Neutrinos vis-à-vis the six-dimensional standard model

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We examine the origin of neutrino masses and oscillations in the context of the six-dimensional standard model. The space-time symmetries of this model explain proton stability and forbid Majorana neutrino masses. The consistency of the six-dimensional theory requires three right-handed neutrinos, and therefore Dirac neutrino masses are allowed. We employ the idea that the smallness of these masses is due to the propagation of the right-handed neutrinos in a seventh, warped dimension. We argue that this class of theories is free of gravitational anomalies. Although an exponential hierarchy arises between the neutrino masses and the electroweak scale, we find that the mass hierarchy among the three neutrino masses is limited by higher-dimension operators. All current neutrino oscillation data, except for the LSND result, are naturally accommodated by our model. In the case of the solar neutrinos, the model leads to the large mixing angle, MSW solution. The mechanism employed, involving three right-handed neutrinos coupled to a scalar in an extra dimension, may explain the features of the neutrino spectrum in a more general class of theories that forbid Majorana masses.

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I. STANDARD MODEL IN SIX DIMENSIONS

The proposal [1] that all the standard model fields access extra spatial dimensions above some energy scale (''universal extra dimensions'') has received considerable attention during the past year. Precision electroweak measurements require only that the compactification scale of universal extra dimensions be above a few hundred GeV, opening up a potentially rich set of signatures, both in additional precision measurements [2,3] and in collider searches [1,3,4].

An especially attractive possibility is that there exist two universal extra dimensions. The six-dimensional standard model is chiral, and the constraints from Lorentz invariance and anomaly cancellation have remarkable consequences. The quarks ($\mathcal{Q},\mathcal{U},\mathcal{D}$) and leptons ($\mathcal{L},\mathcal{E}$) are four-component Weyl fermions of definite chirality, labeled by $+$ and $-$. The cancellation of irreducible gauge anomalies imposes one of the following two chirality assignments consistent with Lorentz invariant Yukawa couplings: $\mathcal{Q}_+, \mathcal{U}_-, \mathcal{D}_-, \mathcal{L}_\mp, \mathcal{E}_\pm$, where generational indices are implicit [5]. The reducible gauge anomalies can be canceled via the Green-Schwarz mechanism as discussed in [6,5,7,8]. Gravitational anomaly cancellation requires that each generation include a gauge singlet fermion $\mathcal{N}_\pm$ with six-dimensional chirality opposite to that of the lepton doublet [6]. In addition, the six-dimensional standard model is the only known theory that constrains the number of fermion generations to be $n_g = 3 \mod 3$, based on the global anomaly cancellation condition [5].

The two universal extra dimensions have to be compactified on an orbifold, so that each of the six-dimensional chiral fermions gives in the effective four-dimensional theory either a left- or a right-handed zero-mode fermion. The simplest orbifold compactifications are either the square T^2/Z_2 or T^2/Z_4 orbifolds.

An intriguing feature of the six-dimensional standard model is that the combination of its Lorentz and gauge symmetries can lead to a sufficient conservation of baryon number, even with the scale of baryon-number violating physics as low as the TeV range [9]. For the T^2/Z_4 orbifold, a Z_8 subgroup of the six-dimensional Lorentz symmetry is exactly preserved. In the case of the square T^2/Z_2 orbifold, the same is true provided the two orbifold fixed points that are exchanged by a 90° rotation in the compactified (transverse) dimensions are physically indistinguishable. The Z_8 symmetry requires that the baryon and lepton numbers, ΔB and ΔL, of any operator in the low-energy four-dimensional Lagrangian obey the selection rule[1]

$$3\Delta B + \Delta L = 0 \mod 8. \tag{1.1}$$

As a result, the proton is very long lived (all $\Delta B = 1$ transitions are governed by very high-dimension operators, and are therefore strongly suppressed), while neutron–anti-neutron oscillations ($\Delta B = 2, \Delta L = 0$) are forbidden. In the lepton sector, there are no neutrino Majorana masses,[2] and more generally neutrino-less double beta decays ($\Delta B = 0, \Delta L = 2$) are forbidden. The absence of Majorana masses follows from the properties of the gamma matrices in six dimensions, namely that the charge conjugation operator does not flip the chirality.

In this paper we study the implications for neutrino physics of the six-dimensional standard model. The mass matrix for the zero-mode neutrinos is induced dominantly by the following dimension-seven Yukawa terms in the six-dimensional Lagrangian:

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[1]The cancellation of anomalies via the Green-Schwarz mechanism requires a (four-dimensional) scalar field that transforms nontrivially under the Z_8. Operators that involve this field can be induced by four-dimensional instanton effects and could result in a violation of the selection rule Eq. (1.1). We expect these effects to be negligible. We thank E. Poppitz for discussions on this point.

[2]The fact that the Z_8 symmetry forbids Majorana masses was not taken into account in Ref. [7]. We note that even if the two universal extra dimensions were compactified on an arbitrary T^2/Z_2 orbifold, an exact Z_4 symmetry would still have prevented any Majorana mass.

$$-\overline{\mathcal{L}}^{i}_{-}\hat{\lambda}^{ii'}_{\mathcal{N}}\mathcal{N}^{i'}_{+}i\sigma_2\mathcal{H}^{*}+\text{H.c.}, \qquad (1.2)$$

where i,i' label the generations, $\mathcal{H}$ is the six-dimensional Higgs doublet, and where we have taken the six-dimensional chirality of $\mathcal{L}$ to be $-$. The ensuing Dirac masses of the three neutrino flavors can accommodate the neutrino oscillation data and all other experimental constraints, with the exception of the Liquid Scintillation Neutrino Detector (LSND) result [10]. It is nevertheless difficult to explain why the eigenvalues of the Yukawa matrix $\hat{\lambda}_{\mathcal{N}}$ are extremely small.

Since the standard model in extra dimensions is an effective theory, breaking down at some scale M_s in the TeV range, it is natural to expect gravity to be strongly coupled there as well. A structure that accommodates the observed weakness of the gravitational interaction should then be added to the universal extra dimensions. The simplest possibilities are that either some number of additional flat dimensions [11] or one additional warped dimension [12] are transverse to the universal ones and are not accessible to standard model fields. Each of these alternatives also provides a possible mechanism for explaining small but finite Dirac neutrino masses, as first proposed in [13] and [14], respectively, by letting the gauge singlet fermions propagate in these extra dimension(s).

We concentrate here on the possibility that the singlet fermions, along with gravity, propagate in a single additional (seventh) warped dimension, with the standard model fields confined to a 5-brane. We adapt and generalize the five-dimensional model of Ref. [14], and examine its consequences for neutrino masses and mixing angles. In Sec. II we present a simple effective theory, involving a single scalar field in the seven-dimensional bulk, that couples to the $\mathcal{N}$ fields. We then discuss, in Sec. III, the global gravitational anomalies in this context and argue that there is no additional constraint on the number of $\mathcal{N}$ fields (contrary to claims made in the literature [14,15]).

We examine the neutrino zero modes in Sec. IV, where we derive their mass matrix in terms of the wave functions of the $\mathcal{N}$ zero modes at the standard model 5-brane. In Sec. V we derive the shape of the scalar vacuum expectation value (VEV), and then find the profiles of the $\mathcal{N}$ zero modes. The effect of the scalar VEV is to concentrate the $\mathcal{N}$ zero modes near the brane opposite to the standard model one, so that the resultant four-dimensional neutrino mass matrix is exponentially suppressed relative to the weak scale. We then show that the mass hierarchy between the different flavors is limited by the presence of higher-dimension operators, which have a flavor mixing effect. This is an important result, especially in view of the often stated existence of a large exponential hierarchy induced between the masses of different flavors, whenever the two chiralities are localized at separate branes [14,16].

The implication, discussed in Sec. VI, is that the ratio between the mass scales associated with the atmospheric and solar neutrinos is not expected to be larger than one to two orders of magnitude. Therefore, the large mixing angle Mikheyev-Smirnov-Wolfenstein (MSW) solution to the solar neutrino problem is a consequence of this model. We discuss the neutrino mixing angles, as well as mass eigenvalues, and show that values compatible with all current neutrino oscillation data, except for the LSND result, emerge naturally. The energy scales associated with the seventh dimension, as well as the universal six dimensions, are such that the singlet-neutrino Kaluza-Klein (KK) modes are too heavy to play a direct role in the observed neutrino oscillations. Similarly, the constraints from astrophysics [17] or cosmology [18] on the mass of the KK neutrinos are not relevant here. In Sec. VII, we discuss the relations among various parameters in the model and draw some conclusions about the expected mass scales that characterize it. In Sec. VIII, we summarize the essential ingredients of our model leading to a viable neutrino mass spectrum, and emphasize that the mechanism applies to a more general class of higher-dimensional models.

II. A WARPED SEVENTH DIMENSION

The seven-dimensional gauge-singlet fermions are Dirac spinors with eight components, denoted by $\mathcal{N}^i(x^M)$, where $i=1,2,3$ labels the generations. The spacetime coordinates, x^M with $M=0,1,\ldots,6$, are labeled as follows: x^0,x^1,x^2,x^3 for the ordinary spacetime, x^4,x^5 for the two additional universal dimensions, and $x^6\equiv z$ for the dimension inaccessible to the standard model fields. We use the following conventions: capitals $M,N\ldots$ (from the middle of the alphabet) denote the seven coordinate indices in a curved background, while capitals $A,B,\ldots$ (from the beginning of the alphabet) denote the corresponding local Lorentz indices. We also use lowercase Greek letters $\alpha,\mu,\ldots$ to refer to the coordinate indices, and lower case Latin letters, $a,m,\ldots$ to refer to the Lorentz indices along the flat universal dimensions.

The $(T^2/Z_4)\times(S^1/Z_2)$ orbifold compactification projects out the unwanted zero modes, and restricts the coordinates to $0\leqslant x^4,x^5\leqslant\pi R_u$ and $0\leqslant z\leqslant\pi r_c$. The six-dimensional standard model fields are localized at $z=\pi r_c$, while the gauge-singlet fields propagate in the whole bulk.

