Conformal vacua and entropy in de Sitter space

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The de Sitter/conformal field theory (dS/CFT) correspondence is illuminated through an analysis of massive scalar field theory in d-dimensional de Sitter space. We consider a one-parameter family of dS-invariant vacua related by Bogolyubov transformations and compute the corresponding Green functions. It is shown that none of these Green functions correspond to the one obtained by analytic continuation from AdS. Among this family of vacua are in (out) vacua which have no incoming (outgoing) particles on $\mathcal{I}^-$ ($\mathcal{I}^+$). Surprisingly, it is shown that in odd spacetime dimensions the in and out vacua are the same, implying the absence of particle production for this state. The correlators of the boundary CFT, as defined by the dS/CFT correspondence, are shown to depend on the choice of vacuum state—the correlators with all points on $\mathcal{I}^-$ vanish in the in vacuum. For dS$_3$ we argue that this bulk vacuum dependence of the correlators is dual to a deformation of the boundary CFT$_2$ by a specific marginal operator. It is also shown that Witten's nonstandard de Sitter inner product (slightly modified) reduces to the standard inner product of the boundary field theory. Next we consider a scalar field in the Kerr-dS$_3$ Euclidean vacuum. A density matrix is constructed by tracing out over modes which are causally inaccessible to a single geodesic observer. This is shown to be a thermal state at the Kerr-dS$_3$ temperature and angular potential. It is further shown that, assuming Cardy's formula, the microscopic entropy of such a thermal state in the boundary CFT precisely equals the Bekenstein-Hawking value of one-quarter the area of the Kerr-dS$_3$ horizon.

I. INTRODUCTION AND SUMMARY

Recently, following earlier work [1–12], a proposal has been made relating quantum gravity in de Sitter space to conformal field theory (CFT) on the spacelike boundary of de Sitter space [13]. The proposal was motivated by an analysis of the asymptotic symmetry group of de Sitter space together with an appropriately crafted analogy to the AdS/CFT correspondence [14,15,16]. Other relevant discussions of quantum gravity in de Sitter space and dS/CFT appear in [17–39].

Unlike the AdS/CFT case, there has been no derivation of the proposed dS/CFT correspondence from string theory. Hopefully, a stringy construction of de Sitter space will be forthcoming. Meanwhile, much has been learned about AdS/CFT by analyzing solutions of the field equations and studying the propagation and interactions of fields, without directly using string theory. In this paper we pursue a parallel approach to dS/CFT, analyzing in some detail massive scalar field theory in de Sitter space. A number of surprising and interesting features emerge. Since this paper contains some rather detailed calculations, for the benefit of the reader we include a summary in this Introduction.

We begin in Sec. II with a discussion of dS-invariant Green functions for a massive scalar, reviewing and generalizing to d dimensions the discussion of [40,41]. We first describe the Green function obtained by analytic continuation from the Euclidean sphere. This is the so-called Euclidean Green function, and it is the two-point function of the scalar field in the Euclidean vacuum. We then construct a family of dS-invariant vacua labeled by a complex parameter α and

compute the Green functions in these α-vacua, which have several peculiarities. Singularities occur at antipodal points which are, however, unobservable since antipodal points are always separated by a horizon. Moreover, these singularities do not affect the scalar commutator, which is independent of α. We also see that the coincident point singularity has two terms, with opposite-signed $i\epsilon$ prescriptions. Hence all of these α-vacua except for the Euclidean vacuum differ from the usual Minkowski vacuum at arbitrarily short distances. We also compute the response of an Unruh detector and find that it is thermal only in the Euclidean vacuum. The dual CFT interpretation of the α-vacua is deferred to Sec. IV.

In relating the AdS/CFT and dS/CFT correspondences, it is natural to consider the particular Green function obtained by "double" analytic continuation from AdS to dS via the hyperbolic plane. We show that the Green function so obtained, while dS-invariant, does *not* correspond to the Green function in any known dS-invariant vacuum.[1] This result underscores the nontriviality of extrapolating from AdS/CFT to dS/CFT.

In Sec. III we consider scalar field theory in spherical coordinates

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[1] We benefited greatly from discussions with M. Spradlin and A. Volovich on this point. There is in fact a four-complex-parameter family of dS-invariant Wightman functions, characterized by the (complex) strengths of the coincident and antipodal poles, as well as the two possible $i\epsilon$ prescriptions at each pole. Only a one-complex-parameter family of these is known to be realizable as two-point vacuum expectation values. Analytic continuation from AdS gives a result which is not realized within this family.

$$\frac{ds^2}{l^2} = -d\tau^2 + \cosh^2\tau d\Omega_{d-1}^2, \qquad (1.1)$$

again generalizing [40,41] to d dimensions. A salient feature of these coordinates is that they cover all of de Sitter space and hence are suitable for studying global properties. The solutions of the massive scalar wave equation are found for arbitrary angular momentum. We then give an explicit construction in terms of these modes of the Bogolyubov transformations relating all the α vacua. Special "in" and "out" vacua are found, which are distinct from the Euclidean vacuum. The in vacuum has no incoming particles on $\mathcal{I}^-$, while the out vacuum has no outgoing particles on $\mathcal{I}^+$. The Bogolyubov transformation between them is computed. Surprisingly, it is found to be trivial in odd dimensions. This means that for the in vacuum of odd-dimensional de Sitter space there is no particle production. This result did not appear in previous analyses, which largely considered the four-dimensional case.

In Sec. IV we specialize to dS_3 and consider the dual two-dimensional CFT (CFT_2) interpretation of these results, along the lines proposed in [13]. We first compute the boundary behavior of the massive scalar Green function as a function of the vacuum parameter α. This behavior is fixed by conformal invariance up to overall constants which are α dependent. The boundary correlators have an especially simple form in the in vacuum. For both points on $\mathcal{I}^-$ (or both on $\mathcal{I}^+$) they vanish.[2] This is related to the fact that on $\mathcal{I}^-$ the spatial kinetic terms vanish and the theory becomes ultralocal. For one point on $\mathcal{I}^-$ and one on $\mathcal{I}^+$ they do not vanish. The simplicity of this behavior suggests that the in vacuum, despite the unphysical singularities, may play an important role in understanding the dS/CFT correspondence.

One way of generating a family of correlators in a CFT is by deforming the theory by a marginal operator. In [13] it was argued that a scalar field of mass m is dual to a pair of CFT operators $\mathcal{O}_\pm$ with conformal weights $1 \pm \sqrt{1-m^2 l^2}$. The composite operator $\mathcal{O}_+\mathcal{O}_-$ always has dimension 2 for any m, exactly what is required for a marginal deformation. We show explicitly for real α that this composite operator deforms the correlators in the same way as shifting α.

In Sec. V we consider the definition of the adjoint in the Hilbert space of the scalar field. In standard treatments of 2D Euclidean conformal field theory, the adjoint of an operator involves a (nonlocal) reflection about the unit circle. This prescription becomes the usual local adjoint when mapped to the cylinder. The "naive" adjoint for a bulk scalar field induces an adjoint in the Euclidean CFT which is local, and hence does not agree with the usual Euclidean CFT adjoint. However, in [17] Witten introduced a modified bulk inner product and corresponding adjoint. We show that, after a modification of the parity operation, Witten's bulk adjoint induces precisely the standard nonlocal Euclidean CFT adjoint. We further show that with the modified adjoint the $SL(2,C)$ generators obey $\mathcal{L}_n^\dagger = \mathcal{L}_{-n}$ (in a standard notation),

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[2]Except for a contact term which is computed.

as opposed to the relation $\mathcal{L}_n^\dagger = \bar{\mathcal{L}}_n$ implied by the naive adjoint.

As in the AdS case one expects that different coordinate systems in dS are relevant for different physical situations. In Sec. VI we consider static coordinates for dS_3, in which the metric is

$$\frac{ds^2}{l^2} = -(1-r^2)dt^2 + \frac{dr^2}{(1-r^2)} + r^2 d\varphi^2, \qquad (1.2)$$

where l is the de Sitter radius. These coordinates do not cover all of dS_3 with a single patch. Nevertheless, they do cover the so-called southern diamond—the region causally accessible to an observer at the "south pole" $r=0$. Moreover, the symmetry generating time evolution of the southern observer is manifest in static coordinates. Hence they appear well adapted to describing the physics accessible to a single observer, as advocated in [42]. $\mathcal{I}^-$ is at $r\to\infty$ and is conformal to a cylinder.

In the (t,r,φ) coordinates, the full dS_3 spacetime can be covered with four patches separated by horizons. We solve the scalar wave equation in each patch and construct global solutions by matching across the horizon. It is shown that the in vacuum on the cylinder and the in vacuum on the sphere are equivalent. A southern density matrix is constructed from the Euclidean vacuum by tracing over modes which are supported only in the northern causal diamond and are thereby unobservable to the southern observer. This is explicitly shown to be a thermal density matrix at temperature $T_{dS} = 1/2\pi l$, with energy measured with respect to the static time coordinate in Eq. (1.2). (This result is implicit in the original work [43].)

In Sec. VII we extend the static coordinate discussion to the Kerr-dS_3 geometry which represents a pair of spinning point masses at the north and south poles of dS_3. This has a Gibbons-Hawking temperature T_{GH} and angular potential Ω_{GH} which depend on the mass and spin. It is shown that, after tracing over northern modes, one obtains a thermal density matrix at precisely temperature T_{GH} and angular potential Ω_{GH}.

According to the dS_3/CFT_2 correspondence the quantum state on a bulk spacelike slice ending on $\mathcal{I}^-$ is dual to a CFT state on the boundary of the spacelike at $\mathcal{I}^-$ [13]. The dS-invariant bulk vacuum should be dual to the $SL(2,C)$ invariant CFT vacuum. For pure de Sitter space, we therefore expect to see a Casimir energy $-c/12$, where $c=3l/2G$ is the central charge of the CFT computed in [13]. We find a two-parameter agreement with this expectation by computing the Brown-York boundary stress tensor in Kerr-dS_3. This generalizes results of [42].

Finally, in Sec. VII we turn to the issue of de Sitter entropy. In the case of Bañados-Teitelboim-Zanelli (BTZ) black holes in AdS_3, the entropy formula can be microscopically derived, including the numerical coefficient, from the properties of the asymptotic symmetry group together with the assumption that the system is described by a consistent, unitary quantum theory of gravity [44]. String theory seems

necessary in order to produce an actual example of such a theory, but the general arguments follow from the stated assumptions independently of the stringy examples. Therefore it is natural to hope that a similar discussion is possible for dS_3. We report here some partial results but not a complete solution of the problem. Related discussions appear in [3,8,45–50].

The main observation is that if we simply assume Cardy's formula for the density of states, then a CFT with $c = 3l/2G$ at temperature T_{GH} and angular potential Ω_{GH} has a microscopic entropy precisely equal to one-quarter the area of the Kerr-dS$_3$ horizon. The two-parameter fit is striking but at present should be regarded as highly suggestive numerology rather than a derivation. For one thing, the dual CFT is unlikely to be unitary [13], and so there is no reason for Cardy's formula to apply. For another thing, it is not clear how a mixed thermal state arises in the dual CFT. The natural CFT state associated with $\mathcal{I}^-$ is the $SL(2,C)$ invariant vacuum, in agreement with the pure nature of the global bulk de Sitter vacuum. A mixed density matrix arises in the bulk only after tracing over the unobservable northern modes. However, tracing over northern modes is a bulk concept. We have not succeeded in finding a natural boundary interpretation of this operation.

We believe this raises a sharp and important question whose answer may lie within the present framework and in particular may not require a stringy construction of de Sitter. What is the meaning, in terms of the dual boundary CFT, of tracing out degrees of freedom which are inaccessible to a single observer?

Two appendixes detail useful properties of hypergeometric functions and de Sitter Green functions. For the rest of the paper we will set $l=1$ unless otherwise stated.

II. GREEN FUNCTIONS

The two-point Wightman function of a free massive scalar can be used to characterize the various de Sitter invariant vacua. In this section we describe these Green functions and their properties. Previous studies of scalar field theory in de Sitter space, largely concentrating on the four-dimensional case, can be found in [40,41,51–60].

A. The Euclidean vacuum and Wightman function

In this section we review the standard Euclidean vacuum and its associated Wightman function. d-dimensional de Sitter space (dS$_d$) is described by the hyperboloid in $(d+1)$-dimensional Minkowski space

$$P(X,X)=1, \tag{2.1}$$

where

$$P(X,X')=\eta_{ab}X^a X'^b, \quad a,b=0,...,d. \tag{2.2}$$

We will use lower case x to denote a d-dimensional coordinate on dS$_d$ and upper case X to denote the corresponding $(d+1)$-dimensional coordinate in the embedding space. The

function $P(x,x')$ is greater than one for timelike separations, equal to one for lightlike separations, and less than one for spacelike separations. In fact, $P(x,x')=\cos\theta$, where θ is the geodesic distance between x and x' for spatial separations, or i times the geodesic proper time difference for timelike separations.

A vacuum state $|\Omega\rangle$ for a free massive scalar in de Sitter space with the mode expansion

$$\phi(x)=\sum_n [a_n\phi_n(x)+a_n^\dagger\phi_n^*(x)] \tag{2.3}$$

can be defined by the conditions

$$a_n|\Omega\rangle=0, \tag{2.4}$$

where a_n and $a_n^\dagger$ as usual obey

$$[a_n,a_m^\dagger]=\delta_{nm}. \tag{2.5}$$

The modes $\phi_n(x)$ satisfy the de Sitter space wave equation

$$(\nabla^2-m^2)\phi_n=0, \tag{2.6}$$

and are normalized with respect to the invariant Klein-Gordon inner product

$$(\phi_n,\phi_m)=-i\int_\Sigma d\Sigma^\mu(\phi_n\overleftrightarrow{\partial}_\mu\phi_m^*)=\delta_{nm}. \tag{2.7}$$

The integral is taken over a complete spacelike slice Σ in dS$_d$ with induced metric h_{ij}, and $d\Sigma^\mu=d^dx\sqrt{h}n^\mu$, where n^μ is the future directed unit normal vector. The norm (2.7) is independent of the choice of this slice. $|\Omega\rangle$ depends on the choice of modes appearing in Eq. (2.3).

