Gravity dual of the chiral anomaly

Igor R. Klebanov and Peter Ouyang
Department of Physics, Princeton University, Princeton, New Jersey 08544

Edward Witten
School of Natural Sciences, Institute for Advanced Studies, Princeton, New Jersey 08540
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We study effects associated with the chiral anomaly for a cascading $SU(N+M)\times SU(N)$ gauge theory using gauge/gravity duality. In the gravity dual the anomaly is a classical feature of the supergravity solution, and the breaking of the $U(1)$ R symmetry down to $\mathbf{Z}_{2M}$ proceeds via the Higgs mechanism.

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I. INTRODUCTION

Many supersymmetric gauge theories exhibit a classical $U(1)$ R symmetry which is broken quantum mechanically to some discrete subgroup. In traditional quantum field theory, this symmetry breaking can be understood as an instanton effect. The purpose of this paper is to explore the analogous effects in the gravity duals to a few field theories exhibiting this phenomenon.

Our analysis relies on the results of recent encouraging progress in extending the gauge theory-supergravity correspondence [1–3] to theories with less than maximal supersymmetry, realized by configurations of D-branes at singularities. For example, we might consider a stack of N coincident D3-branes located at the tip of the singular Calabi-Yau space known as the conifold [4]. This system is dual to an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $SU(N)\times SU(N)$ coupled to chiral superfields (two bifundamentals transforming in the $(\mathbf{N},\overline{\mathbf{N}})$ representation, and their conjugates, which transform in the $(\overline{\mathbf{N}},\mathbf{N})$ representation). The supergravity solution has the geometry $\mathrm{AdS}_5\times \mathrm{T}^{1,1}$, which manifests geometrically the conformal invariance of the gauge theory. The Einstein manifold $T^{1,1}$ possesses a two-cycle and a three-cycle, and we can obtain interesting variations of this theory by wrapping various branes on these cycles [5–8]. In particular, wrapping D5-branes on this two-cycle changes the gauge group to $SU(N)\times SU(N+M)$ and breaks the conformal symmetry.

This supergravity solution with wrapped D5-branes exhibits a host of interesting gauge theory phenomena; for example, the reduction of the five-form flux as the radial coordinate decreases corresponds to a reduction in the size of the gauge groups by a duality cascade [8]. An important feature for our purposes is that the UV *metric* exhibits a $U(1)$ symmetry under rotations of a particular angle β in the tranverse space $T^{1,1}$, which is a geometric realization of the field theoretic R symmetry. However, the Ramond-Ramond (RR) potentials break the $U(1)$ symmetry. It can be shown that there is an unbroken $\mathbf{Z}_{2M}$ subgroup of this $U(1)$ by studying fractional instanton and domain wall probes in the gravity background [9–11]. However, one should not think about this symmetry breaking as an *effect* of these instantons, which do not appear explicitly anywhere in the gravity dual. Rather, we argue that the field theory anomalies are present because the classical supergravity RR potentials are not invariant under the $U(1)$ symmetry. By computing the variation of the RR potentials we obtain the relevant anomaly coefficients, which agree with field theory exactly. Moreover, the anomalous breaking of the global $U(1)$ symmetry appears as spontaneous symmetry breaking in supergravity: the bulk vector field dual to the R-symmetry current of the gauge theory acquires a mass. We will also check the anomaly coefficients for a related $\mathcal{N}=2$ orbifold theory, again showing exact agreement.

II. THE ANOMALY AS NON-INVARIANCE OF THE UV SUPERGRAVITY SOLUTION

Let us recall a few results regarding the supergravity dual of the cascading $SU(N+M)\times SU(N)$ gauge theory. The metric is of the form [8]

$$ds_{10}^2=h^{-1/2}(\tau)dx_{||}^2+h^{1/2}(\tau)ds_6^2, \tag{1}$$

where ds_6^2 is the metric of the deformed conifold. The UV (large τ) limit of this metric was found in [7]:

$$ds_{10}^2=h^{-1/2}(r)dx_{||}^2+h^{1/2}(r)(dr^2+r^2ds_{T^{1,1}}^2). \tag{2}$$

The metric on $T^{1,1}$, the base of the conifold, is

$$ds_{T^{1,1}}^2=\frac{1}{9}\left(2d\beta+\sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2+\frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2+\sin^2\theta_i d\phi_i^2). \tag{3}$$

We define the angle β to range from 0 to 2π; it is related to the angle ψ used in the previous literature by $\psi=2\beta$. The asymptotic form of the warp factor is

$$h(r)=\frac{27\pi(\alpha')^2}{4r^4}\left[g_sN+\frac{3}{2\pi}(g_sM)^2\ln(r/r_0)+\frac{3}{8\pi}(g_sM)^2\right]. \tag{4}$$

For sufficiently small r, this metric is singular; to study IR physics in the gauge theory, one must use the full solution of [8].