The most general metric consistent with six-dimensional Poincaré invariance is diagonal, and warped in the z direction. However, the compactification of the two universal extra dimensions on the T^2/Z_4 orbifold breaks six-dimensional Poincaré invariance, and in general leads to a warp factor for x^4,x^5 different from the warp factor for the familiar uncompactified dimensions. For example, we expect contributions to the stress-energy tensor, due to the Casimir energy of bulk fields, that do not respect the six-dimensional Lorentz invariance. However, we will assume for simplicity that these differences can be neglected.[3] Our main conclusions do not change if we allow for the more general possibility that the warp factors for the uncompactified and compactified universal dimensions are different. Therefore, we consider a diag-

[3]A complete solution that incorporates the gravitational back reaction of Casimir energies or other effects arising from the compactification of the two universal dimensions would involve the specification of a radius stabilization mechanism. We leave such a study for future work.

onal metric G_{MN} that is warped in the z direction, corresponding to a line element

$$ds^2 = G_{MN}\, dx^M dx^N = w^2(z)\, \eta_{\mu\nu} dx^\mu dx^\nu - dz^2, \tag{2.1}$$

where $\mu,\nu = 0,1,\ldots,5$, and $\eta = \mathrm{diag}(+1,-1,\ldots,-1)$ is the six-dimensional Minkowski metric.

Starting in Sec. V we will take the warp factor to have the form [12]

$$w(z) = e^{k(\pi r_c - z)}, \tag{2.2}$$

which is a good approximation whenever the dominant contribution to the bulk stress-energy tensor is due to a bulk cosmological constant. This normalization is chosen so that $w(\pi r_c)=1$, which facilitates the physical interpretation at the standard model brane, located at $z=\pi r_c$. In particular, this choice implies that the coordinate radius R_u is the *proper* radius of the universal extra dimensions as measured by standard model probes. For the AdS metric defined by Eqs. (2.1) and (2.2), the Riemann curvature tensor is $R_{\lambda\rho\sigma\nu} = -k^2(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma})$, using the sign conventions of [19].

So far we have introduced three mass parameters: the inverse coordinate radius $1/R_u$ of the universal extra dimensions (associated with T^2/Z_4), the inverse radius $1/r_c$ of the dimension accessible only to neutral fields, and k. They are all taken to be below the fundamental seven-dimensional mass scale M_*, which, as will be discussed in Sec. VII, is in the TeV range. We will see in Sec. IV that, with the normalization $w(\pi r_c)=1$, the mass scale for the standard model KK modes is set by R_u^{-1}, and thus satisfies the bound $R_u^{-1} \gtrsim 0.5$ TeV, imposed by the electroweak data [1]. The radius r_c of the dimension accessible only to neutral fields is rather loosely constrained by searches for new long-range forces.

In addition to the $\mathcal{N}^i(x^M)$ fermions and the graviton, other fields that are singlets under the standard model gauge group could be present in the warped extra dimension. To describe naturally small neutrino masses it is sufficient to include a single real scalar φ with the dynamics described in the framework of effective field theory. This scalar is thus an effective degree of freedom, and could well represent a composite structure, with the compositeness becoming evident at scales of order M_* and above. The seven-dimensional (effective) action, invariant under both general coordinate and local Lorentz transformations, is then given by

$$\int d^7x \left\{ \sqrt{G}\left[\frac{i}{2}(\bar{\mathcal{N}}^i \Gamma^A e_A^M \hat{D}_M \mathcal{N}^i - \mathrm{H.c.}) + \frac{1}{2} G^{MN}\partial_M\varphi\partial_N\varphi - V_{\varphi,\mathcal{N}}\right] + \delta(z-\pi r_c)\sqrt{-g}\,\mathcal{L}_{\mathrm{SM}} \right\}, \tag{2.3}$$

where the first two terms are kinetic terms in the warped spacetime and the last two terms describe the bulk interactions of the $\mathcal{N}$ and φ fields, and the six-dimensional standard model. Here e_A^M is the inverse vielbein, G is the determinant of the seven-dimensional metric, with $\sqrt{G} = w^6(z)$, and g is the determinant of the six-dimensional induced metric, with $\sqrt{-g} = w^6(\pi r_c)$. The Γ^A are the anti-commuting matrices in seven-dimensional Minkowski space: the gamma matrices of six-dimensional Minkowski space along with $\Gamma^6 = i\Gamma_7$, where $\Gamma_7 = \Gamma^0 \ldots \Gamma^5$ defines six-dimensional chirality via $\mathcal{N}_\pm = \frac{1}{2}(1\pm\Gamma_7)\mathcal{N}$. The fermion covariant derivative in Eq. (2.3), associated with the diagonal metric G_{MN}, is

$$e_A^M \hat{D}_M \mathcal{N} = \begin{cases} w^{-1}(z)[\partial_\alpha + i\,\delta^a{}_\alpha\, \Gamma_a \Gamma_7 (dw/dz)/2]\mathcal{N}, & A=\alpha=0,1,\ldots,5 \\ \partial\mathcal{N}/\partial z, & A=6. \end{cases} \tag{2.4}$$

The bulk interactions preserve the orbifold Z_2 symmetry, defined such that $\mathcal{N}_-^i$ and φ are odd, while $\mathcal{N}_+^i$ are even. They may be organized into a tower of operators of increasing mass dimension:

$$V_{\varphi,\mathcal{N}} = -\Lambda - \frac{1}{2}M_\varphi^2\varphi^2 + \frac{\lambda_\varphi}{4M_*^3}\varphi^4 + \left(\frac{h_{ij}}{M_*^{3/2}}\varphi - \frac{\bar{h}_{ij}}{M_*^{7/2}}\hat{D}^M\partial_M\varphi \right)\bar{\mathcal{N}}^i\mathcal{N}^j + \cdots, \tag{2.5}$$

where Λ is a bulk cosmological constant that needs to be fine-tuned in order to keep the four dimensional sections flat. At the classical level this involves tuning Λ against possible brane tension terms as well as the vacuum energy stored in the φ VEV. The parameter λ_φ is real, and h, $\bar{h}$ are Hermitian matrices. All are dimensionless. The mass-square in the second term of $V_{\varphi,\mathcal{N}}$ is chosen to satisfy $M_\varphi^2 > 0$, so that φ has a nonzero VEV. Both M_φ and k are taken to be well below M_* to justify the use of effective field theory for exploring the vacuum properties of φ. By a flavor transformation it is possible to diagonalize the first term inside the parentheses, so that we can use a basis where

$$h_{ij} = h_i \delta_{ij}, \tag{2.6}$$

with h_i real and positive. The other terms involving $\mathcal{N}^i$ are in general flavor nondiagonal.

The six-dimensional standard model Lagrangian, $\mathcal{L}_{\mathrm{SM}}$, localized at $z=\pi r_c$, includes the kinetic terms for the lepton and Higgs doublets and the Yukawa interactions of $\mathcal{N}^i$:

$$\mathcal{L}_{\rm SM} \supset i\overline{\mathcal{L}^i_-}\Gamma^a e^\alpha_a D_\alpha \mathcal{L}^i_- + g^{\alpha\beta} D_\alpha \mathcal{H}^\dagger D_\beta \mathcal{H} - \left(\frac{\lambda^{ij}_{\mathcal{N}}}{M_*^{3/2}} \overline{\mathcal{L}^i_-}\mathcal{N}^j_+ i\sigma_2 \mathcal{H}^* + \text{H.c.} \right), \tag{2.7}$$

where the induced (inverse) metric and vielbein at the standard model 5-brane are given by

$$g^{\alpha\beta} = w^{-2}(\pi r_c)\,\eta^{\alpha\beta},$$
$$e^\alpha_a = w^{-1}(\pi r_c)\,\delta^\alpha_a, \quad \alpha = 0,1,\ldots,5. \tag{2.8}$$

In Eq. (2.7), D_α are the gauge covariant derivatives, and the Yukawa couplings are again dimensionless. Note that the four-component field $\mathcal{L}_-$ has mass dimension $+5/2$ while the $\mathcal{N}_+$ field (also four-component), being defined in seven dimensions, has mass dimension $+3$.

Before proceeding with the analysis of the neutrino masses, we discuss the consistency of the seven-dimensional theory.

III. GRAVITATIONAL ANOMALIES

We next show that the seven-dimensional model described in the previous section is anomaly free. The reader interested mostly in neutrino phenomenology may wish to move directly to Sec. IV.

The seventh dimension is compactified on a S^1/Z_2 orbifold and the six-dimensional standard model is localized on a 5-brane at one of the two fixed points, while the three singlet neutrino fields propagate in the bulk. It was shown in [5] that if all fields were six-dimensional, the resulting theory would be free of gauge and gravitational anomalies, both *local* and *global*. Letting the neutrino fields propagate in a seventh dimension amounts to adding three infinite towers of KK fields to this theory. Since all gauge fields are localized at the orbifold fixed points, allowing the singlets to propagate in more dimensions cannot introduce any gauge anomalies. Gravity, however, propagates in the bulk and one must consider whether all gravitational anomalies cancel. When coupling fermions to gravity, there can be two types of anomalies: those associated with general coordinate transformations and those associated with local Lorentz transformations. For each of these cases one must distinguish between *local* and *global* anomalies.[4] We analyze first the case of *local* gravitational anomalies. After showing that there are none we turn to the more subtle issue of *global* gravitational anomalies.

A. *Local* gravitational anomalies

A noninvariance of the effective fermion action under local Lorentz transformations would imply that the corresponding stress-energy tensor T^{MN} is not symmetric. This would be incompatible with general covariance and the conservation law $\nabla_M T^{MN} = 0$. Thus, in the presence of local Lorentz anomalies, either general covariance is broken or T^{MN} is not conserved. Anomalies associated with general coordinate transformations, on the other hand, lead directly to $\nabla_M T^{MN} \neq 0$. In either case, the theory that results when gravity becomes dynamical is inconsistent, and it is necessary to ensure that all gravitational anomalies cancel. However, the conditions derived from the requirement of anomaly cancellation for both kinds of transformations are not independent. At least in the case of *local* anomalies, it is possible to shift the anomalies of one kind into anomalies of the other kind by adding suitable local terms to the vacuum functional [20,21]. Thus, we may concentrate only on general coordinate transformations.

