Nonperturbative quenched propagator beyond the infrared approximation

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A new approach to the quenched propagator in QED beyond the IR limit is proposed. The method is based on evolution equations in the proper time.

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I. INTRODUCTION

We propose a new nonperturbative analysis of quenched QED beyond the infrared Bloch-Nordsieck approximation using the worldline formalism. The main idea is to formulate evolution equations in the proper time, inspired by the rather intuitive resulting classical picture. As a warmup we obtain in a simple way the renormalized quenched propagator in the infrared (IR) limit. From there we proceed to improve on this IR approximation, which we achieve in the form of integro-differential equations, our main results. They are of course difficult to solve, but they have an appealing physical interpretation and suggest a simplified numerical analysis.

The paper is organized as follows. In Sec. II we study the infrared behavior of lepton propagators for several cases. First we consider the Bloch-Nordsieck approximation as given by uniform proper time paths, which exhibit the classical image in the worldline. This suggests the use of the proper time evolution equations as a short cut to go beyond the infrared approximation. The idea is carried out for scalar QED and then for the full Dirac algebra. The lesson of this section is a better understanding of renormalization and the fact that the scalar case contains the basic physics, which will be used in Sec. III, where we develop further the proper time formalism to go beyond the infrared limit. The results are summarized in Sec. IV which is devoted to the conclusions.

II. INFRARED BEHAVIOR: THE CLASSICAL IMAGE OF THE WORLDLINE FORMALISM

In this section we will study the classical Bloch-Nordsieck approximation and their extensions for scalar and fermionic QED. Our main point is given in Eq. (12), where an evolution equation is introduced for the quenched propagator. Its solution is the well-known Bloch-Nordsieck (BN) result. The extensions to the scalar and full Dirac algebra are carried out without difficulty.

A. Bloch-Nordsieck approximation

The Bloch-Nordsieck model has been repeatedly used to study the infrared behavior of QED. Here we will consider it from the path integral point of view as a convenient starting point. The model has been extensively studied in the literature [1–5].

In the Bloch-Nordsieck model one substitutes the Dirac gamma matrices by a constant vector $\gamma_\mu \to u_\mu$. This constant vector will be identified to the (constant) velocity of the electron, see below. With this substitution the model is completely integrable and the electron Green function can be cast in terms of elementary functions. The electron propagator in the Bloch-Nordsieck approximation is given by

$$\left[u_\mu\left(i\frac{\partial}{\partial x_\mu}+eA_\mu(x)\right)-m\right]G(x,y)=-\delta(x-y), \qquad (1)$$

where $A_\mu(x)$ is the electromagnetic potential. We will solve this model by introducing a proper time (Schwinger's method [6]) [7,8,5,9]. Define

$$G(x,y)=i\int_0^\infty d\tau G(\tau,x,y), \qquad (2)$$

then $G(\tau,x,y)$ verifies the equation

$$-i\frac{\partial G}{\partial\tau}=\left[u_\mu\left(i\frac{\partial}{\partial x_\mu}+eA_\mu(x)\right)-m\right]G(\tau,x,y)$$
$$=\mathcal{H}G(\tau,x,y), \qquad (3)$$

with initial conditions

$$G(\tau=0,x,y)=\delta(x-y). \qquad (4)$$

The formal solution is given by

$$G(\tau,x,y)=\exp\{-i\tau\mathcal{H}\}\quad\delta(x-y). \qquad (5)$$

Here $\mathcal{H}$ is clearly seen as a Hamiltonian which gives the evolution in the proper time of the Schrödingerlike equation. This formal expression can be cast into a path integral in the proper time. We can write

$$G(\tau,x,y)=N\int Dx(\tau)Dp(\tau)\times\exp\left\{i\int_0^\tau d\tau[p\dot{x}-\mathcal{H}(x,p)]\right\}$$
$$=N\int Dx(\tau)Dp(\tau)\times\exp\left\{i\int_0^\tau d\tau[p\dot{x}-up-euA(x)+m]\right\}. \qquad (6)$$

The electromagnetic field is coupled to the current $j_\mu(z) = u_\mu \int_0^\tau d\tau \delta(z-x(\tau))$ and can be integrated out, since the resulting integral is Gaussian,

$$\langle G(\tau,x,y)\rangle = \int DA_\mu G(\tau,x,y) \times \exp\left\{-i\int dx \frac{1}{4}F^2 + \frac{\lambda}{2}(\partial A)^2\right\}, \qquad (7)$$

where λ is a gauge-fixing parameter and the brackets $\langle\rangle$ mean the average over the gauge fields.