The Wightman function, defined by

$$G_\Omega(x,x')=\langle\Omega|\phi(x)\phi(x')|\Omega\rangle=\sum_n \phi_n(x)\phi_n^*(x'), \tag{2.8}$$

characterizes the vacuum state $|\Omega\rangle$. There is a unique state, the "Euclidean vacuum" $|E\rangle$, whose Wightman function is obtained by analytic continuation from the Euclidean sphere. This state is invariant under the full de Sitter group. In d spacetime dimensions the Wightman function in the state $|E\rangle$ is

$$G_E(x,x')=\langle E|\phi(x)\phi(x')|E\rangle$$

$$=c_{m,d}F\left(h_+,h_-;\frac{d}{2};\frac{1+P(x,x')}{2}\right),$$

$$h_\pm\equiv\frac{d-1}{2}\pm i\mu,$$

$$\mu\equiv\sqrt{m^2-\left(\frac{d-1}{2}\right)^2},$$

$$c_{m,d} \equiv \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{d/2}\Gamma\left(\dfrac{d}{2}\right)}. \qquad (2.9)$$

G_E is real in the spacelike region $P<1$ and singular on the light cone $P=1$. The $i\epsilon$ prescription near the singularity is

$$G_E(x,x') \sim [(t-t'-i\epsilon)^2 - |\vec{x}-\vec{x}'|^2)]^{1-d/2}. \qquad (2.10)$$

Note that this prescription cannot be written in terms of the invariant quantity P alone, which is time-reversal invariant. G_E obeys

$$(\nabla^2 - m^2)G_E(x,x') = 0. \qquad (2.11)$$

In addition to the Wightman function, the Feynman propagator

$$G_F(x,x') = \Theta(t-t')G(x,x') + \Theta(t'-t)G(x',x) \qquad (2.12)$$

and commutator

$$G_C(x,x') = G(x,x') - G(x',x) \qquad (2.13)$$

are also of interest. With the normalization (2.9) G_F obeys

$$(\nabla^2 - m^2)G_F(x,x') = \frac{-i}{\sqrt{-g}}\delta^d(x,x'). \qquad (2.14)$$

B. The MA transform

In this section we describe the Mottola-Allen (MA) transform [40,41], which relates the various de Sitter invariant vacua and Wightman functions to one another. Let $\phi_n^E(x)$ denote the positive frequency modes associated with the Euclidean vacuum. Explicit expressions for ϕ_n^E will be given later (Secs. III C and VI E), but we do not need them now. Let x_A denote the antipodal point to x on the de Sitter hyperboloid (i.e., $X_A = -X$). Then, as will be seen below, the Euclidean modes can be chosen to obey

$$\phi_n^E(x_A) = \phi_n^{E*}(x). \qquad (2.15)$$

Now consider a new set of modes related by the MA transform

$$\tilde{\phi}_n \equiv N_\alpha(\phi_n^E + e^\alpha \phi_n^{E*}), \quad N_\alpha \equiv \frac{1}{\sqrt{1-e^{\alpha+\alpha^*}}}, \qquad (2.16)$$

where α can be any complex number with Re $\alpha<0$. The modes (2.16) can be used to define new operators $\tilde{a}_n$ and $\tilde{a}_n^\dagger$ via a decomposition of the form (2.3). These are related to the Euclidean operators a_n^E and $a_n^{E\dagger}$ by

$$\tilde{a}_n = N_\alpha(a_n^E - e^{\alpha*}a_n^{E\dagger}). \qquad (2.17)$$

This may be rewritten as

$$\tilde{a}_n = \mathcal{U}a_n^E\mathcal{U}^\dagger, \qquad (2.18)$$

where

$$\mathcal{U} = \exp\left\{\sum_n c(a_n^{E\dagger})^2 - \bar{c}(a_n^E)^2\right\},$$

$$c(\alpha) = \frac{1}{4}\left(\ln\tanh\frac{-\text{Re }\alpha}{2}\right)e^{-i\,\text{Im }\alpha}. \qquad (2.19)$$

The vacuum state

$$|\alpha\rangle = \mathcal{U}|E\rangle \qquad (2.20)$$

is annihilated by the $\tilde{a}_n$. The operator $\mathcal{U}$ is unitary, so Eq. (2.20) is properly normalized. In the quantum optics literature, $|\alpha\rangle$ is known as a squeezed state. Equation (2.20) may be formally rewritten as

$$|\alpha\rangle = C\exp\left(\frac{1}{2}e^{\alpha*}\sum_n (a_n^{E\dagger})^2\right)|E\rangle, \qquad (2.21)$$

where C is a constant. Although this expression is not normalizable (so C is technically zero), it is often more convenient than Eq. (2.20).

The Wightman function in the state $|\alpha\rangle$ is

$$G_\alpha(x,x') = \sum_n \tilde{\phi}_n(x)\tilde{\phi}_n^*(x'). \qquad (2.22)$$

Using Eqs. (2.15) and (2.16) this can be rewritten as a sum over Euclidean modes,

$$G_\alpha(x,x') = N_\alpha^2\sum_n [\phi_n^E(x)\phi_n^{E*}(x') + e^{\alpha+\alpha*}\phi_n^E(x')\phi_n^{E*}(x)$$

$$+ e^{\alpha*}\phi_n^E(x)\phi_n^{E*}(x_A') + e^\alpha\phi_n^E(x_A)\phi_n^{E*}(x')], \qquad (2.23)$$

and then evaluated as

$$G_\alpha(x,x') = N_\alpha^2[G_E(x,x') + e^{\alpha+\alpha*}G_E(x',x)$$

$$+ e^{\alpha*}G_E(x,x_A') + e^\alpha G_E(x_A,x')]. \qquad (2.24)$$

Hence it is easy to obtain the $|\alpha\rangle$ Wightman function from the Euclidean one. Since these Wightman functions depend only on the $SO(d,1)$ invariant quantity P (away from the singularities) this construction demonstrates the invariance of the $|\alpha\rangle$ vacua under the connected part of the de Sitter group. Note, however, that if α is not real the collection of modes (2.16) is not mapped into itself by CPT. Therefore the $|\alpha\rangle$ vacua are CPT invariant only for real α.

Of course, since the commutator of two fields is a c number, the commutator function G_C must be the same in all vacua. It is easy to check that the commutator constructed from the two-point function (2.24) has this property.

The Wightman function (2.24) has several peculiarities. First, there are antipodal singularities at $x'=x_A$. However, such antipodal points are separated by a horizon so this singularity is not observable. Second, the singularity at coincident points has a negative frequency component coming

from the second term in Eq. (2.24) (although the commutator is unaffected). This means that for $e^\alpha \neq 0$ the vacuum state does not approach the usual Minkowskian one even at distances much shorter than the de Sitter radius. This "unphysical" behavior was to be expected since the MA transform (2.16) involves arbitrarily high-frequency modes. Despite these peculiarities we will see that these vacua play an interesting role in the dS/CFT correspondence.

C. Analytic continuation from AdS

An alternate way to get a dS Green function is by double analytic continuation from AdS via the hyperbolic plane.[3] In fact, we shall argue that this yields a Green function which differs from any of those discussed in Sec. II B and therefore, as far as we know, is not physically realizable as the Wightman function in any vacuum state. Hence the dS/CFT correspondence is not in any precise sense that we know of the analytic continuation of the AdS/CFT correspondence, and care must be taken in extrapolating from the latter to the former.

AdS$_d$ has a unique $SO(d-1,2)$ invariant vacuum whose scalar Green functions can be obtained as a sum over normalizable eigenmodes. The wave equation allows two possible falloffs (fast and slow) at infinity, but only the fast falloff appears in the Green function. Double analytic continuation from AdS to dS will therefore yield a dS Green function with only one of the two possible falloff rates (which become complex conjugates for large enough m). This cannot be the Euclidean dS Green function, as the latter involves both falloffs. There is a vacuum $|\alpha\rangle$ whose Green function has the required falloff.[4] However from Eq. (2.24) we see that the Green function for every state except $|E\rangle$ has a coincident point singularity with a coefficient larger than that of $|E\rangle$ and containing two terms with opposite-signed $i\epsilon$ prescriptions. However, double analytic continuation from AdS will yield a coincident point singularity with a canonical coefficient and a single $i\epsilon$ prescription. Hence it yields a Green function which is not realized as $\langle\alpha|\phi(x)\phi(x')|\alpha\rangle$ for any α.

D. Particle detection

In this section we discuss particle detection by a geodesic observer in the $|\alpha\rangle$ vacua. We will find a thermal spectrum only for the Euclidean vacuum.

Consider an Unruh detector moving along a timelike geodesic, which couples to the field as

$$\int dt \, m(t) \phi(x(t)), \tag{2.25}$$

where $m(t)$ is an operator acting on the internal states of the detector and the integral is over the proper time along the detector worldline. Without loss of generality we may take the detector to be sitting on the south pole. Let us assume

that the detector has a spectrum of states $|E_i\rangle$ with energies E_i, and define the matrix element $m_{ij} = \langle E_i|m(0)|E_j\rangle$. In the vacuum state $|\alpha\rangle$ the transition rate between the states $|E_i\rangle$ and $|E_j\rangle$ may be evaluated in perturbation theory (see, e.g., the review [62])

$$\dot{P}_\alpha(E_i \to E_j) = |m_{ij}|^2 \int_{-\infty}^{\infty} dt \, e^{-i\Delta E t} G_\alpha(x(t), x(0)), \tag{2.26}$$

where $\Delta E = E_j - E_i$.

First, let us study particle production in the Euclidean vacuum. For two timelike separated points x and x' we have $P(x,x') = \cosh t$ and $P(x_A, x') = -\cosh t$, where t is the proper time between x and x'. We take t to be positive (negative) if x is in the future (past) light cone of x'. As a function of t, the appropriate $i\epsilon$ prescription for the Wightman function is

$$G_E(x,x') = G_E(t - i\epsilon), \tag{2.27}$$

indicating that for positive (negative) t we should go under (over) the branch cut from $P=1$ to $P=\infty$ in Eq. (2.9). As a function in the complex t plane G_E obeys

$$G_E(t) = G_E(-t - 2\pi i), \tag{2.28}$$

$$G_E^*(t) = G_E(\bar{t} - 2\pi i). \tag{2.29}$$

To evaluate $G_E(x',x)$ we must take $t \to -t$:

$$G_E(x',x) = G_E(-t - i\epsilon) = G_E(t + i\epsilon - 2\pi i). \tag{2.30}$$

Similarly, we may evaluate

$$G_E(x,x_A') = G_E(x_A,x') = G_E(t - i\pi). \tag{2.31}$$

The points x and x_A' are spacelike separated, so it is not necessary to insert an $i\epsilon$.

Let us consider the example of $d=3$. As a function of t, the Green function (2.9) has singularities at $t = n\pi i$ for all $n \neq -1$. This may be seen from the alternate form of the Green function presented in Appendix A. Thus in the evaluating Eq. (2.26) we may deform the contour of integration in the complex t plane

$$\int_{-\infty}^{\infty} dt \, e^{-i\Delta E t} G_E(t - i\epsilon)$$

$$= e^{-\pi\Delta E} \int_{-\infty}^{\infty} dt \, e^{-i\Delta E t} G_E(t - i\pi)$$

$$= e^{-2\pi\Delta E} \int_{-\infty}^{\infty} dt \, e^{-i\Delta E t} G_E(t - 2\pi i + i\epsilon). \tag{2.32}$$

The $e^{-\epsilon E}$ terms have been dropped. Using Eq. (2.28) and the second line of Eq. (2.32) we find that the detector response rate (2.26) obeys

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[3]See [2,7,61] for discussions.

[4]It turns out to correspond to the in vacuum discussed below.

$$\frac{\dot{P}_E(E_i \rightarrow E_j)}{\dot{P}(E_j \rightarrow E_i)} = e^{-2\pi\Delta E} \qquad (2.33)$$

in the Euclidean vacuum. This is the condition of detailed balance for a thermal system at the de Sitter temperature

$$T_{dS} = \frac{1}{2\pi}. \qquad (2.34)$$

For a general vacuum state $|\alpha\rangle$ we may use the identities (2.32) to relate the integrals of all four terms in Eq. (2.24). We find

$$\int_{-\infty}^{\infty} dt \, e^{-i\Delta Et} G_\alpha(t-i\epsilon) = N_\alpha^2 |1 + e^{\alpha + \pi\Delta E}|^2$$

$$\times \int_{-\infty}^{\infty} dt \, e^{-i\Delta Et} G_E(t-i\epsilon). \qquad (2.35)$$

So the ratio (2.33) becomes[5]

$$\frac{\dot{P}_\alpha(E_i \rightarrow E_j)}{\dot{P}_\alpha(E_j \rightarrow E_i)} = e^{-2\pi\Delta E} \left| \frac{1 + e^{\alpha + \pi\Delta E}}{1 + e^{\alpha - \pi\Delta E}} \right|^2. \qquad (2.36)$$

We conclude that the detector response is not thermal. In general the detector will not equilibrate. Even though the ratio (2.36) is nonzero, we will see in the next section that there are vacua for which, in a certain sense, there is no particle creation.

III. THE SPHERE

In this section we study scalar field theory on dS_d in global coordinates (τ, Ω). The metric is

$$ds^2 = -d\tau^2 + \cosh^2 \tau \, d\Omega_{d-1}^2, \qquad (3.1)$$

where $d\Omega_{d-1}^2$ is the usual metric on S^{d-1}, parametrized by the coordinates Ω. An important feature of these coordinates is that they cover all of dS_d and hence are suited to a global description of the quantum state.

A. Solutions of the wave equation

In this section we find solutions to the massive wave equation

$$(\nabla^2 - m^2)\phi = 0. \qquad (3.2)$$

This differential equation is separable, with solutions

$$\phi = y_L(\tau) Y_{Lj}(\Omega). \qquad (3.3)$$

The Y_{Lj} are spherical harmonics on S^{d-1} obeying

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[5]This expression was obtained for the case of a scalar with conformal mass in [59].

$$\nabla_{S^{d-1}}^d Y_{Lj} = -L(L+d-2)Y_{Lj}. \qquad (3.4)$$

Here L is a non-negative integer and j is a collective index $(j_1, ..., j_{d-2})$. We will use a nonstandard choice of Y_{Lj}'s, with

$$Y_{Lj}(\Omega_A) = Y_{Lj}^*(\Omega) = (-)^L Y_{Lj}(\Omega). \qquad (3.5)$$

Here Ω_A denotes the point on S^{d-1} antipodal to Ω. In terms of the usual spherical harmonics S_{Lj},

$$Y_{Lj} = \sqrt{\frac{i}{2}} S_{Lj} + (-)^L \sqrt{\frac{-i}{2}} S_{Lj}^*. \qquad (3.6)$$

The functions Y_{Lj} are orthonormal,

$$\int d\Omega \, Y_{Lj}(\Omega) Y_{L'j'}^*(\Omega) = \delta_{LL'} \delta_{jj'}, \qquad (3.7)$$

and complete,

$$\sum_{Lj} Y_{Lj}(\Omega) Y_{Lj}^*(\Omega') = \delta^{d-1}(\Omega, \Omega'). \qquad (3.8)$$

We then have

$$\ddot{y}_L + (d-1)\tanh r \dot{y}_L + \left[m^2 + \frac{L(L+d-2)}{\cosh^2 \tau} \right] y_L = 0. \qquad (3.9)$$

In terms of the coordinate $\sigma = -e^{2\tau}$ this becomes

$$\sigma(1-\sigma)y_L'' + \left[\left(1 - \frac{d-1}{2} \right) - \left(1 + \frac{d-1}{2} \right)\sigma \right] y_L'$$

$$+ \left[\frac{m^2}{4} \frac{1-\sigma}{\sigma} - \frac{L(L+d-2)}{1-\sigma} \right] y_L = 0. \qquad (3.10)$$

Let us make the substitution

$$y_L^{in} = \cosh^L \tau \, e^{\{L + [(d-1)/2] - i\mu\}\tau} x. \qquad (3.11)$$

With

$$\mu = \sqrt{m^2 - \frac{(d-1)^2}{4}}, \qquad (3.12)$$

Eq. (3.10) becomes a hypergeometric equation for x,

$$\sigma(1-\sigma)x'' + [c - (1+a+b)\sigma]x' - abx = 0, \qquad (3.13)$$

with coefficients

$$a = L + \frac{d-1}{2}, \quad b = L + \frac{d-1}{2} - i\mu, \quad c = 1 - i\mu. \qquad (3.14)$$

Let us consider the case of real positive μ, i.e., $2m > (d-1)$. We find that

$$y_L^{\text{in}} = \frac{2^{L+d/2-1}}{\sqrt{\mu}} \cosh^L \tau e^{\{L+[(d-1)/2]-i\mu\}\tau}$$

$$\times F\left(L+\frac{d-1}{2}, L+\frac{d-1}{2}-i\mu; 1-i\mu; -e^{2\tau}\right) \quad (3.15)$$

and its complex conjugate are two linearly independent solutions. The normalization is fixed by demanding that these modes are orthonormal with respect to the inner product (2.7), which is easily evaluated on $\mathcal{I}^-$.

