4}\hat{M}^n(\delta Z^{n,L} - \delta Z^{n,L\dagger}) + \delta\hat{M}^n\right]U^{nT}\Big|_{ll'} \\
&= U^n\left(\frac{1}{2}\delta Z^{n,LT}\hat{M}^n + \frac{1}{2}\hat{M}^n \delta Z^{n,L} + \delta\hat{M}^n\right)U^{nT}\Big|_{ll'} \\
&\quad - \frac{1}{4}U^n[(\delta Z^{n,LT} + \delta Z^{n,L*})\hat{M}^n + \hat{M}^n(\delta Z^{n,L} + \delta Z^{n,L\dagger})]U^{nT}\Big|_{ll'} \\
&= U^n \Sigma^{M*,\rm UV} U^{nT}\Big|_{ll'} + \frac{1}{2}U^n(\Sigma^{LT,\rm UV}\hat{M}^n + \hat{M}^n\Sigma^{L,\rm UV})U^{nT}\Big|_{ll'},
\end{aligned} \tag{3.12}$$

[4]This last result was earlier observed in [14].

where the superscript UV on Σ^M and Σ^L indicates their UV divergent parts, and $l,l'=1,2,\dots,N_G$. In deriving the last step of Eq. (3.12), we have used the relations

$$\Sigma^{M,\mathrm{UV}}=\delta\hat{M}^n+\frac{1}{2}\hat{M}^n\,\delta Z^{n,L*}+\frac{1}{2}\,\delta Z^{n,L\dagger}\hat{M}^n, \tag{3.13}$$

$$\Sigma^{L,\mathrm{UV}}=-\frac{1}{2}(\delta Z^{n,L}+\delta Z^{n,L\dagger}). \tag{3.14}$$

These relations may be straightforwardly obtained with the aid of Eq. (2.4) and (2.8) [see also (4.8) and (4.9) in [14]]. The first term $(U^n\Sigma^{M*,\mathrm{UV}}U^{nT})_{ll'}$ on the RHS of the last equality in Eq. (3.12) vanishes by itself, as a consequence of the absence of the operators $\overline{\nu_{lL}}\nu^C_{l'L}$ from the local Lagrangian. The second term vanishes, only if

$$U^n\hat{M}^n\Sigma^{L,\mathrm{UV}}U^{nT}|_{ll'}=0, \tag{3.15}$$

or equivalently if

$$U^{n*}\Sigma^{L,\mathrm{UV}}U^{nT}|_{\alpha l'}=0, \tag{3.16}$$

with $\alpha=N_G+1,N_G+2,\dots,N_G+N_R$. Equation (3.16) is derived by inserting the unity, $U^{nT}U^{n*}=\mathbf{1}_{N_G+N_R}$, between $\hat{M}^n$ and $\Sigma^{L,\mathrm{UV}}$ in Eq. (3.15), and noticing that $(U^n\hat{M}^nU^{nT})_{li}=M^{n*}_{li}=m^*_{Dl\alpha}$. Employing the analytic expressions for the neutrino self-energies in the Appendix [see also Eq. (4.9) below], it is not difficult to show that Eq. (3.16) is indeed valid. In fact, the vanishing of $(U^{n*}\Sigma^{L,\mathrm{UV}}U^{nT})_{\alpha l'}$ results from the absence of the lepton-number-violating kinetic terms $\overline{\nu_{lL}}i\partial\!\!\!/\nu^C_{\alpha R}$ of dimension 4 from the original Lagrangian in the flavor space. In the SM with right-handed neutrinos, the violation of lepton number occurs softly through the Majorana operators $\overline{\nu_{\alpha R}}\nu^C_{\beta R}$ of dimension 3, which is reflected in the neutrino mass-mixing sum rule (3.1). This completes our proof of Eq. (3.3).

We end our discussion by remarking that our general approach to the mixing-matrix renormalization may be applied to supersymmetric theories as well, e.g. to the unitary matrix [38] that diagonalizes the neutralino mass matrix in the minimal supersymmetric standard model (MSSM). In this case, a convenient framework for renormalization that respects supersymmetry is the so-called modified dimensional reduction ($\overline{\mathrm{DR}}$) scheme [39]. Alternatively, one may work in the $\overline{\mathrm{MS}}$ scheme and translate the results into the $\overline{\mathrm{DR}}$ scheme. As in the Majorana-neutrino case, the particular zero texture in the neutralino mass matrix is protected in the $\overline{\mathrm{DR}}$ scheme, but needs be reinforced by adding appropriate anti-Hermitian constants to the $\overline{\mathrm{DR}}$-renormalized neutralino mixing matrix, if the neutralino masses are renormalized in the OS scheme.

