Lattice extraction of $K \to \pi\pi$ amplitudes to $O(p^4)$ in chiral perturbation theory

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We show that the lattice calculation of $K \to \pi\pi$ and ϵ'/ϵ amplitudes for (8,1) and (27,1) operators to $O(p^4)$ in chiral perturbation theory is feasible when one uses $K \to \pi\pi$ computations at the two unphysical kinematics allowed by the Maiani-Testa theorem, along with the usual (computable) two- and three-point functions, namely $K \to 0$, $K \to \pi$ (with momentum), and K-$\bar{K}$. Explicit expressions for the finite logarithms emerging from our $O(p^4)$ analysis to the above amplitudes are also given.

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I. INTRODUCTION AND MOTIVATION

Recent lattice QCD calculations done by the CP-PACS [1] and RBC [2] Collaborations using domain wall fermions have made significant progress in explaining the $\triangle I = 1/2$ rule in the decay $K \to \pi\pi$, though their results for the direct CP violation parametrized by $\mathrm{Re}(\epsilon'/\epsilon)$ differ rather drastically from experiment. Recall that measurements at CERN [3] and Fermilab [4] have yielded an experimental grand average of $\mathrm{Re}(\epsilon'/\epsilon) = (17.2 \pm 1.8) \times 10^{-4}$. On the theoretical side, both lattice collaborations find a value of $\mathrm{Re}(\epsilon'/\epsilon) \simeq -5 \times 10^{-4}$, a *negative* value, though both groups have made rather severe (uncontrolled) approximations. Given that large cancellations occur between contributions of the strong and the electroweak penguins towards ϵ'/ϵ (cancellations that are not relevant in the calculation of the CP-conserving $K \to \pi\pi$ amplitudes), and given the serious approximations, the disagreement with experiment for ϵ'/ϵ should not be totally unexpected [5,6].

One of these uncontrolled approximations was the use of the quenched approximation, where the fermion determinant in the path integral is set to 1 in order to make the problem tractable on current computers. Another was the use of leading-order chiral perturbation theory to relate unphysical $K \to \pi$ and $K \to |0\rangle$ amplitudes to the physical $K \to \pi\pi$ amplitudes. This method was first proposed by Bernard *et al.* [7]. Because of the difficulty of extracting multihadron decay amplitudes from the lattice, as expressed by the Maiani-Testa theorem [8], it is much easier to compute the two- and three-point functions (i.e., $K \to |0\rangle$ and $K \to \pi$, respectively) and use chiral perturbation theory (ChPT) to extrapolate to the physical matrix elements.

It is likely, however, that next-to-leading-order ChPT will introduce significant corrections ($\sim$30% or more) to the leading-order amplitudes. Furthermore, since final-state (strong) phases cannot arise at tree level in the chiral amplitudes, chiral-loop corrections are essential to enable us to use the measured phases for the $I=0$ and 2 final states as an additional testing ground of the calculational apparatus. Unfortunately, at higher orders in ChPT the number of free parameters that enters the theory (and must be determined from first-principles methods like the lattice) proliferates rapidly. It has been shown by Cirigliano and Golowich [9] that the dominant electroweak penguin contributions [$(8_L,8_R)$'s] to $K \to \pi\pi$ can be recovered at next-to-leading order (NLO) from $K \to \pi$ amplitudes using momentum insertion. Bijnens *et al.* [10] showed how to obtain most of the low-energy constants (LEC's) relevant for the case of the $(8_L,1_R)$'s and $(27_L,1_R)$'s using off-shell $K \to \pi$ Green's functions; not all LEC's could be determined, however.

On the lattice, though, not only $K \to |0\rangle$ and $K \to \pi$ with momentum insertion are calculable, but so is $K \to \pi\pi$ at the two values of unphysical kinematics for which the Maiani-Testa theorem can be bypassed. To recapitulate, despite Maiani-Testa restrictions, direct calculation of $K \to \pi\pi$ on the lattice is accessible at (i) $m_K^{\mathrm{lattice}} = m_\pi^{\mathrm{lattice}}$, where the weak operator inserts energy [11] and (ii) $m_K^{\mathrm{lattice}} = 2m_\pi^{\mathrm{lattice}}$, i.e., at threshold [12]. We will refer to these two special locations as unphysical kinematics point 1 (UK1) and point 2 (UK2), respectively. In this work, we therefore focus on using $K \to |0\rangle$, $K \to \pi$ with momentum insertion and K-$\bar{K}$, along with information from $K \to \pi\pi$ at these two unphysical values of the kinematics which are accessible to the lattice [13]. Thereby, we are able to show that all the relevant $O(p^4)$ LEC's can be recovered for $K \to \pi\pi$ in the physical $(8_L,1_R)$ and $(27_L,1_R)$ cases. Expressions for $O(p^4)$ finite logarithmic contributions to all the processes that may be needed for fitting the lattice data are then given.

The content of the paper is as follows. Section II very briefly recapitulates the formalism of effective four-fermion operators in a standard model calculation. Section III reviews ChPT and the realization of the effective four-quark operators in terms of ChPT operators for weak processes. Section VI presents the results of this paper, showing how to obtain the low-energy constants necessary for physical $K \to \pi\pi$ amplitudes at $O(p^4)$ from quantities which can, in principle, be computed on the lattice. Section V presents the conclusion.

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Finally, the finite logarithmic contributions to the relevant amplitudes are presented in a set of Appendixes.

II. EFFECTIVE FOUR-QUARK OPERATORS

In the standard model, the nonleptonic interactions can be expressed in terms of an effective $\Delta S=1$ Hamiltonian using the operator product expansion [14,15],

$$\langle \pi\pi|\mathcal{H}_{\triangle S=1}|K\rangle=\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^{*}\sum c_i(\mu)\langle \pi\pi|Q_i|K\rangle_\mu , \tag{1}$$

where $c_i(\mu)$ are the Wilson coefficients containing the short-distance perturbative physics, and the matrix elements $\langle \pi\pi|Q_i|K\rangle_\mu$ must be calculated nonperturbatively. The four quark operators are

$$Q_1=\overline{s_a}\gamma_\mu(1-\gamma^5)u_a\overline{u_b}\gamma^\mu(1-\gamma^5)d_b , \tag{2}$$

$$Q_2=\overline{s_a}\gamma_\mu(1-\gamma^5)u_b\overline{u_b}\gamma^\mu(1-\gamma^5)d_a , \tag{3}$$

$$Q_3=\overline{s_a}\gamma_\mu(1-\gamma^5)d_a\sum_q \overline{q_b}\gamma^\mu(1-\gamma^5)q_b , \tag{4}$$

$$Q_4=\overline{s_a}\gamma_\mu(1-\gamma^5)d_b\sum_q \overline{q_b}\gamma^\mu(1-\gamma^5)q_a , \tag{5}$$

$$Q_5=\overline{s_a}\gamma_\mu(1-\gamma^5)d_a\sum_q \overline{q_b}\gamma^\mu(1+\gamma^5)q_b , \tag{6}$$

$$Q_6=\overline{s_a}\gamma_\mu(1-\gamma^5)d_b\sum_q \overline{q_b}\gamma^\mu(1+\gamma^5)q_a , \tag{7}$$

$$Q_7=\tfrac{3}{2}\overline{s_a}\gamma_\mu(1-\gamma^5)d_a\sum_q e_q\overline{q_b}\gamma^\mu(1+\gamma^5)q_b , \tag{8}$$

$$Q_8=\tfrac{3}{2}\overline{s_a}\gamma_\mu(1-\gamma^5)d_b\sum_q e_q\overline{q_b}\gamma^\mu(1+\gamma^5)q_a , \tag{9}$$

$$Q_9=\tfrac{3}{2}\overline{s_a}\gamma_\mu(1-\gamma^5)d_a\sum_q e_q\overline{q_b}\gamma^\mu(1-\gamma^5)q_b , \tag{10}$$

$$Q_{10}=\tfrac{3}{2}\overline{s_a}\gamma_\mu(1-\gamma^5)d_b\sum_q e_q\overline{q_b}\gamma^\mu(1-\gamma^5)q_a . \tag{11}$$

In the effective theory, Q_1 and Q_2 are the current-current weak operators, Q_3-Q_6 are the operators arising from QCD penguin diagrams, while Q_7-Q_{10} are the operators arising from electroweak penguin diagrams.