The integration of the A fields gives a nonlocal Lagrangian for $\langle G\rangle$:

$$\langle G(\tau,x,y)\rangle = N\int DxDp \exp\left\{i\int_0^\tau d\tau(p\dot{x} - pu + m)\right\} \times \exp\left\{-\frac{e^2}{2}\int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \int \frac{dk}{(2\pi)^D} u_\mu u_\nu \times D_{\mu\nu}(k) e^{-ik[x(\tau_1)-x(\tau_2)]}\right\}, \qquad (8)$$

where $D_{\mu\nu}(k)$ is the photon propagator in the gauge λ.

The integral over p gives a delta function, $\delta(\dot{x}_\mu(\tau) - u_\mu))$, which implies that the particle moves with constant velocity, u_μ, as stated previously. We see that the electron in the Bloch-Nordsieck approximation behaves as a classical particle, the only path that contributes to the propagator is the classical path and quantum fluctuations are *exactly* canceled. This fact is most easily seen in the above formalism. The triviality of the quantum corrections is due to the appearance of the δ function, which in turn is related to the triviality of the classical Hamiltonian. For the BN model, the classical path and the uniform velocity path coincide, independently of the external potential $A_\mu(x)$. Alternatively one may notice that Eq. (3) is first order on the variable x. As is well known from the theory of partial differential equations, the solution admits a particle interpretation. This is not true for the cases considered below. The uniform path is only an approximation valid when the momentum interchanged with the gauge fields is sufficiently small such that the path can be considered uniform, i.e., in the infrared domain.

Therefore the nonlocal (NL) term must be evaluated in the classical path alone. It gives

$$N_{\rm NL} = \frac{3-\lambda}{2}\int_0^\tau d\tau_1 d\tau_2 \frac{1}{(\tau_1-\tau_2)^2}. \qquad (9)$$

After renormalization we find

$$\langle G(\tau,p)\rangle = i\tau^a \exp\{-i\tau(m-up)\}, \qquad (10)$$

where $a=(3-\lambda)/(2\pi)\alpha$. Integrating over τ, we obtain finally

$$\langle G(p)\rangle = i\int_0^\infty d\tau \langle G(\tau,p)\rangle = \Gamma(1+a)\frac{i}{(m-up)^{1+a}}. \qquad (11)$$

This is the well-known Bloch-Nordsieck result. In the Yennie-Suura gauge, $\lambda=3$, the electron propagator reduces to the free case. Notice that $\langle G(p,\tau)\rangle$ verifies the homogeneous evolution equation

$$-i\frac{\partial\langle G(p,\tau)\rangle}{\partial\tau} = (up-m)\langle G(p,\tau)\rangle + \frac{a}{\tau}\langle G(p,\tau)\rangle, \qquad (12)$$

which has, of course, Eq. (10) as a solution. This fact will be used later on to go beyond the infrared approximation.

B. Scalar QED

Before considering the spinor electrodynamics, let us study the somewhat simpler case of scalar QED. We will study in detail the renormalization for this case.

