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Spontaneous symmetry breaking and the $p\to 0$ limit

M. Consoli*
Istituto Nazionale di Fisica Nucleare, Sezione di Catania, Corso Italia 57, 95129 Catania, Italy
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I point out a basic ambiguity in the $p\to 0$ limit of the connected propagator in a spontaneously broken phase. This may represent an indication that the conventional singlet Higgs boson, rather than being a purely massive field, might have a gapless branch. This would dominate the energy spectrum for $\mathbf{p}\to 0$ and give rise to a very weak, long-range force. The natural interpretation is in terms of density fluctuations of the "Higgs condensate:" in the region of very long wavelengths, infinitely larger than the Fermi scale, it cannot be treated as a purely classical c-number field.

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I. INTRODUCTION

The ground state of spontaneously broken theories is frequently denoted as the "Higgs condensate." In this view, the name itself (as for the closely related gluon, chiral, . . . ,condensates) indicates that a nonvanishing expectation value of the Higgs field may correspond to a real medium made up by the physical Bose condensation process of elementary spinless quanta whose "empty" vacuum state is not the true ground state of the theory [1]. Noticing that bodies can flow without any apparent friction in such a medium, it is natural to represent the Higgs condensate as a superfluid. In this perspective, such a physical vacuum should support long-wavelength density fluctuations. In fact, the existence of density fluctuations in any known medium is a very general experimental fact, depending on the coherent response of the elementary constituents to disturbances whose wavelength is much larger than their mean free path [2]. This leads to a universal description, the "hydrodynamical regime," that does not depend on the details of the underlying molecular dynamics. By accepting this argument, and quite independently of the Goldstone phenomenon, the energy spectrum of a Higgs condensate should terminate with an "acoustic" branch, say $\tilde{E}(\mathbf{p})=c_s|\mathbf{p}|$ for $\mathbf{p}\to 0$, as for the propagation of sound waves in ordinary media.

However, leaving aside the Goldstone bosons, i.e., for a spontaneously broken one-component $\lambda\Phi^4$ theory, the particle content of the broken phase is usually represented as a single massive field, the (singlet) Higgs boson. Although there is no rigorous proof [3], the Fourier transform of the connected Euclidean propagator is assumed to tend to a finite limit, say $G(p)\to 1/M_h^2$ when the four-momentum $p\to 0$, and the mass squared M_h^2 is related to the quadratic shape of a semiclassical effective potential $V_{\mathrm{NC}}(\phi)$ (NC=nonconvex) at its nontrivial absolute minima, say $\phi=\pm v$. Equivalently, the energy spectrum of the broken phase should tend to a nonzero value, $\tilde{E}(\mathbf{p})\to M_h$, when $\mathbf{p}\to 0$ so that the nonzero quantity $\tilde{E}(0)=M_h$ gives rise to an exponential decay $\sim e^{-\tilde{E}(0)T}$ of the connected Euclidean propagator.

Clearly, by considering the broken phase as a "condensate," the idea of an energy spectrum $\sqrt{\mathbf{p}^2+M_h^2}$ down to $\mathbf{p}=0$ seems unnatural. In fact, for very long wavelengths, one would expect the lowest excitations to arise from small displacements of the condensed quanta that already "pre-exist" in the ground state. Our idea of density fluctuations is motivated by general considerations that should be relevant in any medium and, in particular, in a Bose system at zero temperature. To this end, and for the convenience of the reader, I shall report the following quotations.

(i) "Any quantum liquid consisting of particles with integral spin (such as the liquid isotope ^{4}He) must certainly have a spectrum of this type . . . In a quantum Bose liquid, elementary excitations with small momenta $\mathbf{p}$ (wavelengths large compared with distances between atoms) correspond to ordinary hydrodynamic sound waves, i.e. are phonons. This means that the energy of such quasi-particles is a linear function of their momentum" [4].

(ii) "We now come to the key argument of superfluidity: the only low-energy excitations are phonons. Phonons are excited states of compression, or states involving small displacements of each atom with a resultant change in density" [5].

(iii) "We have seen that low-energy non-phonon excitations are impossible. In other words, there are no possible long-distance movements of the atoms that do not change the density" [5].

After this preliminary introduction, I shall point out that the apparent contradiction between the conventional picture of symmetry breaking and the physical expectation of a superfluid medium with density fluctuations has a precise counterpart in a basic nonperturbative ambiguity for the $p\to 0$ limit of the inverse connected propagator $G^{-1}(p)$. This is a two-valued function when $p\to 0$ and includes the case $G^{-1}(p=0)=0$, as in a gapless theory. This ambiguity, by itself, does not prove that the energy spectrum is actually $\sim c_s|\mathbf{p}|$ for $\mathbf{p}\to 0$, nor does it provide the value of c_s. However, it represents a purely quantum-field theoretical argument in favor of the existence of a gapless branch and of the intuitive picture of the broken phase as a real physical medium with density fluctuations.