The following basis of 1-forms on the compact space is convenient for calculations on the deformed conifold [12]:

$$g^1=\frac{e^1-e^3}{\sqrt{2}}, \quad g^2=\frac{e^2-e^4}{\sqrt{2}},$$
$$g^3=\frac{e^1+e^3}{\sqrt{2}}, \quad g^4=\frac{e^2+e^4}{\sqrt{2}}, \quad g^5=e^5, \tag{5}$$

where

$$e^1\equiv-\sin\theta_1 d\phi_1, \quad e^2\equiv d\theta_1,$$
$$e^3\equiv\cos 2\beta\sin\theta_2 d\phi_2-\sin 2\beta d\theta_2,$$
$$e^4\equiv\sin 2\beta\sin\theta_2 d\phi_2+\cos 2\beta d\theta_2, \tag{6}$$
$$e^5\equiv 2d\beta+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2.$$

In terms of this basis, the Einstein metric on $T^{1,1}$ assumes the form

$$ds^2_{T^{1,1}}=\frac{1}{9}(g^5)^2+\frac{1}{6}\sum_{i=1}^{4}(g^i)^2. \tag{7}$$

The three-form fields are turned on in this background:

$$F_3=\frac{M\alpha'}{2}\omega_3, \quad B_2=\frac{3g_sM\alpha'}{2}\omega_2\ln(r/r_0), \tag{8}$$

$$H_3=dB_2=\frac{3g_sM\alpha'}{2r}dr\wedge\omega_2, \tag{9}$$

where

$$\omega_3=g^5\wedge\omega_2, \tag{10}$$

$$\omega_2=\frac{1}{2}(\sin\theta_1 d\theta_1\wedge d\phi_1-\sin\theta_2 d\theta_2\wedge d\phi_2). \tag{11}$$

A key feature of this solution is that the five-form flux is radially dependent [7]:

$$F_5=\mathcal{F}_5+\star\mathcal{F}_5, \quad \mathcal{F}_5=\frac{1}{2}\pi\alpha'^2N_{eff}(r)\omega_2\wedge\omega_3, \tag{12}$$

with

$$N_{eff}(r)=N+\frac{3}{2\pi}g_sM^2\ln(r/r_0), \tag{13}$$

and the ten-dimensional Hodge dual is defined by $\varepsilon_{txyzr5\theta_1\phi_1\theta_2\phi_2}=\sqrt{-G_{10}}$. In [13] it was shown that

$$\int_{S^2}\omega_2=4\pi, \quad \int_{S^3}\omega_3=8\pi^2. \tag{14}$$

The normalization of the RR 3-form flux is determined by the quantization condition

$$\frac{1}{4\pi^2\alpha'}\int_{S^3}F_3=M. \tag{15}$$

With these results in hand, let us see how the chiral anomaly emerges in supergravity. The asymptotic UV metric (2,3) has a $U(1)$ symmetry associated with the rotations of the angular coordinate β, which appears as the R symmetry of the dual gauge theory. It is crucial, however, that the background value of the RR 2-form C_2 does not have this continuous symmetry. Indeed, although F_3 is $U(1)$ symmetric, there is no smooth global expression for C_2. Locally, we may write for large r,

$$C_2\rightarrow M\alpha'\beta\omega_2. \tag{16}$$

This expression is not single-valued as a function of the angular variable β, but it is single-valued up to a gauge transformation, so that $F_3=dC_2$ is single-valued. In fact, F_3 is completely independent of β. Because of the explicit β dependence, C_2 is not $U(1)$-invariant. Under the transformation $\beta\rightarrow\beta+\epsilon$,

$$C_2\rightarrow C_2+M\alpha'\epsilon\omega_2. \tag{17}$$

A gauge transformation can shift $C_2/(4\pi^2\alpha')$ by an arbitrary integer multiple of $\omega_2/(4\pi)$, so $\beta\rightarrow\beta+\epsilon$ is a symmetry precisely if ϵ is an integer multiple of π/M; because ϵ is anyway only defined mod 2π, a $\mathbf{Z}_{2M}$ subgroup of the $U(1)$ leaves fixed the asymptotic values of the fields, and thus corresponds to a symmetry of the system. This $\mathbf{Z}_{2M}$ respects the asymptotic values of the fields, but in the solution found in [8], it is spontaneously broken in the IR to $\mathbf{Z}_2$, generated by $(-1)^F$, since the full solution does not have $\mathbf{Z}_{2M}$ symmetry. (In that solution, $\mathbf{Z}_{2M}$ is broken to $\mathbf{Z}_2$ by the deformation parameter of the conifold.) The analogous statement in field theory is that instantons break the $U(1)$ down to $\mathbf{Z}_{2M}$, which is then spontaneously broken to $\mathbf{Z}_2$. The domain walls interpolating between these M inequivalent vacua are D5-branes wrapped over the 3-sphere at $\tau=0$ in the solution of [8], and indeed one such 5-brane produces a shift of $C_2/(4\pi^2\alpha')$ by exactly $\omega_2/(4\pi)$ [13].

