Fourth order approximation of the 0^{++} glueball mass of $(2+1)$-dimensional SU(3) lattice gauge theory

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The random-phase approximation is applied to coupled cluster expansions for $(2+1)$-dimensional SU(3) lattice gauge theory. The 0^{++} glueball mass is calculated up to the fourth order. The result agrees with the recent Monte Carlo result.

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I. INTRODUCTION

In an earlier paper [1], we proposed a new scheme to proceed coupled cluster expansions in lattice gauge theory (LGT), that is, to apply the random-phase approximation [2] to coupled cluster expansions. In coupled cluster expansions, the linked clusters used to expand the wave functions consist of Wilson loops at fixed relative orientation and separation [4–6]. When the gauge group is non-Abelian, the bases of linked clusters are overcomplete, related to each other by the unimodular condition of SU(N) gauge group. It is trivial to identify the independent clusters from this overcomplete set when the order of expansions is high, because the linked clusters contain many and complicated Wilson loops as the order of expansions increases [1,3]. We observe that clusters gain at most one Wilson loop when the expansive order increases by 1. If we replace some loop in those clusters whose loops have increased by 1 with its vacuum average value, then all clusters have the same number of loops, i.e., one loop. It is difficult to identify the independent clusters from those consisting of only one Wilson loop. Therefore, high-order expansions are able to do. Using this method, we studied the vacuum state of the $(2+1)$-dimensional SU(3) LGT. The long-wavelength expansion coefficients of the vacuum state show nice scaling behavior and convergence [1]. If this approximation is reasonable, it should also be possible to use this method to study the low-energy exciting states. In this paper, we use this method to calculate the 0^{++} glueball mass of $(2+1)$-dimensional SU(3) LGT.

II. THE APPROXIMATED CLUSTER EXPANSIONS AND THE GLUEBALL MASS

We use the Kogut-Susskind Hamiltonian [7]

$$H=\frac{g^2}{2a}\left[\sum_l E_l^2-\frac{2}{g^4}\sum_p \mathrm{Tr}(U_p+U_p^\dagger)\right], \tag{1}$$

where g is the coupling constant, E_l^a is the ''chromoelectric'' field on the link l, and a is the lattice spacing. The dimensionless coupling constant g is related to the invariant coupling constant e by $g^2=e^2a$ in the case of $(2+1)$-dimensional SU(3) LGT. According to Greensite [8], the ground state of SU(N) lattice gauge fields has the form $e^{R(U)}|0\rangle$, where $|0\rangle$ is the nude vacuum, defined by $E_l^a|0\rangle=0$, and $R(U)$ is a functional of lattice gauge fields U, which may be determined by the eigenvalue equation $He^{R(U)}|0\rangle=\epsilon_\Omega e^{R(U)}|0\rangle$, i.e.,

$$\sum_l \{[E_l^a,[E_l^a,R(U)]]+[E_l^a,R(U)][E_l^a,R(U)]\} -\frac{2}{g^4}\sum_p \mathrm{Tr}(U_p+U_p^\dagger)=\frac{2a}{g^2}\epsilon_\Omega, \tag{2}$$

with ϵ_Ω being the vacuum energy. Equation (2) is difficult to solve exactly. Since the ground state has the form $\exp(S)$ [9], the coupled cluster method (CCM) may be used to find its approximate solution.

In the CCM, the linked clusters which are used to expand the $R(U)$ are products of Wilson loops [5,6,10,11]. As we pointed out in Sec. I, the linked clusters are not completely independent because the elements of the SU(N) gauge group are unimodular. For example, the unimodular condition for the SU(3) group is

$$U_{il}U_{jm}U_{kn}\epsilon_{lmn}=\epsilon_{ijk}, \tag{3}$$

where U is any gauge link variable defined on a link. From this, we may deduce some more convenient relations, such as

$$\begin{aligned}\mathrm{Tr}U^+V&=\mathrm{Tr}U^2V-(\mathrm{Tr}U)\mathrm{Tr}UV+\mathrm{Tr}U^+\mathrm{Tr}V,\\ \mathrm{Tr}UVUQ&=\mathrm{Tr}U\,\mathrm{Tr}VUQ+\mathrm{Tr}U^{+^2}VUQ\\ &\quad-\mathrm{Tr}U^+\mathrm{Tr}U^+VUQ,\end{aligned} \tag{4}$$

where U,V,Q are any Wilson loop. Those relations show that the products of Wilson loops are related to each other and the linked clusters are not completely independent. When the order of expansions is high, the linked clusters may be products of many Wilson loops, so that it is impossible to identify a set of independent clusters from those linked clusters. That was the reason why we calculated the cluster expansions only up to the third order in Refs. [10], [11].