Before reporting any calculation, let us first try to understand why the vacuum of a "pro forma" Lorentz-invariant quantum field theory may be such a kind of medium. This

*Electronic address: consoli@ct.infn.it

question may have several answers. For instance, a fundamental phenomenon such as the macroscopic occupation of the same quantum state (say $\mathbf{p}=0$ in some frame) may represent the operative construction of a "quantum aether" [6,7]. This would be quite distinct from the aether of classical physics whose constituents were assumed to follow definite space-time trajectories. However, it would also be different from the empty space-time of special relativity, assumed at the base of axiomatic quantum field theory to deduce the exact Lorentz covariance of the energy spectrum.

In addition, one should take into account the approximate nature of locality in cutoff-dependent quantum field theories. In this picture, the elementary quanta are treated as "hard spheres," as for the molecules of ordinary matter. Thus the notion of the vacuum as a "condensate" acquires an intuitive physical meaning. For the same reason, however, the simple idea that deviations from Lorentz covariance take place only at the cutoff scale may be incorrect: nonperturbative vacuum condensation may give rise to a hierarchy of scales such that the region of Lorentz covariance is sandwiched both by the high- and low-energy regions.

In fact, in general, an ultraviolet cutoff induces vacuum-dependent reentrant violations of special relativity in the low-energy corner [8]. In the simplest possible case, these extend over a small shell of momenta, say $|\mathbf{p}|<\delta$, where the energy spectrum $\tilde{E}(\mathbf{p})$ may deviate from a Lorentz covariant form and be distorted into a sound-wave shape. However, Lorentz covariance becomes an exact symmetry in the local limit. Therefore, for very large but finite Λ, one expects the scale δ to be naturally infinitesimal in units of the energy scale associated with the Lorentz covariant part of the energy spectrum, say M_h. By introducing dimensionless quantities, the requirement of asymptotic Lorentz covariance introduces a tight infrared-ultraviolet connection since $\epsilon \equiv \delta/M_h \rightarrow 0$ when $t \equiv \Lambda/M_h \rightarrow \infty$. In this sense, formally, $\mathcal{O}(\delta/M_h)$ vacuum-dependent corrections would be equivalent to $\mathcal{O}(M_h/\Lambda)$ effects and these are always neglected when discussing [9] how Lorentz covariance emerges at energy scales that are much smaller than the ultraviolet cutoff. Therefore, in the condensed phase of a cutoff theory, although Lorentz covariance is formally recovered in the local limit, one should expect infinitesimal deviations in an infinitesimal region of momenta.

In this context, one may ask what the word infinitesimal actually means in the physical world. For instance, by assuming $\Lambda=10^{19}$ GeV and $M_h=250$ GeV, a scale $\delta=10^{-5}$ eV, for which $\epsilon \equiv \delta/M_h = 4\times 10^{-17}$, might well represent the physical realization of a formally infinitesimal quantity. If this were the right order of magnitude, the non-Lorentz covariant density fluctuations of the vacuum would start to show up from wavelengths larger than a centimeter up to infinity. These lengths are actually infinitely large as compared to the Fermi scale but have, nevertheless, a physical meaning. At the same time, the associated long-range interactions would have a strength $\epsilon^2 \sim M_h^2/\Lambda^2 = \mathcal{O}(10^{-33})$ relative to the Fermi constant. Although small, this strength is nonvanishing and these interactions can play a physical role on macroscopic distances.

I shall now follow, in Secs. II and III, two different methods to display the ambiguity in the zero-momentum limit of the connected propagator in the broken phase. In Sec. IV I shall present my conclusions and a brief discussion of the most general consequences of my results.

II. THE FUNCTIONAL INTEGRATION OVER THE BACKGROUND FIELD

When discussing spontaneous symmetry breaking, the starting point is the separation of the scalar field into a constant background and a shifted fluctuation field, namely

$$\Phi(x)=\phi+h(x). \tag{1}$$

In order for Eq. (1) to be unambiguous, ϕ denotes the spatial average in a large four-volume Ω

$$\phi=\frac{1}{\Omega}\int d^4x\,\Phi(x) \tag{2}$$

and the limit $\Omega\rightarrow\infty$ has to be taken at the end.

In this way, the full functional measure can be expressed as

$$\int [d\Phi(x)]\cdots=\int_{-\infty}^{+\infty} d\phi \int [dh(x)]\cdots \tag{3}$$

and the functional integration on the right-hand side of Eq. (3) is over all quantum modes with four-momentum $p\neq 0$.