IV. RENORMALIZATION-GROUP EQUATIONS

As an immediate application of our study in Secs. II and III, we derive the gauge-independent RGEs of the mixing matrices B and C in the SM with right-handed neutrinos. With this aim, we first compute their respective CTs δB and δC by means of Eqs. (2.45) and (2.46) in the $\overline{\mathrm{MS}}$ scheme [40]:

$$\begin{aligned}\delta B^{\overline{\mathrm{MS}}}_{li}=&\sum_{l'\neq l}^{N_G}\frac{B_{l'i}}{2(m_l^2-m_{l'}^2)}[(m_l^2+m_{l'}^2)\Sigma^{l,L}_{ll'}\\&+2m_lm_{l'}\Sigma^{l,R}_{ll'}+2(m_l\Sigma^{l,D}_{ll'}+m_{l'}\Sigma^{l,D*}_{l'l})]^{\mathrm{UV}}\\&-\sum_{k\neq i}^{N_G+N_R}\frac{B_{lk}}{2(m_k^2-m_i^2)}[(m_k^2+m_i^2)\Sigma^{n,L}_{ki}+2m_km_i\Sigma^{n,R}_{ki}\\&+2(m_k\Sigma^{n,D}_{ki}+m_i\Sigma^{n,D*}_{ik})]^{\mathrm{UV}},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\delta C^{\overline{\mathrm{MS}}}_{ij}=&\sum_{k\neq i}^{N_G+N_R}\frac{C_{kj}}{2(m_i^2-m_k^2)}[(m_i^2+m_k^2)\Sigma^{n,L}_{ik}+2m_im_k\Sigma^{n,R}_{ik}\\&+2(m_i\Sigma^{n,D}_{ik}+m_k\Sigma^{n,D*}_{ki})]^{\mathrm{UV}}\\&-\sum_{k\neq j}^{N_G+N_R}\frac{C_{ik}}{2(m_k^2-m_j^2)}[(m_k^2+m_j^2)\Sigma^{n,L}_{kj}+2m_km_j\Sigma^{n,R}_{kj}\\&+2(m_k\Sigma^{n,D}_{kj}+m_j\Sigma^{n,D*}_{jk})]^{\mathrm{UV}}.\end{aligned} \tag{4.2}$$

Note that the expressions (4.1) and (4.2) pertain to Dirac neutrinos. In the case of Majorana neutrinos, these expressions are supplemented by the constraints stated in Eq. (2.9), with Σ^D_{ij} replaced by Σ^M_{ij}.

From the analytic results presented in the Appendix, it is straightforward to deduce the UV-divergent parts of the self-energy functions occurring in Eqs. (4.1) and (4.2) in the R_ξ gauge. We start listing the UV-divergent parts of the individual self-energy functions for the charged leptons

$$\begin{aligned}(\Sigma^{l,L}_{ll'})^{\mathrm{UV}}=&\frac{\alpha_w}{16\pi}\Bigg[\delta_{ll'}\Bigg(4s_w^2\xi_\gamma+2\xi_W\\&+\frac{(1-2s_w^2)^2}{c_w^2}\xi_Z+\frac{m_l^2}{M_W^2}\Bigg)+B_{li}B^*_{l'i}\frac{m_i^2}{M_W^2}\Bigg]C_{\mathrm{UV}},\end{aligned} \tag{4.3}$$

$$(\Sigma^{l,R}_{ll'})^{\mathrm{UV}}=\frac{\alpha_w}{16\pi}\delta_{ll'}\left(4s_w^2\xi_\gamma+\frac{4s_w^4}{c_w^2}\xi_Z+\frac{2m_l^2}{M_W^2}\right)C_{\mathrm{UV}}, \tag{4.4}$$

$$(\Sigma_{ll'}^{l,D})^{\rm UV} = -\frac{\alpha_w}{16\pi} m_l \Bigg[\delta_{ll'} \Bigg(4s_w^2(3+\xi_\gamma) - \frac{2s_w^2(1-2s_w^2)}{c_w^2} \times (3+\xi_Z) + \xi_W + \frac{\xi_Z}{2c_w^2} \Bigg) + 2B_{li}B_{l'i}^* \frac{m_i^2}{M_W^2} \Bigg] C_{\rm UV}. \tag{4.5}$$