The scalar propagator, coupled to the electromagnetic current, is given by

$$\{[\partial_\mu - eA_\mu(x)]^2 - m^2\}G(x,y) = \delta(x-y), \qquad (13)$$

as before we introduce the Schwinger proper time

$$i\frac{\partial G(\tau,x,y)}{\partial\tau} = \{[\partial_\mu - eA_\mu(x)]^2 - m^2\}G(x,y). \qquad (14)$$

$G(\tau,x,y)$ has the solution

$$G(\tau,x,y) = \exp\{-i\mathcal{H}\tau\}\delta(x-y) \qquad (15)$$

which can be expressed as a path integral

$$G(\tau,x,y) = N\int Dx(\tau)Dp(\tau) \times \exp\left\{i\int_0^\tau d\tau[p\dot{x} + \mathcal{H}(x,p)]\right\} = N\int Dx(\tau)Dp(\tau) \times \exp\left\{i\int_0^\tau d\tau\{p\dot{x} + [p+eA(x)]^2 - m^2\}\right\}. \qquad (16)$$

Now p appears quadratically and the integral over it is no longer a δ function. As expected, the classical path is not the only contribution to the path integral and quantum corrections are relevant. However, for the moment, we will consider the contribution given by the uniform path alone, as the main contribution in the infrared limit. We first shift the momentum integral $\Pi = p + eA(x)$,

$$G(\tau,x,y)=N\int Dx(\tau)D\Pi(\tau)\times\exp\left\{i\int_0^\tau d\tau[\Pi\dot{x}+\Pi^2-m^2-e\dot{x}A_\mu(x)]\right\}, \tag{17}$$

and as before, the electromagnetic field is coupled to the current $j_\mu(z)=\int d\tau\dot{x}_\mu(\tau)\delta(x(\tau)-z)$. The average over gauge fields gives, therefore, the same nonlocal term:

$$N_{\rm NL}=\int d\tau d\tau'\int\frac{dk}{(2\pi)^D}\dot{x}_\mu(\tau)\dot{x}_\nu(\tau')\frac{D_{\mu\nu}(k)}{k^2}\times e^{ik[x(\tau)-x(\tau')]}. \tag{18}$$

Now, we make the uniform path approximation, $\dot{x}_\mu(\tau)=u_\mu=\text{const}$, which gives

$$N_{\rm NL}=\int d\tau d\tau'\int\frac{dk}{(2\pi)^D}u_\mu u_\nu\frac{D_{\mu\nu}(k)}{k^2}e^{iku(\tau-\tau')}. \tag{19}$$

Integrating over τ and τ' gives

$$N_{\rm NL}=-\int\frac{dk}{(2\pi)^D}u_\mu u_\nu\frac{D_{\mu\nu}(k)}{k^2}\frac{1-\cos(ku\tau)}{(ku)^2}. \tag{20}$$

The integral is IR finite, at low momentum the divergent denominator cancels out with the $1-\cos(ku\tau)$ in the numerator. We will renormalize the ultraviolet divergent k integral using dimensional regularization; expanding around four dimensions, $D=4-\epsilon$, we obtain [5]

$$N_{\rm NL}=-\frac{A}{\epsilon}(\mu^2\tau)^\epsilon, \tag{21}$$

where

$$A=\frac{1}{8\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right)\frac{[1-(-1)^{2-D}]}{3-D}\left(1-\frac{\lambda}{2}(3-D)\right)=A_0+\epsilon A_1+\cdots, \tag{22}$$

and $A_0=-1/(4\pi^2)$. The scalar propagator can be written

$$\langle G(x,y)\rangle=\int\frac{dp}{(2\pi)^D}e^{ip(x-y)}\int_0^\infty d\tau\times\exp-i\tau(m^2-p^2)\exp-e^2\frac{A}{\epsilon}(\mu^2\tau)^\epsilon. \tag{23}$$

Expanding the exponential in powers of e we obtain

$$\langle G(p,\tau)\rangle=\int_0^\infty d\tau\exp-i\tau(m^2-p^2)\times\sum_{n=0}^\infty\left(\frac{-e^2A}{\epsilon}\right)^n(\mu^2\tau)^{n\epsilon}. \tag{24}$$

Each term in the series can be integrated and gives

$$\langle G(p,\tau)\rangle=\frac{-i}{(m^2-p^2)}\sum_{n=0}^\infty\frac{(-e^2A)^n}{n!\,\epsilon^n}\Gamma(1+n\epsilon)\times\left(\frac{2i\mu^2}{m^2-p^2}\right)^{n\epsilon} \tag{25}$$