After integrating out all nonzero quantum modes, the generating functional in the presence of a space-time constant source J is given by

$$Z(J)=\int_{-\infty}^{+\infty} d\phi\, \exp[-\Omega(V_{\mathrm{NC}}(\phi)-J\phi)] \tag{4}$$

where $V_{\mathrm{NC}}(\phi)$ denotes the usual nonconvex (NC) effective potential obtained order by order in the loop expansion. Finally, by introducing the generating functional for connected Green's functions $w(J)$ through

$$\Omega w(J)=\ln\frac{Z(J)}{Z(0)} \tag{5}$$

one can compute the field expectation value

$$\varphi(J)=\frac{dw}{dJ} \tag{6}$$

and the zero-momentum propagator

$$G_J(p=0)=\frac{d^2w}{dJ^2}. \tag{7}$$

In this framework, spontaneous symmetry breaking corresponds to nonzero values of Eq. (6) in the double limit $J\rightarrow\pm 0$ and $\Omega\rightarrow\infty$.

Now, by denoting $\pm v$ the absolute minima of $V_{\rm NC}$ and $M_h^2=V''_{\rm NC}$ its quadratic shape at these extrema, one usually assumes

$$\lim_{\Omega\to\infty}\lim_{J\to\pm 0}\varphi(J)=\pm v \tag{8}$$

and

$$\lim_{\Omega\to\infty}\lim_{J\to\pm 0}G_J(p=0)=\frac{1}{M_h^2}. \tag{9}$$

In this case, the excitations in the broken phase would be massive particles (the conventional Higgs bosons) whose mass M_h is determined by the positive curvature of $V_{\rm NC}$ at its absolute minima. However, at $\varphi=\pm v$, besides the value $1/M_h^2$, one also finds [10]

$$\lim_{\Omega\to\infty}\lim_{J\to\pm 0}G_J(p=0)=+\infty, \tag{10}$$

a result that has no counterpart in perturbation theory.

Let us review how this result emerges from the saddle-point approximation, valid for $\Omega\to\infty$. In this case, we get

$$w(J)=\frac{J^2}{2M_h^2}+\frac{\ln\cosh(\Omega J v)}{\Omega} \tag{11}$$

and

$$\varphi=\frac{dw}{dJ}=\frac{J}{M_h^2}+v\tanh(\Omega J v), \tag{12}$$

$$G_J(p=0)=\frac{d^2w}{dJ^2}=\frac{1}{M_h^2}+\frac{\Omega v^2}{\cosh^2(\Omega J v)}. \tag{13}$$

To determine the zero-momentum propagator in a given background φ, we should now invert J as a function of φ from Eq. (12) and replace it in Eq. (13). However, being interested in the limit $J\to 0$ it is easier to look for the possible limiting behaviors of Eq. (13).

Since both J and Ω are dimensionful quantities, it is convenient to introduce dimensionless variables

$$x\equiv\Omega J v \tag{14}$$

and

$$y\equiv\Omega v^2 M_h^2 \tag{15}$$

so that Eqs. (12) and (13) become

$$\varphi=v\left[\frac{x}{y}+\tanh(x)\right] \tag{16}$$

and

$$G_J(p=0)=\frac{1}{M_h^2}\left[1+\frac{y}{\cosh^2(x)}\right]. \tag{17}$$

In this representation, taking the two limits $\Omega\to\infty$ and $J\to\pm 0$ correspond to choose some path in the two-dimensional space (x,y). The former gives trivially $y\to\infty$. The latter, on the other hand, is equivalent to $x/y\to\pm 0$ since

$$\frac{J}{vM_h^2}=\frac{x}{y} \tag{18}$$

with many alternative possibilities. If we require a nonzero limit of φ this amounts to an asymptotic nonzero value of x. If this value is finite, say $x=x_o$ we get asymptotically

$$\varphi\to v\tanh(x_o) \tag{19}$$

and

$$G_J(p=0)\to\frac{1}{M_h^2}\frac{y}{\cosh^2(x_o)}\to\infty \tag{20}$$

implying the existence of gapless modes for every nonzero value of φ. On the other hand, if $x\to\pm\infty$ we obtain

$$\varphi\to\pm v. \tag{21}$$

In this case $G_J(p=0)$ tend to $1/M_h^2$ (to $+\infty$) depending on whether y diverges slower (faster) than $\cosh^2(x)$.

The above results admit a simple geometrical interpretation in terms of the shape of the effective potential $V_{\rm LT}(\varphi)$ as defined from the Legendre transform (LT) of $w(J)$. After obtaining J as a function of φ from Eq. (12), the inverse zero-momentum propagator in a given background φ is related to the second-derivative of the Legendre-transformed effective potential, namely

$$G_\varphi^{-1}(p=0)=\frac{dJ}{d\varphi}=\frac{d^2V_{\rm LT}}{d\varphi^2}. \tag{22}$$

In this case, Eqs. (19) and (20) require a vanishing result from Eq. (22) when $-v<\varphi<v$. This is precisely what happens since $V_{\rm LT}$ becomes flat in the region enclosed by the absolute minima of the nonconvex effective potential when $\Omega\to\infty$. This is the usual "Maxwell construction" where $V_{\rm LT}(\varphi)=V_{\rm NC}(\pm v)$, for $-v\leqslant\varphi\leqslant v$, and $V_{\rm LT}(\varphi)=V_{\rm NC}(\varphi)$ for $\varphi^2>v^2$.