III. CHIRAL PERTURBATION THEORY

Chiral perturbation theory (ChPT) is an effective quantum field theory where the quark and gluon degrees of freedom have been integrated out, and is expressed only in terms of the lowest mass pseudoscalar mesons [16]. It is a perturbative expansion about small quark masses and small momentum of the low mass pseudoscalars. The effective Lagrangian is made up of complicated nonlinear functions of the pseudoscalar fields, and is nonrenormalizable, making it necessary to introduce arbitrary constants at each order in perturbation theory. In such an expansion, operators of higher order in the momentum (terms with increasing numbers of derivatives) or mass appear at higher order in the perturbative expansion. The most general set of operators at a given order can be constructed out of the unitary chiral matrix field Σ, given by

$$\Sigma=\exp\left[\frac{2i\phi^a\lambda^a}{f}\right], \tag{12}$$

where λ^a are proportional to the Gell-Mann matrices with $\mathrm{tr}(\lambda_a\lambda_b)=\delta_{ab}$, ϕ^a are the real pseudoscalar-meson fields, and f is the meson decay constant in the chiral limit, with f_π equal to 130 MeV in our convention.

At leading order $[O(p^2)]$ in ChPT, the strong Lagrangian is given by

$$\mathcal{L}_{\rm st}^{(2)}=\frac{f^2}{8}\mathrm{tr}[\partial_\mu\Sigma\partial^\mu\Sigma]+\frac{f^2B_0}{4}\mathrm{tr}[\chi^\dagger\Sigma+\Sigma^\dagger\chi], \tag{13}$$

where $\chi=(m_u,m_d,m_s)_{\rm diag}$ and

$$B_0=\frac{m_{\pi^+}^2}{m_u+m_d}=\frac{m_{K^+}^2}{m_u+m_s}=\frac{m_{K^0}^2}{m_d+m_s}.$$

The leading-order weak chiral Lagrangian is given by [17]

$$\begin{aligned}\mathcal{L}_W^{(2)}=&\ \alpha_{88}\,\mathrm{tr}[\lambda_6\Sigma Q\Sigma^\dagger]\\&+\alpha_1\,\mathrm{tr}[\lambda_6\partial_\mu\Sigma\partial^\mu\Sigma^\dagger]+\alpha_2 2B_0\,\mathrm{tr}[\lambda_6(\chi^\dagger\Sigma+\Sigma^\dagger\chi)]\\&+\alpha_{27}t_{kl}^{ij}(\Sigma\partial_\mu\Sigma^\dagger)_i^k(\Sigma\partial^\mu\Sigma^\dagger)_j^l+\mathrm{H.c.},\end{aligned} \tag{14}$$

where t_{kl}^{ij} is symmetric in i, j, and k,l are traceless on any pair of upper and lower indices with nonzero elements $t_{12}^{13}=1$, $t_{22}^{23}=1/2$, and $t_{32}^{33}=-\frac{3}{2}$. Also, Q is the quark charge matrix, $Q=1/3(2,-1,-1)_{\rm diag}$, and $(\lambda_6)_{ij}=\delta_{i3}\delta_{j2}$.

The terms in the weak Lagrangian can be classified according to their chiral transformation properties under $SU(3)_L\times SU(3)_R$. The first term in Eq. (14) transforms as $8_L\times 8_R$ under chiral rotations and corresponds to the electroweak penguin operators Q_7 and Q_8. The next two terms in Eq. (14) transform as $8_L\times 1_R$, while the last transforms as $27_L\times 1_R$ under chiral rotations. All ten of the four-quark operators of the effective weak Lagrangian have a realization in the chiral Lagrangian differing only in their transformation properties and the values of the low-energy constants which contain the nonperturbative dynamics of the theory.

For the transition of interest, $K\to\pi\pi$, the operators can induce a change in isospin of $\frac{1}{2}$ or $\frac{3}{2}$ depending on the final isospin state of the pions. We can then classify the isospin components of the four-quark operators according to their transformation properties [1,2]:

$$Q_1^{1/2},Q_2^{1/2},Q_9^{1/2},Q_{10}^{1/2}:8_L\times 1_R\oplus 27_L\times 1_R;$$
$$Q_1^{3/2},Q_2^{3/2},Q_9^{3/2},Q_{10}^{3/2}:27_L\times 1_R;$$
$$Q_3^{1/2},Q_4^{1/2},Q_5^{1/2},Q_6^{1/2}:8_L\times 1_R;$$
$$Q_7^{1/2},Q_8^{1/2},Q_7^{3/2},Q_8^{3/2}:8_L\times 8_R.$$

Note that Q_3-Q_6 are pure isospin $\frac{1}{2}$ operators. This paper deals only with the $27_L\times 1_R$ and $8_L\times 1_R$ operators. For the treatment of the $8_L\times 8_R$ operators to $O(p^2)$ NLO, see Ref. [9]. At NLO, the strong Lagrangian involves 12 additional operators with undetermined coefficients. These were introduced by Gasser and Leutwyler in [18]. The complete basis of counterterm operators for the weak interactions with $\Delta S=1,2$ was treated by Kambor, Missimer, and Wyler in [19] and [20]. A minimal set of counterterm operators contributing to $K\to\pi$ and $K\to\pi\pi$ for the $(8_L,1_R)$ and $(27_L,1_R)$ cases is given by [17], with the effective Lagrangian

$$\mathcal{L}_W^{(4)}=\sum e_i\mathcal{O}_i^{(8,1)}+\sum d_i\mathcal{O}_i^{(27,1)}, \tag{15}$$

$$\mathcal{O}_1^{(8,1)}=\mathrm{tr}[\lambda_6 S^2],\quad \mathcal{O}_1^{(27,1)}=t_{kl}^{ij}(S)_i^k(S)_j^l,$$
$$\mathcal{O}_2^{(8,1)}=\mathrm{tr}[\lambda_6 S]\mathrm{tr}[S],\quad \mathcal{O}_2^{(27,1)}=t_{kl}^{ij}(P)_i^k(P)_j^l,$$
$$\mathcal{O}_3^{(8,1)}=\mathrm{tr}[\lambda_6 P^2],\quad \mathcal{O}_4^{(27,1)}=t_{kl}^{ij}(L_\mu)_i^k(\{L^\mu,S\})_j^l,$$
$$\mathcal{O}_4^{(8,1)}=\mathrm{tr}[\lambda_6 P]\mathrm{tr}[P],\quad \mathcal{O}_5^{(27,1)}=t_{kl}^{ij}(L_\mu)_i^k([L^\mu,P])_j^l,$$
$$\mathcal{O}_5^{(8,1)}=\mathrm{tr}[\lambda_6[S,P]],\quad \mathcal{O}_6^{(27,1)}=t_{kl}^{ij}(S)_i^k(L^2)_j^l,$$
$$\mathcal{O}_{10}^{(8,1)}=\mathrm{tr}[\lambda_6\{S,L^2\}],\quad \mathcal{O}_7^{(27,1)}=t_{kl}^{ij}(L_\mu)_i^k(L^\mu)_j^l tr[S],$$
$$\mathcal{O}_{11}^{(8,1)}=\mathrm{tr}[\lambda_6 L_\mu S L^\mu],\quad \mathcal{O}_{20}^{(27,1)}=t_{kl}^{ij}(L_\mu)_i^k(\partial_\nu W^{\mu\nu})_j^l, \tag{16}$$
$$\mathcal{O}_{12}^{(8,1)}=\mathrm{tr}[\lambda_6 L_\mu]\mathrm{tr}[\{L^\mu,S\}],\quad \mathcal{O}_{24}^{(27,1)}=t_{kl}^{ij}(W_{\mu\nu})_i^k(W^{\mu\nu})_j^l,$$
$$\mathcal{O}_{13}^{(8,1)}=\mathrm{tr}[\lambda_6 S][L^2],$$
$$\mathcal{O}_{15}^{(8,1)}=\mathrm{tr}[\lambda_6[P,L^2]],$$
$$\mathcal{O}_{35}^{(8,1)}=\mathrm{tr}[\lambda_6\{L_\mu,\partial_\nu W^{\mu\nu}\}],$$
$$\mathcal{O}_{39}^{(8,1)}=\mathrm{tr}[\lambda_6 W_{\mu\nu}W^{\mu\nu}],$$

