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Isotropic representation of the noncommutative 2D harmonic oscillator

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We show that a 2D noncommutative harmonic oscillator has an isotropic representation in terms of commutative coordinates. The noncommutativity in the new mode induces energy level splitting and is equivalent to an external magnetic field effect. The equivalence of the spectra of the isotropic and anisotropic representation is traced back to the existence of the SU(2) invariance of the noncommutative model.

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Recent results obtained in the framework of nonperturbative string theory [1,2], have boosted interest in a deeper understanding of the role played by noncommutative geometry in different sectors of theoretical physics [3]. The inclusion of noncommutativity in quantum field theory can be achieved in two different ways: via a Moyal * product on the space of ordinary functions, or by defining the field theory on a coordinate operator space that is intrinsically noncommutative [4,5]. The equivalence between the two approaches has been nicely described in [6]. While formally well defined, the operator approach is hard to implement in explicit calculations. The analysis of the noncommutative effects is usually performed by expanding the Moyal * product perturbatively, and taking into account additional vertices. In order to get a deeper understanding of the way in which noncommutativity affects quantum field theory one tries to understand these effects first in exactly solvable models of noncommutative quantum mechanics [7].

The difficulty of performing explicit calculations encountered in the operator space formulation of quantum field theory corresponds, in quantum mechanics, to the problem of formulating a Schrödinger equation directly in terms of noncommutative coordinates. The path to follow is to introduce the noncommutativity of coordinates and momenta through the Moyal * product [9]. It turns out that the effect of introducing the * product can be described by suitable shifts of the argument of the wave function [10], or of the Hamiltonian [11]. In order to properly treat the noncommutative variables one needs two commuting Heisenberg algebras [8,12].

In this paper we shall follow an approach where the set of noncommutative coordinates x_i, p_i is expressed as a linear combination of the canonical variables of quantum mechanics α_i, β_i.

As an explicit example we shall study the case of a 2D noncommutative harmonic oscillator. The main result of our work is the description of a the noncommutative system in terms of *new* set of transformations among noncommutative and canonical variables, which we shall name the *isotropic representation*. In this mode, the noncommutative 2D harmonic oscillator has a simple and clear physical interpretation. This representation also exhibits rotational symmetry and leads, in a simple way, to the form of the generator of rotations for the noncommutative representation. Finally, we shall explain the equivalence of the spectra in the two different representations in terms of an SU(2) symmetry.

In order to illustrate the general procedure we start with the set of coordinates and momenta satisfying extended commutators as [13,14]

$$[x_k, x_j] = i\Theta_{kj}, \tag{1}$$

$$[p_k, p_j] = iB_{kj}, \tag{2}$$

$$[x^k, p_j] = i\delta^k{}_j, \tag{3}$$

with Θ_{kj} and B_{kj} antisymmetric matrices characterizing the generalized noncommutativity of the phase space geometry.

We are going to define linear transformations from the noncommutative set of coordinates (x_i, p_i) to a *commutative* set of canonically conjugate coordinates (α_i, β_i). The relation of noncommutative coordinates to conjugate ones is given by

$$x_i = a_{ij}\alpha_j + b_{ij}\beta_j, \tag{4}$$

$$p_i = c_{ij}\beta_j + d_{ij}\alpha_j, \tag{5}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are $N\times N$ transformation matrices. Before going into details of a particular model one needs to determine the conditions that the transformation matrices should satisfy. The resulting conditions from (1),(2),(3) written in matrix form are

$$\mathbf{ab}^T - \mathbf{ba}^T = \mathbf{\Theta}, \tag{6}$$

$$\mathbf{cd}^T - \mathbf{dc}^T = -\mathbf{B}, \tag{7}$$

$$\mathbf{ca}^T - \mathbf{bd}^T = \mathbf{I}. \tag{8}$$

Equations (6),(7),(8) determine the structure of the transformation matrices. Let us apply the above procedure in two dimensions. As a model we choose a noncommutative harmonic oscillator described by the Hamiltonian

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$$H \equiv \tfrac{1}{2}[(p_i)^2 + (x_i)^2]. \tag{9}$$

For simplicity, we have chosen the oscillator mass and frequency to be unity. The 2D representation for $\boldsymbol{\Theta}$ and $\mathbf{B}$ is $\Theta_{ij} \equiv \theta\epsilon_{ij}$, $B_{ij} \equiv B\epsilon_{ij}$. Let us assume diagonality of $\mathbf{a}$ and $\mathbf{c}$ as $a_{ij} \equiv a_{(i)}\delta_{ij}$, $c_{ij} \equiv c_{(i)}\delta_{ij}$. With these assumptions, Eq. (8) imposes the requirement that diagonal elements of the matrices $\mathbf{b}$ and $\mathbf{d}$ must be zero. We are left with eight unknown parameters and six equations. Additional equations can be obtained by requiring that the *mixed* term in the Hamiltonian be zero. This leads to

$$\mathbf{a}^T\mathbf{b} + \mathbf{d}^T\mathbf{c} = 0, \tag{10}$$

which gives two more equations than we need.