Here and in the following, we consider the summation convention over repeated indices in their whole allowed range, unless explicitly stated otherwise. Specifically, charged lepton indices, such as l and l', are summed from 1 to N_G, and neutrino indices, e.g. i,j,k,n, from 1 to N_G+N_R. The divergent pieces of the self-energy functions for Dirac neutrinos are given by

$$(\Sigma_{ij}^{n,L})^{\rm UV} = \frac{\alpha_w}{16\pi} \Bigg[C_{ij} \Bigg(2\xi_W + \frac{\xi_Z}{c_w^2} \Bigg) + B_{li}^* B_{lj} \frac{m_l^2}{M_W^2} + C_{ik}C_{kj} \frac{m_k^2}{M_W^2} \Bigg] C_{\rm UV}, \tag{4.6}$$

$$(\Sigma_{ij}^{n,R})^{\rm UV} = \frac{\alpha_w}{16\pi} C_{ij} \frac{m_i m_j}{M_W^2} C_{\rm UV}, \tag{4.7}$$

$$(\Sigma_{ij}^{n,D})^{\rm UV} = -\frac{\alpha_w}{16\pi} m_i \Bigg[C_{ij} \Bigg(\xi_W + \frac{\xi_Z}{2c_w^2} \Bigg) + 2B_{li}^* B_{lj} \frac{m_l^2}{M_W^2} \Bigg] C_{\rm UV}. \tag{4.8}$$

If the neutrinos are Majorana particles, the UV parts of the self-energy functions then read

$$(\Sigma_{ij}^{n,L})^{\rm UV} = \frac{\alpha_w}{16\pi} \Bigg[C_{ij} \Bigg(2\xi_W + \frac{\xi_Z}{c_w^2} \Bigg) + B_{li}^* B_{lj} \frac{m_l^2}{M_W^2} + C_{ij}^* \frac{m_i m_j}{M_W^2} + C_{ik}C_{kj} \frac{m_k^2}{M_W^2} \Bigg] C_{\rm UV}, \tag{4.9}$$

$$(\Sigma_{ij}^{n,M})^{\rm UV} = -\frac{\alpha_w}{16\pi} \Bigg[(m_i C_{ij} + m_j C_{ij}^*) \Bigg(\xi_W + \frac{\xi_Z}{2c_w^2} \Bigg) + 2(m_i B_{li}^* B_{lj} + m_j B_{li} B_{lj}^*) \frac{m_l^2}{M_W^2} \Bigg] C_{\rm UV}. \tag{4.10}$$

The above analytic results of the UV pole structure of the neutrino self-energies reveal that the RG running of the mixing matrices B and C depends on the nature of neutrinos, namely on whether they are Dirac or Majorana particles. In the $\overline{\rm MS}$ scheme, the μ dependence of B and C may be computed by the beta functions β_B and β_C as

$$\beta_B = \mu \frac{dB}{d\mu} = \lim_{\varepsilon\to 0} \varepsilon g_w \frac{\partial}{\partial g_w} \delta B^{\overline{\rm MS}},$$

$$\beta_C = \mu \frac{dC}{d\mu} = \lim_{\varepsilon\to 0} \varepsilon g_w \frac{\partial}{\partial g_w} \delta C^{\overline{\rm MS}}, \tag{4.11}$$

where we have employed the fact that $\mu dg_w/d\mu = -\varepsilon g_w + \mathcal{O}(g_w^3)$. With the help of Eq. (4.11) and of Eq. (4.1) and (4.2), we obtain the following one-loop beta functions for the Dirac neutrinos:

$$\beta_{B_{li}} = \frac{\alpha_w}{16\pi} \Bigg\{ \sum_{l'\neq l}^{N_G} \frac{m_l^2 + m_{l'}^2}{m_{l'}^2 - m_l^2} B_{lk} B_{l'k}^* B_{l'i} \frac{3m_k^2}{M_W^2} - \sum_{k\neq i}^{N_G+N_R} \frac{B_{lk}}{m_k^2 - m_i^2} \times \Bigg[(m_k^2 + m_i^2) \Bigg(C_{kn}C_{ni} \frac{m_n^2}{M_W^2} - B_{lk}^* B_{li} \frac{3m_l^2}{M_W^2} \Bigg) + 2C_{ki} \frac{m_k^2 m_i^2}{M_W^2} \Bigg] \Bigg\}, \tag{4.12}$$

$$\beta_{C_{ij}} = \frac{\alpha_w}{16\pi} \Bigg\{ \sum_{k\neq i}^{N_G+N_R} \frac{C_{kj}}{m_i^2 - m_k^2} \times \Bigg[(m_i^2 + m_k^2) \Bigg(C_{in}C_{nk} \frac{m_n^2}{M_W^2} - B_{li}^* B_{lk} \frac{3m_l^2}{M_W^2} \Bigg) + 2C_{ik} \frac{m_i^2 m_k^2}{M_W^2} \Bigg] - \sum_{k\neq j}^{N_G+N_R} \frac{C_{ik}}{m_k^2 - m_j^2} \times \Bigg[(m_k^2 + m_j^2) \Bigg(C_{kn}C_{nj} \frac{m_n^2}{M_W^2} - B_{lk}^* B_{lj} \frac{3m_l^2}{M_W^2} \Bigg) + 2C_{kj} \frac{m_k^2 m_j^2}{M_W^2} \Bigg] \Bigg\}. \tag{4.13}$$