The nth term can be written as the nth power of

$$-e^2\frac{A}{\epsilon}e^{\epsilon\log[(2i\mu^2)/(m^2-p^2)]}\Gamma(1+n\epsilon)^{1/n}. \tag{26}$$

Expanding now in powers of ϵ (we assume $n\epsilon\ll1$) we arrive at

$$-e^2\left(\frac{A_0}{\epsilon}+A_1-A_0\gamma+A_0\log[(2i\mu^2)/(m^2-p^2)]+\mathcal{O}(\epsilon)\right), \tag{27}$$

and we can choose the counterterms in such a way to cancel the term $A_0/\epsilon+A_1-A_0\gamma$. The scalar propagator after renormalization is given by

$$\langle G_R(p)\rangle=\frac{i}{m^2-p^2}e^{-a\,\log[\mu^2/(m^2-p^2)]}, \tag{28}$$

which coincides with the Bloch-Nordsieck result, as expected. Notice that care has to be taken in the order of the limit $\epsilon\to0$ and $n\to\infty$ and the integral over τ. We will see next that including the Dirac algebra is under control; this will allow us to go back to the scalar case for the purposes of the present article.

C. Full Dirac algebra

Including the full Dirac algebra is far from trivial. It has been done in Refs. [8,10,11,9].

We will follow the method of Ref. [10]. The Dirac equation is

$$[i\gamma_\mu\partial_\mu-e\gamma_\mu A_\mu(x)-m]G(x,y)=\delta(x-y). \tag{29}$$

Introducing the proper time as before [see Eq. (6) from Ref. [10]]

$$G(x,y)=\int_0^\infty dT\int d\chi\exp\left\{-\frac{i}{2}(m^2T+m\chi)\right\}\times\exp\left\{\frac{i}{2}[(\gamma\Pi)^2T+\gamma\Pi\chi]\right\}, \tag{30}$$

where $\Pi^\mu = p^\mu - eA^\mu(x)$ and χ is a Grassmann variable. Introducing a set of auxiliary Grassmann variables ξ_μ we can write a path integral for the propagator

$$G(x,y)=\exp\left\{\gamma\frac{\partial}{\partial\Gamma}\right\}\int_0^\infty dT\int d\chi\int DxDpD\xi \times\exp\left\{i\int_0^T d\tau(p\dot{x}+i\xi\dot{\xi}-\mathcal{H})-\xi(0)\xi(T)\right\}, \tag{31}$$

where $\mathcal{H}=\frac{1}{2}(-\Pi^2+2ieF_{\mu\nu}\xi^\mu\xi^\nu-(2/T)\Pi_\mu\xi_\mu\chi)$. The ξ_μ verifies the boundary conditions $\xi_\mu(0)+\xi_\mu(T)=\Gamma_\mu$. Γ_μ are auxiliary Grassmann variables which are set to zero after differentiation. Shifting the momentum as before and integrating the resulting Gaussian integral over Π we obtain

$$G(x,y)=\exp\left\{\gamma\frac{\partial}{\partial\Gamma}\right\}\int_0^\infty dT\int d\chi\mathcal{N}(T) \times\exp\left\{-\frac{i}{2}(m^2T+m\chi)\right\} \times\int DxD\xi\exp\{iS[x,\xi]\}, \tag{32}$$

where $S[x,\xi]$ is given by

$$S[x,\xi]=\int_0^T d\tau\left(-\frac{1}{2}\dot{x}^2+i\xi\dot{\xi}-ieF_{\mu\nu}\xi^\mu\xi^\nu-\frac{1}{T}\dot{x}\xi\chi\right) -i\xi(0)\xi(T). \tag{33}$$