B. In and out vacua

We now use the solutions (3.15) to construct in (out) vacua with no incoming (out-going) particles, and find the Bogolyubov transformation relating them. Note that Eq. (3.9) is invariant under time reversal. Hence we obtain another pair of linearly independent solutions by defining

$$y_L^{\text{out}}(\tau) = y_L^{\text{in}*}(-\tau). \quad (3.16)$$

Explicitly,

$$y_L^{\text{out}} = \frac{2^{L+d/2-1}}{\sqrt{\mu}} \cosh^L \tau e^{\{-L-[(d-1)/2]-i\mu\}\tau}$$

$$\times F\left(L+\frac{d-1}{2}, L+\frac{d-1}{2}+i\mu; 1+i\mu; -e^{-2\tau}\right). \quad (3.17)$$

At the past boundary ($\tau \to -\infty$) we find that $F \to 1$ and hence

$$y_L^{\text{in}} \to \frac{2^{d/2-1}}{\sqrt{\mu}} e^{\{[(d-1)/2]-i\mu\}\tau} \quad (3.18)$$

while at the future boundary ($\tau \to \infty$)

$$y_L^{\text{out}} \to \frac{2^{d/2-1}}{\sqrt{\mu}} e^{-\{[(d-1)/2]+i\mu\}\tau}. \quad (3.19)$$

Thus we see that the modes

$$\phi_{Lj}^{\text{in}}(x) = y_L^{\text{in}}(\tau) Y_{Lj}(\Omega),$$

$$\phi_{Lj}^{\text{out}}(x) = y_L^{\text{out}}(\tau) Y_{Lj}(\Omega) \quad (3.20)$$

are positive frequency modes with respect to the global time τ near the asymptotic past and future boundaries, respectively. They represent incoming and outgoing particle states. They define two vacua, $|\text{in}\rangle$ and $|\text{out}\rangle$, which are annihilated by the lowering operators associated to ϕ^{in} and ϕ^{out}, respectively. Physically, $|\text{in}\rangle$ is the state with no incoming particles on $\mathcal{I}^-$ and $|\text{out}\rangle$ is the state with no outgoing particles on $\mathcal{I}^+$.

The Bogolyubov coefficients relating the two sets of modes can be found by using the hypergeometric transformation equations (summarized in Appendix B) and Eq. (3.5). One finds

$$\phi_{Lj}^{\text{in}} = A e^{-2i\theta_L} \phi_{Lj}^{\text{out}} + iB \phi_{Lj}^{\text{out}*}, \quad (3.21)$$

where

$$A = \begin{cases} 1, & d \text{ odd} \\ \coth \pi\mu, & d \text{ even,} \end{cases}$$

$$B = \begin{cases} 0, & d \text{ odd} \\ (-)^{d/2} \operatorname{csch} \pi\mu, & d \text{ even;} \end{cases} \quad (3.22)$$

we have isolated the phase

$$e^{-2i\theta_L} = (-)^{L-[(d-1)/2]} \frac{\Gamma(-i\mu)\Gamma\left(L+\dfrac{d-1}{2}+i\mu\right)}{\Gamma(i\mu)\Gamma\left(L+\dfrac{d-1}{2}-i\mu\right)} \quad (3.23)$$

for later convenience. The coefficients obey $|A|^2 - |B|^2 = 1$ as required for properly normalized modes.

Note that B, the coefficient mixing positive and negative frequency modes, vanishes in odd dimensions. This implies that the two sets of modes define the same vacuum:

$$|\text{in}\rangle = |\text{out}\rangle \quad \text{in odd dimensions.} \quad (3.24)$$

Hence, there is no particle production. If no particles are coming in from $\mathcal{I}^-$, no particles will go out on $\mathcal{I}^+$.[6] This is in contrast with the even-dimensional case for which there is always some particle production.

From Eq. (3.18) it follows that $\phi_{Lj}^{\text{in}} \sim e^{h_-\tau}$ near $\mathcal{I}^-$. In the language of [13], this implies the modes ϕ^{in} are dual to operators of weight h_+ on the boundary. Likewise, $\phi^{\text{in}*}$ are dual to operators of weight h_-. The de Sitter transformations act on the boundary theory as global conformal transformations, which do not mix operators of different weight. We conclude that ϕ^{in} and $\phi^{\text{in}*}$ do not mix under the de Sitter group, so the states $|\text{in}\rangle$ and $|\text{out}\rangle$ are de Sitter invariant.

It is convenient to define the rescaled global modes

$$\tilde{\phi}_{Lj}^{\text{in}}(x) = e^{i\theta_L} y_L^{\text{in}}(\tau) Y_{Lj}(\Omega),$$

$$\tilde{\phi}_{Lj}^{\text{out}}(x) = e^{-i\theta_L} y_L^{\text{out}}(\tau) Y_{Lj}(\Omega). \quad (3.25)$$

This is a trivial phase shift, so $|\text{in}\rangle$ and $|\text{out}\rangle$ are the states annihilated by the lowering operators associated to $\tilde{\phi}^{\text{in}}$ and $\tilde{\phi}^{\text{out}}$, respectively. In this basis the Bogolyubov transformation

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[6]Note, however, that according to Eq. (2.36) an Unruh detector still observes particles.

$$\tilde{\phi}_{Lj}^{in}(x) = A\,\tilde{\phi}_{Lj}^{out}(x) + iB\,\tilde{\phi}_{Lj}^{out*}(x) \qquad (3.26)$$

has the form of an MA transform, and so can be used to define additional de Sitter invariant vacua. The modes (3.25) have the useful property that for any point x

$$\tilde{\phi}_{Lj}^{in}(x_A) = \tilde{\phi}_{Lj}^{out*}(x), \qquad (3.27)$$

where $x_A \sim (-\tau, \Omega_A)$ is the point antipodal to x. In odd dimensions this becomes

$$\tilde{\phi}_{Lj}^{in}(x_A) = \tilde{\phi}_{Lj}^{in*}(x). \qquad (3.28)$$

This implies that in odd dimensions the in vacuum is *CPT* invariant, whereas in even dimensions *CPT* interchanges in and out.

C. The Euclidean vacuum

In this section we construct the Euclidean vacuum $|E\rangle$ in the basis of spherical modes. The Lorentzian de Sitter geometry (3.1) can be continued to Euclidean signature by taking τ to run along the imaginary τ axis, from $\tau = -i\pi/2$ to $\tau = i\pi/2$. The resulting geometry is a round d sphere. We define the upper (lower) Euclidean hemisphere as the portion of this path that lies in the upper (lower) complex τ plane. In particular, the upper (lower) Euclidean pole lies at $\tau = i\pi/2$ ($\tau = -i\pi/2$).

We define positive frequency Euclidean modes to be those that are regular when analytically continued to the lower Euclidean hemisphere. In this section we find these modes in

global coordinates. The Euclidean vacuum $|E\rangle$ is the state that is annihilated by the positive frequency Euclidean modes.

We may rewrite Eq. (3.10) in terms of the variable $z = 1 - \sigma = 1 + e^{2\tau}$, which is well suited to analyzing the behavior of global modes on the Euclidean geometry. Upon substituting

$$y_L^E = \cosh^L \tau e^{\{L + [(d-1)/2] + i\mu\}\tau} x, \qquad (3.29)$$

we obtain the hypergeometric equation

$$z(1-z)\frac{d^2x}{dz^2} + [\hat{c} - (1 + a + b^*)\sigma]\frac{dx}{dz} - ab^*x = 0, \qquad (3.30)$$

with positive integer coefficient

$$\hat{c} = 2L + d - 1. \qquad (3.31)$$

We find the general solution

$$x = CU_1 + DU_2, \qquad (3.32)$$

where

$$U_1 = F\left(L + \frac{d-1}{2}, L + \frac{d-1}{2} + i\mu; 2L + d - 1; z\right). \qquad (3.33)$$

The second solution is given by

$$U_2 = z^{2-2L-d}F\left(1 - L - \frac{d-1}{2}, 1 + i\mu - L - \frac{d-1}{2}; 3 - 2L - d; z\right) \qquad (3.34)$$

if d is odd, and by

$$U_2 = U_1 \ln z + \sum_{k=2-2L-d}^{\infty} Q_k z^k \qquad (3.35)$$

if d is even; the coefficients Q_k are found, e.g., in [63].

The Lorentzian geometry lies on the path from $z = 1$ ($\mathcal{I}^-$) along the real z axis to $z = +\infty$ ($\mathcal{I}^+$). On the throat, at $z = 2$, it intersects with the Euclidean geometry, which lies on a unit circle centered at $z = 1$. The lower (upper) hemisphere corresponds to the lower (upper) half-circle. The Euclidean poles are at $z = 0$. The functions (3.32) have a branch cut from $z = 1$ to $z = +\infty$. Hence, they are not analytic on the whole Euclidean sphere. By choosing the Lorentzian path to run just below the real axis ($z \to z - i\epsilon$), we obtain solutions that are analytic on the lower hemisphere and the entire Lorentzian geometry.

The first solution (3.33) is regular in these regions, whereas the second solution, Eq. (3.34) or (3.35), becomes singular at the lower Euclidean pole, at $z = 0 - i\epsilon$. Hence we

discard the second set of modes and keep the first. The modes can be analytically continued through the branch cut to the upper hemisphere, where they are not expected to be regular.

The normalized Euclidean modes are

$$\phi_{Lj}^E(x) = \frac{1}{f_L\sqrt{e^{2\pi\mu} - 1}}\,y_L^E(\tau)Y_{Lj}(\Omega), \qquad (3.36)$$

where

$$y_L^E = \frac{2^{L+d/2-1}i^{-L+[(d-1)/2]}}{\sqrt{\mu}}\cosh^L \tau e^{\{L+[(d-1)/2]+i\mu\}\tau}$$

$$\times F\left(L + \frac{d-1}{2}, L + \frac{d-1}{2} + i\mu; 2L + d - 1; 1 + e^{2\tau}\right), \qquad (3.37)$$

$$f_L = \frac{\Gamma(2L + d - 1)}{\Gamma\left(L + \frac{d-1}{2}\right)}\left|\frac{\Gamma(i\mu)}{\Gamma\left(L + \frac{d-1}{2} - i\mu\right)}\right|.$$

The Euclidean Green function (2.9) is then given by the mode sum

$$G_E(x,x')=\sum_{L,j} \phi_{Lj}^E(x)\phi_{Lj}^{E*}(x').\qquad(3.38)$$

This expression was given in the four-dimensional case in [40].

D. The $|E\rangle\to|in\rangle$ transformation

In this section we show that the Euclidean and in vacua are MA transforms of each other. Let us again specialize to the case of $2m>(d-1)$. The y^E are then related to the y^{in} by

$$y_L^E=f_L((-)^{L+[(d-1)/2]}e^{-i\theta_L}y_L^{in*}+e^{\pi\mu+i\theta_L}y_L^{in}).\qquad(3.39)$$

So the Euclidean modes are related to the global modes by

$$\phi_{Lj}^E=\frac{1}{\sqrt{1-e^{-2\pi\mu}}}(\tilde{\phi}_{Lj}^{in}+(-)^{(d-1)/2}e^{-\pi\mu}\tilde{\phi}_{Lj}^{in*}),\qquad(3.40)$$

from which it follows, along with Eqs. (3.28) and (3.26), that

$$\phi_{Lj}^E(x_A)=\phi_{Lj}^{E*}(x)\qquad(3.41)$$

in any dimension. This implies the Euclidean vacuum is *CPT* invariant.

Now, Eq. (3.40) may be inverted to give

$$\tilde{\phi}_{Lj}^{in}=\frac{1}{\sqrt{1-e^{-2\pi\mu}}}(\phi_{Lj}^E+(-)^{(d+1)/2}e^{-\pi\mu}\phi_{Lj}^{E*})\qquad(3.42)$$

which is an MA transformation with

$$\alpha=-\pi\mu+i\left(\frac{d+1}{2}\right)\pi.\qquad(3.43)$$

We have thus identified the MA transformation relating the $|in\rangle$ vacuum and the Euclidean vacuum $|E\rangle$.

IV. CFT INTERPRETATION

In this section we interpret the *CPT* invariant (real α) family of bulk de Sitter invariant vacua as a line of marginal deformations of the boundary CFT. A similar interpretation may extend to the case of general complex α but we do not pursue it here. In this and later sections we restrict to the case $d=3$.

A. $\mathcal{I}^\pm$ correlators

In this section we evaluate the various Green functions appearing on the right-hand side of Eq. (2.24) for x and x' on $\mathcal{I}^\pm$, and then put the results together to see how the boundary values of the correlators depend on α. We use global coordi-

nates (τ,Ω), $\Omega=(w,\bar{w})$, where $w=\tan(\theta/2)e^{i\varphi}$ is the complex coordinate on the 2-sphere, so that

$$ds^2=-d\tau^2+4\cosh^2\tau\frac{dwd\bar{w}}{(1+w\bar{w})^2}.\qquad(4.1)$$

The behavior of the correlators at $\mathcal{I}^\pm$ follows from the asymptotic form of the hypergeometric functions. As $|z|\to\infty$ one has (see Appendix B)

$$F(h_+,h_-;\tfrac{3}{2};z)\to c_+(-z)^{-h_+}+c_-(-z)^{-h_-},$$

$$c_\pm=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(h_\mp-h_\pm)}{\Gamma(h_\mp)\Gamma\left(\frac{3}{2}-h_\pm\right)}.\qquad(4.2)$$

This expression is not in general real (unless z is real and negative) because the $h_\pm=1\pm i\mu$ are not real. In spherical coordinates one finds, near $\mathcal{I}^-$,

$$\lim_{\tau,\tau'\to-\infty} P(\tau,\Omega;\tau',\Omega')=-\frac{e^{-\tau-\tau'}|w-w'|^2}{2(1+w\bar{w})(1+w'\bar{w}')}.\qquad(4.3)$$

For $x=(\tau,\Omega)$ and $x'=(\tau',\Omega')$ both on $\mathcal{I}^-$

$$\lim_{\tau,\tau'\to-\infty} G_E(x,x')=e^{h_+(\tau+\tau')}\Delta_+(\Omega,\Omega')$$
$$+e^{h_-(\tau+\tau')}\Delta_-(\Omega;\Omega').\qquad(4.4)$$

$\Delta_\pm$ here is proportional to the two point function for a conformal field of dimension $h_\pm$ on the sphere:

$$\Delta_\pm(\Omega;\Omega')=4^{h_\pm}c_{m,d}c_\pm\left[\frac{(1+w\bar{w})(1+w'\bar{w}')}{|w-w'|^2}\right]^{h_\pm}.\qquad(4.5)$$

We note that $G_E(x,x')=G_E(x',x)$ on $\mathcal{I}^-$ as the points are spacelike separated. We have assumed here, and in the following expressions (unless explicitly stated) that x and x' are not coincident so that contact terms can be ignored.