with $S=2B_0(\chi^\dagger\Sigma+\Sigma^\dagger\chi)$, $P=2B_0(\chi^\dagger\Sigma-\Sigma^\dagger\chi)$, $L_\mu=i\Sigma^\dagger\partial_\mu\Sigma$, and $W^{\mu\nu}=2(\partial_\mu L_\nu+\partial_\nu L_\mu)$.

This list is identical to that of Bijnens *et al.* [10] except for the inclusion of $\mathcal{O}_{35,39}^{(8,1)}$ and $\mathcal{O}_{20,24}^{(27,1)}$, which contain surface terms, and so cannot be absorbed into the other constants for processes which do not conserve 4-momentum at the weak vertex. Since we must use 4-momentum insertion in a number of our amplitudes, these counterterms must be considered, and they are left explicit even in the physical amplitudes. There are additional operators containing surface terms, but it was checked that these counterterms can be absorbed into linear combinations of the above minimal set for all amplitudes considered in this paper.

TABLE I. The divergences in the weak $O(p^4)$ counterterms, e_i's and d_i's, for the (8,1)'s and (27,1)'s, respectively.

e_i	ε_i	ε_i'	d_i	γ_i
1	1/4	5/6	1	−1/6
2	−13/18	11/18	2	0
3	5/12	0	4	3
4	−5/36	0	5	1
5	0	5/12	6	−3/2
10	19/24	3/4	7	1
11	3/4	0	20	1/2
12	1/8	0	24	1/8
13	−7/8	1/2		
15	23/24	−3/4		
35	−3/8	0		
39	−3/16	0		

The $\Delta S=2$ operators are components of the same irreducible tensor [21] under $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$, and so the d_i are the same for both the $\Delta S=1$ and $\Delta S=2$ cases. The operators governing $\Delta S=2$ transitions are obtained from the above $(27_L,1_R)$'s, only with $t_{22}^{33}=t_{33}^{22}=1, t_{kl}^{ij}=0$ otherwise. This is important since some of our information comes from the $K^0\to\overline{K^0}$ amplitude.

The divergences associated with the counterterms have been obtained in [19] and [10]. The subtraction procedure can be defined as

$$e_i=e_i^r+\frac{1}{16\pi^2f^2}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-1-\ln 4\pi)\right]\times 2(\alpha_1\varepsilon_i+\alpha_2\varepsilon_i'), \tag{17}$$

$$d_i=d_i^r+\frac{1}{16\pi^2f^2}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-1-\ln 4\pi)\right]2\alpha_{27}\gamma_i, \tag{18}$$

with the divergent pieces $\varepsilon_i,\varepsilon_i',\gamma_i$ given in Table I.

It is also necessary for the method of this paper to consider the $O(p^4)$ strong Lagrangian, which was first given by Gasser and Leutwyler, $\mathcal{L}_{\mathrm{st}}^{(4)}=\Sigma L_i\mathcal{O}_i^{(\mathrm{st})}$.

The strong $O(p^4)$ operators relevant for this calculation are the following [18]:

$$\mathcal{O}_1^{(\mathrm{st})}=\mathrm{tr}[L^2]^2,$$
$$\mathcal{O}_2^{(\mathrm{st})}=\mathrm{tr}[L_\mu L_\nu]\mathrm{tr}[L^\mu L^\nu],$$
$$\mathcal{O}_3^{(\mathrm{st})}=\mathrm{tr}[L^2L^2],$$
$$\mathcal{O}_4^{(\mathrm{st})}=\mathrm{tr}[L^2]\mathrm{tr}[S],$$
$$\mathcal{O}_5^{(\mathrm{st})}=\mathrm{tr}[L^2S],$$

TABLE II. The divergences in the strong $O(p^4)$ counterterms, Γ_i [18].

i	Γ_i
1	3/32
2	3/16
3	0
4	1/8
5	3/8
6	11/144
8	5/48

$$\mathcal{O}_6^{(\mathrm{st})}=\mathrm{tr}[S]^2,$$

$$\mathcal{O}_8^{(\mathrm{st})}=\frac{1}{2}\mathrm{tr}[S^2-P^2]. \tag{19}$$

The Gasser-Leutwyler counterterms also contribute to the cancellation of divergencies in the expressions relevant to this paper. The subtraction is defined similarly to that of the weak counterterms,

$$L_i=L_i^r+\frac{1}{16\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-1-\ln 4\pi)\right]\Gamma_i, \tag{20}$$

with the divergent parts of the counterterm coefficients given in Table II [18].

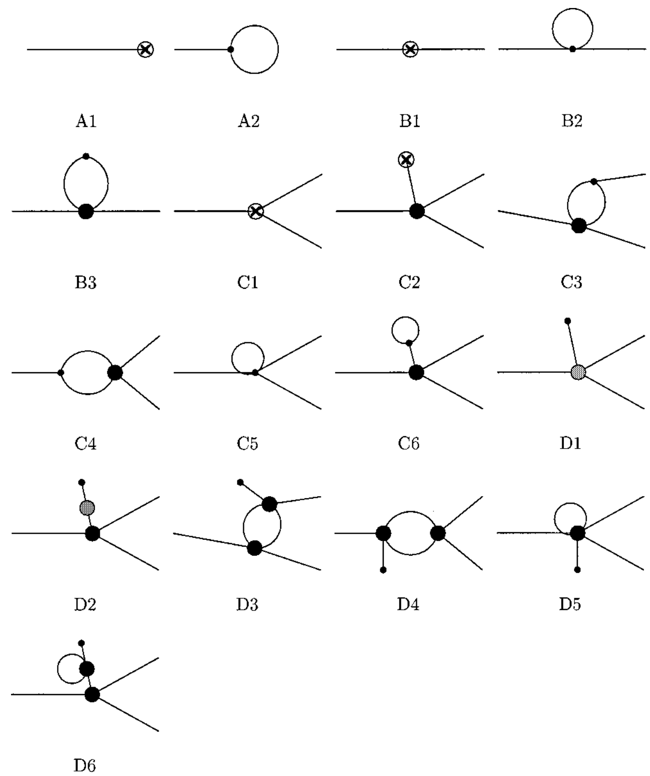

FIG. 1. $O(p^4)$ corrections include tree-level diagrams with insertion of the $O(p^4)$ weak vertices (crossed circles), tree-level diagrams with insertion of $O(p^4)$ strong vertices (lightly shaded circles), one-loop diagrams with insertions of the $O(p^2)$ weak vertices (small filled circles), and the $O(p^2)$ strong vertices (big filled circles). A1 and A2 are for $K\to|0\rangle$. B1–B3 are for $K\to\pi$ and $K\to\bar{K}$. C1–C6 and D1–D6 are for $K\to\pi\pi$.