The complete set of solutions turns out to be

$$c_{11} = \frac{1}{\theta}(na_{11} + a_{22}\sqrt{nt\kappa}), \tag{11}$$

$$c_{22} = -\frac{1}{\theta}(ta_{22} + a_{11}\sqrt{nt\kappa}), \tag{12}$$

$$a_{11}^2 = \frac{\theta}{2n}\left[1 + \frac{1}{\sqrt{1-4ntA^2}}\right], \tag{13}$$

$$a_{22}^2 = \frac{\theta}{2t}\left[-1 + \frac{1}{\sqrt{1-4ntA^2}}\right], \tag{14}$$

$$A \equiv -\frac{\sqrt{nt\kappa}}{nt(1+\kappa+\theta^2)}, \quad \kappa \equiv 1 - B\theta, \tag{15}$$

$$tp = -1, \quad nq = -1. \tag{16}$$

Inserting Eqs. (11),(12),(13), and (14) in the Hamiltonian, one finds

$$H = \tfrac{1}{2}\Omega_1[(\alpha_1)^2 + (\beta_1)^2] + \tfrac{1}{2}\Omega_2[(\alpha_2)^2 + (\beta_2)^2], \tag{17}$$

$$\Omega_1 \equiv (2n)^{-1}[\theta + B + nt\sqrt{4+(\theta-B)^2}], \tag{18}$$

$$\Omega_2 \equiv -(2t)^{-1}[\theta + B - nt\sqrt{4+(\theta-B)^2}]. \tag{19}$$

The Hamiltonian (17) is the representation of a noncommutative 2D harmonic oscillator in terms of two 1D commutative, anisotropic, harmonic oscillators. We shall call this description the *anisotropic representation*. The introduction of the parameters n,t,p,q permits us to consider the complete range of values of the noncommutative parameter B, assuming $\theta > 0$. In fact, the square root in Eq. (11) requires $nt(1 - B\theta) > 0$, which leads to two different ranges: one where $n = t = 1$, $B < 1/\theta$, and the other where $n = -t = 1$, $B > 1/\theta$. Our result agrees with [8] where the two different regions are described as $\kappa > 0$ and $\kappa < 0$. At this point, we shall prove the existence of a different set of solutions for Eqs. (6),(7),(8) which give a particularly nice representation of the 2D noncommutative harmonic oscillator in terms of an *isotropic* oscillator. Let us choose the matrices $\mathbf{a}$ and $\mathbf{c}$ *diagonal*, but with a *single* eigenvalue each. In order to maintain unaltered the total number of free parameters, the matrices $\mathbf{b}$ and $\mathbf{d}$ will be chosen *antisymmetric*:

$$a_{ij} \equiv a\delta_{ij}, \quad c_{ij} \equiv c\delta_{ij}, \tag{20}$$

$$b_{ij} \equiv b\epsilon_{ij}, \quad d_{ij} \equiv d\epsilon_{ij}. \tag{21}$$

The set of Eqs. (6),(7),(8) enables us to solve for three out of the four parameters as

$$b = -\theta/2a, \tag{22}$$

$$c = (2a)^{-1}(1 \pm \sqrt{\kappa}), \quad \kappa \equiv 1 - \theta B, \tag{23}$$

$$d = (a/\theta)(1 \mp \sqrt{\kappa}). \tag{24}$$

The above solutions turn Eq. (8) into

$$(\theta + B) = 0. \tag{25}$$

Thus, Eq. (8) cannot be used to determine the remaining parameter a. At most it can impose a relation between parameters B and θ. Our intention is to work in full generality and, therefore, we shall assume $\theta + B \neq 0$ and drop the condition (10). Thus, the Hamiltonian (9) reads

$$H = h_1(\alpha_i)^2 + h_2(\beta_i)^2 - \frac{\theta + B}{2}\epsilon_{ij}\alpha_i\beta_j, \tag{26}$$

$$h_1 \equiv \frac{a^2}{2}\left[1 + \frac{1}{\theta^2}(1 \mp \sqrt{\kappa})^2\right], \tag{27}$$

$$h_2 \equiv \frac{\theta^2}{8a^2}\left[1 + \frac{1}{\theta^2}(1 \pm \sqrt{\kappa})^2\right]. \tag{28}$$

One can recognize Eq. (26) as the Hamiltonian for the commutative, isotropic, 2D harmonic oscillator with an *additional* term proportional to the two-dimensional angular momentum $L = \epsilon_{ij}\alpha_i\beta_j$. Thus, we shall name this representation of the noncommutative 2D harmonic oscillator the *isotropic representation*. The term linear in the angular momentum remains from the noncommutativity and thus it is important for understanding noncommutative effects. A similar term, in quantum mechanics, results from the coupling of the angular momentum with an external magnetic field.