In Ref. [1], the random-phase approximation is used to circumvent the above problem. In the following, we present a description of it. Suppose we want to expand $R(U)$ to order M,

$$R(U)=\sum_{i=1}^{M} R_i(U), \tag{5}$$

where R_i is a linear combination of the ith-order linked clusters with the coefficients to be determined. Substituting it into Eq. (2) and adopting the truncation scheme of Ref. [5], we get the truncated equation

$$\sum_l \left\{ \left[E_l, \left[E_l, \sum_{i=1}^{M} R_i(U) \right] \right] + \sum_{i+j\leqslant M} [E_l, R_i(U)][E_l, R_j(U)] \right\} - \frac{2}{g^4} \sum_p \mathrm{Tr}(U_p + U_p^\dagger) = \frac{2a}{g^2} \epsilon_\Omega . \tag{6}$$

Generally, the term $[E_l, R_i][E_l, R_j]$ will produce some new clusters which are different from the clusters with order from 1 to $(i+j-1)$. These new clusters are defined as the $(i+j)$th-order clusters [4,5,10]. Therefore, Choosing the R_1 to be

$$R_1 = c_{1,1} G_{1,1} = c_{1,1}(\square + h.c.), \tag{7}$$

with $c_{1,1}$ being a coefficients to be determined, then high-order clusters may be produced by $[E_l, R_i][E_l, R_j]$ automatically. In the usual CCM, these clusters may contain at most $(i+j)$ Wilson loops. However, if we use the RPA, these clusters will contain only one loop. If R_i and R_j are linear combinations of clusters which contain only one Wilson loop, then the new clusters produced by $[E_l, R_i][E_l, R_j]$ will contain at most two loops. Applying the RPA to all those clusters which contain two loops, that is, replacing one of the two Wilson loops with its vacuum average value, we get all new clusters composed of only one Wilson loop, so R_{i+j} is also a linear combination of clusters with only one Wilson loop. Now, the independent bases are obtained directly. As a byproduct, the number of independent bases of high-order expansions is much smaller than that without using the random-phase approximation, hence the calculation is simplified very much.

In the above random-phase approximation, when the size of two Wilson loops is different, we replace the smaller one with its vacuum average and let the larger one remain unchanged. For example,

$$\square\!\square \approx \langle \square \rangle \square \tag{8}$$

There are two reasons to do so. One is that this is the easiest and simplest way to apply the RPA. If we replace the larger loop with its vacuum average, we need to determine many more average values of Wilson loops. Another reason is due to a physical consideration. A glueball has some finite size. Only when the space occupied by the glueball is covered with the Wilson loops used in the expansions is the calculation efficient. So, we replace the smaller loop with its vacuum average value in Eq. (8).

We use the Feynman-Hellman theorem to determine the vacuum average value of a Wilson loop. Let G be some Wilson loop and $\langle G \rangle$ be its vacuum average. Defining $W=(2a/g^2)H$, we make the following change [4]:

$$W \to W + \xi_G (G + G^\dagger). \tag{9}$$

Using $\langle G \rangle = \langle G^\dagger \rangle$, we have

$$2\langle G \rangle = \left. \frac{\partial w_0}{\partial \xi_G} \right|_{\xi_G=0}, \tag{10}$$

where $w_0=(2a/g^2)\epsilon_\Omega$.

Introducing the average value of a Wilson loop in coupled cluster expansions has been proposed in Ref. [4]. However, in their shifted coupled cluster method, clusters containing many Wilson loops still exist, hence their method still suffers from the problem of identifying a set of independent clusters.

Let us turn to the calculation of glueball mass. In the CCM, an excited state may be assumed to be [3,11,6]

$$|F\rangle = F(U)\exp(R(U))|0\rangle, \tag{11}$$

where $F(U)$ is a functional of a lattice gauge field with appropriate symmetries. The 0^{++} glueball state has the same symmetries as the ground state. Let us denote the state with $|F_s\rangle$, then $F_s(U)$ can be expanded by the same clusters as those used in the ground state. The truncated eigenvalue equation with $F_s(U)$ being expanded to order M is

$$\sum_l \left\{ \left[E_l, \left[E_l, \sum_{i=1}^{M} F_i(U) \right] \right] + 2 \sum_{i+j\leqslant M} [E_l, R_i(U)][E_l, F_j(U)] \right\} = \frac{2a}{g^2} \Delta\epsilon_s \sum_{i=1}^{M} F_i(U), \tag{12}$$

where $\Delta\epsilon_s = \epsilon_s - \epsilon_\Omega$ is the mass gap and ϵ_s is defined by $H|F_s\rangle = \epsilon_s |F_s\rangle$. Let the 0^{++} glueball mass be m, then $m=\Delta\epsilon_s$. Applying the RPA to the above calculation will be similar to that of the ground state. However, the situation seems not so simple. The result is not as good as we expect (see the third-order result without overlapping graphs in Fig. 1). When applying the RPA, there are two possible ways to deal with an overlapping Wilson loop. One is using the relation (4) to turn it into unoverlapping loop clusters. For example,

$$\square\!\square = \square\!\square - 2\,\square \tag{13}$$

After the overlapping loop cluster on the left-hand side of Eq. (13) is replaced with the clusters of the right-hand side of Eq. (13), then the RPA is applied. Another way is preserving