IV. $K\to\pi\pi$ AMPLITUDES AT $O(P^4)$

As mentioned before, in this work we will include both $K\to\pi$ with momentum insertion and $K\to\pi\pi$ at the two unphysical kinematics. The complete list of necessary ingredients consists of the two-point functions $K^0\to|0\rangle$, the three point functions $K^0\to\overline{K^0}$, and $K\to\pi$, all with $m_s\neq m_d=m_u$, and the four-point functions $K\to\pi\pi$ at the two values of unphysical kinematics, $m_K=m_\pi$ (requiring energy insertion) and $m_K=2m_\pi$. These two threshold values of the kinematics bypass the Miani-Testa theorem, which states that multihadron final states are not accessible on the lattice at any other kinematics aside from the threshold [8]. At these kinematics, the strong phases are 0, and the effects of final-state interactions vanish. However, these amplitudes at unphysical kinematics do contain information on the $O(p^4)$ low-energy constants, and when combined with information from the other two- and three-point functions mentioned above, all of the $O(p^4)$ low-energy constants necessary for $K\to\pi\pi$ can be obtained. The phases of the amplitude are introduced in ChPT via the one-loop unitarity corrections of the $O(p^2)$ operators.

Because the $K\to\pi$ amplitudes do not conserve four-momentum for $m_s\neq m_d$, it is necessary to allow the weak operator to transfer a four-momentum $q\equiv p_K-p_\pi$, as in [9]. This is also necessary for the case of $K\to\pi\pi, m_K=m_\pi$ [11]. At $O(p^4)$, this requires the inclusion of (potentially many) surface terms in our minimal counterterm operator basis. The number of such additional terms appearing in linearly independent combinations was discovered to be small (four), and thus the method was not invalidated. Also, this method requires the computation of $K\to\pi\pi$ matrix elements at unphysical kinematics because there are LEC's which appear in $K\to\pi\pi$ but do not appear in $K\to\pi$ at all. These are d_5, e_{13}, and e_{15}.

The diagrams which must be evaluated for the $O(p^4)$ corrections are shown in Fig. 1. The diagrams to be evaluated for $K\to|0\rangle$ are A1 and A2. The diagrams to be evaluated for $K\to\pi$ and $K\to\bar{K}$ are B1–B3. C1–C6 and D1–D6 must be evaluated for $K\to\pi\pi$. D1–D6 contain the tadpole vertex of the weak mass $O(p^2)$, (8,1) operator. Also, the renormalization of the external legs via the strong interaction must be taken into account.

A. $(27_L,1_R)$, $\Delta I=3/2$

The counterterms necessary to reconstruct the $O(p^4)(27_L,1_R),\Delta I=3/2,K\to\pi\pi$ amplitudes can be obtained from $K^0\to\overline{K^0}$; $K^+\to\pi^+,\Delta I=3/2$; and $K\to\pi\pi,\Delta I=3/2$ at *either* value of the unphysical kinematics. The expression for $K^0\to\overline{K^0}$ is given by (all masses and decay constants are the bare ones)

$$\langle\overline{K^0}|\mathcal{O}^{(27,1)}_{\Delta S=2}|K^0\rangle_{\rm ct}=\frac{8\alpha_{27}}{f^2}m_K^2-\frac{8}{f^2}[4(d_1^r+d_2^r+d_{20}^r-4d_{24}^r-d_4^r-d_7^r)m_K^4-2(4d_1^r+d_7^r)m_K^2m_\pi^2+4d_1^rm_\pi^4]. \tag{21}$$

Equation (21), as well as all the following amplitudes, include only the tree level $O(p^2)$ and $O(p^4)$ weak counterterm contributions. For brevity, the logarithmic terms as well as the Gasser-Leutwyler L_i counterterms have been omitted in the body of the paper, but are included in a set of Appendixes. It was verified that the divergences in the logarithmic terms cancel those of the counterterms, providing a strong check on the calculation. Note also that for the application of this method most of the Gasser-Leutwyler counterterms must be known, and that an improved determination of the relevant ones could be obtained from a lattice calculation of observables in the purely strong sector, e.g., most can be obtained from the pseudoscalar masses and decay constants.

From the above $K^0\to\overline{K^0}$ amplitude, one can extract the values of d_1^r and d_7^r from a fit to terms quadratic in the quark masses. The other relevant expressions for $K\to\pi\pi,\Delta I=3/2$ are

$$\langle\pi^+|\mathcal{O}^{(27,1),(3/2)}|K^+\rangle_{\rm ct}=-\frac{4}{f^2}\alpha_{27}p_K\cdot p_\pi+\frac{8}{f^2}[2d_2^rm_K^2m_\pi^2+(d_{20}^r-d_4^r-2d_7^r)m_K^2p_K\cdot p_\pi+(d_{20}^r-d_4^r-d_7^r)m_\pi^2p_K\cdot p_\pi-8d_{24}^r(p_K\cdot p_\pi)^2], \tag{22}$$

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle_{\rm ct}=-\frac{8i\alpha_{27}}{f^3}m^2+\frac{16im^4}{f^3}(d_{20}^r-2d_4^r+d_5^r-3d_7^r) \tag{23}$$

for $K\to\pi\pi,m_K=m_\pi=m$, and

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle_{\rm ct}=-\frac{4i\alpha_{27}}{f^3}(m_K^2-m_\pi^2)+\frac{3im_K^4}{2f^3}(2d_2^r+2d_{20}^r-8d_{24}^r-4d_4^r+2d_5^r-9d_7^r), \tag{24}$$

for $K\to\pi\pi,m_{K(\text{1-loop})}=2m_{\pi(\text{1-loop})}$.

From Eq. (22) we get the additional combinations of counterterms d_2^r,d_{24}^r, and $d_4^r-d_{20}^r$. From either expression for $K\to\pi\pi$ at unphysical kinematics we can then obtain $d_4^r-d_5^r$. Along with the tree-level LEC, α_{27}, these five linear combinations $[d_2^r,d_7^r,d_4^r-d_5^r,d_4^r-d_{20}^r,d_{24}^r]$ are sufficient to determine $K\to\pi\pi$ at the physical kinematics as given in Eq. (25). Notice that there is considerable redundancy in determining these coefficients. For example, d_4-d_{20}, d_4-d_5, $d_2\ldots$ occur in several of Eqs. (21)–(24) [22].

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle_{\rm ct}=-\frac{4i\alpha_{27}}{f_Kf_\pi^2}(m_K^2-m_\pi^2)_{(\text{1-loop})}+\frac{4i}{f_Kf_\pi^2}(m_K^2-m_\pi^2)\times[(-d_4^r+d_5^r-4d_7^r)m_K^2+(4d_2^r+4d_{20}^r-16d_{24}^r-4d_4^r-2d_7^r)m_\pi^2]. \tag{25}$$

The logarithmic and Gasser-Leutwyler counterterm contributions to the expressions in this subsection are given in Appendix C. Note, also, that for the cases of physical $K\to\pi\pi$ amplitudes, Eqs. (25), (34), and (35), the pseudoscalar decay constants and masses are the physical (renormalized to one-loop order) ones. For all other amplitudes given in this paper except $K\to\pi\pi$ at physical kinematics, the formulas are in terms of the bare constants. The distinction between bare and renormalized constants is made only in tree-level amplitudes, since making this distinction in the $O(p^4)$ expressions introduces corrections at higher order $[O(p^6)]$ than is considered in this paper.