Notice, however, that the limit of Eq. (22) for $\varphi\to\pm v$ cannot so simply be identified with M_h^2. In fact, even within the "Maxwell construction," this identification requires a strong additional assumption: the derivative in Eq. (22) has to be a left- (or right-) derivative depending on whether we consider the point $\varphi=-v$ (or $\varphi=+v$). Now, this is just a prescription since derivatives depend on the chosen path (unless one deals with infinitely differentiable functions) and, differently from $V_{\rm NC}$, the Legendre transformed $V_{\rm LT}$ is not an infinitely differentiable function in the presence of spontaneous symmetry breaking [11].

Therefore, in general, Eq. (22) leads to multiple solutions at $\varphi=\pm v$. Namely, an exterior derivative for which $G_{\rm ext}^{-1}(0)=M_h^2$ but also a $G_{\rm int}^{-1}(0)=0$, as when approaching the points $\pm v$ from the internal region where the Legendre-transformed potential becomes flat for $\Omega\to\infty$. These two different alternatives correspond to the various limits $y\to\infty$ and $x\to\pm\infty$ in Eq. (17) such that $y/\cosh^2(x)$ tends to zero or infinity.

We observe that the "Maxwell construction," i.e., the replacement $V_{\rm NC}\to V_{\rm LT}$ as a genuine quantum effect, was also discovered in Ref. [12]. Graphically, the resulting effective potential becomes flatter and flatter between $-v$ and $+v$ when removing the infrared cutoff. Numerically, the ratio between left- and right- second derivatives at the absolute minima of $V_{\rm NC}$ is found to diverge in the same limit [13].

I conclude this section with the remark that the singular zero-momentum behavior I have pointed out does not depend at any stage on the existence of a continuous symmetry of the classical potential. As such, there are no differences in a spontaneously broken $O(N)$ theory. Beyond the approximation where the "Higgs condensate" is treated as a purely classical background, one has to perform one more integration over the zero-momentum mode of the condensed σ field. Therefore all ambiguities in computing the inverse propagator of the σ field through Eq. (22) remain. In this sense, the possibility of multiple values for $G_\sigma(p=0)$ has nothing to do with the number of field components.

III. RESUMMATION OF THE TADPOLE GRAPHS

The possibility of a divergent zero-four-momentum propagator in the broken phase, as illustrated in the previous section, is nonperturbative and independent of any diagrammatic analysis. As an additional evidence for the subtle nature of the $p\to 0$ limit of $G(p)$, I shall attempt, however, to isolate the possible origin of this effect in the one-particle reducible zero-momentum tadpole graphs. These enter the usual diagrammatic expansion in the presence of a constant background field and can be considered a manifestation of the quantum nature of the scalar condensate. The expansion we shall consider is defined in terms of the one-particle irreducible (1PI) graphs generated by the nonconvex effective potential $V_{\rm NC}(\phi)$ considered before. In this respect, the tadpole graphs are fully nonperturbative and have to be resummed to all orders. The genuine 1PI interaction graphs, on the other hand, represent perturbative effects and can be considered to any desired order in the loop expansion, without changing qualitatively the conclusions. In the following, after some preliminaries, we shall address the zero-momentum propagator at the absolute minima of $V_{\rm NC}(\phi)$.

Let us start by defining

$$\frac{dV_{\rm NC}}{d\phi}\equiv J(\phi)\equiv\phi T(\phi^2) \tag{23}$$

and $\phi=\pm v\neq 0$ are the solutions of

$$T(\phi^2)=0 \tag{24}$$

with

$$\left.\frac{d^2V_{\rm NC}}{d\phi^2}\right|_{\phi=\pm v}>0. \tag{25}$$

Usually, one defines the h-field propagator from a Dyson sum of 1PI graphs only, say

$$G(p)|_{\rm 1PI}\equiv D(p) \tag{26}$$

where $[D^{-1}(0)\equiv D^{-1}(p=0)]$

$$D^{-1}(0)\equiv\frac{d^2V_{\rm NC}}{d\phi^2}. \tag{27}$$

This provides the conventional definition of M_h^2 through Eq. (27) at $\phi=\pm v$.

In this description one neglects the possible role of the one-particle reducible, zero-momentum tadpole graphs. The reason is that their sum is proportional to the one-point function, i.e., to $J(\phi)$ in Eq. (23) that vanishes by definition at $\phi=\pm v$. However, the zero-momentum tadpole subgraphs are attached to the other parts of the diagrams through zero-momentum propagators so that, in an all-order calculation, their overall contribution vanishes provided the full propagator $G(0)$ is nonsingular at the minima. In this respect, neglecting the tadpole graphs amounts to assuming the regularity of $G(0)$ at $\phi=\pm v$ which is certainly true in a finite-order expansion. In an intuitive analogy, when $\phi\to\pm v$, $J(\phi)$ represents an infinitesimal driving force due to the medium. Thus it will not produce any observable effect unless the mass of a body vanishes in the same limit. The complication in our case is that the mass of our "body," the inverse propagator $G^{-1}(0)$, depends on the medium and on the driving force itself.