Let us now consider the case where x is on $\mathcal{I}^-$ and x' is on $\mathcal{I}^+$. Since the antipodal point to x', namely $x_A'=(-\tau',\Omega_A')=(-\tau',-1/\bar{w},-1/w')$, is on $\mathcal{I}^-$ we may use Eq. (4.4) and the formula

$$P(x,x')=-P(x,x_A').\qquad(4.6)$$

In continuing Eq. (4.4) to positive P we must take care to go above the branch cut, in accord with the $i\epsilon$ prescription for the Wightman function with $\tau'>\tau$. We find

$$\lim_{\substack{\tau\to-\infty\\\tau'\to\infty}} G_E(x,x')=-e^{h_+(\tau-\tau')}e^{-\pi\mu}\Delta_+(\Omega;\Omega_A')$$
$$-e^{h_-(\tau-\tau')}e^{\pi\mu}\Delta_-(\Omega;\Omega_A').\qquad(4.7)$$

To evaluate $G_E(x',x)$ we must go under the branch cut, yielding

$$\lim_{\substack{\tau \to -\infty \\ \tau' \to \infty}} G_E(x',x) = -e^{h_+(\tau-\tau')}e^{\pi\mu}\Delta_+(\Omega;\Omega'_A)$$

$$-e^{h_-(\tau-\tau')}e^{-\pi\mu}\Delta_-(\Omega;\Omega'_A). \quad (4.8)$$

Now we insert these results into formula (2.24) for the Wightman function in the general vacuum state $|\alpha\rangle$. For both points on $\mathcal{I}^-$ one finds

$$\lim_{\tau,\tau' \to -\infty} G_\alpha(x,x') = N_\alpha^2(1-e^{\alpha+\pi\mu})(1-e^{\alpha^*-\pi\mu})$$

$$\times e^{h_+(\tau+\tau')}\Delta_+(\Omega;\Omega')$$

$$+N_\alpha^2(1-e^{\alpha-\pi\mu})(1-e^{\alpha^*+\pi\mu})$$

$$\times e^{h_-(\tau+\tau')}\Delta_-(\Omega;\Omega'). \quad (4.9)$$

On the other hand, for x on $\mathcal{I}^-$ and x' on $\mathcal{I}^+$ we get

$$\lim_{\substack{\tau \to -\infty \\ \tau' \to \infty}} G_\alpha(x,x') = -N_\alpha^2 e^{-\pi\mu}|1-e^{\alpha+\pi\mu}|^2 e^{h_+(\tau-\tau')}$$

$$\times \Delta_+(\Omega;\Omega'_A)$$

$$-N_\alpha^2 e^{\pi\mu}|1-e^{\alpha-\pi\mu}|^2 e^{h_-(\tau-\tau')}$$

$$\times \Delta_-(\Omega;\Omega'_A). \quad (4.10)$$

We see that the boundary correlators depend nontrivially on the choice of vacuum. Since we have taken $|P| \to \infty$, these formulas are valid only for noncoincident points on $\mathcal{I}^\pm$ and omit possible contact terms.

Let us now turn to the interesting special case of the in vacuum, which has $\alpha = -\pi\mu$. For both points on $\mathcal{I}^-$ it follows from Eq. (4.9) that the correlators vanish. On the other hand, for x on $\mathcal{I}^-$ and x' on $\mathcal{I}^+$ we get

$$\lim_{\substack{\tau \to -\infty \\ \tau' \to \infty}} G_{in}(x,x') = -2 \sinh \pi\mu e^{h_-(\tau-\tau')}\Delta_-(\Omega;\Omega'_A),$$

$$(4.11)$$

$$\lim_{\substack{\tau \to -\infty \\ \tau' \to \infty}} G_{in}(x',x) = -2 \sinh \pi\mu e^{h_+(\tau-\tau')}\Delta_+(\Omega;\Omega'_A).$$

When the points on $\mathcal{I}^-$ coincide there is a contact term which can be easily computed by noting that the Wightman function on $\mathcal{I}^-$ reduces to a mode sum over spherical harmonics. This gives

$$\lim_{\tau,\tau' \to -\infty} G_{in}(x,x') = \frac{2}{\mu}e^{h_- \tau + h_+ \tau'}\delta^2(\Omega,\Omega'). \quad (4.12)$$

The situation can be described as follows. As $\mathcal{I}^-$ is approached, the spatial part of the scalar kinetic terms are exponentially suppressed relative to the rest of the action. Neighboring points decouple and the theory becomes ultralo-

cal. It reduces to a harmonic oscillator at each point; hence the vanishing of G_{in}. However, the map defined by propagation from $\mathcal{I}^-$ to $\mathcal{I}^+$ is not ultralocal on the sphere. It introduces nontrivial correlators when one point is on $\mathcal{I}^-$ and the other is on $\mathcal{I}^+$.

Of course, in other vacua—such as the Euclidean vacuum—there are nontrivial $\mathcal{I}^-$ correlators. As will be seen in the next section, the wave functions for these vacua differ from the in vacuum wave function by terms which are nonlocal on $\mathcal{I}^-$. These terms are directly responsible for the nontrivial $\mathcal{I}^-$ correlators.

B. dS vacua as marginal CFT deformations

Now we argue that the dual interpretation of the one-parameter family of dS$_3$ vacua is a one-parameter family of marginal deformations of the CFT. It is convenient to define operators on $\mathcal{I}^-$ and $\mathcal{I}^+$ by

$$\lim_{\tau \to -\infty} \phi(\tau,\Omega) = \phi_+^{in}(\Omega)e^{h_+ \tau} + \phi_-^{in}(\Omega)e^{h_- \tau},$$

$$(4.13)$$

$$\lim_{\tau \to \infty} \phi(\tau,\Omega_A) = \phi_+^{out}(\Omega)e^{-h_+ \tau} + \phi_-^{out}(\Omega)e^{-h_- \tau}.$$

$\phi_\pm^{out}$ has been defined with an antipodal inversion relative to $\phi_\pm^{in}$ so that they transform the same way under conformal transformations [13]. These are position space versions of the creation operators associated to the spherical modes ϕ^{in} and ϕ^{out},

$$\phi_+^{in}(\Omega) = [\phi_-^{in}(\Omega)]^\dagger = \sqrt{\frac{2}{\mu}}\sum_{Lj} a_{Lj}^{in\dagger}Y_{Lj}^*(\Omega),$$

$$(4.14)$$

$$\phi_+^{out}(\Omega) = [\phi_-^{out}(\Omega)]^\dagger = \sqrt{\frac{2}{\mu}}\sum_{Lj} a_{Lj}^{out}Y_{Lj}(\Omega_A).$$

From the asymptotic Green functions (4.12) and (4.11) we find that the only nonzero commutators are

$$[\phi_-^{in}(\Omega),\phi_+^{in}(\Omega')] = [\phi_+^{out}(\Omega),\phi_-^{out}(\Omega')]$$

$$= \frac{2}{\mu}\delta^2(\Omega,\Omega'),$$

$$(4.15)$$

$$[\phi_\pm^{in}(\Omega),\phi_\pm^{out}(\Omega')] = \pm 2 \sinh \pi\mu \Delta_\pm(\Omega,\Omega').$$

The in and out operators are related by a Bogolyubov transformation and hence are not independent. In this section we take $\phi_\pm^{in}$ to be the fundamental operators. At a general point in the bulk ϕ is determined from its value on $\mathcal{I}^-$ via

$$\phi(x) = i\int_{\mathcal{I}^-} d^2x' \sqrt{g}G_C(x,x')\overleftrightarrow{\partial}_{\tau'}\phi(x'). \quad (4.16)$$

In particular, taking x to be on $\mathcal{I}^+$ and using the limiting expression for G_C (which does not depend on α) we find

$$\phi_\pm^{\text{out}}(\Omega) = -\mu \sinh \pi\mu \int d^2\Omega' \Delta_\pm(\Omega,\Omega') \phi_\mp^{\text{in}}(\Omega').$$
(4.17)

This is a position-space version of the Bogolyubov transformation (3.21).[7] We see that the absence of mixing between ϕ^{in} and $\phi^{\text{out}*}$, which seemed so surprising in Sec. III B, follows directly from the asymptotic behavior of the Green functions. We note parenthetically that this implies the identity (verified in [64])

$$(\mu \sinh \pi\mu)^2 \int d^2\Omega'' \Delta_-(\Omega,\Omega'') \Delta_+(\Omega'',\Omega') = \delta^2(\Omega,\Omega').$$
(4.18)

The $|\text{in}\rangle$ vacuum obeys

$$\phi_-^{\text{in}}(\Omega) |\text{in}\rangle = 0.$$
(4.19)

The general $|\alpha\rangle$ vacuum state discussed in Sec. II B can be constructed in terms of the in vacuum as

$$|\alpha\rangle = \exp\left\{ c(\gamma) \frac{\mu}{2} \int d^2\Omega \, \phi_+^{\text{in}} \phi_-^{\text{out}} \right.$$

$$\left. - c(\bar\gamma) \frac{\mu}{2} \int d^2\Omega \, \phi_-^{\text{in}} \phi_+^{\text{out}} \right\} |\text{in}\rangle,$$
(4.20)

where

$$e^\gamma = \frac{\sinh \dfrac{\pi\mu + \alpha}{2}}{\sinh \dfrac{\pi\mu - \alpha}{2}},$$
(4.21)

and the function c is given by Eq. (2.19). This equation may be formally rewritten as

$$|\alpha\rangle = C \exp\left(e^{\gamma^*} \frac{\mu}{4} \int d^2\Omega \, \phi_+^{\text{in}} \phi_-^{\text{out}} \right) |\text{in}\rangle.$$
(4.22)

These vacua obey the manifestly $SL(2,C)$ invariant condition

$$\phi_-^{\text{in}}(\Omega) |\alpha\rangle = -e^{\gamma^*} \phi_-^{\text{out}}(\Omega) |\alpha\rangle.$$
(4.23)

This is most easily seen by applying the representation $\phi_-^{\text{in}} = -(2/\mu)(\delta/\delta\phi_+^{\text{in}})$ of Eq. (4.15) to Eq. (4.22). In particular, the Euclidean vacuum has $\alpha = -\infty$ and therefore obeys

$$\phi_-^{\text{in}}(\Omega) |E\rangle = e^{-\pi\mu} \phi_-^{\text{out}}(\Omega) |E\rangle.$$
(4.24)

Now we consider the boundary field theory. Consider the two operators $\mathcal{O}_\pm$ dual to $\phi_\pm^{\text{in}}$ with conformal weights $h_\pm$. According to the dS/CFT correspondence [13], the dual $\mathcal{O}_\pm$ correlators are determined from the $\phi_\pm^{\text{in}}$ correlator (4.9) as

$$\langle \alpha | \mathcal{O}_\pm(\Omega) \sigma \pm(\Omega') | \alpha \rangle$$

$$= -\frac{\mu^2}{2} N_\alpha^2 (1 - e^{\alpha \pm \pi\mu})(1 - e^{\alpha^* \mp \pi\mu})$$

$$\times \Delta_\pm(\Omega,\Omega').$$
(4.25)

The commutators (4.15) also imply the contact terms

$$\langle \alpha | \mathcal{O}_-(\Omega) \mathcal{O}_+(\Omega') | \alpha \rangle = \frac{1}{1 - e^{\gamma + \gamma^*}} \frac{\mu}{2} \delta^2(\Omega,\Omega'),$$
(4.26)

$$\langle \alpha | \mathcal{O}_+(\Omega) \mathcal{O}_-(\Omega') | \alpha \rangle = \frac{e^{\gamma + \gamma^*}}{1 - e^{\gamma + \gamma^*}} \frac{\mu}{2} \delta^2(\Omega,\Omega').$$

From the CFT point of view this is an unusual contact term prescription in that it depends on the operator ordering.

What is the CFT origin of the parameter α? Usually a one-parameter family of correlators corresponds to a line of marginal deformations generated by a dimension (1, 1) operator. Indeed, $\mathcal{O}_+\mathcal{O}_-$ is a dimension (1, 1) operator. Let us consider adding this operator to the two-dimensional CFT action with real coefficient λ. At linear order this perturbs the correlators according to the formula

$$\delta_\lambda \langle \alpha | \mathcal{O}_+(\Omega) \mathcal{O}_+(\Omega') | \alpha \rangle$$

$$= -\langle \alpha | \frac{\lambda}{2} \int d\Omega'' \{\mathcal{O}_+(\Omega'')\mathcal{O}_-(\Omega''), \mathcal{O}_+(\Omega)\mathcal{O}_+(\Omega')\} | \alpha \rangle$$

$$= -4\mu\lambda \coth \gamma \langle \alpha | \mathcal{O}_+(\Omega) \mathcal{O}_+(\Omega') | \alpha \rangle.$$
(4.27)

Let us take α to be real, so that $\langle \alpha | \mathcal{O}_+\mathcal{O}_+ | \alpha \rangle$ is a monotonically increasing function of α. Then the variation of the two-point function as $\alpha \to \alpha + \epsilon$ is proportional to the deformation (4.27), which may be integrated to determine λ as a function of α.

This strongly suggests that the family of *CPT* and $SO(d,1)$ invariant vacuum states are marginal deformations of the boundary CFT generated by the (1, 1) operator $\mathcal{O}_+\mathcal{O}_-$. The two-point functions of these CFTs can all be made equivalent by rescaling operators, except for the special case $\alpha = -\pi\mu$. So, in principle, from this analysis alone the CFTs with $\alpha \neq \pi\mu$ might all be equivalent. In order to complete the argument one should check that the three-point function is not invariant under such rescalings. This has been shown in [65].[8]

V. $\mathcal{CPT}$ AND THE INNER PRODUCT

In this section we discuss various choices of norm for the Hilbert space of a real scalar field on dS$_3$, or equivalently the definition of the adjoint. The first naive choice one might make is

$$\phi^\dagger(x) = \phi(x).$$
(5.1)

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[7] In fact, expression (4.17) is singular for $\Omega = \Omega'$ and so is really defined by Eq. (3.21).

[8] We thank Greg Moore for discussions on this point.

However, Witten [17] argues that this choice may not be well defined for full quantum gravity outside of perturbation theory. An alternate norm is proposed [17] which involves path integral evolution form $\mathcal{I}^-$ to $\mathcal{I}^+$ together with $\mathcal{CPT}$ conjugation. In this section we will explicitly compute this norm for a free scalar and find, after a slight modification involving the form of $\mathcal{P}$, that it has a very natural boundary interpretation: it yields the Zamolodchikov metric for the boundary CFT.