The way that the asymptotic behavior of C_2 transforms under the $U(1)$ generator is dual to the way that in field theory a $U(1)_R$ transformation shifts the Θ-angles of the two gauge groups by opposite amounts. The Θ-angles are given by

$$\Theta_1-\Theta_2=\frac{1}{\pi\alpha'}\int_{S^2}C_2, \quad \Theta_1+\Theta_2\sim C, \tag{18}$$

where C is the RR scalar, which vanishes for the case under consideration. Using the fact that $\int_{S^2}\omega_2=4\pi$, we find that the small $U(1)$ rotation induces

$$\Theta_1=-\Theta_2=2M\epsilon. \tag{19}$$

We can compare Eq. (19) with our expectations from the field theory. The conventionally normalized Θ terms for the gauge theory action are

$$\int d^4x\left(\frac{\Theta_1}{32\pi^2}F^a_{ij}\tilde{F}^{aij}+\frac{\Theta_2}{32\pi^2}G^b_{ij}\tilde{G}^{bij}\right), \tag{20}$$

where F^a_{ij} and G^b_{ij} are the field strengths of $SU(N+M)$ and $SU(N)$ respectively. If we assume that ϵ is a function of the 4 world volume coordinates x^i, then the terms linear in ϵ in the dual gauge theory are

$$\int d^4x\left[-\epsilon\partial_i J^i+\frac{M\epsilon}{16\pi^2}(F^a_{ij}\tilde{F}^{aij}-G^b_{ij}\tilde{G}^{bij})\right], \tag{21}$$

where J^i is the chiral R current. The appearance of the second term is due to the non-invariance of C_2 under the $U(1)$ rotation. Varying with respect to ϵ, we therefore find the equation

$$\partial_i J^i=\frac{M}{16\pi^2}(F^a_{ij}\tilde{F}^{aij}-G^b_{ij}\tilde{G}^{bij}). \tag{22}$$

This is precisely the anomaly equation for this theory. Indeed, the effective number of flavors for the $SU(N+M)$ factor is $2N$, and each one carries R charge 1/2. The chiral fermions which are their superpartners have R charge $-1/2$ while the gluinos have R charge 1. Therefore, the anomaly coefficient is $M/16\pi^2$. An equivalent calculation for the $SU(N)$ gauge group with $2(N+M)$ flavors produces the opposite anomaly, in agreement with the holographic result (21).

The upshot of the calculation presented above is that the chiral anomaly of the $SU(N+M)\times SU(N)$ gauge theory is encoded in the ultraviolet (large r) behavior of the dual classical supergravity solution. No additional fractional D-instanton effects are needed to explain the anomaly. Thus, as often occurs in the gauge/gravity duality, a quantum effect on the gauge theory side turns into a classical effect in supergravity.

III. THE ANOMALY AS SPONTANEOUS SYMMETRY BREAKING IN AdS_5

Let us look for a deeper understanding of the anomaly from the dual gravity point of view. On the gauge theory side, the R symmetry is global, but in the gravity dual it as usual becomes a gauge symmetry, which must not be anomalous, or the theory would not make sense at all. Rather, we will find that the gauge symmetry is spontaneously broken: the 5D vector field dual to the R current of the gauge theory "eats" the scalar dual to the difference of the theta angles and acquires a mass.[1] A closely related mechanism was observed in studies of RG flows from the dual gravity point of view [15,16]. There R-current conservation was violated not through anomalies but by turning on relevant perturbations or expectation values for fields. In these cases it was shown [15,16] that the 5D vector field dual to the R current acquires a mass through the Higgs mechanism. We will show that symmetry breaking through anomalies can also have the bulk Higgs mechanism as its dual.

In the absence of fractional branes there are no background three-form fluxes, so the $U(1)$ R symmetry is a true symmetry of the field theory. Because the R symmetry is realized geometrically by invariance under a rigid shift of the angle β, it becomes a local symmetry in the full gravity theory, and the associated gauge fields $A=A_\mu dx^\mu$ appear as fluctuations of the ten-dimensional metric and RR four-form potential [17,18]. The natural metric ansatz is of the familiar Kaluza-Klein form:

$$ds^2=h(r)^{-1/2}(dx_n dx^n)+h(r)^{1/2}r^2\left[\frac{dr^2}{r^2}+\frac{1}{9}(g^5-2A)^2+\frac{1}{6}\sum_{r=1}^{4}(g^r)^2\right], \tag{23}$$

where $h(r)=L^4/r^4$, and $L^4=\frac{27}{16}(4\pi\alpha'^2 g_s N)$. It is convenient to define the one-form $\chi=g^5-2A$, which is invariant under the combined gauge transformations

$$\beta\to\beta+\lambda,\quad A\to A+d\lambda. \tag{24}$$

The equations of motion for the field A_μ appear as the $\chi\mu$ components of Einstein's equations,

$$R_{MN}=\frac{g_s^2}{4\cdot 4!}\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS}. \tag{25}$$