To complete the solution it is appropriate to work in polar coordinates where the Schrödinger equation reads

$$\left[h_2\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right) - h_2 r^2 + \frac{i}{2}(\theta + B)\frac{\partial}{\partial\phi}\right]\psi(r,\phi) = E\psi(r,\phi). \tag{29}$$

Equation (29) admits solutions in terms of generalized Laguerre polynomials as

$$\psi_{n_r m}(z,\phi) = Nz^{|m|/2}L_{n_r}^{|m|}(z)\exp(-\tfrac{1}{2}z + im\phi),$$
$$z \equiv \sqrt{(h_1/h_2)}r^2, \tag{30}$$

$$L_n^s(z)=z^{-s}\exp(z)\frac{d_n}{dz^n}[z^{n+s}\exp(-z)], \tag{31}$$

where N is the proper normalization constant, n_r is the radial quantum number, and m is the magnetic quantum number. The spectrum of the system is

$$E_{n_r m}=2\sqrt{h_1h_2}(2n_r+|m|+1)+\tfrac{1}{2}m(\theta+B) \tag{32}$$

with the quantum numbers taking values $n_r=0,1,2,\ldots$, $m=0,\pm1,\pm2,\ldots$. The spectrum (32), in the special case $B=0$, was studied in [10].

Using the definitions (27),(28) one finds the frequency in the isotropic case to be

$$\omega\equiv 2\sqrt{h_1h_2}=\tfrac{1}{2}\sqrt{4+(\theta-B)^2}. \tag{33}$$

Equation (33) shows that ω is independent of the parameter a. This parameter simply induces a harmless (global) rescaling of the radial coordinate.

We point out that the spectrum (32) clearly displays the fact that the noncommutativity parameters play the same role as an external magnetic field $\mathcal{H}\equiv\theta+B$. The result explains the choice made in [9] as corresponding to the absence of the magnetic field and thus to energy level degeneracy.

As we have shown, solutions (22),(23),(24) and (11),(12),(13),(14) lead to two representations of the noncommutative harmonic oscillator. We would like to show that the spectra of the two modes are identical. Let us rewrite Eq. (32) in the following way:

$$E_{n_+n_-}=[\omega+(\theta+B)/2](n_++\tfrac{1}{2})+[\omega-(\theta+B)/2]\times(n_-+\tfrac{1}{2}), \tag{34}$$

$$n_r\equiv n_+(m-|m|)/2,\quad m\equiv m_+-n_-. \tag{35}$$

The energy spectrum (34) matches that of the Hamiltonian (26), provided one identifies the parameters as

$$\omega=\tfrac{1}{2}(\Omega_1+\Omega_2),\quad \theta+B=\Omega_1-\Omega_2. \tag{36}$$

The advantage of the representation in terms of the isotropic oscillator is that it offers a clear identification of the noncommutativity as a magnetic field effect. On the other hand, the equivalence to the anisotropic representation shows that the magnetic field can be simulated by the frequency difference of the anisotropic oscillators. A similar conclusion in a different context and in terms of chiral oscillators was found in [15,16].

The equivalence of the spectra displays that the two descriptions of the noncommutative harmonic oscillator are equivalent, in spite of the asymmetry with respect to rotations. The generator of rotations, i.e., angular momentum, is defined by the commutators

$$[L,\alpha_k]=i\epsilon_{kj}\alpha_j,\quad [L,\beta_k]=i\epsilon_{kj}\beta_j. \tag{37}$$

Therefore, our definition of the isotropic representation is motivated by the fact that the Hamiltonian (29) commutes with L. Let us express the angular momentum operator in terms of the noncommutative coordinates (x,p) with the help of Eqs. (22),(23),(24). One finds that the noncommutative form of L, call it J, is

$$J=\frac{1}{\kappa}\left(\epsilon_{ij}x_ip_j+\frac{\theta}{2}p_i^2+\frac{B}{2}x_i^2\right). \tag{38}$$

The additional terms take into account the noncommutativity in θ and B. Equation (38) has the form found in [8]. J is the representation of the angular momentum in the space of noncommutative coordinates. In fact, it satisfies

$$[J,x_k]=i\epsilon_{kl}x_l,\quad [J,p_k]=i\epsilon_{kl}p_l. \tag{39}$$

In both representations the angular momentum commutes with the Hamiltonian. Thus, both commutative (in terms of α and β) and noncommutative (in terms of x and p) representations are isotropic.