Before delving into details it is instructive to recall an isomorphic discussion of norms which arises in the standard treatment of Euclidean CFT. Consider the mode expansion for a free boson on the Lorentzian cylinder (ignoring zero modes)

$$X(\sigma^+,\sigma^-)=i\sum_n\left(\frac{\alpha_n}{n}e^{-2\pi in\sigma^+}+\frac{\bar{\alpha}_n}{n}e^{2\pi in\sigma^-}\right). \quad (5.2)$$

Using $\alpha^\dagger_{-n}=\alpha_n$ one finds

$$X^\dagger(\sigma^+,\sigma^-)=X(\sigma^+,\sigma^-). \quad (5.3)$$

On the other hand, the standard mode expansion on the complex Euclidean plane is

$$X(z,\bar{z})=i\sum_n\frac{1}{n}\left(\frac{\alpha_n}{z^n}+\frac{\bar{\alpha}_n}{\bar{z}^n}\right). \quad (5.4)$$

Using $\alpha^\dagger_{-n}=\alpha_n$ one now finds

$$X^\dagger(z,\bar{z})=-i\sum_n\frac{1}{n}\left(\frac{\alpha_{-n}}{\bar{z}^n}+\frac{\bar{\alpha}_{-n}}{z^n}\right)$$

$$=X\left(\frac{1}{\bar{z}},\frac{1}{z}\right). \quad (5.5)$$

In this case the adjoint relates X at points in the Euclidean plane reflected across the unit circle. In particular the norm of the state created by $X(z,\bar{z})$ or any other operator is just the two-point function, and hence is the Zamolodchikov norm.

Returning now to dS$_3$, the naive adjoint rule (5.1) induces an adjoint in the Euclidean boundary CFT of the form $X^\dagger(z,\bar{z})=X(z,\bar{z})$. On the other hand, we will show that the modified Witten adjoint gives precisely (5.5). We further consider the dS$_3$ $SL(2,C)$ isometry generators $\mathcal{L}_n$, $\bar{\mathcal{L}}_n$, for $n=0,\pm1$. It is shown that $\mathcal{L}^\dagger_n=\bar{\mathcal{L}}_n$ for the naive adjoint, but $\mathcal{L}^\dagger_n=\mathcal{L}_{-n}$ for the modified adjoint. Although we take $d=3$, much of the following discussion carries over simply to higher dimensions.

A. Continuous and discrete symmetries of de Sitter space

dS$_3$ can be represented by the hyberboloid

$$X^+X^-+z\bar{z}=l^2 \quad (5.6)$$

in flat Minkowski space. The isometries of dS$_3$ are then inherited from the $SL(2,C)$ Lorentz isometries of Minkowski space. The six generators can be written as combinations $\vec{J}$

$+i\vec{K}$ of rotations and boosts together with their complex conjugates. We denote the associated Killing vectors by ζ_n and $\bar{\zeta}_n$ for $n=0,\pm1$. The past and future horizons of an observer world line at $z=0$ are located on the hyperboloid at $X^+X^-=0$. We denote the Killing vectors preserving this horizon as

$$\zeta_0+\bar{\zeta}_0=X^+\partial_+-X^-\partial_-,$$

$$\zeta_0-\bar{\zeta}_0=\bar{z}\partial_{\bar{z}}-z\partial_z. \quad (5.7)$$

The four additional Killing vectors are

$$\zeta_1=X^+\partial_z-\bar{z}\partial_-,$$

$$\zeta_{-1}=X^-\partial_{\bar{z}}-z\partial_+,$$

$$\bar{\zeta}_1=X^+\partial_{\bar{z}}-z\partial_-,$$

$$\bar{\zeta}_{-1}=X^-\partial_z-\bar{z}\partial_+. \quad (5.8)$$

They obey the Lie bracket relation

$$[\zeta_m,\zeta_n]=(n-m)\zeta_{m+n}. \quad (5.9)$$

In addition, we consider the two discrete symmetries parity and time reversal

$$PX^\pm=X^\pm, \quad Pz=-z,$$

$$TX^\pm=X^\mp, \quad Tz=z. \quad (5.10)$$

In terms of the global coordinates (τ,Ω), P takes a point $\Omega=(\theta,\varphi)$ on the 2-sphere to the point $P\Omega=(\theta,\pi+\varphi)$ and T takes τ to $-\tau$.

Our choice of parity P in Eq. (5.10) reflects all the coordinates about an observer at the south pole. An alternate choice is $Pz=\bar{z}$ which reflects only one coordinate. This is the choice employed in [17], motivated by the fact that the corresponding $\mathcal{CPT}$ operation is known to be an exact field theory symmetry, after taking a flat space limit of dS$_3$. We shall indicate below how the results are modified if this definition of P is employed.

B. $\mathcal{CPT}$

We now compute the action of the discrete symmetries $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ on the field operators. We consider a real scalar field so that $\mathcal{C}$ is trivial. We wish to find Hilbert space operators $\mathcal{P}$ and $\mathcal{T}$ that implement Eq. (5.10) on $\phi(x)$ as

$$\mathcal{P}\phi(x)\mathcal{P}=\phi(Px), \quad \mathcal{T}\phi(x)\mathcal{T}=\phi(Tx). \quad (5.11)$$

As usual $\mathcal{T}=UK$ is an antilinear operator which combines a unitary operator U with complex conjugation K of functions.

The mode expansions for ϕ in terms of the ϕ^{in} and ϕ^{out} modes are

$$\phi(\tau,\Omega)=\sum_{L,j}[a^{\text{in}}_{Lj}y^{\text{in}}_L(\tau)Y_{Lj}(\Omega)$$

$$+b^{\text{in}}_{Lj}y^{\text{in}*}_L(\tau)Y^*_{Lj}(\Omega)], \quad (5.12)$$

$$\phi(\tau,\Omega) = \sum_{L,j} \left[a_{Lj}^{\text{out}} y_L^{\text{out}}(\tau) Y_{Lj}(\Omega) \right.$$
$$\left. + b_{Lj}^{\text{out}} y_L^{\text{out}*}(\tau) Y_{Lj}^*(\Omega) \right]. \quad (5.13)$$

We have written lowering and raising operators as a's and b's, respectively, and are not assuming here that $a^\dagger = b$.

We define the action of $\mathcal{P}$ by

$$\mathcal{P} a_{Lj}^{\text{in}} \mathcal{P} = (-)^j a_{Lj}^{\text{in}}, \quad \mathcal{P} b_{Lj}^{\text{in}} \mathcal{P} = (-)^j b_{Lj}^{\text{in}}, \quad (5.14)$$

and similarly for the out operators. Since $Y_{Lj}(P\Omega) = (-)^j Y_{Lj}(\Omega)$ this definition reproduces Eq. (5.11). We define the action of $\mathcal{T}$ by

$$\mathcal{T} a_{Lj}^{\text{in}} \mathcal{T} = (-)^L a_{Lj}^{\text{out}}, \quad \mathcal{T} b_{Lj}^{\text{in}} \mathcal{T} = (-)^L b_{Lj}^{\text{out}}. \quad (5.15)$$

At the same time it acts as complex conjugation on functions. The wave functions appearing in Eq. (5.12) transform as

$$Y_{Lj}^*(\Omega) = (-)^L Y_{Lj}(\Omega),$$
$$y_L^{\text{in}*}(\tau) = y_L^{\text{out}}(-\tau). \quad (5.16)$$

Putting this together gives

$$\mathcal{T}\phi(\tau,\Omega)\mathcal{T} = \sum_{L,j} \left[a_{Lj}^{\text{out}} y_L^{\text{out}}(-\tau) Y_{Lj}(\Omega) \right.$$
$$\left. + b_{Lj}^{\text{out}} y_L^{\text{out}*}(-\tau) Y_{Lj}^*(\Omega) \right]$$
$$= \phi(-\tau,\Omega), \quad (5.17)$$

as required.

We wish to consider the action of $\mathcal{CPT}$ on the in and out field operators ϕ_+^{in} and ϕ_+^{out} defined by Eq. (4.13). Using Eqs. (5.14)–(5.16) these obey

$$\mathcal{PT}\phi_+^{\text{in}}(\Omega)\mathcal{PT} = \phi_-^{\text{out}}(P\Omega_A),$$
$$\mathcal{PT}\phi_+^{\text{out}}(\Omega)\mathcal{PT} = \phi_-^{\text{in}}(P\Omega_A). \quad (5.18)$$

C. The Witten inner product and modifications

Following Witten [17], we now describe a modified inner product. First we construct a bilinear pairing between states on $\mathcal{I}^-$ and states on $\mathcal{I}^+$. We will consider asymptotic states[9] on $\mathcal{I}^\pm$:

$$|\Psi^{\text{in}}\rangle = \int \Psi^{\text{in}}(\Omega)\phi_+^{\text{in}}(\Omega)|\text{in}\rangle,$$
$$\quad (5.19)$$
$$|\Psi^{\text{out}}\rangle = \int \Psi^{\text{out}}(\Omega)\phi_-^{\text{out}}(\Omega)|\text{in}\rangle,$$

where $\Psi^{\text{in}}(\Omega)$ and $\Psi^{\text{out}}(\Omega)$ are functions on the 2-sphere. Using Eq. (4.17), the out state can be expressed as a linear combination of in states

$$|\Psi^{\text{out}}\rangle = -\mu \sinh \pi\mu \int \int \Psi^{\text{out}}(\Omega')\Delta_-(\Omega',\Omega)$$
$$\times \phi_+^{\text{in}}(\Omega)|\text{in}\rangle. \quad (5.20)$$

This corresponds to evolving the state $|\Psi^{\text{out}}\rangle$ backwards from $\mathcal{I}^+$ to $\mathcal{I}^-$, and defines the bilinear pairing

$$(\Psi^{\text{out}}|\Psi^{\text{in}}) = -\mu \sinh \pi\mu$$
$$\times \int \int \Psi^{\text{out}}(\Omega')\Delta_-(\Omega',\Omega)\Psi^{\text{in}}(\Omega).$$
$$\quad (5.21)$$

We now use the pairing to define an inner product on $\mathcal{I}^-$ that is antilinear in the first argument. Note that applying $\mathcal{CPT}$ to a state on $\mathcal{I}^-$ gives us a state on $\mathcal{I}^+$:

$$\mathcal{CPT}|\Psi^{\text{in}}\rangle = \int \Psi^{\text{in}*}(P\Omega_A)\phi_-^{\text{out}}(\Omega)|\text{in}\rangle \quad (5.22)$$

to which we may apply the pairing (5.21). We find the inner product between two states on $\mathcal{I}^-$

$$\langle\Psi^{\text{in}}|\Upsilon^{\text{in}}\rangle = -\mu \sinh \pi\mu \int \int \Psi^{\text{in}*}(\Omega)$$
$$\times \Delta_-(P\Omega_A,\Omega')\Upsilon^{\text{in}}(\Omega'). \quad (5.23)$$

For free field theory the norm (5.23) implies the adjoint relations[10]

$$\phi^\dagger(x) = \phi(PTx),$$
$$\phi_\pm^{\text{in}\dagger}(\Omega) = \phi_\pm^{\text{out}}(P\Omega_A). \quad (5.24)$$

This may look strange at first but is in fact precisely the usual norm employed for a Euclidean CFT. Note that P coupled with the antipodal map is reflection about the equator, so that

$$PA(z,\bar{z}) = \left(\frac{1}{z}, \frac{1}{z}\right), \quad (5.25)$$

as in Eq. (5.5). For states constructed by acting with operators on $\mathcal{I}^-$, it therefore follows that the norm is simply the two-point function. Hence Eq. (5.23) gives the Zamolodchikov metric on the boundary CFT.

Formula Eq. (5.23) in fact remains valid for any choice of P. Using $Pz = \bar{z}$ as in [17], one finds instead of Eq. (5.25), $PA(z,\bar{z}) = (-1/z, -1/z)$. The adjoint then involves rotation by π about $z = \pm i$ rather than reflection across the unit disc.

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[9]Of course, these states are linear in b^{in} and b^{out}. The general asymptotic states will take a more complicated form.

[10]It is intriguing that this adjoint relates degrees of freedom separated by a horizon.

D. Adjoints of the $SL(2,C)$ generators

The quantum generators of the symmetries (5.7) and (5.8) are as usual given by

$$\mathcal{L}_z = \int_\Sigma d\Sigma^\mu \mathcal{T}_{\mu\nu} \zeta_n^\nu,$$

$$\bar{\mathcal{L}}_n = \int_\Sigma d\Sigma^\mu \mathcal{T}_{\mu\nu} \bar{\zeta}_n^\nu, \qquad (5.26)$$

for any complete spacelike slice Σ. We choose Σ to be the throat $X^+ = X^-$ because it is mapped to itself under both P and T. For a massive scalar,

$$\mathcal{T}_{\mu\nu}(x) = \partial_\mu \phi(x) \partial_\nu \phi(x) - \tfrac{1}{2} g_{\mu\nu} [(\nabla \phi(x))^2 + m^2 \phi^2(x)]. \qquad (5.27)$$

With the ordinary inner product, $\mathcal{T}_{\mu\nu}$ is Hermitian, and one finds $\mathcal{L}_n^\dagger = \bar{\mathcal{L}}_n$. With the modified inner product, one has

$$\mathcal{L}_n^\dagger = \int d\Sigma^\mu(x) \mathcal{T}_{\mu\nu}(PTx) \bar{\zeta}_n^\nu(x). \qquad (5.28)$$

We then consider a coordinate transformation $x' = PTx$. One finds

$$\mathcal{L}_n^\dagger = \int d\Sigma^\mu(PTx') \mathcal{T}_{\mu\nu}(x') \bar{\zeta}_n^\nu(PTx'). \qquad (5.29)$$

Using the relations

$$\zeta_n(PTx') = -\bar{\zeta}_{-n}(x'),$$

$$d\Sigma^\mu(PTx') = -d\Sigma^\mu(x'), \qquad (5.30)$$

it follows that

$$\mathcal{L}_n^\dagger = \mathcal{L}_{-n}. \qquad (5.31)$$

In [13] the $SL(2,C)$ isometries of dS$_3$ were conjectured to extend to a full Virasoro symmetry of the full quantum gravity (not just a free scalar). This naturally acted not on closed spacelike slices but on asymptotically flat slices ending on $\mathcal{I}$. It would be interesting to compute the adjoints of these generators.

VI. THE CYLINDER

In this section we study scalar field theory in static coordinates. Again for simplicity we specialize to dS$_3$, although we expect the higher dimensional cases to be similar. The metric is

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{(1-r^2)} + r^2 d\varphi^2. \qquad (6.1)$$

This metric is singular at the horizons $r=1$, which divides dS$_3$ into four regions. There are two regions with $0 \le r < 1$ corresponding to the causal diamonds of observers at the

north and south poles. We shall refer to these as the northern and southern diamonds. There are two more regions with $1 < r < \infty$ containing $\mathcal{I}^+$ and $\mathcal{I}^-$ which we shall refer to as the future and past triangles. On $\mathcal{I}^\pm$, where $r \to \infty$, the spatial metric approaches $r^2(dt^2 + d\varphi^2)$ and hence is conformal to the cylinder.