B. $(8_L,1_R)+(27_L,1_R)$, $\Delta I=1/2$

The counterterms necessary to reconstruct the $O(p^4)$ $[(8_L,1_R)+(27_L,1_R)]$, $\Delta I=1/2$, $K\to\pi\pi$ amplitudes, relevant for operators such as $Q_2^{1/2}$, can be obtained using d_i's obtained from the $[(27,1);\Delta I=3/2]$ case given above along with information from $K^0\to|0\rangle;K^+\to\pi^+,\Delta I=1/2$; and $K\to\pi\pi,\Delta I=1/2$ at *both* unphysical kinematics. For $K^0\to|0\rangle$, we have

$$\langle 0|\mathcal{O}^{(8,1)}|K^0\rangle_{\rm ct}=\frac{4i\alpha_2}{f}(m_K^2-m_\pi^2)-\frac{8i}{f}[2(-e_1^r-e_2^r+e_5^r)m_K^4+(2e_1^r+e_2^r-2e_5^r)m_K^2m_\pi^2+e_2^rm_\pi^4], \tag{26}$$

$$\langle 0|\mathcal{O}^{(27,1)}|K^0\rangle_{\rm ct}=-\frac{48i}{f}d_1^r(m_K^2-m_\pi^2)^2. \tag{27}$$

Given the previously obtained value of d_1^r from the $\Delta I=\frac{3}{2}$ case, we can obtain e_2^r and $e_1^r-e_5^r$ from $K^0\to|0\rangle$. The other relevant expressions are

$$\langle\pi^+|\mathcal{O}^{(27,1),(1/2)}|K^+\rangle_{\rm ct}=-\frac{4}{f^2}\alpha_{27}p_K\cdot p_\pi-\frac{8}{f^2}[6d_1^rm_K^4-2(3d_1^r+d_2^r)m_K^2m_\pi^2+(-d_{20}^r+d_4^r-3d_6^r+2d_7^r)m_K^2p_K\cdot p_\pi+(-d_{20}^r+d_4^r+3d_6^r+d_7^r)\times m_\pi^2p_K\cdot p_\pi+8d_{24}^r(p_K\cdot p_\pi)^2], \tag{28}$$

$$\langle\pi^+|\mathcal{O}^{(8,1),(1/2)}|K^+\rangle_{\rm ct}$$
$$=\frac{4}{f^2}\alpha_1 p_K\cdot p_\pi-\frac{4}{f^2}\alpha_2 m_K^2-\frac{8}{f^2}[2(e_1^r+e_2^r-e_5^r)\times m_K^4+(e_2^r+2e_3^r+2e_5^r)m_K^2m_\pi^2+(2e_{35}^r-2e_{10}^r)m_K^2p_K\cdot p_\pi+(2e_{35}^r-e_{11}^r)m_\pi^2p_K\cdot p_\pi-8e_{39}^r(p_K\cdot p_\pi)^2], \tag{29}$$

as well as

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle_{\rm ct}=-8i\frac{\alpha_{27}}{f^3}m^2+16i\frac{m^4}{f^3}(d_{20}^r-2d_4^r+d_5^r-3d_7^r), \tag{30}$$

$$\langle\pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle_{\rm ct}=8i\frac{\alpha_1}{f^3}m^2+4i\frac{\alpha_2}{f^3}m^2+8i\frac{m^4}{f^3}(2e_1^r+4e_{10}^r+2e_{11}^r+4e_{15}^r+3e_2^r-4e_{35}^r-2e_5^r), \tag{31}$$

for $K\to\pi\pi, m_K=m_\pi=m$, and

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle_{\rm ct}=-4i\frac{\alpha_{27}}{f^3}(m_K^2-m_\pi^2)+\frac{3i}{2}\frac{m_K^4}{f^3}\times(6d_1^r+2d_2^r+2d_{20}^r-8d_{24}^r-4d_4^r+2d_5^r+12d_6^r-9d_7^r), \tag{32}$$

$$\langle\pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle_{\rm ct}=4i\frac{\alpha_1}{f^3}(m_K^2-m_\pi^2)+\frac{3i}{2}\frac{m_K^4}{f^3}\times(-2e_1^r+6e_{10}^r+e_{11}^r-4e_{13}^r+4e_{15}^r-4e_2^r-2e_3^r-4e_{35}^r+8e_{39}^r) \tag{33}$$

for $K\to\pi\pi, m_{K(\text{1-loop})}=2m_{\pi(\text{1-loop})}$.

From expressions (28) and (29), one can obtain the leading-order LEC's α_1 and α_2, as well as the linear combinations $e_{39}^r, e_1^r+e_3^r, e_{10}^r-e_{35}^r+\frac{3}{2}d_6^r$, and $2e_{10}^r-e_{11}^r+6d_6^r$. From Eqs. (30) and (31), for UK1 one can then obtain $e_{11}^r+2e_{15}^r-3d_6^r$. Making use of all the input thus obtained into Eqs. (32) and (33) for UK2 yields $e_{13}^r-\frac{3}{2}d_6^r$. These 14 linear combinations (namely $d_1^r, d_2^r, d_7^r, d_4^r-d_5^r, d_4^r-d_{20}^r, d_{24}^r, e_2^r, e_1^r-e_5^r, e_1^r+e_3^r, e_{39}^r, e_{10}^r-e_{35}^r+\frac{3}{2}d_6^r, 2e_{10}^r-e_{11}^r+6d_6^r, e_{11}^r+2e_{15}^r-3d_6^r, e_{13}^r-\frac{3}{2}d_6^r$) are sufficient to reconstruct the physical $K\to\pi\pi, \Delta I=\frac{1}{2}$ amplitudes for operators such as $Q_1^{1/2}, Q_2^{1/2}$, etc.,

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle_{\rm ct}=-4i\frac{\alpha_{27}}{f_Kf_\pi^2}(m_K^2-m_\pi^2)_{(\text{1-loop})}+4i\frac{1}{f_Kf_\pi^2}(m_K^2-m_\pi^2)\times[(-d_4^r+d_5^r+9d_6^r-4d_7^r)m_K^2+2(6d_1^r+2d_2^r+2d_{20}^r-8d_{24}^r-2d_4^r-6d_6^r-d_7^r)m_\pi^2], \tag{34}$$

$$\langle\pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle_{\rm ct}=4i\frac{\alpha_1}{f_Kf_\pi^2}(m_K^2-m_\pi^2)_{(\text{1-loop})}+8i\frac{1}{f_Kf_\pi^2}(m_K^2-m_\pi^2)\times[(e_{10}^r-2e_{13}^r+e_{15}^r)m_K^2+(-2e_1^r+2e_{10}^r+e_{11}^r+4e_{13}^r-4e_2^r-2e_3^r-4e_{35}^r+8e_{39}^r)m_\pi^2]. \tag{35}$$

The logarithmic and Gasser-Leutwyler counterterm contributions to the amplitudes presented in this subsection are given in Appendix D.

C. $(8_L,1_R)$, $\Delta I=1/2$

The case of pure (8,1) operators, i.e., $Q_{3,4,5,6}$, is simpler than the previous case of mixed $\Delta I=1/2$ operators; note also that phenomenologically, pure (8,1)'s are the most important. This is clearly a special case of the previous one for which $(27_L,1_R)$ contributions are irrelevant. For the physical $K\to\pi\pi$ reaction at $O(p^4)$, Eq. (35), eight new linear combinations are needed: $e_2^r, e_1^r-e_5^r, e_3^r+e_5^r, e_{35}^r-e_{10}^r, 2e_{35}^r-e_{11}^r, e_{39}^r, e_{11}^r+2e_{15}^r, e_{13}^r$.