For this reason, I shall try to control the full propagator in a small region of ϕ values around the minima by including all zero-momentum tadpole graphs, and finally take the limit $\phi\to\pm v$. I observe that the problem of tadpole graphs was considered in Ref. [14] where the emphasis was mainly to find an efficient way to rearrange the perturbative expansion. Here I shall attempt a nonperturbative all-order resummation of the various effects to check the regularity of $G(0)$ for $\phi\to\pm v$.

I shall approach the problem in two steps. In a first step, I shall consider the contributions to the propagator by including all possible insertions of zero-momentum lines in the internal part of the graphs, i.e., inside 1PI vertices. At this stage, however, the external zero-momentum propagators to the sources maintain their starting value $D(0)$ at $J=0$. This approximation gives rise to an auxiliary inverse propagator given by

$$G^{-1}_{\rm aux}(p)=D^{-1}(p)-\varphi z\Gamma_3(p,0,-p)+\frac{(\varphi z)^2}{2!}\Gamma_4(0,0,p,-p)-\frac{(\varphi z)^3}{3!}\Gamma_5(0,0,0,p,-p)+\cdots, \tag{28}$$

where

$$z\equiv T(\phi^2)D(0) \tag{29}$$

represents the basic one-tadpole insertion. Equation (28) can be easily checked diagrammatically starting from the tree approximation where

$$V_{\rm tree} = \frac{1}{2} r \phi^2 + \frac{\lambda}{4!} \phi^4, \tag{30}$$

$$D^{-1}(p) = p^2 + r + \frac{\lambda \phi^2}{2}, \tag{31}$$

and $\Gamma_3(p,0,-p) = \lambda \phi$, $\Gamma_4(0,0,p,-p) = \lambda$ (with all Γ_n vanishing for $n > 4$).

Now, by using the relation of the zero-momentum 1PI vertices with the effective potential at an arbitrary ϕ,

$$\Gamma_n(0,0,\ldots,0) = \frac{d^n V_{\rm NC}}{d\phi^n}, \tag{32}$$

we can express the auxiliary zero-four-momentum inverse propagator of Eq. (28) as

$$G^{-1}_{\rm aux}(0) = \left. \frac{d^2 V_{\rm NC}}{d\phi^2} \right|_{\phi_{\rm aux} = \phi(1-z)}. \tag{33}$$

For $\phi \to \pm v$, this partial resummation of tadpole graphs gives a vanishing contribution to the inverse propagator so that $G^{-1}_{\rm aux}(0) \to M_h^2$ as determined from the quadratic shape of $V_{\rm NC}$ at its absolute minima. However, for arbitrary ϕ even this partial resummation produces nonperturbative modifications of the zero-momentum propagator. For instance, as one can check with the tree-level potential, there are values of ϕ where $D^{-1}(0)$ is positive but $G^{-1}_{\rm aux}(0)$ is negative.

The second step consists of including now all possible tadpole corrections on each external zero-momentum line. In fact, in a diagrammatic expansion, a single external zero-momentum leg gives rise to a new infinite hierarchy of graphs, each producing another infinite number of graphs and so on. Despite the apparent complexity of the task, the final outcome of this computation can be cast in a rather simple form, at least on a formal ground. The point is that this infinite class of graphs can be included into a redefinition of the two basic expansion parameters entering the tadpole resummation: the source function J and the zero-momentum propagator $D(0)$.

In fact, to any finite order in J, I can rearrange the expansion for the zero-momentum propagator (all Γ_n are evaluated at zero external momenta)

$$G(0) = D(0) + J\Gamma_3 D^3(0) + \frac{3J^2\Gamma_3^2}{2} D^5(0) - \frac{\Gamma_4 J^2}{2} D^4(0) + \mathcal{O}(J^3) \tag{34}$$

in terms of a modified source

$$\tilde{J} = J - \frac{J^2 \Gamma_3}{2} D^2(0) + \mathcal{O}(J^3) \equiv \phi \tilde{T}(\phi^2) \tag{35}$$

in such a way that the formal power series for the exact inverse zero-momentum propagator can be expressed as

$$G^{-1}(0) = \left. \frac{d^2 V_{\rm NC}}{d\phi^2} \right|_{\hat{\phi} = \phi(1-\tau)} \tag{36}$$

with

$$\tau \equiv \tilde{T}(\phi^2) G(0), \tag{37}$$

i.e., as in Eq. (33) with the replacement $z \to \tau$. As a result, after including tadpole graphs to all orders, one finds multiple solutions for the zero-four-momentum propagator that differ from Eq. (27) even when $\phi \to \pm v$.