In the case of gauge theories in three or five dimensions, any local gauge noninvariance, which is necessarily localized at orbifold boundary points, can always be cancelled by a bulk Chern-Simons term [22,23], provided the anomalies in the lower dimensional effective theory vanish.[5] We now argue that this is also the case for *local* general coordinate anomalies in seven dimensions.[6] We follow the argument given in [23] for the spin-1 case. The idea is to calculate the one-loop contributions to the covariant divergence of the seven-dimensional stress-energy tensor in the six-dimensional effective theory. If we regularize in such a way as to produce the covariant form of the anomaly, it is possible to perform the calculation in any convenient gauge.

For the analysis of anomalies, it is sufficient to consider small fluctuations about a flat background

$$ds^2 = [\eta_{\alpha\beta} + h_{\alpha\beta}(x,z)]dx^\alpha dx^\beta - dz^2, \tag{3.1}$$

where $h_{\mu\nu} \ll 1$. In Eq. (3.1) we took advantage of the gauge freedom[7] to set $G_{\mu z} = 0$ and $G_{zz} = -1$. We also choose the vielbein as follows: $e_{a\alpha} = (\eta_{a\alpha} + \frac{1}{2}h_{a\alpha})$, $e_{az} = e_{z\alpha} = 0$ and $e_{zz} = 1$. The fact that we take a symmetric vielbein means

[4]Note that the word "local" can have two different meanings. In the context of "local Lorentz" transformations it means that the Lorentz group has been gauged, the standard usage in gauge theories. There can be, however, local Lorentz transformations that are continuously connected to the identity as well as local Lorentz transformations that are not. It is customary to refer to the transformations of the first kind as *local* and to those of the second kind as *global*. We use *italic fonts* whenever we want to emphasize the distinction between the transformations that are continuously connected to the identity and those that are not.

[5]The mathematical relation between Chern-Simons forms in odd dimensions and anomalies in even dimensions was discussed in [24,25].

[6]In five dimensions there are no *local* gravitational anomalies: the triangle diagrams always vanish.

[7]The invariance of the line element Eq. (3.1) under the reflection $z \to -z$ requires $G_{\alpha\beta}$ and G_{zz} to be even, while $G_{\mu z}$ should be odd. For consistency, the infinitesimal parameters of a general coordinate transformation ζ^α (ζ^z) should be even (odd). Although these boundary conditions do not allow the zero mode of G_{zz} to be gauged away, this "radion" mode has vectorlike couplings in our theory so that it does not contribute to the anomaly, and we do not include it here.

that the stress-energy tensor is symmetric. (In this gauge the absence of local Lorentz anomalies is explicit.)

The action for a fermion Ψ in the background Eq. (3.1) becomes

$$S=\frac{i}{2}\int d^7x\, e\overline{\Psi}\Gamma^A e_A^M \hat{D}_M\Psi+\text{H.c.}$$
$$=\int d^7x\Big[i\overline{\Psi}\Gamma^\alpha\partial_\alpha\Psi+\frac{i}{2}\overline{\Psi}\Gamma^6\overleftrightarrow{\partial}_z\Psi - h_{\alpha\beta}T^{\alpha\beta}+\mathcal{O}(h^2)\Big], \tag{3.2}$$

where

$$T^{MN}=\frac{i}{4}[\overline{\Psi}\Gamma^{(M}\overleftrightarrow{\partial}^{N)}\Psi-\eta^{MN}(\overline{\Psi}\Gamma^\mu\overleftrightarrow{\partial}_\mu\Psi+\overline{\Psi}\Gamma^6\overleftrightarrow{\partial}_z\Psi)], \tag{3.3}$$

and all components with an index along the seventh dimension vanish. Here we use the notation $\overline{\Psi}\overleftrightarrow{\partial}\Psi\equiv\overline{\Psi}\partial\Psi-(\partial\overline{\Psi})\Psi$, and in the second line of Eq. (3.2) as well as in Eq. (3.3) it is understood that all indices are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$.

We now expand the fermion fields in KK modes,

$$\Psi_\pm(x,z)=\sum_n \psi_\pm^{(n)}(x)\xi_n^\pm(z), \tag{3.4}$$

where $\psi_\pm(x,z)=\frac{1}{2}(1\pm\Gamma_7)\psi(x,z)$, with $\Gamma_7=-i\Gamma^6$. The KK wave functions $\xi_n^\pm(z)$, which can be taken real, are solutions to

$$\partial_z\xi_n^\pm=\pm m_n\xi_n^\mp, \tag{3.5}$$

where $\xi_n^+(z)$ and $\xi_n^-(z)$ satisfy Neumann and Dirichlet boundary conditions, respectively, and are normalized as

$$\int_0^{\pi r_c}dz\,\xi_n^\pm(z)\xi_{n'}^\pm(z)=\delta_{nn'}. \tag{3.6}$$

The result of replacing the mode expansion Eq. (3.4) in the action Eq. (3.2) is

$$S=\sum_n\int d^6x\Big[\overline{\psi}^{(n)}(i\Gamma^\alpha\partial_\alpha-m_n)\psi^{(n)} -\sum_{n'}h_{nn'}^{\alpha\beta\pm}T_{\alpha\beta\pm}^{(n,n')}+\mathcal{O}(h^2)\Big], \tag{3.7}$$

where

$$T_{\alpha\beta\pm}^{(n,n')}=\frac{1}{4}[i\overline{\psi}_\pm^{(n)}\Gamma_{(\alpha}\overleftrightarrow{\partial}_{\beta)}\psi_\pm^{(n')}-\eta_{\alpha\beta}(i\overline{\psi}_\pm^{(n)}\Gamma^\mu\overleftrightarrow{\partial}_\mu\psi_\pm^{(n')} -m_n\overline{\psi}_\mp^{(n)}\psi_\pm^{(n')}-m_{n'}\overline{\psi}_\pm^{(n)}\psi_\mp^{(n')})],$$

and

$$h_{nn'}^{\alpha\beta\pm}(x)=\int_0^{\pi r_c}dz\,\xi_n^\pm(z)\xi_{n'}^\pm(z)h^{\alpha\beta}(x,z). \tag{3.8}$$

The action Eq. (3.7) corresponds to the six-dimensional theory of an infinite number of fermion fields that couple (chirally) to background fields $h_{nn'}^{\alpha\beta\pm}$ with standard gravitational couplings. Note that when $h_{\alpha\beta}$ is z independent, the resulting couplings are vectorlike (except for those of the zero-mode fermion) due to the normalization condition Eq. (3.6), which is the same for both chiralities.

One can calculate now the relevant square diagram [26] with one insertion of the operator Eq. (3.3), with the seven-dimensional fermions replaced by their KK mode expansions. Performing then the same manipulations as in [23], and adding the contribution of a brane fermion (in our model these are the electrically neutral component of the $\mathcal{L}_-^i$, while the corresponding bulk fermions, labeled generically in this section by Ψ, are the $\mathcal{N}^i$) to compensate for the zero mode projected out by the orbifold boundary conditions, one can finally write

$$\int d^6x\,\zeta_M\nabla_N T^{MN}=\frac{1}{2}[\delta(z-\pi r_c)-\delta(z)]\frac{1}{4\pi^4}\times\int\left\{\frac{1}{288}\text{Tr}[v_\zeta R]\text{Tr}[R^2]+\frac{1}{360}\text{Tr}[v_\zeta R^3]\right\}. \tag{3.9}$$

Here we used a compact differential form notation: $R^\alpha{}_\beta=\frac{1}{2}R^\alpha{}_{\beta\mu\nu}dx^\mu\wedge dx^\nu$, where $R^\alpha{}_{\beta\mu\nu}(x,z)$ is the Riemann tensor calculated from the background Eq. (3.1), but with the indices running only from 0 to 5. The traces are taken over the indices that are not saturated by differentials. Also, $(v_\zeta)^\alpha{}_\beta=\partial_\beta\zeta^\alpha$, where the $\zeta^\alpha(x,z)$ can be thought of as the infinitesimal parameters of a general coordinate transformation. Due to general covariance, the result Eq. (3.9) holds in any gauge. In addition, we are allowed to replace the Riemann tensor in Eq. (3.9) by its exact, nonlinear expression, so that the final result holds in an arbitrary background.

The covariant anomaly given in Eq. (3.9) does not satisfy the Wess-Zumino consistency conditions and therefore cannot be obtained from the general coordinate variation of a functional of the metric. There is a standard procedure to obtain the consistent anomaly by adding local terms to the stress-energy tensor [20]. The resulting anomaly $Q_6^1(v_\zeta,\Gamma,R)[\delta(z-\pi r_c)-\delta(z)]/2$, where $\Gamma^\alpha{}_\beta=\Gamma^\alpha{}_{\beta\mu}dx^\mu$ is the connection 1-form, is related to the variation of a 7-form $Q_7(\Gamma,R)$ that can be added to the seven-dimensional action by

$$\delta_{v_\zeta}\int Q_7(\Gamma,R)=\int dQ_6^1(v_\zeta,\Gamma,R). \tag{3.10}$$

The Chern-Simons secondary characteristic class $Q_7(\Gamma,R)$ involves traces over all seven dimensional indices.[8] How-

[8] $Q_7(\Gamma,R)$ satisfies $dQ_7(\Gamma,R)=[1/(2\pi)^4]\{\frac{1}{288}(\text{Tr}[R^2])^2+\frac{1}{360}\text{Tr}[R^4]\}$, where R is the seven-dimensional Riemann curvature two-form. For a pedagogical exposition see [21].

ever, only the six-dimensional components contribute to the right-hand side of Eq. (3.10) due to the orbifold boundary conditions on the metric and on the infinitesimal parameters ζ^M.