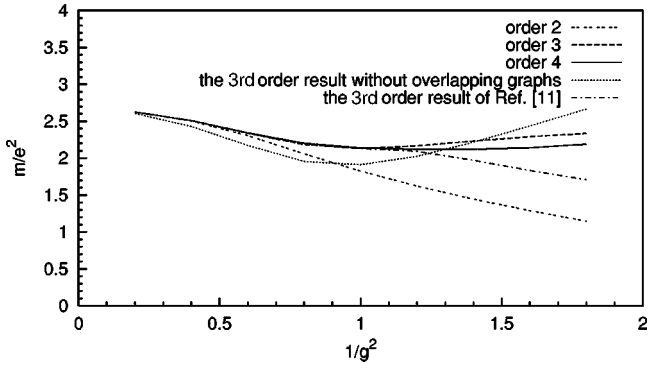

FIG. 1. m/e^2 as a function of $1/g^2$. The curves denoted by ''order 2, 3, and 4'' are the results obtained by using the RPA and preserving overlapping Wilson loop clusters. The curve denoted by ''the 3rd order result without overlapping graphs'' is the third order result obtained by using the RPA but not preserving overlapping Wilson loop clusters.

the clusters with one overlapping Wilson loop. In the expansion of the ground state, we use the former, because it is simpler than the latter one. However, it is probably too simple for the glueball state. A glueball is a localized object. We expect the detailed structure of clusters to play a more important role in a glueball state than in the vacuum state. Therefore, we take the latter one in the expansion of the glueball wave function.

III. THE RESULTS AND DISCUSSIONS

In Fig. 1, we present the results of m/e^2 against $1/g^2$ from the second order to the fourth order. For comparison, we also plot the third order result without using the random-phase approximation, taken from Ref. [11], and the third order result using the random-phase approximation but not preserving the overlapping loop clusters. From the figure, we see that the results obtained by using the RPA and preserving the overlapping loop clusters show better scaling behavior. That the third order and fourth order results are close to each other shows that the result of our method converges rapidly.

From the fourth order result, we obtain the 0^{++} glueball mass

$$\frac{m}{e^2} = 2.13 \pm 0.10, \tag{14}$$

where the error is the standard error in taking the average of the fourth order data in the scaling region $1.2 \leqslant 1/g^2 \leqslant 1.7$. The value $1/g^2 \sim 1.2$ at which the scaling begins is close to that in $(2+1)$-dimensional U(1) [12] and SU(2) [6] LGT, and almost the same as that in Ref. [1]. Having finished the paper, we found recent Monte Carlo research about the mass spectrum of SU(N) LGT in $(2+1)$ dimensions [13]. For 0^{++} of SU(3), Teper gave

$$\frac{m}{\sqrt{\sigma}} = 4.329(41) \quad \text{and} \quad \frac{\sqrt{\sigma}}{e^2} = 0.5530(20), \tag{15}$$

where σ is the string tension. Equation (15) gives $m/e^2 = 2.39$. Our approximation result is reasonable.

Since this method is simple and the number of clusters is much smaller than that in using the usual CCM, it is able to extrapolate this method to the more realistic cases, e.g., SU(3) LGT in $(3+1)$ dimensions.

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[1] P. Hui, X. Y. Fang, and Q. Z. Chen, Phys. Rev. D **62**, 034505 (2000).
[2] P. A. Wolff, Phys. Rev. **120**, 814 (1960); T. Izuyama, D. Kim, and R. Kubo, J. Phys. Soc. Jpn. **18**, 1025 (1963).
[3] D. Schütte, W. H. Zheng, and C. J. Hamer, Phys. Rev. D **55**, 2974 (1997).
[4] C. H. Llewellyn Smith and N. J. Watson, Phys. Lett. B **302**, 463 (1993).
[5] S. H. Guo, Q. Z. Chen, and L. Li, Phys. Rev. D **49**, 507 (1994).
[6] Q. Z. Chen, S. H. Guo, W. H. Zheng, and X. Y. Fang, Phys. Rev. D **50**, 3564 (1994).
[7] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[8] J. P. Greensite, Nucl. Phys. **B166**, 113 (1980).
[9] H. Kümmel, K. H. Lührmann, and J. G. Zabolitzky, Phys. Rep. **36**, 1 (1978); J. Arponen, Ann. Phys. (N.Y.) **151**, 311 (1983).
[10] Q. Z. Chen, X. Q. Luo, and S. H. Guo, Phys. Lett. B **341**, 349 (1995).
[11] Q. Z. Chen, X. Q. Luo, S. H. Guo, and X. Y. Fang, Phys. Lett. B **348**, 560 (1995).
[12] X. Y. Fang, J. M. Liu, and S. H. Guo, Phys. Rev. D **53**, 1523 (1996).
[13] M. J. Teper, Phys. Rev. D **59**, 014512 (1999).