The situation is similar to solving the following equation:

$$f^{-1}(x) = 1 + x^2 - g^2 x^2 f(x). \tag{38}$$

For $x \to 0$ there are two distinct limiting behaviors: (a) $f(x) \to 1$ and (b) $f(x) \sim 1/g^2 x^2 \to +\infty$. However, only the former solution is recovered in a finite number of iterations from

$$f_0(x) = \frac{1}{1+x^2} \tag{39}$$

for $g^2 = 0$. In the case of $\lambda\Phi^4$ theory, deriving the gapless mode from tadpole resummation corresponds to the (b) type of behavior.

In this sense, the resummation in Eq. (36) is only formal since, with this procedure, the function $\tilde{J}(\phi)$ is always determined as a finite-order polynomial up to higher orders in J, as it happens in perturbation theory. However, this is not so important since the possibility of a singular zero-four-momentum propagator at $\phi = \pm v$ does not depend on the form of $\tilde{J}(\phi)$ but only on its vanishing at $\phi = \pm v$. As we shall see, this vanishing reflects simple geometrical properties of the nonconvex effective potential $V_{\rm NC}$.

To understand this point, we first observe that traditionally, for $\phi \to \pm v$, tadpole resummation has been considered to be irrelevant. Namely, in the literature the inverse propagator is always defined from Eq. (27) that neglects the tadpole graphs altogether. As anticipated in the Introduction, this is the main motivation to relate the physical Higgs boson mass to the quadratic shape of $V_{\rm NC}$. Therefore, after including the tadpole graphs, Eq. (27) should also be contained in Eq. (36), at least as a particular solution where $G(0) = D(0)$. To this end, however, when $J \to 0$ and $\phi \to \pm v$, also the modified source $\tilde{J}$ has to vanish. The alternative possibility, i.e., that the full $\tilde{J}$ remains nonzero when $\phi \to \pm v$, would produce a drastic result. In fact, after tadpole resummation, an inverse propagator as in Eq. (27) would never be recovered from Eq. (36), even as a particular solution. This would also contradict the indications of the previous section where we have found evidence for both regular and singular values of the zero-four-momentum propagator at $\phi = \pm v$.

Adopting the natural point of view that the modified source $\tilde{J}$ (and so $\tilde{T}$) vanishes when $\phi \to \pm v$, Eq. (36) provides a regular solution $G^{-1}_{\rm reg}(0) = D^{-1}(0)$ for which

$$\lim_{\phi \to \pm v} \tau = \overline{\tau} = 0 \tag{40}$$

and a singular solution $G^{-1}_{\text{sing}}(0) = 0$ such that

$$\lim_{\phi \to \pm v} \tau = \overline{\tau} \neq 0. \tag{41}$$

As an example, let us consider the situation of $V_{\text{NC}} \equiv V_{\text{tree}}$ in Eq. (30) which is equivalent to re-summing tree-level tadpole graphs to all orders (i.e., no loop diagrams). In this case the regular solution is $G^{-1}_{\text{reg}}(0) = \lambda v^2/3$, while the singular solution is

$$\lim_{\phi^2 \to v^2} G^{-1}_{\text{sing}}(0) = \frac{\lambda v^2}{2}\left[\overline{\tau^2} - 2\overline{\tau} + \frac{2}{3}\right] = 0 \tag{42}$$

which implies limiting values $\overline{\tau} = 1 \pm 1/\sqrt{3}$.

In general, beyond the tree-approximation, finding the singular solution $G^{-1}_{\text{sing}}(0) = 0$ at $\phi = \pm v$ is equivalent to determine that value of $\hat{\phi}^2 \equiv v^2(1-\overline{\tau})^2$ where $d^2 V_{\text{NC}}/d\phi^2 = 0$. For instance, in the case of the Coleman-Weinberg effective potential

$$V_{\text{NC}}(\phi) = \frac{\lambda^2 \phi^4}{256\pi^2}\left(\ln\frac{\phi^2}{v^2} - \frac{1}{2}\right) \tag{43}$$

the required values are $\overline{\tau} = 1 \pm e^{-1/3}$. In principle, such solutions exist in any approximation to V_{NC} due to the very general properties of the shape of a nonconvex effective potential.

IV. CONCLUSIONS AND OUTLOOK

In principle, a medium can support different types of excitations. For instance, the energy spectrum of superfluid ^{4}He is considered to arise from the combined effect of two types of excitations, phonons and rotons, whose separate energy spectra ‘‘match,’’ giving rise to a complicated pattern [15]. In this sense, there is a unique spectrum but phonons and rotons can be considered different ‘‘particles’’ reflecting different aspects of the superfluid helium wave function [5].