The five-form flux will also fluctuate when we activate the Kaluza-Klein gauge field; indeed, the unperturbed $\tilde{F}_5$ of Eqs. (12) is not self-dual with respect to the gauged metric (23). An appropriate ansatz to linear order in A is

$$\tilde{F}_5=dC_4=\frac{1}{g_s}d^4x\wedge dh^{-1}+\frac{\pi\alpha'^2N}{4}\left[\chi\wedge g^1\wedge g^2\wedge g^3\wedge g^4-dA\wedge g^5\wedge dg^5+\frac{3}{L}\star_5 dA\wedge dg^5\right]. \tag{26}$$

The five-dimensional Hodge dual $\star_5$ is defined with respect to the AdS_5 metric $ds_5{}^2=h^{-1/2}dx_n dx^n+h^{1/2}dr^2$. It is straightforward to show that the supergravity field equation $d\tilde{F}_5=0$ implies that the field A satisfies the equation of motion for a massless vector field in AdS_5 space:

$$d\star_5 dA=0. \tag{27}$$

[1] The connection between anomalies in a D-brane field theory and spontaneous symmetry breaking in string theory was previously noted in [14] (and probably elsewhere in the literature).

Using the identity $dg^5\wedge dg^5=-2g^1\wedge g^2\wedge g^3\wedge g^4$, we can check that the expression for C_4 is[2]

$$C_4=\frac{1}{g_s}h^{-1}d^4x+\frac{\pi\alpha'^2N}{2}\bigg[\beta g^1\wedge g^2\wedge g^3\wedge g^4 -\frac{1}{2}A\wedge dg^5\wedge g^5-\frac{3}{2r}h^{-1/4}\star_5 dA\wedge g^5\bigg]. \qquad (28)$$

Another way to see that A is a massless vector in AdS_5 is to consider the Ricci scalar for the metric (23)

$$R=R(A=0)-\frac{h^{1/2}r^2}{9}F_{\mu\nu}F^{\mu\nu} \qquad (29)$$

so that on reduction from ten dimensions the five-dimensional supergravity action will contain the action for a massless vector field.

The story changes when we add wrapped D5-branes. As described in Sec. II, the 5-branes introduce M units of RR flux through the three-cycle of $T^{1,1}$. In what follows, we work in the limit $(g_sM)^2\ll g_sN$, so that it is consistent to expand the metric and five-form to quadratic order in g_sM and expand the three-forms and one-forms to linear order in g_sM. At this order we may consistently take the dilaton and axion to vanish. Now, with the addition of 5-branes, the new wrinkle is that the RR three-form flux of Eqs. (8) is not gauge-invariant with respect to shifts of β (24). To restore the gauge invariance, we introduce a new field $\theta\sim\int_{S^2}C_2$:

$$F_3=dC_2=\frac{M\alpha'}{2}(g^5+2\partial_\mu\theta dx^\mu)\wedge\omega_2 \qquad (30)$$

so that F_3 is invariant under the gauge transformation $\beta\to\beta+\lambda,\theta\to\theta-\lambda$. Let us also define $W_\mu=A_\mu+\partial_\mu\theta$. In terms of the gauge invariant forms χ and $W=W_\mu dx^\mu$,

$$F_3=\frac{M\alpha'}{2}(\chi+2W)\wedge\omega_2. \qquad (31)$$

From Eq. (31) we can immediately see how the anomaly will appear in the gravity dual. Assuming that the Neveu-Schwarz–Neveu-Schwarz (NS-NS) three form is still given by Eq. (9), we find that up to terms of order g_sM^2/N the three-form equation implies

$$d\star_5W=0\Rightarrow\frac{L^2}{r^2}\partial_iW^i+\frac{1}{r^5}\partial_r r^5W_r=0 \qquad (32)$$

which is just what one would expect for a massive vector field in five dimensions. To a four dimensional observer, however, a massive vector field would satisfy $\partial_iW^i=0$. Thus in the field theory one cannot interpret the $U(1)$ symmetry breaking as being spontaneous, and the additional W_r term in Eq. (32) appears in four dimensions to be an anomaly.

[2]This expression was independently derived by D. Berenstein.

Another way to see that the vector field becomes massive is to compute its equation of motion. To do this calculation precisely, we should derive the $\chi\mu$ components of Einstein's equations, and also find the appropriate expressions for the five-form and metric up to quadratic order in g_sM and linear order in fluctuations. This approach is somewhat nontrivial and we present it in an Appendix for the interested reader. A more heuristic yet enlightening approach is to consider the type IIB supergravity action to quadratic order in W, ignoring the 5-form field strength contributions:

$$S=-\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_{10}}\bigg[R_{10}-\frac{g_s{}^2}{12}|F_3|^2\bigg]+\cdots \qquad (33)$$

$$\sim-\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_{10}}\bigg[-\frac{h^{1/2}r^2}{9}F_{\mu\nu}F^{\mu\nu} -\left(\frac{g_sM\alpha'}{2}\right)^2\frac{36}{hr^4}W_\mu W^\mu\bigg]+\cdots. \qquad (34)$$