Correspondingly, the one-loop beta functions for the Majorana neutrinos are given by

$$\beta_{B_{li}}=\frac{\alpha_w}{16\pi}\Bigg\{\sum_{l'\neq l}^{N_G}\frac{m_l^2+m_{l'}^2}{m_{l'}^2-m_l^2}B_{lk}B^*_{l'k}B_{l'i}\frac{3m_k^2}{M_W^2}-\sum_{k\neq i}^{N_G+N_R}\frac{B_{lk}}{m_k^2-m_i^2}\Bigg[(m_k^2+m_i^2)\Bigg(C^*_{ki}\frac{m_km_i}{M_W^2}+C_{kn}C_{ni}\frac{m_n^2}{M_W^2}-B^*_{lk}B_{li}\frac{3m_l^2}{M_W^2}\Bigg)$$
$$+2m_km_i\Bigg(C_{ki}\frac{m_km_i}{M_W^2}+C^*_{kn}C^*_{ni}\frac{m_n^2}{M_W^2}-B_{lk}B^*_{li}\frac{3m_l^2}{M_W^2}\Bigg)\Bigg]\Bigg\}, \tag{4.14}$$

$$\beta_{C_{ij}}=\frac{\alpha_w}{16\pi}\Bigg\{\sum_{k\neq i}^{N_G+N_R}\frac{C_{kj}}{m_i^2-m_k^2}\Bigg[(m_i^2+m_k^2)\Bigg(C^*_{ik}\frac{m_im_k}{M_W^2}+C_{in}C_{nk}\frac{m_n^2}{M_W^2}-B^*_{li}B_{lk}\frac{3m_l^2}{M_W^2}\Bigg)$$
$$+2m_im_k\Bigg(C_{ik}\frac{m_im_k}{M_W^2}+C^*_{in}C^*_{nk}\frac{m_n^2}{M_W^2}-B_{li}B^*_{lk}\frac{3m_l^2}{M_W^2}\Bigg)\Bigg]-\sum_{k\neq j}^{N_G+N_R}\frac{C_{ik}}{m_k^2-m_j^2}$$
$$\times\Bigg[(m_k^2+m_j^2)\Bigg(C^*_{kj}\frac{m_km_j}{M_W^2}+C_{kn}C_{nj}\frac{m_n^2}{M_W^2}-B^*_{lk}B_{lj}\frac{3m_l^2}{M_W^2}\Bigg)+2m_km_j\Bigg(C_{kj}\frac{m_km_j}{M_W^2}+C^*_{kn}C^*_{nj}\frac{m_n^2}{M_W^2}-B_{lk}B^*_{lj}\frac{3m_l^2}{M_W^2}\Bigg)\Bigg]\Bigg\}. \tag{4.15}$$

It is worth commenting again on the fact that the beta functions β_B and β_C in Eqs. (4.12)–(4.15) become gauge independent in the $\overline{\rm MS}$ scheme, only after the gauge-dependent tadpole terms proportional to M_H^2 have been added to the self-energy functions Σ^D (or Σ^M). To the best of our knowledge, the beta functions β_B and β_C represent the most general results pertaining to the one-loop RG running of the mixing matrices B and C in the existing literature of the SM with isosinglet neutrinos. However, we should remark that the derived RGEs for β_B and β_C are only valid for energies larger than the heaviest neutrino mass. We have not considered the threshold effects due to the decoupling [41] of the heavy neutrinos, as these effects highly depend on the particular low-energy structure of the model [42–45,41] and will therefore be studied elsewhere.