The essential point is that, for superfluid ^{4}He, the existence of two types of excitations was first deduced theoretically by Landau on the basis of very general arguments [16]. According to this original idea, there are phonons with energy $E_{\text{ph}}(\mathbf{p}) = v_s|\mathbf{p}|$ and rotons with energy $E_{\text{rot}}(\mathbf{p}) = \Delta + \mathbf{p}^2/2\mu$. Only later, it was experimentally discovered that there is a single energy spectrum $E(\mathbf{p})$. This is made up by a continuous matching of these two different parts and is dominated by phonons for $\mathbf{p} \to 0$ where the rotons become unphysical.

Let us now consider our analysis of the zero-momentum propagator. Its two-valued nature may be the indication that something similar happens in the broken phase, in agreement with the intuitive picture of the Higgs vacuum as a superfluid system. In fact, the existence of both a $G_a^{-1}(0) \equiv M_h^2$ and a $G_b^{-1}(0) \equiv 0$ implies that there would be two possible types of excitations with the same quantum numbers but different energies when the three-momentum $\mathbf{p} \to 0$: a massive one, with $\tilde{E}_a(\mathbf{p}) \to M_h$ and a gapless one, with $\tilde{E}_b(\mathbf{p}) \to 0$ that, *a priori*, can both propagate (and interfere) in the broken-symmetry phase. In analogy with ^{4}He, I would conclude that the latter dominates the exponential decay $\sim e^{-\tilde{E}_b(\mathbf{p})T}$ of the connected Euclidean correlator for $\mathbf{p} \to 0$ so that the massive excitation becomes unphysical in the infrared region. Therefore, differently from the simplest perturbative indications, in a (one-component) spontaneously broken $\lambda\Phi^4$ theory there would be no energy-gap associated with the ‘‘Higgs mass’’ M_h, as in a genuine massive single-particle theory where the relation

$$\tilde{E}_a(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M_h^2} \tag{44}$$

remains true for $\mathbf{p} \to 0$. Rather, the far-infrared region would be dominated by gap-less collective excitations whose typical energy spectrum for $\mathbf{p} \to 0$,

$$\tilde{E}_b(\mathbf{p}) \equiv c_s|\mathbf{p}| \tag{45}$$

depends on an unknown parameter c_s. This, according to the arguments given in the Introduction, would represent the ‘‘sound velocity’’ for the density fluctuations of the superfluid scalar condensate.

By their very nature, these density fluctuations represent non-Lorentz covariant effects and, following the discussion given in the Introduction, should be restricted to an infinitesimal region of momenta $|\mathbf{p}| < \delta$ with $\delta/M_h = \mathcal{O}(M_h/\Lambda)$, Λ being the ultraviolet cutoff of the theory.

As discussed in the Introduction, the strength of the associated long-range interactions is also expected to be infinitesimal. In this framework, one may consider possible viable phenomenological frameworks, of the type presented in Ref. [18], where the massive branch dominates at higher momenta as it would happen in a superfluid system where $\Delta = \mu \equiv M_h$. The superfluid analogy is further supported by the observation [1] that, as for the interatomic ^{4}He-^{4}He potential, the low-energy limit of cutoff $\lambda\Phi^4$ is a theory of quanta with a short-range repulsive core and a long-range attractive tail. The latter originates from ultraviolet-finite parts of higher loop graphs [1] that give rise to a $-\lambda^2 e^{-2mr}/r^3$ attraction, m being the mass of the elementary condensing quanta. Differently from the usual ultraviolet divergences, this finite part cannot be reabsorbed into a standard redefinition of the tree-level, repulsive $+\lambda\delta^{(3)}(\mathbf{r})$ contact potential and is essential for a physical description of the condensation process when approaching the phase transition limit $m \to 0$ where the symmetric vacuum at $\langle \Phi \rangle = 0$ becomes unstable.

Of course, one may object that I have not provided a calculation of the energy spectrum. Admittedly, this is a weak point of my analysis that just concentrates on the zero-momentum propagator. At the same time, it will not be so easy to improve on it. In fact, for a full calculation of the energy spectrum, one should improve on the usual covariant generalization of the Bogolubov method used in Ref. [1]. This approximation, where the creation and annihilation operators $a_{\mathbf{p}=0}$, $a^{\dagger}_{\mathbf{p}=0}$ for the elementary quanta in the $\mathbf{p}=0$

mode are simply replaced by the c-number $\sqrt{N}$, is equivalent to treat the scalar condensate as a purely classical background field ϕ entering the quadratic Hamiltonian for the shifted fluctuation field $h(x)$. In this case, the energy spectrum is just $\sqrt{\mathbf{p}^2+M_h^2}$ and, therefore, the existence of a gapless excitation branch for $\mathbf{p}\to 0$ could not be discovered there.