Unlike the global coordinates, static coordinates do not smoothly cover all of dS$_d$. However, they are well suited to describing the physics associated to an observer who can access a single causal diamond. The Killing vector $\partial/\partial t$ is manifest in static coordinates, but is future-directed only in the southern diamond; it is past-directed in the northern diamond and spacelike in the past and future triangles. In the following we solve the scalar wave equation in the four regions. Then we patch the solutions together to get a global solution over all of dS$_3$ by matching at the horizons. We further show explicitly that tracing the Euclidean vacuum over the Hilbert space of the northern modes leads to a thermal density matrix in the southern diamond.

A. The wave equation

The equation of motion for a scalar field of mass m is $(\nabla^2 - m^2)\phi = 0$. In static coordinates, this becomes

$$\left[-\frac{1}{1-r^2} \partial_t^2 + \frac{1}{r} \partial_r(1-r^2)r\partial_r + \frac{1}{r^2} \partial_\varphi^2 - m^2 \right]\phi = 0. \qquad (6.2)$$

The equation separates, so that a general solution can be expanded

$$\phi(t,r,\varphi) = \int_0^\infty d\omega \sum_{j=-\infty}^\infty a_{\omega j}\phi_{\omega j} + b_{\omega j}\phi^{\omega j}$$

$$+ a_{\omega_j}^\dagger \phi_{\omega j}^* + b_{\omega j}^\dagger \phi^{\omega j *}, \qquad (6.3)$$

where

$$\phi_{\omega j} = f_{\omega j}(r)e^{-i\omega t + ij\varphi}, \qquad \phi^{\omega j} = f^{\omega j}(r)e^{-i\omega t + ij\varphi}, \qquad (6.4)$$

and $f_{\omega j}(r)$, $f^{\omega j}(r)$ are two linearly independent solutions of the radial equation

$$(1-r^2)\frac{d^2 f_{\omega j}}{dr^2} + \left(\frac{1}{r} - 3r\right)\frac{df_{\omega j}}{dr}$$

$$+ \left(\frac{\omega^2}{1-r^2} - \frac{j^2}{r^2} - m^2\right)f_{\omega j} = 0. \qquad (6.5)$$

B. The northern and southern diamonds

A solution smooth near $r=0$ is given by

$$\phi_{\omega j}^S = f_{\omega j}(r)e^{-i\omega t + ij\varphi},$$

$$f_{\omega j}(r) \equiv r^{|j|}(1-r^2)^{i\omega/2}F(a,b;c;r^2),$$

$$a \equiv \frac{1}{2}(|j| + i\omega + h_+), \qquad (6.6)$$

$$b \equiv \frac{1}{2}(|j| + i\omega + h_-),$$

$$c \equiv 1 + |j|.$$

We have not normalized this solution, although the necessary factor follows from computations below. The superscript S denotes that this solution is in the southern diamond. One can show from the transformation formulas for hypergeometric functions (see Appendix B) that

$$f^*_{\omega j} = f_{-\omega j} = f_{\omega j}. \tag{6.7}$$

Similarly we may define northern modes

$$\phi^N_{\omega j} = f_{\omega j}(r) e^{-i\omega t + i j\varphi}. \tag{6.8}$$

It is convenient to use the time coordinate t both in the northern and in the southern diamond. Although this coordinate system does not uniquely label points on all of dS_3, there will be no confusion since we denote northern functions with a superscript N. The coordinate t runs forward in the southern diamond and backward in the northern diamond. Hence for $\omega > 0$ the modes (6.6) are positive frequency and (6.8) are negative frequency.

Near the horizon, for $r \to 1$, one can show (see Appendix B for details) that Eq. (6.6) becomes

$$\phi^S_{\omega j} \to e^{-i\omega t + i j\varphi} \Gamma(1 + |j|) \left[\frac{\Gamma(-i\omega)}{\Gamma\left(\frac{1}{2}(|j| - i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(|j| - i\omega + h_-)\right)} (1 - r^2)^{i\omega/2} \right.$$

$$\left. + \frac{\Gamma(i\omega)}{\Gamma\left(\frac{1}{2}(|j| + i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(|j| + i\omega + h_-)\right)} (1 - r^2)^{-\omega/2} \right]. \tag{6.9}$$

In order to analyze the flux across the horizons it is useful to introduce Kruskal coordinates

$$r = \frac{1 + UV}{1 - UV},$$

$$t = \frac{1}{2} \ln\left(-\frac{U}{V}\right), \tag{6.10}$$

in which

$$ds^2 = \frac{1}{(1 - UV)^2} [-4 dU dV + (1 + UV)^2 d\varphi^2]. \tag{6.11}$$

$U > 0$ and $V < 0$ in the southern diamond. The future (past) horizon is at $V = 0$ ($U = 0$). In contrast to the static coordinates, Kruskal coordinates are nonsingular at the horizon.

The modes (6.9) become, for $r \to 1$ ($UV \to 0$):

$$\phi^S_{\omega j} \to e^{i j\varphi} [\alpha_{\omega j}(-V)^{i\omega} + \alpha^*_{\omega j} U^{-i\omega}], \tag{6.12}$$

where we define the complex constants

$$\alpha_{\omega j} \equiv \frac{\Gamma(1 + |j|) \Gamma(-i\omega) 2^{i\omega}}{\Gamma\left(\frac{1}{2}(|j| - i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(|j| - i\omega + h_-)\right)}$$

$$= \alpha^*_{-\omega j}. \tag{6.13}$$

The first term in Eq. (6.12) is incoming flux across the past horizon, while the second is outgoing flux across the future horizon. A similar analysis in the northern diamond with $U < 0$, $V > 0$ gives for $r \to 1$:

$$\phi^N_{\omega j} \to e^{i j\varphi} [\alpha_{\omega j} V^{i\omega} + \alpha^*_{\omega j}(-U)^{-i\omega}]. \tag{6.14}$$

The northern and southern modes are simply related by

$$\phi^S_{\omega j}(-U, -V) = \phi^N_{\omega j}(U, V). \tag{6.15}$$

The second family of solutions is given by

$$\phi^{\omega j} = \ln(r^2) \phi_{\omega j} + e^{-i\omega t + i j\varphi} r^{|j|} (1 - r^2)^{i\omega/2}$$

$$\times \sum_{n = -|j|}^{\infty} A_n r^{2n}, \tag{6.16}$$

where the coefficients A_n are given in, e.g., Eq. 15.5.19 of [66]. These modes are singular at $r = 0$ for all j and hence are excluded.

C. The past and future triangles

Let us analyze the behavior of the modes in the past triangle (which includes $\mathcal{I}^-$ but not $\mathcal{I}^+$) where $r^2 > 1$. A complex solution of Eq. (6.2) is

$$\phi^{in+}_{\omega j} = f^+_{\omega j}(r) e^{-i\omega t + i j\varphi},$$

$$f^+_{\omega j}(r) \equiv r^{-h_+} \left(1 - \frac{1}{r^2}\right)^{i\omega/2} F\left(a, 1 - a^*; h_+; \frac{1}{r^2}\right). \tag{6.17}$$

Using properties of the hypergeometric functions one finds that $f_{\omega j}^+$ is invariant under $\omega \to -\omega$, but is not real. Therefore the second solution of Eq. (6.2) is obtained by complex conjugation:

$$\phi_{\omega j}^{\text{in}-} = (f_{\omega j}^+(r))^* e^{-i\omega t + ij\varphi}. \tag{6.18}$$

This is equivalent to replacing h_+ with h_- in Eq. (6.17).

Near $\mathcal{I}^-$ we find

$$\phi_{\omega j}^{\text{in}\pm} \sim r^{-h_\pm}. \tag{6.19}$$

In the past triangle the coordinate r is timelike and past-directed, so that the ϕ^{in-} are positive frequency for $m^2 > 1$.

Near the horizon, for $r \to 1$, we find

$$\phi_{\omega j}^{\text{in}+} \to e^{-i\omega t + ij\varphi}\Gamma(h_+)\left[\frac{\Gamma(-\omega)}{\Gamma\left(\frac{1}{2}(|j| - i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(-|j| - i\omega + h_+)\right)}(r^2 - 1)^{i\omega/2}\right.$$
$$\left. + \frac{\Gamma(i\omega)}{\Gamma\left(\frac{1}{2}(-|j| + i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(|j| + i\omega + h_+)\right)}(r^2 - 1)^{-i\omega/2}\right]. \tag{6.20}$$

The relation between static and Kruskal coordinates in the past triangle is

$$r = \frac{1 + UV}{1 - UV},$$

$$t = \frac{1}{2}\ln\left(\frac{U}{V}\right). \tag{6.21}$$

U and V are both negative in this region. The boundary with the northern (southern) diamond is at $V = 0$ ($U = 0$). The near horizon behavior (6.20) becomes

$$\phi_{\omega j}^{\text{in}+} \to e^{ij\varphi}[\beta_{\omega j}(-V)^{i\omega} + \beta_{-\omega j}(-U)^{-i\omega}], \tag{6.22}$$

where

$$\beta_{\omega j} \equiv \frac{\Gamma(h_+)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(|j| - i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(-|j| - i\omega + h_+)\right)}. \tag{6.23}$$

Similarly one finds near $r = 1$ that

$$\phi_{\omega j}^{\text{in}-} \to e^{ij\varphi}[\beta_{-\omega j}^*(-V)^{i\omega} + \beta_{\omega j}^*(-U)^{-i\omega}]. \tag{6.24}$$

One may also define modes in the future triangle by

$$\phi_{\omega j}^{\text{out}+} = f_{\omega j}^+(r)e^{-i\omega t + ij\varphi},$$

$$\phi_{\omega j}^{\text{out}-} = (f_{\omega j}^+(r))^* e^{-i\omega t + ij\varphi}. \tag{6.25}$$

Near $\mathcal{I}^+$ we find

$$\phi_{\omega j}^{\text{out}+} \sim r^{-h_\pm}. \tag{6.26}$$

In the future triangle the coordinate r is future-directed, so that the $\phi^{\text{out}+}$ are positive frequency.

The relation between static and Kruskal coordinates in the future triangle is again given by Eq. (6.21), which means that t increases to the south (north) in the future (past) triangle. U and V are both positive in this region. The boundary of the future triangle with the northern (southern) diamond is at $U = 0$ ($V = 0$). Near the horizons ($UV = 0$) the ϕ^{out} modes obey

$$\phi_{\omega j}^{\text{out}+} = e^{ij\varphi}[\beta_{\omega j}V^{i\omega} + \beta_{-\omega j}U^{-i\omega}],$$

$$\phi_{\omega j}^{\text{out}-} = e^{ij\varphi}[\beta_{-\omega j}^* V^{i\omega} + \beta_{\omega j}^* U^{-i\omega}]. \tag{6.27}$$

The past and future modes are simply related by

$$\phi_{\omega j}^{\text{out}\pm}(U,V) = \phi_{\omega j}^{\text{in}\pm}(-U, -V). \tag{6.28}$$

D. Matching across the horizon

In Secs. VI B and VI C we have described solutions in the past and future triangles as well as the northern and southern diamonds. By matching fluxes across the horizon, these may be extended to global solutions over all of dS$_3$. For example the $(-V)^{i\omega}$ $((-U)^{-i\omega})$ terms in the past modes (6.22) and (6.24) carry flux into the southern (northern) diamond. The continuation of Eqs. (6.22) and (6.24) into these regions is obtained by matching to Eq. (6.12) along $U = 0$ and to Eq. (6.14) along $V = 0$. Matching across the horizon again then yields the future mode.

Henceforth we shall use the symbol $\phi^{\text{in}\pm}$ to denote the global solution so constructed. Similarly, $\phi^{\text{out}\pm}$ will denote the global solution agreeing with Eq. (6.27) in the future triangle. We may also construct global solution ϕ^S (ϕ^N) that agree with the modes (6.6) [(6.8)] in the southern (northern) diamond—these solutions vanish in the northern (southern) diamond.

From the matching procedure outlined above we find that these modes obey

$$\begin{pmatrix} \phi^S_{\omega j} \\ \phi^N_{\omega j} \end{pmatrix} = \mathbf{A}_{\omega j} \begin{pmatrix} \phi^{\text{in}+}_{\omega j} \\ \phi^{\text{in}-}_{\omega j} \end{pmatrix} = \mathbf{A}^*_{\omega j} \begin{pmatrix} \phi^{\text{out}-}_{\omega j} \\ \phi^{\text{out}+}_{\omega j} \end{pmatrix}, \qquad (6.29)$$

where

$$N_{\omega j} \mathbf{A}_{\omega j} = \begin{pmatrix} \alpha_{\omega j}\beta^*_{\omega j} & -\alpha_{\omega j}\beta_{-\omega j} \\ -\alpha^*_{\omega j}\beta^*_{-\omega j} & \alpha^*_{\omega j}\beta_{\omega j} \end{pmatrix} \qquad (6.30)$$

and

$$N_{\omega j} \equiv (\beta_{\omega j}\beta^*_{\omega j} - \beta_{-\omega j}\beta^*_{-\omega j}) = -\frac{\mu}{\omega}. \qquad (6.31)$$

Reversing the signs of U and V and using $\sigma_x \mathbf{A}\sigma_x = \mathbf{A}^*$, one finds that the second equation in Eq. (6.29) follows from the first. The Bogolyubov transformation from $\mathcal{I}^-$ to $\mathcal{I}^+$ then follows from Eq. (6.29) as

$$\begin{pmatrix} \phi^{\text{out}-}_{\omega j} \\ \phi^{\text{out}+}_{\omega j} \end{pmatrix} = \mathbf{B}_{\omega j} \begin{pmatrix} \phi^{\text{in}+}_{\omega j} \\ \phi^{\text{in}-}_{\omega j} \end{pmatrix}, \qquad (6.32)$$

where

$$\mathbf{B}_{\omega j} = \sigma_x \mathbf{A}^{-1}_{\omega j} \sigma_x \mathbf{A}_{\omega j} = \begin{pmatrix} \dfrac{\alpha_{\omega j}\beta^*_{\omega j}}{\alpha^*_{\omega j}\beta_{\omega j}} & 0 \\ 0 & \dfrac{\alpha^*_{\omega j}\beta_{\omega j}}{\alpha_{\omega j}\beta^*_{\omega j}} \end{pmatrix}. \qquad (6.33)$$

As with the spherical modes of Sec. III B, the Bogolyubov transformation (6.32) is trivial. The vacuum $|\text{in}\rangle$ defined by the modes ϕ^{in} is identical to the vacuum $|\text{out}\rangle$ defined by the ϕ^{out}.

E. Euclidean modes on the cylinder

In this section, following [67], we write the Euclidean modes as linear combinations of northern and southern modes.