V. CONCLUSIONS

We have revisited the problem of gauge dependence that occurs in the renormalization of mixing matrices, within the context of two generic frameworks: (i) the quark sector of the SM and (ii) the leptonic sector of the SM with isosinglet neutrinos. Although we confirmed the earlier observations [26,12,13] that an on-shell renormalized mixing matrix contains gauge-dependent terms, we have observed, however, that these terms are UV finite and have a pure dispersive form. Because of this last fact, we have found that these naive gauge-dependent terms can always be absorbed into the definition of a manifestly gauge-invariant, but physically equivalent, mixing matrix, where the latter is evaluated within a generalized $\overline{\rm MS}$ scheme of renormalization. This generalized scheme of renormalization is obtained by adding gauge-independent anti-Hermitian constants to a gauge-invariant, $\overline{\rm MS}$-renormalized mixing matrix [cf. Eqs. (2.25) and (3.10)]. In this way, the different renormalization schemes proposed in the literature [6,14,26,12,13,27,28], including the OS scheme [6,14], may be described for appropriate choices of the anti-Hermitian constants.

Our generalized $\overline{\rm MS}$ approach to the renormalization of mixing matrices may also be conveniently applied to maintain fundamental global or local symmetries of the unrenormalized Lagrangian. For instance, our approach may be utilized to protect the texture-zero structure of the Majorana-neutrino mass matrix (2.34) or similar constrained structures of predictive neutrino-mass models. Even though such additional symmetries are automatically preserved in the $\overline{\rm MS}$ scheme, they become distorted by UV-finite terms and so need to be reinforced, if the renormalized masses are evaluated within other renormalization schemes, such as the frequently adopted OS scheme. Most importantly, our approach of mixing-matrix renormalization may be applied to supersymmetric theories as well. In this case, the corresponding generalized $\overline{\rm DR}$ approach may be used to renormalize the mixing matrices that occur in the chargino and neutralino sectors [38], as well as in the squark and Higgs scalar sectors [46,47,27] of the MSSM.

As a byproduct of our study, we have derived in Sec. IV the gauge-independent RGEs for the $\overline{\rm MS}$-renormalized mixing matrices in the SM with isosinglet neutrinos. The so-derived RGEs are valid for energies that are higher than the mass of the heaviest of the heavy neutrinos. We have not taken into account the decoupling effects due to heavy neutrino thresholds, since they crucially depend on the low-energy structure of the model. However, they prove important to properly describe the RG running of the observed neutrino masses and the neutrino-oscillation angles at lower energies. We plan to return to this phenomenologically interesting topic in the near future.

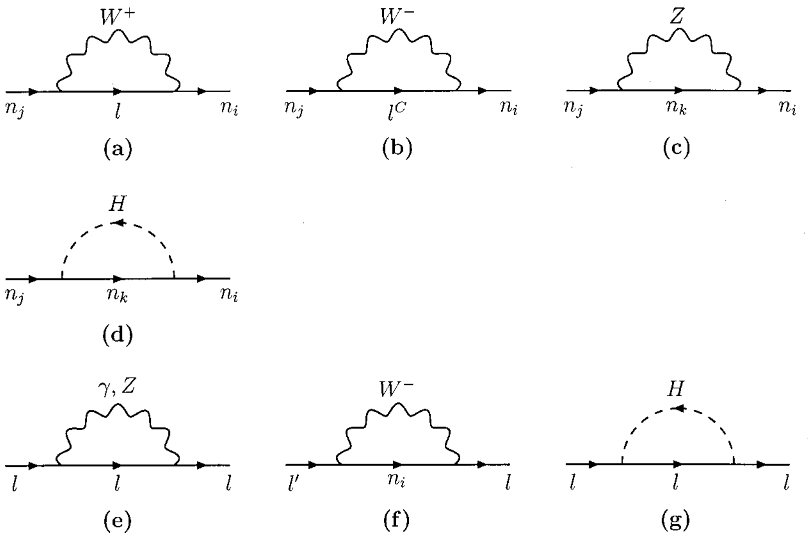

FIG. 1. Feynman graphs contributing to (a)–(d) neutral and (e)–(g) charged lepton self-energies in the unitary gauge. If neutrinos are Dirac particles, the graph (b) is absent.

APPENDIX: NEUTRAL AND CHARGED LEPTON SELF-ENERGIES

Here, we present analytic expressions for the neutrino and charged-lepton self-energies in the renormalizable R_ξ gauge. The Feynman diagrams that contribute to the neutrino self-energies are shown in Figs. 1(a)–1(d), while the corresponding graphs giving rise to charged-lepton self-energies are displayed in Figs. 1(e)–1(g). Our analytic results are expressed in terms of the usual Pasarino-Veltman one-loop functions [48]:

$$\{B_0;B_\mu\}(p^2,m_1^2,m_2^2)=(2\pi\mu)^{4-D}\int\frac{d^Dk}{i\pi^2}\times\frac{\{1;k_\mu\}}{(k^2-m_1^2)[(k+p)^2-m_2^2]},\tag{A1}$$