However, in Ref. [1] it was noticed that the Bogolubov method does not allow for a straightforward extrapolation down to $\mathbf{p}=0$. This is due to an ambiguity relating the original creation and annihilation operators to their Bogolubov-transformed counterpart in the limit $\mathbf{p}\to 0$. Requiring continuity of the massive energy spectrum down to $\mathbf{p}=0$ is equivalent to replacing $a_{\mathbf{p}=0}$ and $a^{\dagger}_{\mathbf{p}=0}$ with $\sqrt{N}$. This choice is the second-quantization analog of "freezing" $\phi=\pm v$ without performing the functional integration in Sec. II or without first resumming the zero-momentum tadpole graphs in Sec. III. In this case, the only solution is $G^{-1}(0)=M_h^2$.

In this sense, one should first include a genuine operator part for $a_{\mathbf{p}=0}$ [19], say $a_{\mathbf{p}=0}=\sqrt{N}-\hat{\xi}$ with $\langle\hat{\xi}\rangle=\alpha$ and $|\alpha|^2\ll N$. This introduces new contributions as the three-linear couplings $a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}}\hat{\xi},\ldots,$ the four-linear couplings $a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}}\hat{\xi}^{\dagger}\hat{\xi},\ldots,$ whose effects should be preliminarly computed in analogy with the zero-momentum tadpole graphs considered in Sec III. The new contributions produce corrections to the standard Bogolubov massive spectrum computed in Ref. [1].

Now, the stationarity value of α, say $\alpha=\bar{\alpha}$, has to be determined after minimizing the energy density and does not necessarily correspond to $\alpha=0$. This can easily be understood by noticing that, after minimization, the number of particles in the condensate is

$$\langle a^{\dagger}_{\mathbf{p}=0}a_{\mathbf{p}=0}\rangle=N-2\,\mathrm{Re}(\bar{\alpha})\sqrt{N}+\mathcal{O}(\bar{\alpha}^2). \qquad (46)$$

Therefore α describes dynamical rearrangements of the total number of particles between the condensate and the states with $\mathbf{p}\neq 0$. By dynamical, we mean that a nonzero α changes the fraction of particles in the condensate and that this modification is not an overall change of N. The latter does not depend on the occurrence of Bose condensation but concerns the infinite volume limit of any system with a given particle density.

With an intuitive term, a nonzero $\bar{\alpha}$ produces a depletion of the condensate, i.e., additional contributions to the ground state wave function as $(\mathbf{p},-\mathbf{p})$, $(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2),\ldots$. These are needed for the dynamical equilibrium in the presence of interactions and extend over a typical region of momenta, say $|\mathbf{p}|<\delta$, such that $\epsilon\equiv\delta/M_h\to 0$ when $\bar{\alpha}\to 0$. In this limit, in fact, the Bogolubov spectrum $\sqrt{\mathbf{p}^2+M_h^2}$ applies to the whole range of momenta producing an exact Lorentz-covariant theory (with the possible exception of the zero-measure set $\mathbf{p}=0$).

Therefore, taking into account my discussion in the Introduction, I conclude that the limit of vanishing interactions, $\bar{\alpha}\to 0$, has to correspond to the continuum limit $t=\Lambda/M_h\to\infty$ of cutoff $\lambda\Phi^4$ theory, i.e., "triviality," in full agreement with all rigorous results [3]. However, my analysis shows that, in the broken-symmetry phase, the approach to the continuum theory is more subtle than what is generally believed. In fact, usually, one just considers the deviations from "triviality" to be perturbative corrections to a free massive theory, without any qualitative change for $\mathbf{p}\to 0$.

I end up by mentioning that the existence of gapless modes in the broken symmetry phase finds some support in the results of numerical simulations. These have been performed [20] in the low-temperature phase of a one-component 4D Ising model. The lattice data for the exponential decay of the connected correlator show that, by simply increasing the lattice size, one finds smaller and smaller values of the energy gap $\tilde{E}(\mathbf{p}=0)$. Namely, by using the same lattice parameters that on a 20^4 lattice give [21,22] $\tilde{E}(0)=0.3912\pm 0.0012$, the results of Ref. [20] were $\tilde{E}(0)=0.3826\pm 0.0041$ on a 24^4 lattice, $\tilde{E}(0)=0.3415\pm 0.0085$ on a 32^4, lattice and $\tilde{E}(0)=0.303\pm 0.026$ on a 40^4 lattice.

However, in Ref. [20] the extraction of the energy-gap was obtained from a fit to a single exponential $\sim e^{-\tilde{E}(0)T}$ for all time-slices. Although the quality of the fit was quite good, strictly speaking, due to the contamination from higher excited states, the energy-gap should be obtained from the exponential decay for asymptotic T. In this case, by restricting to large T only, a good signal-to-noise ratio requires very high statistics (say 10 mil of sweeps) so that on large 32^4 and 40^4 lattices, one has to wait for very long running times. For this reason, the numerical evidence from lattice simulations, although promising, is still inconclusive and suggests the need for additional efforts by other groups.

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