The terms quadratic in quark mass for $K\to 0$, Eq. (26), yield e_2^r and $e_1^r-e_5^r$. A similar fit to $K^+\to\pi^+$, Eq. (29), then leads to $e_3^r+e_5^r, e_{35}^r-e_{10}^r, 2e_{35}^r-e_{11}^r$, and e_{39}^r. Using this for $K\to\pi\pi$ at UK1, Eq. (31), yields $e_{11}^r+2e_{15}^r$ and $K\to\pi\pi$ at UK2, Eq. (33), may be fitted to give e_{13}^r. While determining these coefficients is expected to be quite demanding, it is useful to note that several of them are obtained via more than one measurement. Note, in particular, that the term linear in quark mass, α_2, originating from operator mixing occurs in $K\to 0$, in $K\to\pi$, and also in $K\to\pi\pi$ at UK1 where the operator injects energy.

The logarithmic and Gasser-Leutwyler counterterm contributions to the amplitudes presented in this subsection are a subset of those given in Appendix D.

V. CONCLUSION

This paper presents all of the counterterm and finite logarithm contributions to $K^0\to|0\rangle$, $K^0\to\overline{K^0}$, and $K\to\pi$ with momentum insertion, and $K\to\pi\pi$ (at two values of unphysical kinematics) to $O(p^4)$ in ChPT for the $(27_L,1_R)$ and $(8_L,1_R)$ operators. It demonstrates that these quantities are sufficient to fully determine $K\to\pi\pi$ to $O(p^4)$ at the physical kinematics. It should be emphasized that this calculation was done in full ChPT, and that these arguments do not necessarily apply to the quenched theory. In fact, it is quite likely that some of the $K\to\pi\pi$ matrix elements suffer from large corrections due to the quenched approximation; this possibilty has recently been raised in Ref. [23] for the case of Q_6. Indeed, we have done a fit to the quenched RBC data [2] for $Q_7^{3/2}$ and $Q_8^{3/2}$ using the next-to-leading order ChPT prediction of Cirigliano and Golowich [9] and have found a poor fit (χ^2/d.o.f.≈ 2). Thus, the data tend to disfavor a large coefficient for the chiral log term that is predicted by full ChPT. A simple quadratic fit with the coefficient of the log term set to 0 yielded a much better fit (χ^2/d.o.f.≈ 0.1). These arguments suggest that an unquenched lattice calculation is probably necessary in order to correctly extract the $O(p^4)$ counterterms from the lattice. It is clearly important to see whether this extraction procedure, especially including $K\to\pi\pi$ at the two unphysical kinematics, can be extended to the case of $O(p^4)$ quenched ChPT.

In closing, we briefly want to remind the reader that two other interesting methods have been proposed recently [24,25] for lattice extraction of $K\to\pi\pi$ amplitudes. We believe it is important to use all the methods in order to obtain reliable information on this important process.

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APPENDIX A

Appendixes B–D contain the finite logarithm and Gasser-Leutwyler counterterm $O(p^4)$ contributions to the amplitudes presented in this paper. They were calculated using the FEYNCALC package [28] written for the MATHEMATICA [29] system. These expressions involve the regularized Veltman-Passarino basis integrals A and B [30]:

$$A(m^2)=\frac{1}{16\pi^2 f^2}m^2\ln\frac{m^2}{\mu^2}$$
$$=\lim_{d\to 4}\frac{1}{i}\left[\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2-m^2}+2im^2\overline{\lambda}\right], \tag{A1}$$

$$B(q^2,m_1^2,m_2^2)$$
$$=\lim_{d\to 4}\frac{1}{i}\left[\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{1}{[(l+q)^2-m_1^2](l^2-m_2^2)}+2i\overline{\lambda}\right]$$
$$=\int_0^1 dx\frac{1}{(4\pi)^2}\{1+\ln[-x(1-x)q^2+xm_1^2+(1-x)m_2^2]$$
$$-\ln\mu^2\}, \tag{A2}$$

where

$$\overline{\lambda}=\frac{1}{16\pi^2}\left[\frac{1}{d-4}-\frac{1}{2}(\ln 4\pi-\gamma_E+1)\right]. \tag{A3}$$

APPENDIX B

At one-loop order, the pseudoscalar decay constants and masses are renormalized such that $f_{\pi,K}=f[1+(\Delta f_{\pi,K}/f)]$ and $m^2_{\pi,K(\text{1-loop})}=m^2_{\pi,K}[1+(\Delta m^2_{\pi,K}/m^2_{\pi,K})]$. The corrections are given by

$$\frac{\Delta f_\pi}{f}=-2A(m_\pi^2)-A(m_K^2)+\frac{8}{f^2}(2m_K^2+m_\pi^2)L_4+\frac{8}{f^2}m_\pi^2L_5, \tag{B1}$$

$$\frac{\Delta f_K}{f}=-\frac{3}{4}A(m_\pi^2)-\frac{3}{2}A(m_K^2)-\frac{3}{4}A(m_\eta^2)$$
$$+\frac{8}{f^2}(2m_K^2+m_\pi^2)L_4+\frac{8}{f^2}m_K^2L_5. \tag{B2}$$

$$\frac{\Delta m_\pi^2}{m_\pi^2}=A(m_\pi^2)-\frac{1}{3}A(m_\eta^2)+\frac{16}{f^2}[(-L_4+2L_6)2m_K^2$$
$$+(-L_4-L_5+2L_6+2L_8)m_\pi^2], \tag{B3}$$

$$\frac{\Delta m_K^2}{m_K^2}=\frac{2}{3}A(m_\eta^2)+\frac{16}{f^2}[(-2L_4-L_5+4L_6+2L_8)m_K^2$$
$$+(-L_4+2L_6)m_\pi^2]. \tag{B4}$$

For degenerate quark masses at one-loop order, $m^2_{K(\text{1-loop})}=m^2_{\pi(\text{1-loop})}=m^2[1+(\Delta m^2/m^2)]$, $f_\pi=f_K=f[1+(\Delta f/f)]$,

$$\frac{\Delta m^2}{m^2}=\frac{2}{3}A(m^2)+\frac{16m^2}{f^2}(-3L_4-L_5+6L_6+2L_8), \tag{B5}$$

$$\frac{\Delta f}{f}=-3A(m^2)+\frac{8m^2}{f^2}(3L_4+L_5). \tag{B6}$$

APPENDIX C

The logarithmic corrections for the quantities relevant for the determination of the (27,1), $\Delta I=3/2\, K\to\pi\pi$ amplitudes in this paper are given by

$$\langle \overline{K^0}|\mathcal{O}^{(27,1)}_{\Delta S=2}|K^0\rangle_{\log}=\frac{8\alpha_{27}}{f^2}\Bigg[-\frac{2}{16\pi^2 f^2}m_K^4-8m_K^2A(m_K^2)+\left(\frac{1}{2}m_\pi^2-\frac{13}{2}m_K^2\right)A(m_\eta^2)+\left(-\frac{7}{2}m_K^2-\frac{1}{2}m_\pi^2\right)A(m_\pi^2)-2\frac{\Delta f_K}{f}m_K^2+\Delta m_K^2\Bigg], \tag{C1}$$

$$\langle \pi^+|\mathcal{O}^{(27,1),(3/2)}|K^+\rangle_{\log}=-\frac{4\alpha_{27}}{f^2}p_K\cdot p_\pi\Bigg[-2p_K\cdot p_\pi B(q^2,m_K^2,m_\pi^2)-\frac{3}{2}A(m_\eta^2)-7A(m_K^2)-\frac{15}{2}A(m_\pi^2)-\frac{\Delta f_K}{f}-\frac{\Delta f_\pi}{f}\Bigg], \tag{C2}$$