As before the gauge fields appear linearly only and the integration over them can be done exactly,

$$\langle G(x,y)\rangle=\exp\left\{\gamma\frac{\partial}{\partial\Gamma}\right\}\int_0^\infty dT\int d\chi\mathcal{N}(T) \times\int DxD\xi\exp\{iS_{\rm eff}\}, \tag{34}$$

where the action $S_{\rm eff}$ is given by

$$S_{\rm eff}=\int_0^T d\tau\left(-\frac{1}{2}(\dot{x})^2+i\xi\dot{\xi}\right) \times\frac{-e^2}{2}\int_0^T d\tau_1 d\tau_2\int\frac{dk}{(2\pi)^D}\frac{D_{\mu\nu}(k)}{k^2} \times(\dot{x}^1_\mu+2\xi^1_\mu k\xi^1)(\dot{x}^2_\mu-2\xi^2_\mu k\xi^2)e^{-ik(x^1-x^2)}, \tag{35}$$

$x^1=x(\tau_1)$, and so on.

We will calculate the action for the uniform path, as done before.

From the classical equations of movement for the free case $(e=0)$ we get that the uniform path in our case is given by

$$x_\mu(\tau)=x_\mu(0)+p_\mu\tau+\frac{\tau}{T}\xi_\mu(0)\chi, \qquad \xi_\mu(\tau)=\xi_\mu(0)-\frac{\tau}{2iT}p_\mu\chi. \tag{36}$$

Introducing this solution in the path integral and after some algebra we arrive at

$$\langle G(x,y)\rangle=\exp\left\{\gamma\frac{\partial}{\partial\Gamma}\right\}\int\frac{dp}{(2\pi)^4}e^{ip(x-y)} \times\int_0^\infty dT\int d\chi\exp\left\{\frac{-i}{2}(m^2T+m\chi)\right\} \times\exp\left\{\frac{i}{2}p^2T+ip\xi(0)\chi -e^2\frac{A}{\epsilon}(\mu Tp)^\epsilon\left(1+\frac{p\xi(0)\chi}{p^2T}\epsilon\right)\right\}, \tag{37}$$

where A is given as before Integrating over χ and applying the $\partial/\partial\Gamma$ we arrive at

$$\langle G(x,y)\rangle=\int\frac{dp}{(2\pi)^4}e^{ip(x-y)}\int_0^\infty dT\exp\left\{\frac{-i}{2}(m^2T)\right\} \times\exp\left\{\frac{i}{2}p^2T-e^2\frac{A}{\epsilon}(\mu Tp)^\epsilon\right\} \times\left(\frac{-i}{2}(m+p\gamma)+\frac{e^2A}{2\epsilon}(\mu Tp)^\epsilon\epsilon\frac{p\gamma}{p^2T}\right). \tag{38}$$

If we set $e=0$ we obviously reproduce the free fermion propagator. For $e\neq 0$ care must be taken in the regularization procedure since the integration over T and the limit $\epsilon\rightarrow 0$ do not commute. If we boldly take the limit first and integrate we arrive at

$$\langle G(p)\rangle=Z\frac{(\mu p)^a}{(p^2-m^2)^a}\Gamma(1+a)\frac{1}{p^2-m^2} \times\left[m+p\gamma\left(1+\frac{p^2-m^2}{2p^2}\right)\right], \tag{39}$$

where Z is the renormalization constant $Z=\exp(-e^2/\epsilon A+\cdots)$.

Taking correctly the limit after the integration we arrive at

$$\langle G(p)\rangle=Z\frac{(\mu p)^a}{(p^2-m^2)^a}\Gamma(1+a)\frac{1}{p^2-m^2}(m+p\gamma), \tag{40}$$

which is again the Bloch-Nordsieck result and reproduces the IR logarithms upon expansion on powers of a.

III. BEYOND INFRARED BEHAVIOR: PROPER TIME EVOLUTION EQUATIONS

We will now write down the evolution equations for a scalar coupled to a scalar field. We will use this simpler case to demonstrate that the evolution equations allow us to go beyond the infrared domain. The obvious next step would be to construct evolution equations whose perturbative expansion give exact results up to a given order. Let us start by writing down equations valid up to first order. Further generalizations to the spinor case and higher orders will be left for future work.