In Kruskal coordinates the southern modes (6.6) in the southern diamond are of the form

$$\phi^S_{\omega j} = f_{\omega j}(UV) e^{ij\varphi} \left(-\frac{V}{U} \right)^{i\omega/2} \qquad (6.34)$$

for $U>0$, $V<0$, and vanish for $U<0$, $V>0$. The northern modes in the northern diamond are of the same form

$$\phi^N_{\omega j} = f_{\omega j}(UV) e^{ij\varphi} \left(-\frac{V}{U} \right)^{i\omega/2}, \qquad (6.35)$$

but have support for $U<0$, $V>0$ instead of $U>0$, $V<0$. We wish to find a linear combination of Eqs. (6.34) and (6.35) which

is analytic in the lower complex U and V planes.[11] This can be accomplished by analytically continuing the southern modes (6.34) to the northern diamond along the contour

$$U \to e^{-i\gamma}U, \quad V \to e^{i\gamma}V, \qquad (6.36)$$

taking γ from 0 to π. Notice that the product UV is independent of γ, so that the continuation of the southern mode (6.34) is

$$e^{-\pi\omega} f_{\omega j}(UV) e^{ij\varphi} \left(-\frac{V}{U} \right)^{i\omega/2}. \qquad (6.37)$$

Comparing with Eq. (6.35) we see that the linear combination

$$\phi^E_{\omega j} = \phi^S_{\omega j} + e^{-\pi\omega}\phi^N_{\omega j} \qquad (6.38)$$

is analytic in the lower half of the complex U and V planes. Since t runs backwards in the northern diamond, this is a linear combination of positive and negative frequency modes. A second linear combination

$$\phi^{E'}_{\omega j} = (\phi^N_{\omega j})^* + e^{-\pi\omega}(\phi^S_{\omega j})^* \qquad (6.39)$$

is also analytic in the lower half plane. Both ϕ^E and $\phi^{E'}$ are positive frequency for $\omega > 0$.

F. MA transform to Euclidean modes

In this section we will show that the $|\text{in}\rangle$ vacuum on the cylinder is the same as the $|\text{in}\rangle$ vacuum on the sphere by showing that it is an MA transform of the Euclidean vacuum with $\alpha = -\pi\mu$. This result is anticipated by the fact that the dual CFTs should be simply related by the conformal transformation from the sphere to the cylinder. Nevertheless, it provides a useful check on our constructions.

The first step is to redefine $\phi^{\text{in}\pm}$ in order to simplify the expression for $\mathbf{A}$ in Eq. (6.29). Let

$$\tilde{\phi}^{\text{in}+}_{\omega j} = i^j \frac{\alpha_{\omega j}\beta^*_{\omega j}}{N_{\omega j}} \phi^{\text{in}+}_{\omega j},$$

$$\tilde{\phi}^{\text{in}-}_{\omega j} = (-i)^j \frac{\alpha^*_{\omega j}\beta_{\omega j}}{N_{\omega j}} \phi^{\text{in}-}_{\omega j}. \qquad (6.40)$$

Then Eq. (6.29) becomes

$$\begin{pmatrix} \phi^S_{\omega j} \\ \phi^N_{\omega j} \end{pmatrix} = (-i)^j \begin{pmatrix} 1 & q \\ (-)^j q & (-)^j \end{pmatrix} \begin{pmatrix} \tilde{\phi}^{\text{in}+}_{\omega j} \\ \tilde{\phi}^{\text{in}-}_{\omega j} \end{pmatrix}, \qquad (6.41)$$

with

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[11]Euclidean modes were defined earlier to be regular on the lower Euclidean hemisphere ($\tau^{\text{Re}}=0, (-\pi/2) \leq \tau^{\text{Im}} \leq 0$). Explicit transformation of coordinates shows that sgn $U^{\text{Im}}=$ sgn $V^{\text{Im}}=$ sgn τ^{Im}. The lower pole, $\tau = -i(\pi/2)$, maps to a single point, $U=V=-i$, independently of θ. Smooth curves through this pole remain smooth in the U and V planes. Thus, modes that are analytic and bounded in the lower half U and V planes will be regular on the lower Euclidean hemisphere.

$$q \equiv (-)^{j+1} \frac{\alpha_{\omega j} \beta_{-\omega j}}{\alpha^*_{\omega j} \beta_{\omega j}} = - \frac{(-)^j + e^{\pi(\omega + \mu)}}{e^{\pi \omega} + (-)^j e^{\pi \mu}}. \quad (6.42)$$

It follows that the Euclidean modes obey

$$\phi^E_{\omega j} = \phi^S_{\omega j} + e^{-\pi \omega} \phi^N_{\omega j}$$

$$= (-i)^j \frac{e^{\pi \omega} - e^{-\pi \omega}}{e^{\pi \omega} + (-)^j e^{\pi \mu}} (\tilde{\phi}^{\text{in}+}_{\omega j} - e^{\pi \mu} \tilde{\phi}^{\text{in}-}_{\omega j}). \quad (6.43)$$

Inverting this relation, one recovers $\alpha = -\pi \mu$.

G. The thermal state

Let us summarize the southern and northern mode expansions:

$$\phi^S(t,r,\varphi) = \int_0^\infty d\omega \sum_{j=-\infty}^\infty a^S_{\omega j} \phi^S_{\omega j} + (a^S_{\omega j})^\dagger (\phi^S_{\omega j})^*,$$
$$(6.44)$$

$$\phi^N(t,r,\varphi) = \int_0^\infty d\omega \sum_{j=-\infty}^\infty a^N_{\omega j} \phi^N_{\omega j} + (a^N_{\omega j})^\dagger (\phi^N_{\omega j})^*.$$

Here we take the modes ϕ^S and ϕ^N to be normalized with respect to the Klein-Gordon inner product (2.7). The Fock space in the southern diamond is constructed with lowering operators $a^S_{\omega j}$ and raising operators $(a^S_{\omega j})^\dagger$. The Fock space in the northern diamond is constructed with lowering operators $(a^N_{\omega j})^\dagger$ and raising operators $a^N_{\omega j}$.

The modes (6.38) and (6.39) annihilate the Euclidean vacuum, $|E\rangle$. This allows us to express $|E\rangle$ as a superposition of states in the northern and southern Fock spaces [68]:

$$|E\rangle = \prod_{\omega=0}^\infty \prod_{j=-\infty}^\infty (1 - e^{-2\pi\omega})^{1/2}$$

$$\times \exp[e^{-\pi\omega}(a^S_{\omega j})^\dagger a^N_{\omega j}] |S\rangle \otimes |N\rangle$$

$$= \prod_{\omega,j} (1 - e^{-2\pi\omega})^{1/2}$$

$$\times \sum_{n_{\omega j}=0}^\infty e^{-\pi\omega n_{\omega j}} |n_{\omega j},S\rangle \otimes |n_{\omega j},N\rangle. \quad (6.45)$$

Here $|S\rangle$ and $|N\rangle$ are the southern and northern vacua, and

$$|n_{\omega j},S\rangle = (n_{\omega j}!)^{-1/2} [(a^S_{\omega j})^\dagger]^{n_\omega} |S\rangle,$$

$$|n_{\omega j},N\rangle = (n_{\omega j}!)^{-1/2} [a^N_{\omega j}]^{n_{\omega j}} |N\rangle. \quad (6.46)$$

Only the southern diamond is causally accessible to an observer at the south pole. The quantum state in this region is described by a density matrix ρ^S, which is obtained from a global state by tracing over the field modes in the northern diamond. For the Euclidean vacuum (6.45) we obtain

$$\rho^S_E = \text{tr}_N |E\rangle\langle E| = \prod_{\omega,j} \left[(1 - e^{-2\pi\omega}) \right.$$

$$\left. \times \sum_{n_{\omega j}} e^{-2\pi\omega n_{\omega j}} |n_{\omega j},S\rangle\langle n_{\omega j},S| \right]. \quad (6.47)$$

Recall that the Killing vector $\xi^\mu \partial_\mu = \partial_t$ is everywhere timelike and future directed in the southern diamond. Neglecting gravitational back-reaction of the field modes, this allows us to define a Hamiltonian for the southern modes:

$$\mathcal{M} = \int_{\Sigma^S} d\Sigma^\mu \mathcal{T}_{\mu\nu} \xi^\nu = \int_0^\infty d\omega \sum_{j=-\infty}^\infty (a^S_{\omega j})^\dagger a^S_{\omega j} \omega,$$
$$(6.48)$$

where $\mathcal{T}$ is the stress tensor of the scalar field. Here Σ^S is a $t = \text{const}$ Cauchy surface in the southern diamond with normal vector is $n^\mu_\Sigma \partial_\mu = (1 - r^2)^{-1/2} \partial_t$. This definition of energy is natural for the observer at the south pole. For later use, we also define the angular momentum $\mathcal{J}$ as the conserved charge associated with the Killing vector $v^\mu \partial_\mu = -\partial_\varphi$:

$$\mathcal{J} = \int_{\Sigma^S} d\Sigma^\mu \mathcal{T}_{\mu\nu} v^\nu = \int_0^\infty d\omega \sum_{j=-\infty}^\infty (a^S_{\omega j})^\dagger a^S_{\omega j} j. \quad (6.49)$$

With respect to the Hamiltonian $\mathcal{M}$, the southern state (6.47) becomes a thermal density matrix

$$\rho^S_E = C \exp\left(-\frac{\mathcal{M}}{T} \right) \quad (6.50)$$

with temperature $T = 1/2\pi$; $C = \Pi(1 - e^{-2\pi\omega})$ is a normalization factor.

VII. KERR–de SITTER

In this section we generalize the discussion of the previous sections to the three-dimensional Kerr–de Sitter solution, which represents a spinning point mass in dS_3.

A. Static coordinates

The Kerr–de Sitter metric describes the gravitational field of a point particle whose mass and spin are parametrized by $1 - M$ and J:

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + N^\varphi dt)^2. \quad (7.1)$$

The lapse and shift functions are

$$N^2 = M - r^2 + \frac{16G^2 J^2}{r^2}, \quad N^\varphi = -\frac{4GJ}{r^2}. \quad (7.2)$$

The lapse function vanishes for one positive value of r:

$$r_+ = \frac{1}{2}(\sqrt{\tau} + \sqrt{\bar{\tau}}), \quad (7.3)$$

where

$$\tau \equiv M + i(8GJ). \tag{7.4}$$

This is the cosmological event horizon surrounding an observer at $r=0$. It has a Bekenstein-Hawking entropy [69,70] of

$$S = \frac{\pi r_+}{2G} = \frac{\pi}{4G}(\sqrt{\tau} + \sqrt{\bar{\tau}}). \tag{7.5}$$

B. Kerr-dS$_3$ as a quotient of dS$_3$

In $2+1$ dimensions, there is no black hole horizon for Kerr–de Sitter because the "black hole" degenerates to a conical singularity at the origin. This is best seen by writing the metric as an identification of de Sitter [71]. Let us define $\mu = r_+$ and $\alpha = 4GJ/r_+$, so that

$$M = \mu^2 - \alpha^2, \quad J = \frac{\mu\alpha}{4G}. \tag{7.6}$$

The coordinate transformation

$$\tilde{t} = \mu t + \alpha\varphi,$$

$$\tilde{\varphi} = \mu\varphi - \alpha t, \tag{7.7}$$

$$\tilde{r} = \sqrt{\frac{r^2 + \alpha^2}{\mu^2 + \alpha^2}}$$

changes the Kerr–de Sitter metric to the vacuum form

$$ds^2 = -(1-\tilde{r}^2)d\tilde{t}^2 + \frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2 d\tilde{\varphi}^2, \tag{7.8}$$

but with a nonstandard coordinate identification. In empty de Sitter space, $(\tilde{t}, \tilde{r}, \tilde{\varphi} + 2\pi n)$ labels the same point for all integer n. In the presence of a particle, the points

$$(\tilde{t}, \tilde{r}, \tilde{\varphi}) + 2\pi n(\alpha, 0, \mu) \tag{7.9}$$

are identified instead.

C. Kerr-dS$_3$ temperature and angular potential

In this section we consider a scalar field in Kerr-dS$_3$. The cylinder mode solutions found for de Sitter space in Sec. VI are also solutions in Kerr–de Sitter space, after the substitutions $t \to \tilde{t}$, $r \to \tilde{r}$, and $\varphi \to \tilde{\varphi}$ are performed. For the modes to remain single valued, the angular momentum j must be non-integer:

$$j = \frac{n + \omega\alpha}{\mu}, \quad n \text{ integer}. \tag{7.10}$$

The mode analysis carries over trivially. In particular, the Euclidean modes (6.38) and (6.39) take the same form in Kerr–de Sitter space.

Analogues of Eqs. (6.48) and (6.49) define conserved charges associated with the Killing vectors $\tilde{\xi}^\mu \partial_\mu = \partial_{\tilde{t}}$ and $\tilde{\upsilon}^\mu \partial_\mu = -\partial_{\tilde{\varphi}}$:

$$\tilde{\mathcal{M}} = \int_{\Sigma^S} d\Sigma^\mu \mathcal{T}_{\mu\nu}\tilde{\xi}^\nu = \int_0^\infty d\omega \sum_{j=-\infty}^\infty (a^S_{\omega j})^\dagger a^S_{\omega j}\omega,$$

$$\tilde{\mathcal{J}} = \int_{\Sigma^S} d\Sigma^\mu \mathcal{T}_{\mu\nu}\tilde{\upsilon}^\nu = \int_0^\infty d\omega \sum_{j=-\infty}^\infty (a^S_{\omega j})^\dagger a^S_{\omega j}j,$$

$$\tag{7.11}$$

where $\mathcal{T}_{\mu\nu}$ is the matter stress tensor. Here the hypersurface Σ^S is defined, for example, by the normal vector

$$n^\mu_{\Sigma^S}\partial_\mu = \frac{\tilde{r}}{\sqrt{1-\tilde{r}^2}}\frac{\mu}{r}\partial_{\tilde{t}} + \frac{\sqrt{1-\tilde{r}^2}}{\tilde{r}}\frac{\alpha}{r}\partial_{\tilde{\phi}}. \tag{7.12}$$

(For $\alpha > 0$, Σ^S is not a spacelike surface near the origin; this does not affect the definition of conserved quantities.) The expressions for $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{J}}$ nevertheless take the same form as $\mathcal{M}$ and $\mathcal{J}$ in de Sitter space. The Euclidean state, restricted to the southern diamond ($\tilde{r}<1$), is a density matrix

$$\rho^S_E = C\exp(-2\pi\tilde{\mathcal{M}}). \tag{7.13}$$

In the (t,r,φ) coordinates, the asymptotic metric of Kerr–de Sitter space takes a standard form near $\mathcal{I}$ (detailed in Sec. VII D). In order to compare conserved quantities of different space-times, we must use the Killing vectors ∂_t and ∂_φ to measure energy and angular momentum.[12] The corresponding conserved charges are related to $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{J}}$ by a linear transformation. Using Eq. (7.7) one finds

$$\tilde{\mathcal{M}} = \frac{\mu}{\mu^2 + \alpha^2}\mathcal{M} + \frac{\alpha}{\mu^2 + \alpha^2}\mathcal{J}. \tag{7.14}$$