This is clearly the action for a massive four-dimensional vector field, which has as its equation of motion

$$\partial_\mu(hr^7F^{\mu\nu})=\tilde{m}^2hr^7W^\nu \qquad (35)$$

which in differential form notation is $d(h^{7/4}r^7\star_5 dW)=-\tilde{m}^2h^{7/4}r^7\star_5W$. From the action (34), we see that the mass-squared is given by

$$\tilde{m}^2=(g_sM\alpha')^2\frac{81}{2h^{3/2}r^6}. \qquad (36)$$

In the limit $(g_sM)^2\ll g_sN$ which we are using, this becomes

$$\tilde{m}^2=\frac{4}{\alpha'3^{1/2}\pi^{3/2}}\frac{(g_sM)^2}{(g_sN)^{3/2}}. \qquad (37)$$

This result, however, ignores the subtlety of the type IIB action in presence of the self-dual 5-form field. As it turns out, the "honest" calculation including the 5-form field, presented in the Appendix, changes Eq. (37) by a factor of 1/3. Thus in the limit $(g_sM)^2\ll g_sN$, the mass-squared reduces (with the correct factor) to

$$m^2=\frac{1}{3}\tilde{m}^2=\frac{4}{\alpha'(3\pi)^{3/2}}\frac{(g_sM)^2}{(g_sN)^{3/2}}. \qquad (38)$$

The nonvanishing vector mass is consistent with gauge invariance because the massless vector field A has eaten the scalar field θ, spontaneously breaking the gauge symmetry, as advertised. It is interesting that the anomaly appears as a bulk effect in AdS space, in contrast to previously studied examples [3,19] where anomalies arose from boundary terms.

It is instructive to compare Eq. (38) with the "universal" mass-squared of the vector field found in the 5-dimensional gauged supergravity description. In [15,16] it was shown that the 5D vector field associated with a $U(1)_R$ symmetry acquires a mass in the presence of a symmetry-breaking relevant perturbation, and that this mass is related in a simple way to the warp factor of the geometry.[3] It is conventional to write the 5D gauged supergravity metric in the form

$$\tilde{G}_{\mu\nu}dx^\mu dx^\nu = e^{2T(q)}\eta_{ij}dx^i dx^j + dq^2. \tag{39}$$

The result of [15] is that $m^2 = -2T''$. To relate the 5D metric (39) to the 10D metric (1) we must normalize the 5D metric so that the graviton has a canonical kinetic term. Doing this carefully we find

$$\tilde{G}_{\mu\nu}dx^\mu dx^\nu = (hr^4/L^4)^{5/6}(h^{-1/2}\eta_{ij}dx^i dx^j + h^{1/2}dr^2). \tag{40}$$

The factor $(hr^4/L^4)^{5/6}$ arises due to the radial dependence of the size of $T^{1,1}$ through the usual Kaluza-Klein reduction. The radial variables q and r are related, at leading order in g_sM^2/N, by

$$\log(r) \sim \frac{q}{L} - \frac{g_sM^2}{2\pi N}\left(\frac{q}{L}\right)^2. \tag{41}$$

We can also show that $-2T = -2\log(r) +$(terms which do not affect the mass to leading order in g_sM^2/N), so now computing the mass-squared by the prescription of [15] we indeed reproduce Eq. (38). We consider this an interesting check on the laborious calculation presented in the Appendix which also makes contact between our 10D methods and the 5D gauged supergravity results of [15,16].

The appearance of a mass implies that the R-current operator should acquire an anomalous dimension. From Eq. (38) it follows that

$$(mL)^2 = \frac{2(g_sM)^2}{\pi(g_sN)}. \tag{42}$$

Using the AdS conformal field theory (CFT) correspondence we find that the dimension of the current J^μ dual to the vector field W^μ is

$$\Delta = 2 + \sqrt{1+(mL)^2}. \tag{43}$$

Therefore, the anomalous dimension of the current is

$$\Delta - 3 \approx (mL)^2/2 = \frac{(g_sM)^2}{\pi(g_sN)}. \tag{44}$$

We can obtain a rough understanding of this result by considering the relevant weak coupling calculation in the gauge theory. The leading correction to the current-current two-point function comes from the three-loop Feynman diagram composed of two triangle diagrams glued together, and the resulting anomalous dimension γ_J is quadratic in M and N. γ_J must vanish when $M=0$, and it must be invariant under the map $M\to -M, N\to N+M$, which simply interchanges the two gauge groups. Thus, the lowest order piece of the anomalous dimension will be of order $(g_sM)^2$. Our supergravity calculation predicts that this anomalous dimension is corrected at large g_sN by an extra factor of $1/(g_sN)$. Of course, it would be interesting to understand this result better from the gauge theory point of view.