$$\langle \pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle_{\log}=-8i\frac{\alpha_{27}}{f^3}m^2\Bigg[\frac{-3m^2}{16\pi^2f^2}\left(5\ln\frac{m^2}{\mu^2}+1\right)-\frac{3\Delta f}{f}+\frac{\Delta m^2}{m^2}\Bigg] \tag{C3}$$

for $K\to\pi\pi$, $m_K=m_\pi=m$, and

$$\langle \pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle_{\log}=-3i\frac{\alpha_{27}}{f^3}m_K^2\Bigg[\frac{-m_K^2}{12}\frac{1}{16\pi^2f^2}\left(114\ln\frac{m_K^2}{\mu^2}+31\ln 5-148\ln 2-16\cot^{-1}2+46\right)-\frac{\Delta f_K}{f}-\frac{2\Delta f_\pi}{f}+\frac{4}{3m_K^2}(\Delta m_K^2-\Delta m_\pi^2)\Bigg] \tag{C4}$$

for $K\to\pi\pi$, $m_{K(\text{1-loop})}=2m_{\pi(\text{1-loop})}$. The logarithmic corrections to the physical $\Delta I=3/2\, K\to\pi\pi$ amplitude (included for completeness) are given by

$$\langle \pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle_{\log}=-4i\frac{\alpha_{27}}{f_Kf_\pi^2}\Bigg[-\frac{1}{12}m_K^4\left(1-\frac{m_K^2}{m_\pi^2}\right)B(m_\pi^2,m_K^2,m_\eta^2)+m_K^2\left(\frac{5}{4}\frac{m_K^4}{m_\pi^2}-\frac{13}{4}m_K^2+2m_\pi^2\right)\times B(m_\pi^2,m_K^2,m_\pi^2)+(m_K^4-3m_\pi^2m_K^2+2m_\pi^4)B(m_K^2,m_\pi^2,m_\pi^2)-\frac{1}{4}m_K^2\left(\frac{m_K^2}{m_\pi^2}+3\right)\times A(m_\eta^2)+\left(\frac{-m_K^4}{m_\pi^2}-4m_K^2+4m_\pi^2\right)A(m_K^2)+\left(\frac{5}{4}\frac{m_K^4}{m_\pi^2}-\frac{45}{4}m_K^2+12m_\pi^2\right)A(m_\pi^2)\Bigg], \tag{C5}$$

where the imaginary part of expression (C5) is given by

$$\mathrm{Im}(i\langle \pi^+\pi^-|\mathcal{O}^{(27,1),(3/2)}|K^0\rangle)=-\frac{2\alpha_{27}}{f_Kf_\pi^2}\frac{1}{16\pi f^2}\sqrt{1-4\frac{m_\pi^2}{m_K^2}}(m_K^2-m_\pi^2)(m_K^2-2m_\pi^2). \tag{C6}$$

APPENDIX D

The logarithmic corrections for the quantities relevant for the determination of the [(8,1)+(27,1)], $\Delta I=1/2\, K\to\pi\pi$ amplitudes [as well as the pure (8,1) amplitudes, neglecting the (27,1) expressions] are given by

$$\langle 0|\mathcal{O}^{(8,1)}|K^0\rangle_{\log}=\frac{4i\alpha_2}{f}(m_K^2-m_\pi^2)\left[-\frac{5}{6}A(m_\eta^2)-3A(m_K^2)-\frac{3}{2}A(m_\pi^2)-\frac{\Delta f_K}{f}\right]$$
$$-\frac{4i\alpha_1}{f}\left[\frac{1}{6}(m_\pi^2-4m_K^2)A(m_\eta^2)-m_K^2A(m_K^2)+\frac{3}{2}m_\pi^2A(m_\pi^2)\right], \tag{D1}$$

$$\langle 0|\mathcal{O}^{(27,1)}|K^0\rangle_{\log}=\frac{6i\alpha_{27}}{f}[(m_\pi^2-4m_K^2)A(m_\eta^2)+4m_K^2A(m_K^2)-m_\pi^2A(m_\pi^2)], \tag{D2}$$

$$\langle\pi^+|\mathcal{O}^{(27,1),(1/2)}|K^+\rangle_{\log}=-\frac{4\alpha_{27}}{f^2}\left[\frac{1}{8}\left(-\frac{(7m_K^2-m_\pi^2)(m_K^2-m_\pi^2)^2}{q^2}+7m_K^4-6m_K^2m_\pi^2-m_\pi^4-36(p_K\cdot p_\pi)^2\right.\right.$$
$$\left.-6(3m_K^2-5m_\pi^2)p_K\cdot p_\pi\right)B(q^2,m_K^2,m_\eta^2)+\frac{1}{8}\left(\frac{3(m_K^2+m_\pi^2)(m_K^2-m_\pi^2)^2}{q^2}-3(m_K^2-m_\pi^2)^2\right.$$
$$\left.+20(p_K\cdot p_\pi)^2-6(m_K^2+m_\pi^2)p_K\cdot p_\pi\right)B(q^2,m_K^2,m_\pi^2)+\frac{3}{8}\left(\frac{7m_K^4-8m_K^2m_\pi^2+m_\pi^4}{q^2}\right.$$
$$\left.+9m_K^2-m_\pi^2-10p_K\cdot p_\pi\right)A(m_\eta^2)-\left(\frac{3m_K^2(m_K^2-m_\pi^2)}{q^2}+3m_K^2+10p_K\cdot p_\pi\right)$$
$$\left.\times A(m_K^2)+\frac{3}{8}\left(\frac{m_K^4-m_\pi^4}{q^2}-m_K^2+m_\pi^2-6p_k\cdot p_\pi\right)A(m_\pi^2)-\left(\frac{\Delta f_K}{f}+\frac{\Delta f_\pi}{f}\right)p_K\cdot p_\pi\right], \tag{D3}$$

$$\langle\pi^+|\mathcal{O}^{(8,1),(1/2)}|K^+\rangle_{\log}=\frac{4\alpha_1}{f^2}\left[\frac{1}{72}\left(\frac{(7m_K^2-m_\pi^2)(m_K^2-m_\pi^2)^2}{q^2}-7m_K^4+6m_K^2m_\pi^2+m_\pi^4+36(p_K\cdot p_\pi)^2+6(3m_K^2-5m_\pi^2)p_K\cdot p_\pi\right)\right.$$
$$\times B(q^2,m_K^2,m_\eta^2)+\frac{1}{8}\left(\frac{3(m_K^2+m_\pi^2)(m_K^2-m_\pi^2)^2}{q^2}-3(m_K^2-m_\pi^2)^2+20(p_K\cdot p_\pi)^2\right.$$
$$\left.-6(m_K^2+m_\pi^2)p_K\cdot p_\pi\right)B(q^2,m_K^2,m_\pi^2)+\frac{1}{24}\left(\frac{-7m_K^4+8m_K^2m_\pi^2-m_\pi^4}{q^2}-9m_K^2+m_\pi^2-30p_K\cdot p_\pi\right)$$
$$\times A(m_\eta^2)-\frac{1}{12}\left(\frac{m_K^4+4m_K^2m_\pi^2-5m_\pi^4}{q^2}+11m_K^2+5m_\pi^2+30p_K\cdot p_\pi\right)$$
$$\left.\times A(m_K^2)+\frac{3}{8}\left(\frac{m_K^4-m_\pi^4}{q^2}-m_K^2+m_\pi^2-6p_K\cdot p_\pi\right)A(m_\pi^2)-\left(\frac{\Delta f_K}{f}+\frac{\Delta f_\pi}{f}\right)p_K\cdot p_\pi\right]$$
$$-\frac{4\alpha_2}{f^2}m_K^2\left[\frac{1}{12}\left(\frac{(m_K^2-m_\pi^2)^2}{q^2}-m_K^2-m_\pi^2+6p_K\cdot p_\pi\right)B(q^2,m_K^2,m_\eta^2)\right.$$
$$+\frac{1}{4}\left(\frac{3(m_K^2-m_\pi^2)^2}{q^2}-3(m_K^2+m_\pi^2)+10p_K\cdot p_\pi\right)B(q^2,m_K^2,m_\pi^2)-\frac{1}{12}\left(\frac{3(m_K^2-m_\pi^2)}{q^2}+7\right)$$
$$\left.\times A(m_\eta^2)-\frac{1}{2}\left(\frac{m_K^2-m_\pi^2}{q^2}+5\right)A(m_K^2)+\frac{3}{4}\left(\frac{m_K^2-m_\pi^2}{q^2}-3\right)A(m_\pi^2)-\frac{\Delta f_k}{f}-\frac{\Delta f_\pi}{f}\right], \tag{D4}$$