Thus we obtain a density matrix

$$\rho^S_E = C\exp\left(-\frac{\mathcal{M} + \Omega\mathcal{J}}{T}\right) \tag{7.15}$$

at temperature and angular potential

$$T = \frac{\mu^2 + \alpha^2}{2\pi\mu}, \quad \Omega = \frac{\alpha}{\kappa}. \tag{7.16}$$

For later convenience it is useful to rewrite the density matrix (7.15) in terms of the complex inverse temperature

$$\beta \equiv \frac{1+i\Omega}{T} = \frac{2\pi}{\sqrt{\bar{\tau}}}, \tag{7.17}$$

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[12]We are choosing the normalization of the timelike Killing vector to be fixed at $\mathcal{I}$, as is appropriate for a CFT description. By normalizing at $\tilde{r}=0$ instead, one would obtain the apparent temperature seen by a local observer [72].

and the complex charges

$$\mathcal{L}_0 = \frac{1}{2}(\mathcal{M} - i\mathcal{J}), \quad \bar{\mathcal{L}}_0 = \frac{1}{2}(\mathcal{M} + i\mathcal{J}). \qquad (7.18)$$

These charges are constructed from the complex Killing vector fields

$$\zeta_0 = \frac{1}{2}(\partial_t + i\partial_\varphi), \quad \bar{\zeta}_0 = \frac{1}{2}(\partial_t - i\partial_\varphi). \qquad (7.19)$$

Then the density matrix of the scalar field in the southern diamond takes the form

$$\rho_E^S = C \exp(-\beta\mathcal{L}_0 - \bar{\beta}\bar{\mathcal{L}}_0). \qquad (7.20)$$

D. The boundary stress tensor and Virasoro charges

In this section we define, compute, and interpret the Brown-York boundary stress tensor in static coordinates, following [42]. In the static coordinates $\mathcal{I}^\pm$ is at $r \to \infty$. The metric takes the asymptotic form

$$ds^2 = -\frac{dr^2}{r^2} + \left(r^2 - \frac{M}{2}\right)dw\,d\bar{w} + \frac{\tau}{4}dw^2 + \frac{\bar{\tau}}{4}d\bar{w}^2 + \mathcal{O}\left(\frac{1}{r^4}\right), \qquad (7.21)$$

with

$$w \equiv \varphi + it. \qquad (7.22)$$

Since $w \sim w + 2\pi$, the boundary is a cylinder with conformal metric

$$ds_{\text{conf}}^2 = dw\,d\bar{w}. \qquad (7.23)$$

dS$_3$ has an infinite number of asymptotic symmetries, whose associated bulk vector fields ζ generate the conformal group on $\hat{\mathcal{I}}^-$ [13]. With each of these symmetries there is an associated charge. A general procedure for constructing such charges for spacelike slices ending on a boundary was given in [73], adapted to AdS in [74], and adapted to dS in [13]. For dS$_3$ in planar coordinates, $\mathcal{I}^-$ is a plane and the charges are

$$L_n = \frac{1}{2\pi i} \oint dz\, T_{zz} z^{n+1},$$

$$\bar{L}_n = -\frac{1}{2\pi i} \oint d\bar{z}\, T_{\bar{z}\bar{z}} \bar{z}^{n+1}, \qquad (7.24)$$

where T_{zz} is the boundary stress tensor given by [73,74,13]

$$T_{\mu\nu} = \frac{1}{4G}[K_{\mu\nu} - (K+1)\gamma_{\mu\nu}]. \qquad (7.25)$$

Here $\gamma_{\mu\nu}$ is the induced metric on the boundary, and the extrinsic curvature is defined by $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu}$ with n^μ the future-directed unit normal. The contour integral is over the S^1 boundary of $\mathcal{I}^-$ in planar coordinates at $|z| = \infty$. The

Abbott-Deser (AD) mass [75] is proportional to $L_0 + \bar{L}_0$. The complex coordinates on the boundary cylinder in Eq. (7.21) are related to those of the plane by

$$z = e^{-iw}. \qquad (7.26)$$

In the preceding section charges $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$ were constructed for weak scalar field excitations on a fixed de Sitter background. These can be related to the weak field limit of L_0 and $\bar{L}_0$ by using the conservation equation [73]

$$\frac{1}{2\pi}\nabla^\mu T_{\mu\nu} = n^\mu T_{\mu\nu}, \qquad (7.27)$$

which states that the failure of $T_{\mu\nu}$ to be conserved is given by the matter flux across the boundary. Contracting both sides of Eq. (7.27) with a Killing vector ζ and integrating over a disc Σ_C spanning a contour C on $\mathcal{I}^-$ yields

$$\frac{1}{2\pi}\int_C d\sigma^\mu T_{\mu\nu}\zeta^\nu = \int_{\Sigma_C} d\Sigma^\mu \mathcal{T}_{\mu\nu}\zeta^\nu, \qquad (7.28)$$

where $d\sigma^\mu$ is the normal boundary volume element normal to the curve C. Comparing with Eqs. (7.11), (7.18), and (7.19), we see that integrand on the right-hand side of this expression for L_0 ($\bar{L}_0$) agrees with that in the expression for $\mathcal{L}_0$ ($\bar{\mathcal{L}}_0$).[13] Of course when the fields are not weak there are gravitational corrections to the bulk expressions.

The cylinder charges corresponding to Eq. (7.24) are[14]

$$H_n = -\frac{1}{2\pi}\int_0^{2\pi} dw\, T_{ww} e^{-inw} \qquad (7.29)$$

and its complex conjugate. We have used the symbol H_n rather than L_n because on the cylinder (7.29) includes a Casimir energy contribution for H_0. We will be interested in H_0, which is the charge associated with the vector field

$$\zeta_0 = \frac{1}{2}(\partial_t + i\partial_\varphi). \qquad (7.30)$$

For $r \to \infty$ one finds

$$T_{ww} = \frac{1}{4G}\gamma_{ww} = \frac{\tau}{16G}. \qquad (7.31)$$

Integrating around the cylinder then gives

$$H_0 = -\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, T_{ww} = -\frac{c}{24}\tau, \qquad (7.32)$$

and similarly

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[13]Our sign convention in Eq. (7.24) was chosen so that in the weak field limit $\mathcal{H}$ reduces to the integral of the scalar stress energy density, without a relative minus sign. This convention agrees with [13,42,62], but differs by a sign from [46,47].

[14]A minus sign arises in this expression from the relative orientation of the z and w contours.

$$\bar{H}_0 = -\frac{c}{24}\bar{\tau}. \tag{7.33}$$

For later convenience we have written these expressions in terms of the dS$_3$ central charge

$$c = \frac{3l}{2G}, \tag{7.34}$$

where we have restored the factor of the de Sitter radius l. However, so far our discussions have been purely classical.

We note for pure de Sitter space ($M=1$ and $J=0$) $H_0 = -c/24$. This has a nice interpretation in the dual field theory on the boundary, as discussed in [42].[15] According to [13], the bulk gravity state on the slice $t=\infty$ in planar coordinates is dual to a CFT state on the S^1 boundary of $\mathcal{I}^-$ (i.e., where the slice $t=\infty$ intersects $\mathcal{I}^-$) at $z=\infty$. This state is the wave functional produced by fixing boundary conditions on the S^1 and then doing the CFT path integral over the disk. This should give the $SL(2,C)$ invariant ground state of the CFT. Transforming from planar to static coordinates in the bulk is then dual to the conformal mapping from the plane to the cylinder. This mapping should produce, via the Schwarzian in the stress tensor transformation law, the Casimir energy $-c/24$ for a CFT with central charge c on a circle of radius 1. Indeed this agrees beautifully with the fact that the boundary stress tensor vanishes in planar coordinates but gives $H_0 = -c/24$ in static coordinates. We note for future reference that the state so constructed on $\mathcal{I}^-$ is a pure state with no entropy.

The agreement with the CFT picture persists for general τ. Equation (7.32) is then precisely the Casimir energy from conformal mapping from the plane to a cone. We note also that as M decreases, the energy H_0 increases, in accord with the expectation that a positive deficit angle has a positive mass.

VIII. ENTROPY

In this section we discuss the conditions under which the entropy (7.5) might be microscopically derived from a 2D CFT. Related discussions have appeared in [8,45,46].

Consider the canonical partition function of a 2D CFT with complex potential β,

$$Z = \int dL_0 d\bar{L}_0 \rho(L_0,\bar{L}_0) e^{-\beta L_0 - \bar{\beta}\bar{L}_0}, \tag{8.1}$$

where ρ is the density of states. We wish to evaluate this in the saddle point approximation. Let us assume that we are in a regime where the thermodynamic approximation is valid, and we can use Cardy's formula [76] for the density of states[16]

$$\rho(L_0,\bar{L}_0) = \exp\left[2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\bar{L}_0 - \frac{c}{24}\right)}\right]. \tag{8.2}$$

When β is complex, Eq. (8.1) has a complex saddle point at $L_0 = (\pi^2 c/6\beta^2) + (c/24)$.[17] Evaluating the integral at the saddle point and using $S = (1 - \beta\partial_\beta - \bar{\beta}\partial_{\bar{\beta}})\ln Z$ gives

$$S = \frac{\pi^2 c}{3\beta} + \frac{\pi^2 c}{3\bar{\beta}}. \tag{8.3}$$

If we now use the formula

$$c = \frac{3l}{2G}, \tag{8.4}$$

for the central charge of the boundary CFT, together with the formula

$$\beta = \frac{2\pi}{\sqrt{M - i(8GJ)}}, \tag{8.5}$$

derived in Sec. VI C for the complex temperature of Kerr-dS$_3$, the microscopic formula (8.3) reproduces exactly the macroscopic formula (7.5) for the Bekenstein-Hawking entropy of Kerr-dS$_3$.

This yields a two-parameter fit relating the area of the Kerr-dS$_3$ horizon to the number of microstates of a 2D CFT. However, with our current understanding, this should be regarded as highly suggestive numerology rather than a derivation of the entropy. One problem is that the dual CFT is not unitary, and hence is not obligated to obey Cardy's formula. A second problem is that we have not specified where the CFT density matrix resides whose entropy is being computed. In most discussions—including ours—the quantum state on global de Sitter is in a pure state. Furthermore its dual—as discussed at the end of Sec. VII D—is the $SL(2,C)$ invariant CFT vacuum. A density matrix arises only after tracing over a correlated but unobservable sector. We saw in Sec. VI C that for a scalar field in the (pure) Euclidean vacuum state, a thermal density matrix arises after a northern trace over the Hilbert space in the unobservable northern diamond. One might expect that the quantum state of the boundary CFT would also become thermal after performing a similar trace. However, it is not clear to us exactly what a northern trace corresponds to in the boundary CFT on $\mathcal{I}^\pm$.

It appears that de Sitter entropy arises when attention is restricted to the true observables in the theory. The boundary CFT includes information about correlators at acausal separations that do not directly correspond to observable data. It is a challenging and important problem to understand what are the true observables in the language of the boundary CFT.

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[15]An alternate interpretation was given in [46].

[16]Since we are working in the canonical picture rather than the microcanonical picture, the final formula for the entropy is unaffected by the shift of L_0 in the exponent.

[17]For pure dS$_3$ this is $L_0 = c/12$, as in [8].

ACKNOWLEDGMENTS

APPENDIX A: ALTERNATE FORMS OF GREEN FUNCTIONS ON dS$_d$

In this appendix we present several alternate expressions for the Green functions. First, let us consider a de Sitter invariant vacuum $|\Omega\rangle$, so that the wave equation for $G_\Omega(x,x')$ becomes

$$(1-P^2)\partial_P^2 G - dP\partial_P G - m^2 G = 0, \qquad (A1)$$

where P is related to the geodesic distance $\theta(x,x')$ by

$$P = \cos\theta. \qquad (A2)$$

Note that if G_{d,m^2} solves Eq. (A1) in d dimensions for mass-squared m^2, then $\partial_P G_{d,m^2}$ solves Eq. (A1) in $d+2$ dimensions with mass-squared m^2+d. This gives an iterative procedure for constructing Green's functions in all dimensions. We find

$$G_{3+2n,m^2} = \partial_P^n G_{3,m^2+1-(n+1)^2}$$

$$G_{2+2n,m^2} = \partial_P^n G_{2,m^2-n(n+1)}, \qquad (A3)$$

where n is a positive integer.

Let us first consider odd d. For $d=3$, if we let

$$G_{3,m^2} = \frac{\chi}{\sin\theta} \qquad (A4)$$

then χ satisfies

$$\partial_\theta^2 \chi + (1-m^2)\chi = 0. \qquad (A5)$$

So the general solution in three dimensions is

$$G_{3,m^2} = \frac{A\sinh\mu(\pi-\theta) + B\sinh\mu\theta}{\sin\theta}, \qquad (A6)$$

where $\mu = \sqrt{m^2-1}$ and A and B are arbitrary constants. The first term gives the usual short distance singularity for the Euclidean vacuum—with the correct normalization, it gives the usual expression (2.9). The second term is present for the transformed vacuum states $|\alpha\rangle$, and has the antipodal singularities mentioned in Sec. II B. From Eq. (A3) we can obtain an expression for the Green functions in higher dimensions,

$$G_{d,m^2} = \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n-m+2i\mu)}{\Gamma(m+1+2i\mu)} \frac{A\sinh(2\mu-in+2im)(\pi-\theta) + B\sinh(2\mu-in+2im)\theta}{\sin^{d-2}\theta} \qquad (A7)$$

where $n = \frac{1}{2}(d-3)$ and

$$\mu = \sqrt{m^2 - \left(\frac{d-1}{2}\right)^2}. \qquad (A8)$$

We have absorbed an overall normalization into the constants A and B. As a function of θ, G has isolated singularities but no branch cuts. However, $\theta = \cos^{-1}P$ has a branch cut from $P=1$ to ∞ along the real axis, across which $\theta(P)$ changes sign. When expressed as a function of P, G will likewise have a branch cut.

For even d, we start with the $d=2$ solution in terms of Legendre functions

$$G_{2,m^2} = AP_\nu(\cos\theta) + BQ_\nu(\cos\theta), \qquad (A9)$$

where $\nu(\nu+1) = -m^2$. So

$$G_{d,m^2} = AP_\nu^{(n)}(\cos\theta) + BQ_\nu^{(n)}(\cos\theta), \qquad (A10)$$

where $n = \frac{1}{2}(d-2)$ and $\nu(\nu+1) = n(n+1) - m^2$. Here, $P_\nu^{(n)}$ is an associated Legendre function, the nth derivative of the Legendre function.

APPENDIX B: PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

We collect a few relevant facts about hypergeometric functions. More details may be found in, e.g., [66]. The formula

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z) \qquad (B1)$$

relates hypergeometric functions of z with different values of parameters, as in Eq. (6.7). To relate hypergeometric functions of different variables we use

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}F\left(a,a+1-c;a+1-b;\frac{1}{z}\right) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b,b+1-c;b+1-a;\frac{1}{z}\right)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;1+a+b-c;1-z)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z). \tag{B2}$$

These give us the Bogolyubov relations (3.21) and (3.39), respectively. Since $F(a,b;c;0)=1$ these equations also fix the behavior of $F(a,b;c;z)$ as $z\to\infty$ and $z\to 1$, as in Eqs. (4.2), (6.9), and (6.20).

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