The modified (consistent) form of Eq. (3.9) matches precisely with Eq. (3.10). We note that $Q_7(\Gamma,R)$ is odd under parity (defined as reflection through the $z=0$ hyperplane). Therefore, if we define the orbifold theory by starting from a compactification on the circle S^1, the coefficient of the Chern-Simons term must change sign when crossing $z=0$ (and $z=\pi r_c$), so that the theory is invariant under the reflection that is used in the orbifold projection. Because of this discontinuity, the gauge variation of such a term gives rise to delta-function singularities as in Eq. (3.9). Alternatively, we can think of the S^1/Z_2 orbifold as a compactification on an interval (the half circle) with certain boundary conditions imposed at the end points. In this picture the coefficient of the Chern-Simons term is constant and the compensating anomaly comes from the boundary contributions. In either picture, it is clearly possible to cancel the noninvariance of the original fermion effective action by including the seven-dimensional Chern-Simons form $Q_7(\Gamma,R)$. We therefore assume that this Chern-Simons term is present so that the vacuum functional is invariant under *local* coordinate transformations.

B. *Global* gravitational anomalies

There still remains the question of *global* general coordinate (local Lorentz) transformations [26–28]. If there are diffeomorphisms not continuously connected to the identity, the previous analysis is not enough to ensure that the theory is invariant under such transformations. We phrase the following analysis in terms of general coordinate transformations, but the same arguments apply for the case of local Lorentz transformations.

If $W(G)$ denotes the fermion determinant in the presence of a background metric G, we have in general

$$\frac{W(G^\rho)}{W(G)}=e^{i\delta_{[\rho]}}, \tag{3.11}$$

where G^ρ denotes the metric obtained from G under a representative ρ of one of the disconnected classes of diffeomorphisms, and $\delta_{[\rho]}$ is a phase that depends on the class to which ρ belongs.[9] We are specifically interested in the diffeomorphism classes of $\mathcal{M}=S^4\times(T^2/Z_4)\times(S^1/Z_2)$. The key observation is that S^1/Z_2 is diffeomorphic to an interval and any diffeomorphism of $\mathcal{M}$ onto $\mathcal{M}$ can be continuously deformed into one which is trivial in the interval S^1/Z_2:

$$x'^{\alpha}=f^{\alpha}(x), \quad z'=z, \tag{3.12}$$

where x^α denote the coordinates in $S^4\times(T^2/Z_4)$. We can therefore restrict attention to coordinate transformations of the type (3.12). It is then convenient to perform a KK decomposition for all fields, including the background metric, and analyze the resulting six-dimensional theory. Regarding the background, we note that under the limited class of diffeomorphisms (3.12) the affine connection transforms as

$$\Gamma'^{\lambda}_{\alpha\beta}=\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\beta}}\Gamma^{\rho}_{\tau\sigma}+\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\alpha}\partial x'^{\beta}}, \tag{3.13}$$

while all other components transform simply as tensors [the second term in Eq. (3.13) would vanish if any of the indices α,β,λ were along the seventh dimension parametrized by z]. Furthermore, if we perform a suitable KK mode expansion, only the zero mode of $\Gamma^{\lambda}_{\alpha\beta}$ is affected by the second term in Eq. (3.13); all other KK modes are true tensors under Eq. (3.12). (When referring to the gravitational background field, a zero mode is *defined* to be independent of the higher-dimensional coordinate.)

We next argue that if there are no *global* anomalies in a purely zero-mode gravitational background, then there are no *global* anomalies even in the presence of the higher gravity KK modes. The reason for this is that the group of disconnected diffeomorphisms is finite, at least for the case of the n-sphere S^n [30]. It follows that for any element ρ in the group there exists a (smallest) integer N such that ρ^N is the identity element, and therefore the phase in Eq. (3.11) associated with ρ must be an integer multiple of $2\pi/N$. If this phase vanishes when the higher gravity KK modes are turned off, and we turn them on smoothly, the phase must remain zero, unless it changes discontinuously. Since the higher gravity KK modes are just like background "matter" fields in the appropriate tensor representation of Eq. (3.12), we consider this very unlikely.

Now, in a zero-mode gravity background, the theory in question is just the six-dimensional standard model with the addition of three infinite towers of massive neutrino KK modes, which have *vectorlike* couplings to the background gravity field. We also note that the Chern-Simons term that is needed to cancel the *local* anomalies is invariant under Eq. (3.12) when the higher gravity KK modes are turned off. Therefore, the fermion effective action in a zero-mode gravity background is invariant under general coordinate transformations. From the argument given in the previous paragraph it follows that there are no *global* anomalies in an *arbitrary* gravitational background. It is worth pointing out that in the presence of the higher gravity KK modes, the Chern-Simons term $Q_7(\Gamma,R)$ is not invariant even under the restricted class of diffeomorphisms Eq. (3.12). This noninvariance must be canceled by the rest of the terms involving the higher gravity KK modes.

[9] We note that the group of disconnected diffeomorphisms of the n-sphere S^n is trivial for $n\leq 5$. This follows from the absence of "exotic" $(n+1)$-spheres for $n+1\leq 6$ [29]. There are therefore no constraints on the number of bulk neutrinos in the popular five-dimensional S^1/Z_2 orbifolds from *global* anomaly considerations. For higher dimensional theories the situation is not as straightforward, as indicated by the fact that there are 28 disconnected components on S^6 and two on S^7. See [30] for some other higher dimensional cases.

We conclude that adding a seventh dimension compactified on S^1/Z_2 to the six-dimensional standard model and letting the neutrinos propagate in the bulk introduces neither *local* nor *global* gravitational anomalies. Turning the argument around, we can say that the consistency constraints on the number of neutrinos in the seven-dimensional model are the same as in the six-dimensional standard model analyzed in [5], namely it is necessary to include one singlet neutrino per generation. From the point of view of anomaly cancellation it is immaterial whether these neutrinos are bulk or brane fields.

IV. NEUTRINO MASSES

We now return to the action of Eq. (2.3), which leads to Dirac neutrino masses. In order to study the implications for neutrino oscillations it is sufficient to analyze the zero-mode spectrum.

The zero modes of $\mathcal{N}_-^i$ are projected out by the orbifold boundary conditions. The KK decomposition along the warped dimension that includes the zero-mode $\mathcal{N}$ fields is given by

$$\mathcal{N}_+^i(x^\alpha,z)=\frac{1}{\sqrt{\pi r_c}}\sum_{n=0}^{\infty}\mathcal{N}_+^{i(n)}(x^\alpha)\xi_i^n(z), \tag{4.1}$$

where the index $\alpha=0,1,\ldots,5$ labels the universal dimensions. The $\xi_i^n(z)$ form a complete set of orthogonal (dimensionless) functions on the $[0,\pi r_c]$ interval, satisfying Neumann boundary conditions appropriate for even fields. They are chosen to obey the ortho-normality conditions

$$\frac{1}{\pi r_c}\int_0^{\pi r_c}dz\, w^5(z)\xi_i^{n*}(z)\xi_i^{n'}(z)=\delta_{nn'}, \tag{4.2}$$

which ensure the canonical normalization of the six-dimensional kinetic terms for $\mathcal{N}_+^{i(n)}$.

We now adopt a warp factor chosen to be unity at the standard model brane, as in Eq. (2.2). All kinetic terms for the standard model fields are then automatically canonically normalized. Keeping the zero modes with respect to the seventh dimension only, and integrating over z, the six-dimensional effective Lagrangian [see Eq. (2.3)] is

$$\mathcal{L}_{6D}=i\overline{\mathcal{L}}_-^i\Gamma^\alpha D_\alpha\mathcal{L}_-^i+D_\alpha\mathcal{H}^\dagger D^\alpha\mathcal{H}+i\overline{\mathcal{N}}_+^{i(0)}\Gamma^\alpha\partial_\alpha\mathcal{N}_+^{i(0)} -\left(\frac{\lambda_{\mathcal{N}}^{ij}\xi_j^0(\pi r_c)}{\sqrt{\pi r_c}M_*^{3/2}}\overline{\mathcal{L}}_-^i\mathcal{N}_+^{j(0)}i\sigma_2\mathcal{H}^*+\text{H.c.}\right)+\cdots, \tag{4.3}$$

where $\alpha=0,1,\ldots,5$. Note that we do not need to distinguish between coordinate and Lorentz indices anymore, and all the indices are raised and lowered with the flat metric $\eta_{\alpha\beta}$. Equation (4.3) shows that the mass scale for the standard model KK modes is set by the inverse *proper* radius $1/R_u$ [or $1/(w(\pi r_c)R_u)$ for an arbitrary normalization of the warp factor].

Integrating out the universal extra dimensions, the Dirac neutrino mass matrix induced after electroweak symmetry breaking is

$$M_\nu^{ij}=\frac{\lambda_{\mathcal{N}}^{ij}v_h}{\pi R_u M_*\sqrt{\pi r_c M_*}}\xi_j^0(\pi r_c), \tag{4.4}$$

where $v_h=174$ GeV is the Higgs VEV, and the denominator represents the square-root of the volume of the $(T^2/Z_4)\times(S^1/Z_2)$ orbifold. As we will see in the next section, the neutrino mass eigenvalues are largely determined by the hierarchy among the $\xi_j^0(\pi r_c)$, while the mixing angles are determined by the flavor structure of the couplings $\lambda_{\mathcal{N}}^{ij}$ and $\overline{h_{ij}}$.

V. ZERO-MODE PROFILES OF GAUGE-SINGLET FERMIONS

In this section we derive the profiles of the neutrino zero modes, which determine the neutrino mass matrix according to Eq. (4.4). These depend on the VEV of φ and therefore our first task is to determine the solution to the φ equation of motion that follows from the $\mathcal{N}$-independent part of Eqs. (2.3) and (2.5).

A. The bulk VEV

We will be interested in a region of parameter space where the φ field VEV varies slowly in the bulk of the 7th dimension (with the exception of two narrow regions close to the branes), so that to a good approximation it simply gives a contribution to the bulk cosmological constant. Thus, we use the explicit form for the warp factor, Eq. (2.2).

