

Thorny spheres and black holes with strings

V. P. Frolov*

Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1

D. V. Fursaev†

Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, 141 980 Dubna, Russia

D. N. Page‡

Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1

(Received 22 December 2001; published 9 May 2002)

We consider thorny spheres, that is, 2-dimensional compact surfaces which are everywhere locally isometric to a round sphere S^2 except for a finite number of isolated points where they have conical singularities. We use thorny spheres to generate, from a spherically symmetric solution of the Einstein equations, new solutions which describe spacetimes pierced by an arbitrary number of infinitely thin cosmic strings radially directed. Each string produces an angle deficit proportional to its tension, while the metric outside the strings is a locally spherically symmetric solution. We prove that there can be arbitrary configurations of strings provided that the directions of the strings obey a certain equilibrium condition. In general this equilibrium condition can be written as a force-balance equation for string forces defined in a flat 3-space in which the thorny sphere is isometrically embedded, or as a constraint on the product of holonomies around strings in an alternative 3-space that is flat except for the strings. In the case of small string tensions, the constraint equation has the form of a linear relation between unit vectors directed along the string axes.

DOI: 10.1103/PhysRevD.65.104029

PACS number(s): 04.70.Bw, 04.20.Gz

I. INTRODUCTION

Recently [1] it was demonstrated that cosmic strings attached radially to a black hole can be used for very effective energy mining from black holes. There were also found sets of exact solutions of the Einstein equations which describe a black hole with infinitely thin radial cosmic strings [2] and generalize the results of [3,4]. For such solutions a regular round sphere is changed to a sphere with a number of conical singularities on it with angle deficits $\mu = 8\pi\tilde{\mu}$, where $\tilde{\mu}$ is the dimensionless¹ cosmic string tension.

A characteristic property of the configurations studied in [2] is that the positions of the conical singularities on a sphere form a regular symmetric structure. The number of types of these configurations is restricted. There are three configurations which are related to platonic solids and one family of configurations which looks like a “double pyramid.” In the latter case the number of conical singularities (and hence the strings) is not restricted.

In physical applications one can always assume that the string tension is very small. For example, for strings which appear in grand unified theories (GUTs) the tension is 10^{-6} , while for electroweak strings it is 10^{-34} . Finding all possible static radial string configurations for small μ without any

additional *a priori* symmetry assumptions is our first goal in the present paper. We shall demonstrate that such configurations exist for any number $n < 4\pi/\mu$ of strings, and in the general case they do not possess any symmetry. Nevertheless there always exists a vector force-balance constraint equation

$$\mathbf{F} = \sum_{k=1}^n \mathbf{F}_k = 0, \quad (1.1)$$

which for $\sum_{i=1}^n \mu_k \ll 1$ is approximated by

$$\sum_{k=1}^n \mu_k \mathbf{n}_k = 0. \quad (1.2)$$

The sum is taken over all singular points with corresponding angular deficits μ_k . In the given approximation the position of each point is characterized by the unit vector \mathbf{n}_k on a smooth sphere S^2 .

We also studied in detail the case when the string tension is not small, and this is our second main goal. We demonstrated that there can exist configurations with an arbitrary number of strings n , provided the total angular deficit is less than 4π . The constraint equations which again must be satisfied are now more involved. We demonstrate that these relations can be written as a constraint on the products of the elements of the holonomy group representing the conical singularities. The constraint equations are more involved since the corresponding operators of the holonomy group do not commute.

This analysis requires knowledge of different geometrical properties of an object which we called a *thorny sphere*. A thorny sphere is a compact 2-dimensional surface which has n points with conical singularities, and away from these

*Email address: frolov@phys.ualberta.ca

†Email address: fursaev@thsun1.jinr.ru

‡Email address: don@phys.ualberta.ca

¹We work in the system of units $\hbar = G = c = 1$. In these units the string tension $\tilde{\mu}$ is dimensionless and corresponds to the combination $G\hat{\mu}/c^2$, where $\hat{\mu}$ is the tension measured in physical units.

points is everywhere locally isometric to a unit sphere S^2 . If l_k is a length of a circle of radius r_k around the singular point k , then $\mu_k = 2\pi - \lim_{r_k \rightarrow 0} (l_k/r_k)$ is its angle deficit.