where the Minkowski space is extended to $D=4-2\varepsilon$ dimensions and μ is the so-called 't Hooft mass scale. Also, we adopt the frequently-used 4-dimensional convention for the Minkowskian metric $g^{\mu\nu}$: $g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$. The one-loop functions B_0 and B_μ, defined in Eq. (A1), are given by

$$B_0(p^2,m_1^2,m_2^2)=C_{\mathrm{UV}}-\ln\left(\frac{m_1m_2}{\mu^2}\right)+2+\frac{1}{p^2}\left[(m_2^2-m_1^2)\ln\left(\frac{m_1}{m_2}\right)+\lambda^{1/2}(p^2,m_1^2,m_2^2)\times\cosh^{-1}\left(\frac{m_1^2+m_2^2-p^2}{2m_1m_2}\right)\right],\tag{A2}$$

$$B_\mu(p^2,m_1^2,m_2^2)=p_\mu B_1(p^2,m_1^2,m_2^2),\tag{A3}$$

with $C_{\mathrm{UV}}=1/\varepsilon-\gamma_E+\ln 4\pi$, $\lambda(x,y,z)=(x-y-z)^2-4yz$ and

$$B_1(p^2,m_1^2,m_2^2)=\frac{m_2^2-m_1^2}{2p^2}(B_0(p^2,m_1^2,m_2^2)-B_0(0,m_1^2,m_2^2))-\frac{1}{2}B_0(p^2,m_1^2,m_2^2).\tag{A4}$$

The one-loop function $B_0(p^2,m_1^2,m_2^2)$ evaluated at $p^2=0$ simplifies to

$$B_0(0,m_1^2,m_2^2)=C_{\mathrm{UV}}-\ln\left(\frac{m_1m_2}{\mu^2}\right)+1+\frac{m_1^2+m_2^2}{m_1^2-m_2^2}\ln\left(\frac{m_2}{m_1}\right).\tag{A5}$$

From this last expression, a useful identity relating the arguments of the B_0-function at $p^2=0$ may easily be derived

$$B_0(0,m_1^2,m_2^2)=\frac{m_1^2}{m_1^2-m_2^2}B_0(0,m_1^2,m_1^2)-\frac{m_2^2}{m_1^2-m_2^2}B_0(0,m_2^2,m_2^2)+1.\tag{A6}$$

Equation (A6) may be successfully employed to check the gauge independence of physical quantities.

We first derive analytic expressions for the case of Dirac singlet neutrinos. In the R_ξ gauge, these are given by

$$\Sigma_{ij}^{n,L}(p^2)=-\frac{\alpha_w}{8\pi}\Bigg\{B_{li}^*B_{lj}\Bigg[2B_1(p^2,m_l^2,M_W^2)+B_0(p^2,m_l^2,M_W^2)+1-\xi_W B_0(p^2,m_l^2,\xi_W M_W^2)+\frac{p^2-m_l^2}{M_W^2}(B_1(p^2,M_W^2,m_l^2)$$
$$-B_1(p^2,\xi_W M_W^2,m_l^2))+\frac{m_l^2}{M_W^2}B_1(p^2,m_l^2,\xi_W M_W^2)\Bigg]+\frac{1}{2c_w^2}C_{ki}^*C_{kj}\Bigg[2B_1(p^2,m_k^2,M_Z^2)+B_0(p^2,m_k^2,M_Z^2)+1$$
$$-\xi_Z B_0(p^2,m_k^2,\xi_Z M_Z^2)+\frac{p^2-m_k^2}{M_Z^2}(B_1(p^2,M_Z^2,m_k^2)-B_1(p^2,\xi_Z M_Z^2,m_k^2))+\frac{m_k^2}{M_Z^2}(B_1(p^2,m_k^2,\xi_Z M_Z^2)$$
$$+B_1(p^2,m_k^2,M_H^2))\Bigg]\Bigg\}, \tag{A7}$$

$$\Sigma_{ij}^{n,R}(p^2)=-\frac{\alpha_w}{8\pi}\Bigg[B_{li}^*B_{lj}\frac{m_i m_j}{M_W^2}B_1(p^2,m_l^2,\xi_W M_W^2)+\frac{1}{2c_w^2}C_{ki}^*C_{kj}\frac{m_i m_j}{M_Z^2}(B_1(p^2,m_k^2,\xi_Z M_Z^2)+B_1(p^2,m_k^2,M_H^2))\Bigg], \tag{A8}$$

$$\Sigma_{ij}^{n,D}(p^2)=-\frac{\alpha_w}{8\pi}m_i\Bigg\{B_{li}^*B_{lj}\frac{m_l^2}{M_W^2}B_0(p^2,m_l^2,\xi_W M_W^2)+\frac{1}{2c_w^2}C_{ki}^*C_{kj}\frac{m_k^2}{M_Z^2}(B_0(p^2,m_k^2,\xi_Z M_Z^2)-B_0(p^2,m_k^2,M_H^2))$$
$$+\frac{1}{2}C_{ij}\Bigg[\xi_W(1+B_0(0,\xi_W M_W^2,\xi_W M_W^2))+\frac{1}{2c_w^2}\xi_Z(1+B_0(0,\xi_Z M_Z^2,\xi_Z M_Z^2))\Bigg]\Bigg\}. \tag{A9}$$