The negative mass-squared of φ implies that a nonzero VEV for φ is energetically favored, but at the same time φ is an odd field under the orbifold identification, and therefore its VEV must satisfy the boundary conditions

$$\langle\varphi(0)\rangle=\langle\varphi(\pi r_c)\rangle=0. \tag{5.1}$$

In terms of the rescaled VEV,

$$f(z)=\frac{\langle\varphi(z)\rangle}{M_*^{3/2}}, \tag{5.2}$$

which has mass dimension $+1$, the equation of motion is

$$\frac{d^2f}{dz^2}-6k\frac{df}{dz}=\lambda_\varphi f^3-M_\varphi^2 f, \tag{5.3}$$

subject to the boundary conditions $f(0)=f(\pi r_c)=0$. We assume $k>0$. In Eq. (5.3) we have neglected possible higher dimension operators. This is justified as long as the effective field theory description is valid, that is as long as k and M_φ are well below M_*.

Equation (5.3) describes the motion of a particle in the potential

$$V(f)=-\frac{\lambda_\varphi}{4}f^4+\frac{M_\varphi^2}{2}f^2, \tag{5.4}$$

in the presence of an *anti*-friction term proportional to k. Thus, we are looking for trajectories in which the particle starts at the bottom of the potential $f=0$ with some initial velocity, climbs the potential up to a certain point and then rolls down back to $f=0$. The antifriction term puts energy into the system, so it is conceivable that for a sufficiently large k, no matter how small the initial velocity, the particle will gain enough energy to overcome the potential barrier; in this case the only solution that satisfies the boundary conditions is the trivial one $f(z)=0$.

We first determine the restrictions in parameter space for nontrivial solutions to exist. In order to do this it is convenient to set $\tilde{f}(z)=\sqrt{\lambda_\varphi}M_\varphi^{-1}\,e^{-3kz}f(z)$, so that Eq. (5.3) becomes

$$\frac{d^2\tilde{f}}{dz^2}=M_\varphi^2 e^{6kz}\tilde{f}^{\,3}-(M_\varphi^2-9k^2)\tilde{f}, \tag{5.5}$$

describing now frictionless motion in a potential whose slope decreases with time. For $k\geqslant M_\varphi/3$, the "motion" starts at $z=0$ from the *maximum* $\tilde{f}=0$ of a continuosly decreasing potential, so that $\tilde{f}(z)\equiv 0$ is the only solution satisfying $\tilde{f}(\pi r_c)=0$. Therefore, Eq. (5.5) can have a nontrivial solution which satisfies $\tilde{f}(0)=\tilde{f}(\pi r_c)=0$ only if

$$k<\frac{M_\varphi}{3}. \tag{5.6}$$

This necessary condition is not sufficient for the existence of solutions with $\tilde{f}(0)=\tilde{f}(\pi r_c)=0$. Another necessary condition can be derived as follows. If we neglect the first term on the right-hand side of Eq. (5.5) the fictitious particle feels just a harmonic oscillator potential and returns to the origin after a "time" $z=\pi(M_\varphi^2-9k^2)^{-1/2}$. The effect of the neglected term is always to increase the oscillation period. Hence, the boundary condition $\tilde{f}(\pi r_c)=0$ requires

$$\frac{1}{r_c^2}<M_\varphi^2-9k^2. \tag{5.7}$$

In the Appendix we prove that the conditions (5.6) and (5.7) are also sufficient for the existence of nontrivial solutions. We conclude that the VEV $\langle\varphi(z)\rangle$ is nonzero in the interval $0<z<\pi r_c$ for a substantial range of values of M_φ and k.

For the special case of a flat and large z dimension, $k\ll 1/(\pi r_c)\ll M_\varphi$, the solution is given approximately by [31]

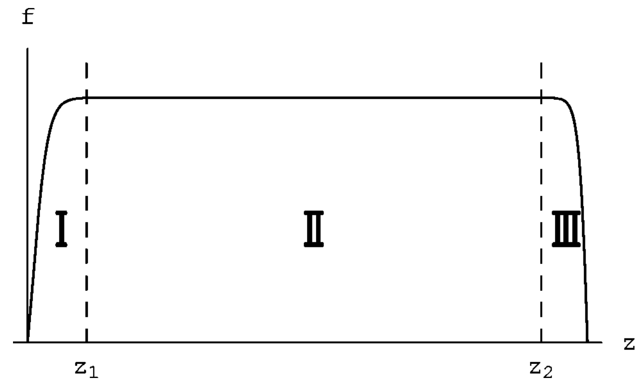

FIG. 1. Scalar profile when $M_\varphi/k=10$ and $\pi r_c M_\varphi=70$. The plateau is at $f\approx M_\varphi/\sqrt{\lambda_\varphi}$. The points z_1 and z_2 define the boundaries between regions I, II and III. In the main text we consider the case where $z_1/z_2\ll 1$.

$$f(z)\approx\frac{M_\varphi}{\sqrt{\lambda_\varphi}}\tanh\frac{M_\varphi z}{\sqrt{2}}\tanh\frac{M_\varphi(\pi r_c-z)}{\sqrt{2}}\left[1+\mathcal{O}\left(\frac{k}{M_\varphi}\right)+\mathcal{O}(e^{-\pi r_c M_\varphi})\right]. \tag{5.8}$$

This solution is essentially constant except in the region of size $\sim M_\varphi^{-1}$ around the endpoints. We will in general be interested in the parameter range in which k is not negligible, but where $1/(\pi r_c)\ll M_\varphi$. The latter hierarchy will need to be only one to two orders of magnitude to explain the smallness of the neutrino masses relative to the weak scale. As we show in the Appendix, the solutions in this case are qualitatively similar to the flat case $k=0$. We show a typical numerical solution in Fig. 1.

B. Bulk fermions

In the presence of the φ VEV, the $\mathcal{N}$ fields have nontrivial profiles along the z dimension. The three $\mathcal{N}$ zero modes defined in Eq. (4.1) are a solution to the set of equations ($i,j=1,2,3$, with i fixed and j summed over)

$$\frac{d\xi_i^0}{dz}=\left(3k\delta_{ij}-h_i\delta_{ij}f-\frac{\overline{h_{ij}}}{M_*^2}f''\right)\xi_j^0, \tag{5.9}$$

where we have again neglected possible higher dimension operators in Eq. (2.5). We also set $f''=d^2f/dz^2$ where f was defined in Eq. (5.2). It will be useful to factor out the leading order solution in powers of M_φ/M_*, namely the solution in the absence of the last term in Eq. (5.9), by defining new functions $c_i(z)$ through

$$\xi_i^0(z)=c_i(z)e^{3kz-h_iS(z)}, \tag{5.10}$$

where

$$S(z)\equiv\int_0^z d\zeta\, f(\zeta). \tag{5.11}$$

The c_i's satisfy the following differential equations

$$\frac{dc_i}{dz} = -\frac{\overline{h_{ij}}}{M_*^2} f'' e^{(h_i-h_j)S(z)} c_j, \tag{5.12}$$

which we now solve in the limit $\pi r_c M_\varphi \gg 1$.

Given the general features of the f profile discussed in the previous subsection, it is convenient to separate the analysis in the three regions shown in Fig. 1. We first note that in region II, f'' is exponentially small and therefore all the c_i's remain essentially constant throughout it. The differential equations (5.12) are nontrivial in regions I and III. In region I the integral Eq. (5.11) is of order $\lambda_\varphi^{-1/2}$. In region III this integral is much larger, so that the important features of the $c_i(z)$ are determined in this region as follows: for $z_2 < z < \pi r_c$ the integral expression for $S(z)$, Eq. (5.11), can be replaced, to a good approximation, by $S(\pi r_c)$, which is itself of order $\lambda_\varphi^{-1/2} \pi r_c M_\varphi$. Therefore, in this region there will be an exponential hierarchy among the various terms on the right-hand side of Eqs. (5.12), provided $h_i \lambda_\varphi^{-1/2} \pi r_c M_\varphi \gg 1$. Without loss of generality we can assume the ordering $h_1 > h_2 > h_3 > 0$. If we keep only the leading terms, then Eqs. (5.12) in region III reduce to

$$\frac{dc_3}{dz} \simeq -\frac{\overline{h_{33}}}{M_*^2} f'' c_3 \tag{5.13}$$

$$\frac{dc_i}{dz} \simeq -\frac{\overline{h_{i3}}}{M_*^2} f'' e^{(h_i-h_3)S(\pi r_c)} c_3 \quad \text{for} \;\; i=1,2. \tag{5.14}$$

We first solve for ξ_3^0. It will be sufficient to work to zeroth order in M_φ/M_*, so that from Eq. (5.13) we have $c_3(z) = \text{const}$. Imposing the normalization condition Eq. (4.2), and in the limit $1/(\pi r_c) \ll k < M_\varphi/3$, where we can evaluate $S(z)$ by setting $f(z) = M_\varphi/\sqrt{\lambda_\varphi}$ throughout the region of integration, we find from Eq. (5.10) that

$$\xi_3^0(\pi r_c) \simeq \sqrt{\pi r_c (2\tilde{M}_\varphi - k)}\, e^{-\pi r_c (\tilde{M}_\varphi - k/2)}, \tag{5.15}$$

where only one combination of the parameters in the Lagrangian for φ appears,

$$\tilde{M}_\varphi \equiv \frac{h_3 M_\varphi}{\sqrt{\lambda_\varphi}}. \tag{5.16}$$

We will see in the next section that the expression Eq. (5.15) will lead to exponential suppression of neutrino masses provided only that $\tilde{M}_\varphi > k/2$.