$$\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle_{\log}=-8i\frac{\alpha_{27}}{f^3}m^2\left[-3m^2\frac{1}{16\pi^2f^2}\left(5\ln\frac{m^2}{\mu^2}+1\right)-\frac{3\Delta f}{f}+\frac{\Delta m^2}{m^2}\right] \tag{D5}$$

for $K\to\pi\pi$, $m_K=m_\pi=m$,

$$\langle \pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle_{\log}=8i\frac{\alpha_1}{f^3}m^2\left[-\frac{1}{6}m^2\frac{1}{16\pi^2f^2}\left(50\ln\frac{m^2}{\mu^2}-37\right)-\frac{3\Delta f}{f}+\frac{\Delta m^2}{m^2}\right]+4i\frac{\alpha_2}{f^3}m^2\left[-2m^2\frac{1}{16\pi^2f^2}\times\left(\frac{101}{9}\ln\frac{m^2}{\mu^2}-\frac{47}{9}\right)-\frac{3\Delta f}{f}+\frac{\Delta m^2}{m^2}+\frac{32m^2}{f^2}(2L_1+2L_2+L_3+2L_4+2L_6+L_8)\right] \tag{D6}$$

for $K\rightarrow\pi\pi$, $m_K=m_\pi=m$,

$$\langle \pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle_{\log}=-3i\frac{\alpha_{27}}{f^3}m_K^2\left[\frac{-m_K^2}{24}\frac{1}{16\pi^2f^2}\left(456\ln\frac{m_K^2}{\mu^2}-20(5+\ln 16)-7\ln 5+232\cot^{-1}2\right)-\frac{\Delta f_K}{f}-\frac{2\Delta f_\pi}{f}+\frac{4}{3m_K^2}(\Delta m_K^2-\Delta m_\pi^2)\right], \tag{D7}$$

for $K\rightarrow\pi\pi$, $m_{K(1\text{-loop})}=2m_{\pi(1\text{-loop})}$, and

$$\langle \pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle_{\log}=3i\frac{\alpha_1}{f^3}m_K^2\left[\frac{m_K^2}{72}\frac{1}{16\pi^2f^2}\left(-518\ln\frac{m_K^2}{\mu^2}-209\ln 5+700\ln 2+184\cot^{-1}2+80\right)-\frac{\Delta f_K}{f}-\frac{2\Delta f_\pi}{f}+\frac{4}{3m_K^2}(\Delta m_k^2-\Delta m_\pi^2)\right]+12i\frac{\alpha_2}{f^5}m_K^4(4L_4-L_5+8L_6+4L_8) \tag{D8}$$

for $K\rightarrow\pi\pi$, $m_{K(1\text{-loop})}=2m_{\pi(1\text{-loop})}$. The logarithmic corrections to the physical $\Delta I=\frac{1}{2}$, $K\rightarrow\pi\pi$ amplitude are given by

$$\begin{aligned}\langle \pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle_{\log}=-4i\frac{\alpha_{27}}{f_Kf_\pi^2}\Bigg[&-\frac{2}{3}m_K^4\left(\frac{m_K^2}{m_\pi^2}-1\right)B(m_\pi^2,m_K^2,m_\eta^2)+m_K^2\left(\frac{m_K^4}{2m_\pi^2}-\frac{5}{2}m_K^2+2m_\pi^2\right)B(m_\pi^2,m_K^2,m_\pi^2)\\&+(-2m_K^4+3m_K^2m_\pi^2-m_\pi^4)B(m_K^2,m_\pi^2,m_\pi^2)+m_\pi^2(m_K^2-m_\pi^2)B(m_K^2,m_\eta^2,m_\eta^2)\\&+\left(\frac{2m_K^4}{m_\pi^2}-\frac{15}{2}m_K^2+\frac{9}{2}m_\pi^2\right)A(m_\eta^2)+\left(\frac{-5m_K^4}{2m_\pi^2}-\frac{11}{2}m_K^2+10m_\pi^2\right)\\&\times A(m_K^2)+\left(\frac{m_K^4}{2m_\pi^2}-3m_K^2+\frac{3}{2}m_\pi^2\right)A(m_\pi^2)\Bigg],\end{aligned} \tag{D9}$$

$$\begin{aligned}\langle \pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle_{\log}=4i\frac{\alpha_1}{f_Kf_\pi^2}\Bigg[&\frac{1}{6}m_K^4\left(\frac{m_K^2}{m_\pi^2}-1\right)B(m_\pi^2,m_K^2,m_\eta^2)+\frac{1}{2}m_K^2\left(\frac{m_K^4}{m_\pi^2}-5m_K^2+4m_\pi^2\right)B(m_\pi^2,m_K^2,m_\pi^2)\\&-(2m_K^4-3m_K^2m_\pi^2+m_\pi^4)B(m_K^2,m_\pi^2,m_\pi^2)-\frac{1}{9}m_\pi^2(m_K^2-m_\pi^2)B(m_K^2,m_\eta^2,m_\eta^2)\\&-\frac{1}{2}\left(\frac{m_K^4}{m_\pi^2}+m_\pi^2\right)A(m_\eta^2)+(5m_\pi^2-3m_K^2)A(m_K^2)+\frac{1}{2}\left(\frac{m_K^4}{m_\pi^2}-6m_K^2+3m_\pi^2\right)A(m_\pi^2)\Bigg]\\&+\frac{64i}{f^5}\alpha_2(m_K^2-m_\pi^2)[2m_K^2L_4+(-4L_4-L_5+8L_6+4L_8)m_\pi^2].\end{aligned} \tag{D10}$$

The imaginary parts of expressions (D9) and (D10) are given by

$$\mathrm{Im}(i\langle\pi^+\pi^-|\mathcal{O}^{(27,1),(1/2)}|K^0\rangle)=\frac{2\alpha_{27}}{f_K f_\pi^2}\frac{1}{16\pi f^2}\sqrt{1-4\frac{m_\pi^2}{m_K^2}}(m_K^2-m_\pi^2)(2m_K^2-m_\pi^2), \tag{D11}$$

$$\mathrm{Im}(i\langle\pi^+\pi^-|\mathcal{O}^{(8,1),(1/2)}|K^0\rangle)=-\frac{2\alpha_1}{f_K f_\pi^2}\frac{1}{16\pi f^2}\sqrt{1-4\frac{m_\pi^2}{m_K^2}}(m_K^2-m_\pi^2)(2m_K^2-m_\pi^2). \tag{D12}$$

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