Note that in Eq. (A9) the p^2-independent terms represent the ξ-dependent part of the tadpole contributions. These contributions are crucial, as they restore the gauge independence in the RG running of neutrino masses and mixing angles.

Next, we present analytic expressions for Majorana-neutrino self-energies in the R_ξ gauge:

$$\Sigma_{ij}^{n,L}(p^2)=-\frac{\alpha_w}{8\pi}\Bigg\{B_{li}^*B_{lj}\Bigg[2B_1(p^2,m_l^2,M_W^2)+B_0(p^2,m_l^2,M_W^2)+1-\xi_W B_0(p^2,m_l^2,\xi_W M_W^2)+\frac{p^2-m_l^2}{M_W^2}(B_1(p^2,M_W^2,m_l^2)$$
$$-B_1(p^2,\xi_W M_W^2,m_l^2))+\frac{m_l^2}{M_W^2}B_1(p^2,m_l^2,\xi_W M_W^2)\Bigg]+B_{li}B_{lj}^*\frac{m_i m_j}{M_W^2}B_1(p^2,m_l^2,\xi_W M_W^2)$$
$$+\frac{1}{2c_w^2}C_{ki}^*C_{kj}\Bigg[2B_1(p^2,m_k^2,M_Z^2)+B_0(p^2,m_k^2,M_Z^2)+1-\xi_Z B_0(p^2,m_k^2,\xi_Z M_Z^2)+\frac{p^2-m_k^2}{M_Z^2}(B_1(p^2,M_Z^2,m_k^2)$$
$$-B_1(p^2,\xi_Z M_Z^2,m_k^2))\Bigg]+\frac{1}{2c_w^2}\frac{1}{M_Z^2}(m_iC_{ki}+m_kC_{ki}^*)(m_kC_{kj}+m_jC_{kj}^*)\times(B_1(p^2,m_k^2,\xi_Z M_Z^2)+B_1(p^2,m_k^2,M_H^2))\Bigg\}, \tag{A10}$$

$$\Sigma_{ij}^{n,M}(p^2)=-\frac{\alpha_w}{8\pi}\Bigg\{\frac{m_l^2}{M_W^2}(m_iB_{li}^*B_{lj}+m_jB_{li}B_{lj}^*)B_0(p^2,m_l^2,\xi_W M_W^2)-\frac{1}{2c_w^2}\Bigg[m_kC_{ki}C_{kj}(3B_0(p^2,m_k^2,M_Z^2)-2$$
$$+\xi_Z B_0(p^2,m_k^2,\xi_Z M_Z^2))-\frac{m_k}{M_Z^2}(m_iC_{ki}^*+m_kC_{ki})(m_kC_{kj}+m_jC_{kj}^*)\times(B_0(p^2,m_k^2,\xi_Z M_Z^2)-B_0(p^2,m_k^2,M_H^2))\Bigg]$$
$$+\frac{1}{2}(m_iC_{ij}+m_jC_{ij}^*)\Bigg[\xi_W(1+B_0(0,\xi_W M_W^2,\xi_W M_W^2))+\frac{1}{2c_w^2}\xi_Z(1+B_0(0,\xi_Z M_Z^2,\xi_Z M_Z^2))\Bigg]\Bigg\}. \tag{A11}$$

Likewise, we included in Eq. (A11) the ξ-dependent part of the tadpole contributions.