Naively, one might think that for the other two generations there will be a greater exponential suppression controlled by their larger Yukawa couplings $h_{1,2}$. We now show that this is not the case, due to the presence of the higher dimension operators that couple $\mathcal{N}^{1,2}$ to $\mathcal{N}^3$ in Eq. (2.5). To see this we need to solve for $c_{1,2}(z)$ in region III to first order in M_φ^2/M_*^2. By using the zeroth order solution for $c_3(z) \simeq c_3(\pi r_c)$ in Eq. (5.14) we obtain

$$c_i(z) \simeq c_i(z_2) - c_3(\pi r_c) \frac{\overline{h_{i3}}}{M_*^2} e^{(h_i-h_3)S(\pi r_c)} [f'(z) - f'(z_2)], \quad \text{for} \;\; i=1,2, \tag{5.17}$$

from which we find

$$\frac{\xi_i^0(\pi r_c)}{\xi_3^0(\pi r_c)} \simeq \frac{c_i(z_2)}{c_3(\pi r_c)} e^{-(h_i-h_3)S(\pi r_c)} - \frac{\overline{h_{i3}}}{M_*^2} [f'(\pi r_c) - f'(z_2)]. \tag{5.18}$$

But the first term here is exponentially small compared to the second term. Furthermore, $f'(z_2)$ is exponentially small compared to $f'(\pi r_c)$ and we finally obtain

$$\frac{\xi_i^0(\pi r_c)}{\xi_3^0(\pi r_c)} \simeq -\frac{\overline{h_{i3}}}{M_*^2} f'(\pi r_c). \tag{5.19}$$

Because $f(z)$ varies from $\mathcal{O}(\lambda_\varphi^{-1/2} M_\varphi)$ to zero over a distance of order $1/M_\varphi$, it follows that $-f'(\pi r_c) = \mathcal{O}(\lambda_\varphi^{-1/2} M_\varphi^2)$. This provides the promised result. The zero-mode wave functions $\xi_{1,2}^0(\pi r_c)$ are suppressed compared to $\xi_3^0(\pi r_c)$, but only by a quantity

$$\epsilon \equiv -\frac{f'(\pi r_c)}{M_*^2} \sim \mathcal{O}\left(\frac{M_\varphi^2}{\sqrt{\lambda_\varphi} M_*^2}\right). \tag{5.20}$$

We note that in Eq. (5.18) the leading term in $1/M_*$ is exponentially suppressed while the subleading term is not, so that it dominates as long as $\overline{h_{i3}}$ is not extremely small compared with unity. The reader might wonder whether this signals a breakdown in the effective theory description, which relies on the convergence of the expansion in $1/M_*$. There is no reason to worry: the term suppressed by M_* in Eq. (5.18) comes actually from the *leading* flavor off-diagonal operator. All other terms suppressed by higher powers of M_* give just small corrections to ϵ.

VI. NEUTRINO OSCILLATIONS

We are now equipped with all the tools necessary for analyzing the neutrino mass spectrum and the ensuing neutrino oscillations.

A. Neutrino mass matrix

As we discussed in the introduction (Sec. I), the neutrino masses are of the Dirac type, the Majorana masses being forbidden by the symmetry under rotations of the two universal extra dimensions. Below the electroweak scale, the effective four-dimensional theory contains three left-handed neutrinos, ν_L^i, which are the neutral zero modes of $\mathcal{L}_-^i$, and three right-handed neutrinos, N_R^i, which are the zero modes of $\mathcal{N}_+^{i(0)}$ with respect to the two universal dimensions. In the weak eigenstate basis,

$$\overline{\nu_L} M_\nu N_R + \text{H.c.}, \tag{6.1}$$

the neutrino mass matrix derived in Eqs. (4.4), (5.15) and (5.19) is given by

$$M_\nu = m_0 \begin{pmatrix} \epsilon\, \overline{h}_{13}\lambda_{\mathcal{N}}^{11} & \epsilon\, \overline{h}_{23}\lambda_{\mathcal{N}}^{12} & \lambda_{\mathcal{N}}^{13} \\ \epsilon\, \overline{h}_{13}\lambda_{\mathcal{N}}^{21} & \epsilon\, \overline{h}_{23}\lambda_{\mathcal{N}}^{22} & \lambda_{\mathcal{N}}^{23} \\ \epsilon\, \overline{h}_{13}\lambda_{\mathcal{N}}^{31} & \epsilon\, \overline{h}_{23}\lambda_{\mathcal{N}}^{32} & \lambda_{\mathcal{N}}^{33} \end{pmatrix}, \tag{6.2}$$

where $\epsilon \ll 1$ is defined in Eq. (5.20), and the scale of the neutrino masses is set by

$$m_0 \equiv v_h e^{-\pi r_c(\tilde{M}_\varphi - k/2)} \left(\frac{1}{\pi R_u M_*}\right)\left(\frac{2\tilde{M}_\varphi - k}{M_*}\right)^{1/2}. \tag{6.3}$$

As we will see in Sec. VII, the factors in parentheses are expected to provide a suppression of no more than one to two orders of magnitude, so that the neutrino mass scale is explained by the first exponential.

The neutrino mass matrix is diagonalized by unitary transformations:

$$U_L^\dagger M_\nu U_R = \mathrm{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}). \tag{6.4}$$

The unitary matrix describing neutrino oscillations, U_L, and the physical neutrino masses, m_{ν_i}, may be found by expanding in powers of $\epsilon = \mathcal{O}(\lambda_\varphi^{-1/2} M_\varphi^2/M_*^2)$. The largest physical neutrino squared-mass is

$$m_{\nu_3}^2 = m_0^2(|\lambda_{\mathcal{N}}^{13}|^2 + |\lambda_{\mathcal{N}}^{23}|^2 + |\lambda_{\mathcal{N}}^{33}|^2)[1 + \mathcal{O}(\epsilon^2)]. \tag{6.5}$$

It is convenient to use the following identity, valid up to corrections of order ϵ^2:

$$M_\nu = \begin{pmatrix} \tilde{l}_3 & -(l_{23}\tilde{l}_1)^* & l_{13} \\ 0 & \sqrt{|l_{13}|^2 + |l_{33}|^2} & l_{23} \\ -\tilde{l}_1 & -(l_{23}\tilde{l}_3)^* & l_{33} \end{pmatrix} \begin{pmatrix} \epsilon\, \tilde{m}_{11} & \epsilon\, \tilde{m}_{12} & 0 \\ \epsilon\, \tilde{m}_{21} & \epsilon\, \tilde{m}_{22} & 0 \\ 0 & 0 & m_{\nu_3} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -\epsilon\theta_1 \\ 0 & 1 & -\epsilon\theta_2 \\ \epsilon\theta_1 & \epsilon\theta_2 & 1 \end{pmatrix}, \tag{6.6}$$

where we use the following notation:

$$l_{ij} = \frac{\lambda_{\mathcal{N}}^{ij}}{(|\lambda_{\mathcal{N}}^{13}|^2 + |\lambda_{\mathcal{N}}^{23}|^2 + |\lambda_{\mathcal{N}}^{33}|^2)^{1/2}}, \quad i,j = 1,2,3,$$

$$\tilde{l}_i = \frac{\lambda_{\mathcal{N}}^{i3*}}{(|\lambda_{\mathcal{N}}^{13}|^2 + |\lambda_{\mathcal{N}}^{33}|^2)^{1/2}}, \quad i = 1,3,$$

$$\theta_i = \overline{h}_{i3} \sum_{j=1}^{3} l_{ji}^* l_{j3}^*, \quad i = 1,2. \tag{6.7}$$

The first matrix that appears in Eq. (6.6) can be shown to be unitary, while the last matrix is unitary to leading order in ϵ^2. The elements of the block-diagonal mass matrix shown in Eq. (6.6) are given by

$$\tilde{m}_{1i} = m_{\nu_3}\overline{h}_{i3}(l_{1i}\tilde{l}_3^* - l_{3i}\tilde{l}_1^*),$$

$$\tilde{m}_{2i} = m_{\nu_3}\overline{h}_{i3}[\tilde{l}_1(l_{13}l_{2i} - l_{23}l_{1i}) + \tilde{l}_3(l_{33}l_{2i} - l_{23}l_{3i})], \quad i = 1,2. \tag{6.8}$$

Equation (6.6) shows that the m_{ν_1} and m_{ν_2} physical neutrino masses are of order ϵm_{ν_3}, and generically are nondegenerate. We choose $m_{\nu_1} < m_{\nu_2}$. These can be computed straightforwardly by diagonalizing $\tilde{m}$ (the 2×2 matrix whose elements are given by $\tilde{m}_{ij}$):

$$\epsilon^2 \tilde{m}\tilde{m}^\dagger = V \mathrm{diag}(m_{\nu_1}^2, m_{\nu_2}^2) V^\dagger, \tag{6.9}$$

where V is a unitary 2×2 matrix. Its V_{ij} elements depend only on ratios of $\lambda_{\mathcal{N}}^{ij}$'s and on $|\overline{h}_{13}/\overline{h}_{23}|$. This dependence can be computed straightforwardly using Eq. (6.8), but is cumbersome and we do not display it here.

The unitary 3×3 matrix U_L is then given by the product of the first matrix on the right-hand side of Eq. (6.6) with

$$\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.10}$$

The third column entries of U_L,

$$U_L^{i3} = \frac{\lambda_{\mathcal{N}}^{i3}[1 + \mathcal{O}(\epsilon^2)]}{(|\lambda_{\mathcal{N}}^{13}|^2 + |\lambda_{\mathcal{N}}^{23}|^2 + |\lambda_{\mathcal{N}}^{33}|^2)^{1/2}}, \quad i = 1,2, \tag{6.11}$$

are relevant for atmospheric neutrino oscillations, as discussed below. The first two columns of U_L have entries that also depend (to leading order in ϵ) only on ratios of $\lambda_{\mathcal{N}}^{ij}$'s and on $|\overline{h}_{13}/\overline{h}_{23}|$. The ratio of the U_L^{11} and U_L^{12} entries, which is relevant for solar neutrino oscillations, is given by

$$\left|\frac{U_L^{12}}{U_L^{11}}\right| = \left|\frac{V_{21}\tilde{l}_3^* - V_{11}\tilde{l}_1 l_{23}}{V_{11}^*\tilde{l}_3^* - V_{21}^*\tilde{l}_1 l_{23}}\right|. \tag{6.12}$$

This ratio is typically of order unity if most of the $\lambda_{\mathcal{N}}^{ij}$'s have the same order of magnitude. In particular, this is true for the phenomenologically favored case discussed below, which is near the bimaximal mixing of three neutrinos [32].

B. Predictions and experimental constraints

The neutrino oscillation data constrains the differences of neutrino squared-masses. With three neutrinos, the atmospheric and solar neutrino data can be fit nicely (the LSND data cannot be accommodated).