As before we start with the scalar propagator as given in Eq. (15) in the proper time formalism, but now for a scalar field coupled to a scalar field $\mathcal{H}=\partial_\mu^2-m^2+g\phi(x)$. The propagator verifies the evolution equation

$$-i\frac{\partial G}{\partial\tau}=[(\partial_\mu^2-m^2)+g\phi(x)]G(\tau,x,y), \tag{41}$$

where g is the coupling constant to the scalar field $\phi(x)$. We want to average over the field ϕ with a weight given by the free action:

$$\langle\mathcal{O}\rangle=\int d\phi\mathcal{O}\exp\left\{i\int dx\mathcal{L}_\phi\right\}, \tag{42}$$

$\mathcal{L}_\phi=\phi(\partial_\mu^2-m_\phi^2)\phi$. Therefore we get

$$-i\frac{\partial\langle G\rangle}{\partial\tau}=(\partial_\mu^2-m^2)\langle G\rangle+g\langle\phi(x)G(\tau,x,y)\rangle. \tag{43}$$

In the infrared limit the average $\langle\phi(x)G(\tau,x,y)\rangle$ is simply given by $a/\tau G$, as shown above by Eq. (12). But in general it can obviously not be written as a simple operator acting on G. So one has to estimate the value of $\langle\phi(x)G(\tau,x,y)\rangle$ by introducing as before a path integral representation for G,

$$G(\tau,x,y)=N\int DxDp \times\exp\left\{-i\int_0^\tau d\tau p\dot{x}+p^2-m^2+g\phi(x)\right\}, \tag{44}$$

then

$$\langle\phi(x)G(\tau,x,y)\rangle=\int D\phi DxDp\,\phi(x) \times\exp\left\{-i\int_0^\tau d\tau p\dot{x}+p^2-m^2+g\phi(x)\right\} \times\exp\left\{i\int dx\,\phi(x)(p^2-m_\phi^2)\phi(x)\right\}, \tag{45}$$

introducing the current $j_0(z)=g\int d\tau\delta(z-x(\tau))$ the average can be written

$$\langle\phi(x)G(\tau,x,y)\rangle=\frac{\delta}{\delta j(x)}\int DxDp \times\exp\left\{-i\int_0^\tau d\tau p\dot{x}+p^2-m^2\right\} \times\int D\phi\exp\left\{i\int dx\,\phi(p^2-m_\phi^2)\phi +j\phi\right\}, \tag{46}$$

where the derivative is evaluated at $j=j_0$. The integral over ϕ is Gaussian and after performing the functional derivative we arrive at

$$\langle\phi(x)G(\tau,x,y)\rangle=\int DxDp\exp\left\{-i\int_0^\tau d\tau p\dot{x}+p^2-m^2\right\} \times\exp\left\{\frac{ig^2}{2}\int d\tau_1 d\tau_2 D(x(\tau_1),x(\tau_2))\right\}g \times\int_0^\tau d\tau_0 D(x,x(\tau_0)), \tag{47}$$

where $D(x,y)$ is the free scalar propagator of field ϕ. This expression can be simplified further if we make some approximations and constitutes the starting point to obtain evolution equations. To start with one can neglect the nonlocal term in the exponential since it is order g^2. Then it is straightforward to evaluate the functional integral since all integrals are Gaussian. After some algebra we arrive at

$$\langle\phi(x)G(\tau,x,y)\rangle=g\int_0^\tau d\tau_0\int dk_0 D(k_0)e^{ik_0x} \times\int DxDp\exp\left\{-i\int_0^\tau d\tau p^2 +p\dot{x}-m^2+jx\right\}, \tag{48}$$

where $j(\tau)=-k_0\delta(\tau-\tau_0)$. The integral over x can be done and gives now a δ function $\delta(\dot{p}-j)$, i.e., the path is still uniform but now presents a jump at $\tau=\tau_0$ where the momentum changes from p to $p-k_0$. The electron emits a hard photon of momentum k_0. Evaluating the action at this specific path gives the total contribution, in this approximation

$$\langle\phi(x)G(\tau,x,y)\rangle=g\int_0^\tau d\tau_0\int dk_0 D(k_0)e^{i(k_0-p)(x-y)} \times\exp\{-i[(p-k_0)^2(\tau-\tau_0)-m^2(\tau-\tau_0) +p^2\tau_0-m^2\tau_0]\}. \tag{49}$$