Finally, for completeness, we give the charged-lepton transition amplitudes for $l'\to l$:

$$\Sigma^{l,L}_{ll'}(p^2)=-\frac{\alpha_w}{8\pi}\Bigg\{2s_w^2\delta_{ll'}\Bigg[2B_1(p^2,m_l^2,\mu_\gamma^2)+B_0(p^2,m_l^2,\mu_\gamma^2)+1-\xi_\gamma B_0(p^2,m_l^2,\xi_\gamma\mu_\gamma^2)+\frac{p^2-m_l^2}{\mu_\gamma^2}(B_1(p^2,\mu_\gamma^2,m_l^2)$$
$$-B_1(p^2,\xi_\gamma\mu_\gamma^2,m_l^2))\Bigg]+\frac{(1-2s_w^2)^2}{2c_w^2}\delta_{ll'}\Bigg[2B_1(p^2,m_l^2,M_Z^2)+B_0(p^2,m_l^2,M_Z^2)+1-\xi_Z B_0(p^2,m_l^2,\xi_Z M_Z^2)$$
$$+\frac{p^2-m_l^2}{M_Z^2}(B_1(p^2,M_Z^2,m_l^2)-B_1(p^2,\xi_Z M_Z^2,m_l^2))\Bigg]+\delta_{ll'}\frac{1}{2c_w^2}\frac{m_l^2}{M_Z^2}(B_1(p^2,m_l^2,\xi_Z M_Z^2)+B_1(p^2,m_l^2,M_H^2))$$
$$+B_{li}B^*_{l'i}\Bigg[2B_1(p^2,m_i^2,M_W^2)+B_0(p^2,m_i^2,M_W^2)+1-\xi_W B_0(p^2,m_i^2,\xi_W M_W^2)+\frac{p^2-m_i^2}{M_W^2}$$
$$\times(B_1(p^2,M_W^2,m_i^2)-B_1(p^2,\xi_W M_W^2,m_i^2))+\frac{m_i^2}{M_W^2}B_1(p^2,m_i^2,\xi_W M_W^2)\Bigg]\Bigg\}, \tag{A12}$$

$$\Sigma^{l,R}_{ll'}(p^2)=-\frac{\alpha_w}{8\pi}\Bigg\{2s_w^2\delta_{ll'}\Bigg[2B_1(p^2,m_l^2,\mu_\gamma^2)+B_0(p^2,m_l^2,\mu_\gamma^2)+1-\xi_\gamma B_0(p^2,m_l^2,\xi_\gamma\mu_\gamma^2)+\frac{p^2-m_l^2}{\mu_\gamma^2}$$
$$\times(B_1(p^2,\mu_\gamma^2,m_l^2)-B_1(p^2,\xi_\gamma\mu_\gamma^2,m_l^2))\Bigg]+\frac{2s_w^4}{c_w^2}\delta_{ll'}\Bigg[2B_1(p^2,m_l^2,M_Z^2)+B_0(p^2,m_l^2,M_Z^2)$$
$$+1-\xi_Z B_0(p^2,m_l^2,\xi_Z M_Z^2)+\frac{p^2-m_l^2}{M_Z^2}(B_1(p^2,M_Z^2,m_l^2)-B_1(p^2,\xi_Z M_Z^2,m_l^2))\Bigg]$$
$$+\delta_{ll'}\frac{1}{2c_w^2}\frac{m_l^2}{M_Z^2}(B_1(p^2,m_l^2,\xi_Z M_Z^2)+B_1(p^2,m_l^2,M_H^2))+B_{li}B^*_{l'i}\frac{m_l m_{l'}}{M_W^2}B_1(p^2,m_i^2,\xi_W M_W^2)\Bigg\}, \tag{A13}$$

$$\Sigma^{l,D}_{ll'}(p^2)=-\frac{\alpha_w}{8\pi}m_l\Bigg\{2s_w^2\delta_{ll'}(3B_0(p^2,m_l^2,\mu_\gamma^2)-2+\xi_\gamma B_0(p^2,m_l^2,\xi_\gamma\mu_\gamma^2))-\frac{s_w^2(1-2s_w^2)}{c_w^2}\delta_{ll'}(3B_0(p^2,m_l^2,M_Z^2)-2$$
$$+\xi_Z B_0(p^2,m_l^2,\xi_Z M_Z^2))+\delta_{ll'}\frac{1}{2c_w^2}\frac{m_l^2}{M_Z^2}(B_0(p^2,m_l^2,\xi_Z M_Z^2)-B_0(p^2,m_l^2,M_H^2))$$
$$+B_{li}B^*_{l'i}\frac{m_i^2}{M_W^2}B_0(p^2,m_i^2,\xi_W M_W^2)+\frac{1}{2}\delta_{ll'}\Bigg[\xi_W(1+B_0(0,\xi_W M_W^2,\xi_W M_W^2))$$
$$+\frac{1}{2c_w^2}\xi_Z(1+B_0(0,\xi_Z M_Z^2,\xi_Z M_Z^2))\Bigg]\Bigg\}. \tag{A14}$$

Apart from the ξ-dependent part of the Higgs tadpoles included in Eqs. (A14), Eqs. (A12)–(A14) agree well with those presented in [12].

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