The atmospheric neutrino data require a larger mass splitting compared to the solar data so that $m_{\nu_3}^2$ is determined to leading order in ϵ^2. The range for the mass-square difference $(\Delta m^2)_{\rm atm}$ given by the global fit [33] to the data obtained in atmospheric neutrino experiments and in the CHOOZ reactor experiment is

$$m_{\nu_3}^2 \approx (\Delta m^2)_{\rm atm}^{\rm exp} = (1.5-6.0)\times 10^{-3}\ \ {\rm eV}^2 \quad {\rm at}\ \ 99\%\ \ {\rm C.L.} \tag{6.13}$$

As long as the combination of Yukawa couplings shown in Eq. (6.5) is not smaller than unity by many orders of magnitude, Eq. (6.3) requires

$$\pi r_c \left(\tilde{M}_\varphi - \frac{k}{2} \right) \approx 30, \tag{6.14}$$

where we anticipate that the terms in parentheses in Eq. (6.3) are of order unity.

The solar neutrino oscillations are controlled by the mass-square difference of the lighter neutrinos:

$$(\Delta m^2)_{\rm solar} = m_{\nu_2}^2 - m_{\nu_1}^2. \tag{6.15}$$

Based on the reasonable assumptions that $|\overline{h}_{23}|,|\overline{h}_{23}| = \mathcal{O}(1)$, and that most Yukawa couplings $\lambda_{\mathcal{N}}^{ij}$ have the same order of magnitude, we find the following prediction for the solar neutrino oscillation scale:

$$\frac{(\Delta m^2)_{\rm solar}}{(\Delta m^2)_{\rm atm}^{\rm exp}} \approx \mathcal{O}(\epsilon^2), \tag{6.16}$$

where ϵ is given by Eq. (5.20). In the absence of fine-tuning, the φ scalar mass-squared can be no smaller than the one-loop contribution in the seven-dimensional theory: $\delta M_\varphi^2 \sim \mathcal{O}[\lambda_\varphi M_*^2/(128\pi^3)]$. Also, we expect that the effective theory description starts breaking down when M_φ approaches M_*. This leads to an allowed range for ϵ [see Eq. (5.20)]:

$$\mathcal{O}(10^{-3}\sqrt{\lambda_\varphi}) \lesssim \epsilon \lesssim \mathcal{O}(10^{-1}/\sqrt{\lambda_\varphi}), \tag{6.17}$$

with the values near the upper end being preferred if the mass parameter M_φ is not too much smaller than the fundamental scale M_*. If λ_φ is of order unity, the ratio shown in Eq. (6.16) is then of order 10^{-2}. The generic prediction of our model for the solar neutrino scale then becomes

$$(\Delta m^2)_{\rm solar} \approx 10^{-5} - 10^{-4}\ \ {\rm eV}^2. \tag{6.18}$$

This prediction fits well the range currently allowed for the large-mixing angle MSW solution to the solar neutrino problem [34]. The large uncertainties due to the unknown values of various parameters do not allow us to rule out the long oscillation wavelength solution, whose fit to the data prefers $(\Delta m^2)_{\rm solar} \approx 10^{-7}$ eV^2. On the other hand, the vacuum oscillation solution requires $(\Delta m^2)_{\rm solar} \approx 10^{-10}$ eV^2, so that it is disfavored within our model.

The elements of the unitary matrix U_L are constrained by the solar, atmospheric and reactor neutrino experiments. The element U_L^{13} is most tightly constrained by the global fit to the CHOOZ and solar neutrino data, which gives $(U_L^{13})^2 < 6.5\times 10^{-2}$ at 99% C.L. [35]. Applied to our case, this translates into a mild restriction:

$$\frac{|\lambda_{\mathcal{N}}^{13}|}{\sqrt{|\lambda_{\mathcal{N}}^{23}|^2 + |\lambda_{\mathcal{N}}^{33}|^2}} < 0.26. \tag{6.19}$$

The atmospheric neutrino data strongly favor pure $\nu_\mu \leftrightarrow \nu_\tau$ oscillations with a mixing angle satisfying $\sin^2 2\theta_{23} > 0.83$ at the 99% confidence level [36]. The expression of this mixing angle in terms of the Yukawa couplings is the same in our model as in the five-dimensional model of Grossman and Neubert [14], and gives

$$0.65 < \left| \frac{\lambda_{\mathcal{N}}^{23}}{\lambda_{\mathcal{N}}^{33}} \right| < 1.55. \tag{6.20}$$

This is not a particularly strong constraint, given that the Yukawa couplings are typically expected to be of the same order of magnitude. However, if the march of the atmospheric neutrino data towards *maximal* $\nu_\mu \leftrightarrow \nu_\tau$ mixing continues, then it will be necessary to find a more detailed explanation for why the ratio of Yukawa couplings shown above is so close to unity.

Finally, the global fits of the large mixing angle MSW solution to the solar neutrino problem require [34]

$$0.2 < \left| \frac{U_L^{12}}{U_L^{11}} \right|^2 \lesssim 1. \tag{6.21}$$

This constraint is naturally accommodated by order unity values of the parameters entering the first two columns of the matrix shown in Eq. (6.12).

In summary, we can explain the hierarchy between the electroweak scale and the scale relevant for the atmospheric neutrino data by the first exponential factor in Eq. (6.3). We can further explain the small hierarchy between this scale and the solar neutrino scale associated with the large-mixing angle MSW solution based on the typical size of the higher-dimension operators controlling neutrino flavor mixing. Furthermore, the currently allowed ranges for the mixing angles are natural if the various couplings of our model are of order unity.

VII. MASS PARAMETERS AND WARPING

Having observed that the smallness of the neutrino mass scale relative to the electroweak scale is explained by Eq. (6.3) with the exponent given by Eq. (6.14), we now comment on the other mass scales of the model. These are the scale M_* at which the seven-dimensional theory, including its six-dimensional component, breaks down, the parameter k related to the seven-dimensional cosmological constant, the size R_u of the universal extra dimensions, and the size r_c of the warped dimension. We also introduced the mass M_φ of the bulk scalar that localizes the neutrino fields away from the standard model brane, and the Higgs doublet mass parameter M_H. These scales give rise to other, derived scales such as the electroweak scale v_h.

There are several important relations among the above parameters. We assume a normalization of the warp factor as in Eq. (2.2). (For a different normalization it is only necessary to interpret R_u in the following formulas as the proper radius of the universal extra dimensions at the standard model brane.) The first relation follows from the fact that the observed standard model gauge couplings, collectively denoted by g_4 (as well as the top Yukawa coupling) are of order one. Writing a typical six-dimensional standard model gauge coupling in $\mathcal{L}_{\rm SM}$ of Eq. (2.3) as g_6/M_*, where g_6 is dimensionless, the observed four-dimensional gauge coupling is given by

$$1\sim g_4=(\pi R_u M_*)^{-1}g_6. \tag{7.1}$$

If the standard model gauge interactions become strong at the scale M_*, then $g_6^2\sim 128\pi^3$ [37]. In this case, the product $R_u M_*$ is of order $\sqrt{128\pi}$. For a range of values of $g_6>1$, M_* will be somewhat above $1/R_u$, providing a finite range of validity for the six-dimensional standard model. In the following we assume that $\mathcal{O}(1)<R_u M_*\lesssim\mathcal{O}(10)$.[10]

A second relation follows from naturalness considerations with respect to the Higgs doublet mass. The Higgs doublet mass parameter M_H must be below M_* for the effective theory description to be valid. However, on naturality grounds, M_H cannot be much smaller than the one loop corrections. They can be roughly estimated by cutting off the quartically divergent one-loop integrals, such as the one arising from the quartic Higgs self-interaction, at the breakdown scale of the effective, six-dimensional theory. This yields

$$M_*>M_H\gtrsim\delta M_H\approx\sqrt{\frac{\lambda_6}{128\pi^3}}M_*, \tag{7.2}$$

where λ_6 is a dimensionless coupling in the six-dimensional theory. Using this estimate, it follows that the Higgs VEV is given by

$$v_h=[\pi R_u M_*\lambda_6^{-1/2}]M_H\gtrsim\frac{1}{\sqrt{128\pi}}(R_u M_*)^2 R_u^{-1}. \tag{7.3}$$

If $R_u M_*\gtrsim\mathcal{O}(1)$ then Eq. (7.3) can be read as an upper bound on $1/R_u$ in the TeV range. But the electroweak precision measurements impose a lower bound $1/R_u\gtrsim 0.5$ TeV [1]. It then follows from $R_u M_*\lesssim\mathcal{O}(10)$ that M_* is in the TeV range.

Finally we note that the parameters M_φ and k are expected to satisfy relations analogous to Eq. (7.2) (though, as noted in Sec. V A, we must have $k<M_\varphi/3$). Our discussion of the solar and atmospheric neutrino data in Sec. VI A indicated that M_φ must be approximately an order of magnitude below M_*, which is consistent with a naturality estimate analogous to Eq. (7.2), if the corresponding coupling λ_φ is of order unity or smaller.

There are now various possibilities depending on how large the warping is. First suppose that the warping is no more than mild: $e^{\pi r_c k}\sim 1$. In this case, the weakness of gravity must be attributed to some suppression that lies beyond the seven-dimensional theory presented here, such as a few other flat dimensions accessible only to gravity along the lines of [11]. It is important to note that the mechanism for suppressing the neutrino masses presented here is independent of any such extension. From Eq. (6.14), we now have $\pi r_c\tilde{M}_\varphi\approx 30$. With $\tilde{M}_\varphi$ roughly an order of magnitude below M_*, the inverse size of the seventh dimension, $1/\pi r_c$, is of order 10 GeV.