IV. THE $\mathcal{N}=2$ SUPERSYMMETRIC $\mathbf{Z}_2$ ORBIFOLD

Encouraged by the agreement of field theory and supergravity on the conifold, let us examine another example to see how the same physical ideas apply in a different system. In this section we will study the $\mathcal{N}=2$ version of the conifold theory; it has gauge group $SU(N+M)\times SU(N)$ and is dual to a supergravity solution on an orbifold $S^5/\mathbf{Z}_2$ [5,6,20,21]. (After the completion of this work, we learned of a very similar analysis of this orbifold system which appeared earlier in [22].) The supergravity solution may be constructed as follows. We start with the space $\mathbf{R}^{1,5}\times\mathbf{R}^4/\mathbf{Z}_2$ where the orbifold is given by the identification $x_{6,7,8,9}\sim -x_{6,7,8,9}$. Then we add N coincident D3-branes, which we choose to be tangent to the 0123 directions; the resulting space has the geometry $\mathrm{AdS}_5\times S^5/\mathbf{Z}_2$. To add fractional branes, we may take M D5-branes and wrap them on the vanishing two-cycle of the orbifold $\mathbf{R}^4/\mathbf{Z}_2$. These fractional branes are "pinned" to the orbifold fixed plane.

It is possible to identify the corresponding gauge theory by standard orbifold techniques [23]. The field content is in fact almost identical to that of the conifold theory, but there is an additional pair of adjoint chiral multiplets corresponding to the motion of D-branes along the orbifold fixed plane. These extra multiplets combine with the vector multiplet in the $\mathcal{N}=1$ theory to form an $\mathcal{N}=2$ vector multiplet. It is convenient to define $(x_4+ix_5)/(2\pi\alpha') \equiv \Phi = |\Phi|e^{i\beta}$. Rotations of the phase of Φ are dual to the $U(1)$ R-symmetry in the gauge theory.

To compute the anomaly for the $SU(N+M)$ gauge factor, notice that there are now $2N+(N+M)=3N+M$ effective flavors, which have R-charge $-1/3$. Combining this with the contribution from the gluinos, we find that the anomaly coefficient is $2M/3/16\pi^2$. We would like to compare this with a computation from supergravity. Equations (16),(17) are also satisfied for the orbifold. To identify properly the relation between β and the R symmetry, note that the field Φ has R charge 2/3; thus a shift of $\beta\to\beta+\epsilon$ actually shifts the $U(1)_R$ by $\frac{3}{2}\epsilon$. This will change the first term in Eq. (21) by a factor of $\frac{3}{2}$ and give an anomaly coefficient $2M/3/16\pi^2$ in agreement with the gauge theory expectation.

A very interesting generalization of this theory was studied by Graña and Polchinski [24], and also by Bertolini *et al.* [22], who added D7-branes wrapped on the 01236789 directions; an analogous solution with D7-branes on the conifold is not currently known. The extra D7-branes allow excitations of 3-7 strings and, depending on how the 7-branes are wrapped, will add N_{7+} flavors coupled to the $SU(N+M)$ gauge group and N_{7-} flavors coupled to the $SU(N)$ gauge group. The total number of 7-branes is $N_{7+}+N_{7-}$. For a small rotation $\beta\to\beta+\epsilon$, the corresponding Θ terms are

[3] We are grateful to O. DeWolfe and K. Skenderis for pointing out the relevance of this work to the present calculation.

$$\Theta_1 = 2\epsilon(M + \tfrac{1}{2}N_{7+}) \tag{45}$$

$$\Theta_2 = 2\epsilon(-M + \tfrac{1}{2}N_{7-}). \tag{46}$$

and the associated anomaly coefficient is $1/16\pi^2(2M/3 + N_{7+}/6 - N_{7-}/6)$.

We can reproduce the same result for the anomaly by a supergravity calculation, using the results of [20] and [24,22]. It is helpful to think about the D3-branes on the orbifold fixed plane as a combination of a wrapped D5-brane and anti-D5-brane, each of which carries half a unit of D3-brane charge. By considering the Chern-Simons term in the action for a probe 5-brane

$$\pm\mu_5\int(2\pi\alpha')\mathcal{F}_2^{\pm}\wedge C_4 = \pm\frac{\mu_3}{2\pi}\int\mathcal{F}_2^{\pm}\wedge C_4 = +\frac{1}{2}\mu_3\int C_4, \tag{47}$$

we see that the field strength on the 5-brane worldvolume will satisfy

$$\mathcal{F}_2^{\pm} = \pm\frac{1}{4}\omega_2 \tag{48}$$

where the upper sign refers to a D5-brane and the lower sign to an anti-D5-brane. Now let us add the D7-branes. The supergravity solution has RR scalar and two-form potentials given by [22,24]

$$C_0 = \frac{\beta}{2\pi}(N_{7+} + N_{7-}) \tag{49}$$

$$C_2 = \alpha'\beta\omega_2\left(M + \frac{N_{7+} - N_{7-}}{4}\right). \tag{50}$$

To find the Θ terms for the dual gauge theory, we just need to look at the Chern-Simons terms in the actions for a probe D5-brane and anti-D5-brane. For a D5-brane [whose excitations are in the $SU(N+M)$ gauge group] we find that

$$\frac{1}{2\pi\alpha'}\int(C_2 + 2\pi\alpha' C_0\mathcal{F}_2^{+}) = 2\beta(M + \tfrac{1}{2}N_{7+}). \tag{51}$$

Comparison with Eqs. (18) and (19) shows that gravity reproduces the field theory expectation for Θ_1 given in Eq. (45). The analogous computation for an anti-D5-brane will reproduce Eq. (46). Thus the anomaly as computed from supergravity agrees exactly with the field theory calculation.