We have separated the contributions from $\tau-\tau_0$ and τ_0 to make it clear that they are both the free scalar propagators, i.e.,

$$\langle\phi(x)G(\tau,x,y)\rangle=g\int_0^\tau d\tau_0\int dk_0 D(k_0)e^{ik_0(x-y)}G_0(\tau_0,p)\times G_0(\tau-\tau_0,p-k_0). \tag{50}$$

This result is valid up to first order on g. Then the evolution equation, valid up to this order is

$$-i\frac{\partial\langle G\rangle}{\partial\tau}=(p^2-m^2)\langle G\rangle+g^2\int_0^\tau d\tau_0\int dk_0 D(k_0)G_0(\tau_0,p)\times G_0(\tau-\tau_0,p-k_0). \tag{51}$$

The solution of this equation up to order g^2 gives the exact one-loop scalar propagator, as it should. A natural extension of this result, in analogy with the Dyson-Schwinger equation, is to substitute the free propagator, G_0, by the dressed one, $\langle G\rangle$, inside the integral. We arrive, therefore, at our final evolution equation

$$-i\frac{\partial\langle G\rangle}{\partial\tau}=(p^2-m^2)\langle G\rangle+g^2\int_0^\tau d\tau_0\int dk_0 D(k_0)\langle G(\tau_0,p)\rangle\times\langle G(\tau-\tau_0,p-k_0)\rangle. \tag{52}$$

The precise nature of the higher order contributions included in our conjectured generalization will be clear only after the complete QED formulation, technically very involved, has been completed, as well as a numerical analysis, but they are qualitatively understood as we discuss in detail separately in the next section. Of course, Eq. (52) gives the exact result to first order and in the IR limit it reduces to the Bloch-Nordsieck case.

IV. CONCLUSIONS

We have arrived at an expression for the quenched propagator beyond the infrared domain. It has a clear physical interpretation as one can see by looking at the diagrams resummed by the above equation. From direct inspection, one sees that the vertex function is evaluated exactly at the one-loop level, and that there are an infinity number of such hard loops. Therefore, hard photons do not connect propagators with different hard vertex, i.e., we have some kind of rainbow expansion, where overlapping loops are not included, as shown by Fig. 1. But there are in addition an infinity number of infrared photons which join any two points in the propagator and make the infrared limit exact. Since the proper time in the diagram is related to the virtuality of the propagator and we have, in the above equation, an integral from 0 to the upper limit τ, we expect that higher order loops are calculated only up to some minimum virtuality; i.e., one expects an ordering in the virtuality of the diagrams that contribute to the above equation.

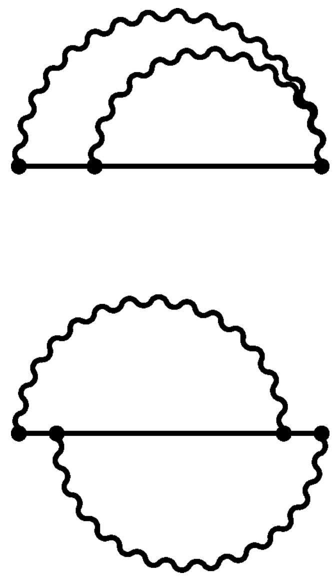

FIG. 1. Diagrams included (upper graph) and not included (lower graph) on the evolution equation

New insights in the nonperturbative regime are always welcome, even if quenched. In fact, a better understanding of quenched approximations is essential for lattice simulations. The nonpositron approximation is also useful in analytic attempts to fundamental problems, like the stability of relativistic QED, where renormalization is one of the main problems [12]. Our result in Eq. (52) can have many practical applications, such as a systematic way to calculate hard loop corrections with exponentiation of soft photons. They will depend on our ability to solve or approximate analytically the above equation. Of course, Eq. (52) constitutes a new approach and can serve in any case as a starting point for new numerical analysis.

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