Let us return, now, to the anisotropic representation. We shall call the canonical coordinates of this representation Q_i,P_i for easier distinction from other representations. Rotations are still generated by the angular momentum operator given by $L=\epsilon_{ij}Q_iP_j$. On the other hand, one can verify that $[H,L]\neq0$ implies the absence of rotational symmetry. The isotropic and anisotropic representations have different symmetry properties with respect to rotations. Therefore, rotational symmetry cannot cause the equivalence of their spectra. We would like to identify the symmetry which leaves the spectra unchanged. For this purpose, let us express the angular momentum operator of the isotropic representation in terms of the coordinates Q_i and P_i. We write down the relations among the coordinates of the two representations:

$$\alpha_1=\frac{1}{\sqrt{2}}\left(\frac{h_2}{h_1}\right)^{1/4}(Q_1-P_2),\quad \alpha_2=-\frac{1}{\sqrt{2}}\left(\frac{h_2}{h_1}\right)^{1/4}(Q_2-P_1), \tag{40}$$

$$\beta_1=\frac{1}{\sqrt{2}}\left(\frac{h_1}{h_2}\right)^{1/4}(Q_2+P_1),\quad \beta_2=-\frac{1}{\sqrt{2}}\left(\frac{h_1}{h_2}\right)^{1/4}(Q_1+P_2). \tag{41}$$

Transforming the angular momentum operator of the isotropic representation into the coordinate system (Q_i,P_i), and renaming it $\hat{L}$, we get

$$\hat{L}=\tfrac{1}{2}(Q_2^2+P_2^2-Q_1^2-P_1^2). \tag{42}$$

Furthermore, one can define additional operator $\overline{L}$ as

$$\overline{L}=-(Q_1Q_2+P_1P_2). \tag{43}$$

One can verify that the operators $L,\hat{L},\overline{L}$ form an SU(2) algebra [12]. The equivalence of the spectra in the anisotropic and isotropic representations must, therefore, be the result of the invariance of the Hamiltonian with respect to the above SU(2) group. To show the SU(2) invariance of the

Hamiltonian let us calculate the sum of the squares of the three operators $L,\hat{L},\overline{L}$ in the anisotropic representation. One finds

$$L^2+\hat{L}^2+\overline{L}^2=\tfrac{1}{4}[Q_i^2+P_i^2]^2-1\equiv C^2-1. \quad (44)$$

The result shows the existence of the operator C which permits the following expression of the Hamiltonian:

$$H=\tfrac{1}{2}(\Omega_1+\Omega_2)C-\tfrac{1}{2}(\Omega_1-\Omega_2)\hat{L}. \quad (45)$$

The expression (45) shows that the set of commuting operators needed to describe the spectrum is $H,C,\hat{L}$. Equation (45) exhibits the representation invariant SU(2) symmetry of the Hamiltonian. The eigenvalues of these operators can be described in terms of two quantum numbers n_- and n_+, associated with the operators $Q_2^2+P_2^2$ and $Q_1^2+P_1^2$, respectively. Choosing a representation corresponds to writing the operators C and $\hat{L}$ in terms of the appropriate coordinates. In doing so, $\hat{L}$ is the angular momentum operator in the isotropic representation assuring rotational symmetry. One can verify that Eq. (45) reproduces Eqs. (26) and (9).

In this paper we have shown the existence of an isotropic representation of the noncommutative harmonic oscillator which goes hand in hand with the already known anisotropic representation. These two representations are different if seen from the point of view of rotational symmetry. The reason for this symmetry breaking can be traced back to the choice of the ansatz for the transformation matrices connecting commutative and noncommutative coordinates. Different eigenvalues of the transformation matrices break rotational symmetry explicitly. Nonetheless, the two representations describe the same physical system, as the equivalence of the spectra shows. The symmetry of the spectrum, for any choice of the ansatz, is the SU(2) symmetry described in the discussion above. The isotropic representation has the advantage of giving clear physical meaning to the effect of noncommutativity as being equivalent to an external magnetic field. There may be other representations of the noncommutative system corresponding to different solutions for the transformation matrices, but they should all be equivalent to one of the two forms of the Hamiltonian described in this paper.

Finally, we would like to correct the generally accepted, but not completely appropriate, use of the term magnetic field when referring *only* to the noncommutative parameter B. This is motivated by the fact that the noncommutative momentum turns into a covariant one in terms of canonical coordinates. However, the role of coordinates and momenta is equivalent in phase space, and thus it is clear that the parameter θ plays the same role as B. This is displayed in Eq. (26). Therefore, the parameter θ equally deserves the name of "magnetic field" [16].

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