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APPENDIX: THE VECTOR FIELD EQUATION OF MOTION

In this appendix we derive the equation of motion of the vector field W by considering the RR five-form and metric fluctuations at second order in g_sM, and their corresponding supergravity equations of motion; the approach given in Sec. III is not strictly correct, because there is no good way to incorporate the self-duality constraint of the five-form in the action. From the equations of motion, we find that the vector field W is massive, confirming the heuristic arguments in Sec. III.

Before proceeding with the computation, it is worthwhile to record some useful identities. A convenient definition is

$$d^3x^i = \frac{1}{6}\eta^{ij}\varepsilon_{jklm}dx^k\wedge dx^l\wedge dx^m, \tag{A1}$$

and some useful relations are

$$\star(dr\wedge g^1\wedge g^2\wedge g^3\wedge g^4) = -\frac{12}{h^2r^3}d^4x\wedge g^5 \tag{A2}$$

$$\star(dx^i\wedge g^1\wedge g^2\wedge g^3\wedge g^4) = -\frac{12}{hr^3}d^3x^i\wedge dr\wedge g^5 \tag{A3}$$

$$\star(dx^i\wedge dr\wedge g^5\wedge dg^5) = \frac{3}{hr}d^3x^i\wedge dg^5 \tag{A4}$$

$$(\star_5 dA)\wedge dr = (\partial_r A_i - \partial_i A_r)d^3x^i dr h^{-3/4} \tag{A5}$$

$$d(v_i d^3x^i)\wedge dr = \partial_i v_i d^4x\wedge dr \tag{A6}$$

$$\frac{1}{4}d(f_{ij}\varepsilon_{ijkl}dx^k\wedge dx^l)\wedge dr = -\partial_i f_{ij}d^3x^j\wedge dr. \tag{A7}$$

We also define

$$b \equiv \frac{3}{2\pi}\frac{g_sM^2}{N}. \tag{A8}$$

In our perturbative expansion, the fields other than the five-form and metric are given by Eqs. (9),(31), and $C=\Phi=0$. The equation of motion for W appears in two places: the equations of motion for the five-form, $d\tilde{F}_5 = H_3\wedge F_3$ and $\tilde{F}_5 = \star\tilde{F}_5$, and the $\mu\chi$ components of Einstein's equations,

$$R_{\mu\chi} = \frac{g_s^2}{4\cdot 4!}\tilde{F}_{\mu PQRS}\tilde{F}_{\chi}^{\ PQRS} + \frac{g_s^2}{4}\tilde{F}_{\mu PQ}\tilde{F}_{\chi}^{\ PQ}. \tag{A9}$$

It turns out that the correct four-form potential is

$$C_4=\frac{1}{g_s}h^{-1}d^4x+\frac{\pi\alpha'^2N}{2}(\beta+\theta)g^1\wedge g^2\wedge g^3\wedge g^4-\frac{\pi\alpha'^2}{4}(N_{eff}+NbY)\left(W\wedge g^5\wedge dg^5+\frac{3}{h^{1/4}r}\star_5 dW\wedge g^5\right)$$
$$-3\frac{\pi\alpha'^2}{4}\frac{Nb}{L^4}r^2W_i d^3x^i\wedge g^5-\frac{\pi\alpha'^2}{16}NbrW_r g^1\wedge g^2\wedge g^3\wedge g^4. \quad (A10)$$

The logic that leads to this potential is that eventually we will want an equation of motion for W roughly of the form (35). Requiring self-duality of $\tilde{F}_5$ will give such an equation with a mass term if there are terms of the form $W\wedge g^1\wedge g^2\wedge g^3\wedge g^4$ and its dual in $\tilde{F}_5$, and this four-form potential (A10) allows for such terms in all natural ways. For the moment Y is a numerical free parameter. So the five-form is given by

$$\tilde{F}_5=dC_4+B_2\wedge F_3 \quad (A11)$$

$$=\frac{1}{g_s}d^4x\wedge dh^{-1}+\frac{\pi\alpha'^2}{4}N_{eff}\chi\wedge g^1\wedge g^2\wedge g^3\wedge g^4-\frac{\pi\alpha'^2}{4}(N_{eff}+NbY)\left[dW\wedge g^5\wedge dg^5-\frac{3}{h^{1/4}r}\star_5 dWdg^5\right]$$
$$+\frac{\pi\alpha'^2}{4}NbW_i\left[dx^i\wedge\frac{dr}{r}\wedge g^5\wedge dg^5+\frac{3r^2}{L^4}d^3x^i\wedge dg^5\right]-3\frac{\pi\alpha'^2}{4}Nb\partial_iW_r\left[\frac{r}{12}dx^i\wedge g^1\wedge g^2\wedge g^3\wedge g^4-\frac{r^2}{L^4}d^3x^i\wedge dr\wedge g^5\right]$$
$$-3\frac{\pi\alpha'^2}{4}\frac{Nb}{L^4}r^2\partial_iW_i d^4x\wedge g^5-\frac{\pi\alpha'^2}{16}Nb\frac{1}{r^4}\partial_r(r^5W_r)dr\wedge g^1\wedge g^2\wedge g^3\wedge g^4$$
$$-\frac{\pi\alpha'^2}{4}(N_{eff}+NbY)\frac{3}{h^2r^8}d(h^{7/4}r^7\star_5 dW)\wedge g^5+\frac{3\pi\alpha'^2}{2}\frac{Nb}{L^4}rW_i d^3x^i\wedge dr\wedge g^5+\frac{\pi\alpha'^2}{4}NbW_r dr\wedge g^1\wedge g^2\wedge g^3\wedge g^4$$
$$-\frac{\pi\alpha'^2}{2}NbYW\wedge g^1\wedge g^2\wedge g^3\wedge g^4. \quad (A12)$$