A more interesting possibility is that $\pi k r_c\gg 1$. In this case it is possible to explain the weakness of gravity in the manner of Randall and Sundrum [12] within the seven-dimensional model. There are, however, some differences arising from the existence of the universal extra dimensions. The four-dimensional Planck mass is now related to M_* by

$$M_{\rm Pl}^2\approx\frac{M_*^5(\pi R_u)^2}{4k}e^{4\pi r_c k}. \tag{7.4}$$

Suppose that k is on the order of (but somewhat less than) M_*. Then if $1/R_u$ and M_* are in the TeV range as expected from the previous considerations,[11] it follows from Eq. (7.4) that $e^{2\pi r_c k}\sim 10^{15}$, which translates into $\pi k r_c\sim 20$. (Note that there is an extra factor of two in the exponent compared to the five-dimensional Randall-Sundrum model.) We then see from Eq. (6.14) that $\pi\tilde{M}_\varphi r_c\sim 40$, and the inverse size of the seventh dimension, $1/\pi r_c$, is again of order 10 GeV. The lightest spin-2 KK mode with momentum along the warped dimension has a mass of about $4k$, which is roughly of the same order as the mass of the first KK modes of the standard model fields. It will be interesting to see which KK modes will be discovered first if this model is realized in nature.

VIII. CONCLUSIONS

We have presented a higher-dimensional mechanism for generating a realistic neutrino mass spectrum. The smallness of the neutrino masses compared with the electroweak scale is explained by an exponential suppression of the right-handed neutrino wave functions on the standard-model brane. The hierarchy between the mass scales associated

[10]An interpretation based on the AdS/CFT correspondence [38] may be useful if $R_u M_*<1$, in which case the present six-dimensional description is not applicable. We note that the Z_8 symmetry that lies at the heart of the remarkable proton decay suppression pointed out in [9] and which also forbids Majorana masses, is expected to remain valid. Thus, even though it would not be possible to talk about a six-dimensional standard model, the six-dimensional structure still has observable effects in the low-energy four dimensional theory.

[11]Note that we have chosen to measure all mass scales at the standard model brane with respect to the corresponding induced metric. Had we measured them with respect to a metric rescaled by $e^{\pi k r_c}$ as in Ref. [12], we would have concluded that M_* and the other "fundamental" parameters of the seven dimensional theory are of order M_P. The cutoff on the effective six dimensional standard model would, however, remain in the TeV range, being given by $M_* e^{-\pi k r_c}$. None of the physical conclusions described here would change.

with the $\nu_\mu \leftrightarrow \nu_\tau$ and $\nu_e \leftrightarrow \nu_\mu$ transitions, measured by the atmospheric and solar neutrino data, respectively, is limited by the effect of flavor-nondiagonal, higher-dimension operators. As a result, the mass scale of the solar neutrino oscillations fits well the large-mixing-angle MSW solution. Furthermore, the neutrino mixing angles are naturally large if no large hierarchies between the neutrino Yukawa couplings occur. This is an important result in view of the fact that a majority of the models in the literature (for a recent review see [39]) can accommodate only the small-mixing-angle MSW or vacuum solutions to the solar neutrino problem, which are less favored by the data. In addition, the seemingly ''maximal'' mixing required by the atmosperic neutrino data is consistent with our mechanism for a reasonably large range of parameters.

While our mechanism has been developed in the framework of the six-dimensional standard model, it is worth pointing out that it relies fundamentally on four ingredients that could naturally be present in a more general class of higher-dimensional theories:

(i) Three right-handed neutrinos;

(ii) A symmetry structure forbidding Majorana neutrino masses;

(iii) A spatial dimension compactified on S^1/Z_2 and accessible to the right-handed neutrinos but not to the standard model fields;

(iv) A bulk (effective) scalar field which is odd under the Z_2 orbifold transformation, has a VEV, and couples to the right-handed neutrinos.

Ingredients (i) and (ii) are automatically present in the six-dimensional standard model, as required by the six-dimensional gravitational anomaly cancellation [5,6] and the Z_8 rotational symmetry of the two universal extra dimensions [9]. We were then led to consider the six-dimensional standard model localized in a seventh dimension satisfying (iii) and (iv).

More generally, ingredient (ii) could be enforced by lepton number conservation in a variety of models, and its experimental test is the absence of neutrinoless double-beta decay. Ingredients (iii) and (iv) could be present in $4+1$-dimensional models, as suggested in Ref. [31] as a possible source of fermion mass hierarchies. We have generalized this construction by allowing a warped metric. We have analyzed the conjectured restriction on the number of gauge-singlet fermions [14], and have found that in $6+1$ dimensions the local and global gravitational anomalies cancel independently of the number of right-handed neutrinos, provided the fermion content is free of six-dimensional anomalies. (In $4+1$ dimensions no restrictions arise because there are no *local* or *global* gravitational anomalies in four or five dimensions.)

Finally, it is worth noting that the model presented here has another intriguing feature. In the most appealing version of the model, the extra dimension that leads to an exponential suppression of the right-handed neutrino wave functions also solves the hierarchy problem along the lines of Randall and Sundrum [12], while explaining proton stability based on six-dimensional Lorentz invariance as in Ref. [9]. In this case collider searches at the TeV scale will reveal graviton Kaluza-Klein modes in addition to the Kaluza-Klein modes of the standard model fields.

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APPENDIX

In this appendix we prove that the real scalar φ has a nonzero VEV whenever the parameters k, M_φ and r_c satisfy the conditions (5.6) and (5.7). The rescaled VEV $u(z) = \sqrt{\lambda_\varphi} M_\varphi^{-1} f(z)$ must satisfy the z-independent equation

$$\frac{d^2u}{dz^2} = 6k\frac{du}{dz} + M_\varphi^2(u^3 - u). \tag{A1}$$

This equation describes the mechanical motion in the potential $-M_\varphi^2(u^2-1)^2/4$, and in the presence of an *anti*-friction term proportional to k. The z coordinate plays the role of ''time''variable.

We will prove the existence of a solution to Eq. (A1), satisfying the boundary conditions $u(0) = u(\pi r_c) = 0$, by analyzing the ''flows'' in the equivalent first order system:

$$\frac{du}{dz} = P_u,$$
$$\frac{dP_u}{dz} = 6kP_u + M_\varphi^2(u^3 - u). \tag{A2}$$

In the (u, P_u) plane, the solution sought corresponds to a flow that starts somewhere on the $u=0$ line and, after time πr_c, comes back to this line (because of the symmetry $u \to -u$, we can restrict $u \geqslant 0$.) More precisely, we shall prove that for any $k < M_\varphi/3$, there are flows starting and ending on the $u=0$ line, with total elapsed times (πr_c) ranging from π/ω to infinity, where

$$\omega \equiv \sqrt{M_\varphi^2 - 9k^2}. \tag{A3}$$

The system (A2) has two fixed points, $(u, P_u) = (0,0)$ and $(1,0)$, near which it may be linearized. In the vicinity of $(0,0)$, Eqs. (A2) have the following explicit solution satisfying the initial conditions $(u, P_u) = (0, P_0 > 0)$:

$$u(z) = \frac{P_0}{\omega} e^{3kz} \sin \omega z,$$
$$P_u(z) = u(z)(3k + \omega \tan^{-1} \omega z). \tag{A4}$$

Since u vanishes at $z = \pi/\omega$, the existence of solutions is established for a separation between the branes of πr_c

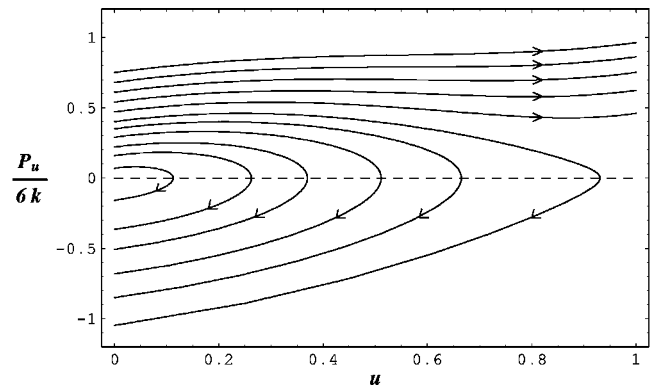

FIG. 2. The flow diagram when $M_\varphi/3k=4$.

$\rightarrow\pi/\omega$ [see Eq. (5.7)], at least when $P_0\rightarrow 0$ such that the linear approximation is reliable.

A similar linear analysis around (1,0) shows that there is an attractive flow that approaches the fixed point (1,0) from the region $u<1$, $P_u>0$, and a repulsive flow in the region $u<1$, $P_u<0$. For either one of them, the total time is infinite. This shows that adjacent flows that come arbitrarily close to (1,0) will spend an arbitrarily long time in its vicinity. Note that whenever the flow enters the region $u<1$, $P_u<0$, it will be driven to the $u=0$ line, since the only possibility is to roll down the slope of the potential.

We now prove the possibility of reaching these flows from a point $(0,P_u>0)$, based on the following remarks. First, Eq. (A2) shows that along the line $P_u=0$, and for $u<1$, the trajectories flow vertically downward (see Fig. 2). Second, the trajectories passing through the $u=1$ line flow into the region $u>1$ whenever $P_u>0$, and then they are driven to $u\rightarrow\infty$ due to the negative slope of the potential. Tracing these trajectories back in time, it follows from the first remark that they necessarily cross the $u=0$, $P_u\geqslant 0$ half-line. However, they cannot cross the origin because the flows cannot stop except at fixed points, and the point (0,0) cannot be reached [for small enough $P_u>0$ the trajectories stay only in the vicinity of (0,0); see Eqs. (A4)]. Note that the flows cannot cross due to the uniqueness of the solutions to ordinary differential equations such as Eq. (A2). Third, tracing back in time the trajectory attracted to the fixed point (1,0), the same argument as above shows that it crosses the line $u=0$, $P_u>0$. This "critical" trajectory, which starts at some critical $(0,P_u>0)$ and is attracted to (1,0), sets a boundary between qualitatively different flows. Finally, from any point $(u,0)$, $u<1$, the trajectory traced back in time crosses the line $u=0$ at some $0<P_u<P_u$. Thus, a point $(u,0)$ with u arbitrarily close to 1 corresponds to a trajectory that spends an arbitrarily long time in the vicinity of (1,0).

We have shown so far that there are solutions for πr_c close to either π/ω or infinity. By continuity, it follows that there are solutions to Eq. (A1), satisfying the boundary conditions $u(0)=u(\pi r_c)=0$, for any $r_c>1/\omega$. This completes the proof that Eqs. (5.6) and (5.7) are necessary and sufficient conditions for the existence of a nonzero φ VEV.

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