The expression for the five-form (A12) is not self-dual with respect to the metric (23), because of the last four terms in (A12). We may remedy the situation by correcting the metric at order b:

$$ds^2=h(r)^{-1/2}(dx_n dx^n)+h(r)^{1/2}r^2\left[\frac{dr^2}{r^2}+\frac{1}{9}\left(g^5-2A+\frac{3b}{2}W_r dr\right)^2+\frac{1}{6}\sum_{r=1}^{4}(g^r)^2\right]. \quad (A13)$$

Then by shifting χ appropriately to make the first line of $\tilde{F}_5$ self-dual with respect to the new metric, there will be an additional contribution of the form $-(3\pi\alpha'^2/8)NbW_r dr\wedge g^1\wedge g^2\wedge g^3\wedge g^4$. The new terms in the metric (A13) do not affect the two-form or scalar equations at our order in perturbation theory. Now, the self-duality constraint for these remaining terms gives an equation of the right form for W, with mass-squared

$$m^2=\frac{4b}{L^2}(\tfrac{1}{2}+2Y). \quad (A14)$$

On the other hand, when we compute Einstein's equations (A9) we find that

$$\partial_iF^{ir}=-\frac{2b}{L^2}W^r(-\tfrac{1}{2}+2Y) \quad (A15)$$

with corresponding mass-squared

$$m^2=\frac{2b}{L^2}(\tfrac{1}{2}-2Y) \quad (A16)$$

so that $Y=-1/12$. Finally, we obtain the true mass of the W field,

$$m^2=\frac{4b}{3L^2} \quad (A17)$$

which is of the same form as Eq. (36) but differs by a factor of 1/3.

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] I.R. Klebanov and E. Witten, Nucl. Phys. **B536**, 199 (1998).
[5] S.S. Gubser and I.R. Klebanov, Phys. Rev. D **58**, 125025 (1998).
[6] I.R. Klebanov and N. Nekrasov, Nucl. Phys. **B574**, 263 (2000).
[7] I.R. Klebanov and A. Tseytlin, Nucl. Phys. **B574**, 123 (2000).
[8] I.R. Klebanov and M. Strassler, J. High Energy Phys. **08**, 052 (2000).
[9] J. Maldacena and C. Nunez, Phys. Rev. Lett. **86**, 588 (2001).
[10] A. Loewy and J. Sonnenschein, J. High Energy Phys. **08**, 007 (2001).
[11] Jaume Gomis, Nucl. Phys. **B624**, 181 (2002).
[12] R. Minasian and D. Tsimpis, Nucl. Phys. **B572**, 499 (2000).
[13] C.P. Herzog, I.R. Klebanov, and P. Ouyang, ''Remarks on the Warped Deformed Conifold,'' hep-th/0108101.
[14] O. Aharony, S. Kachru, and E. Silverstein, Nucl. Phys. **B488**, 159 (1997).
[15] M. Bianchi, O. DeWolfe, D.Z. Freedman, and K. Pilch, J. High Energy Phys. **01**, 021 (2001).
[16] M. Bianchi, D.Z. Freedman, and K. Skenderis, ''Holographic Renormalization,'' hep-th/0112119.
[17] H.J. Kim, L.J. Romans, and P. van Nieuwenhuizen, Phys. Rev. D **32**, 389 (1985).
[18] A. Ceresole, G. Dall'Agata, R. D'Auria, and S. Ferrara, Phys. Rev. D **61**, 066001 (2000); A. Ceresole, G. Dall'Agata, and R. D'Auria, J. High Energy Phys. **11**, 009 (1999).
[19] M. Henningson and K. Skenderis, J. High Energy Phys. **07**, 023 (1998).
[20] J. Polchinski, Int. J. Mod. Phys. A **16**, 707 (2001).
[21] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta, and I. Pesando, J. High Energy Phys. **02**, 014 (2001).
[22] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, and R. Marotta, Nucl. Phys. **B621**, 157 (2002).
[23] M.R. Douglas and G.W. Moore, ''D-branes, Quivers, and ALE Instantons,'' hep-th/9603167.
[24] M. Graña and J. Polchinski, Phys. Rev. D (to be published), hep-th/0106014.