In Sec. II we study thorny spheres isometrically embedded in a Euclidean 3-space to derive one form of the constraint equation. In Sec. III we show how the thorny sphere can be obtained from a regular round sphere by a set of reconstructions (*elementary deformations*), starting with an arbitrary triangulation of S^2 . In Sec. IV we describe methods of mapping a thorny sphere onto a unit sphere with cuts. Constraint equations are derived in Sec. V from a consistency condition of these maps and from the holonomy group of a 3-space that is flat except for the strings. The special case of small angle deficits is also considered in this section. Concrete examples of thorny spheres with 3, 4, and general n conical singularities are studied in detail in Sec. VI. Topological aspects of the problem are the subject of Sec. VII. Finally, in Sec. VIII, we demonstrate how thorny spheres can be used to construct static solutions of the Einstein equations with n radial cosmic strings.

II. THORNY SPHERES EMBEDDED IN FLAT EUCLIDEAN SPACE AND THE NUMBER OF FREE PARAMETERS

A. Intrinsic geometry of a thorny sphere

We shall first consider the large class of thorny spheres M^2 (e.g., all those with three or more conical singularities, all of which have positive deficit angles) that are both isometrically embeddable into Euclidean 3-space (in a unique manner, up to overall translations and rotations) and also have their geometries uniquely defined by the edge lengths of geodesic triangulations with the vertices at the conical singularities. If N_f , N_e and $N_v = n \geq 3$ are the number of triangles, edges and vertices for the triangulation, then the Euler theorem gives

$$N_f - N_e + N_v = 2. \quad (2.1)$$

Since each triangle has 3 edges and each edge belongs to two triangles, we have $N_e = 3N_f/2$. (Thus the total number of triangles is always even.) From this relation and the Euler theorem, we get

$$N_v = n, \quad N_e = 3n - 6, \quad N_f = 2n - 4. \quad (2.2)$$

Except at the conical singularities at the vertices of the triangles, the thorny sphere has constant Gaussian curvature $K = \frac{1}{2}R$ (with R being the Ricci scalar curvature) which we shall take to be unity, and hence each triangle can be isometrically mapped to a spherical triangle on the unit sphere.

Consider a spherical triangle with vertices 1,2,3. Its edges are parts of great circles on the sphere. Denote by l_1 , l_2 and l_3 the lengths of the triangle, and by γ_1 , γ_2 and γ_3 its interior angles at the vertices 1,2,3, respectively. We assume that the edge l_k is opposite to the k th vertex. For given lengths of the edges, the angles are uniquely defined, assuming as we shall that they are all less than π (which is indeed the case when the deficit angles are all positive). In particular one has

$$\cos \gamma_1 = \frac{\cos l_1 - \cos l_2 \cos l_3}{\sin l_2 \sin l_3}. \quad (2.3)$$

The triangle can also be specified by its angles. The lengths of the edges can be determined by using the relation

$$\cos l_1 = \frac{\cos \gamma_2 \cos \gamma_3 + \cos \gamma_1}{\sin \gamma_2 \sin \gamma_3}. \quad (2.4)$$

We also shall use the following expression for the area A of a spherical triangle:

$$A = \gamma_1 + \gamma_2 + \gamma_3 - \pi. \quad (2.5)$$

Because the edges of the triangles are geodesics of the thorny sphere, adjacent triangles match without producing any singularities along the edges, except at the vertices. There one gets a deficit angle μ_k that is 2π minus the sum of the interior angles of the triangles at that vertex.

Thus the entire geometry of each M^2 is uniquely determined by the $N_e = 3n - 6$ edge lengths of the triangulation, which can all be specified independently, within an open set of the $(3n - 6)$ -parameter space that is restricted by certain inequalities (e.g., triangular inequalities). (For $n = 2$, there is no triangulation, and $3n - 6 = 0$, but there is a one-parameter family of thorny spheres of arbitrary deficit angle $\mu < 2\pi$. See Appendix B.)

If the deficit angles are all positive, the thorny sphere M^2 has non-negative Gaussian curvature K everywhere, unit curvature everywhere away from the conical singularities and Dirac delta-function curvature at the singularities. The contribution of these two parts of the curvature to the Gauss-Bonnet theorem is

$$\int_{M^2} K dA = A + \sum_{k=1}^n \mu_k = 4\pi. \quad (2.6)$$

Therefore a thorny sphere can be isometrically rigidly (i.e., uniquely up to overall translations and rotations) embedded as a convex surface M^2 in 3-dimensional Euclidean space [5,6].

B. Gaussian normal map

Let \mathbf{n} be the outward normal to the embedded surface at each point. One can then map each point of M^2 to a corresponding point of a unit round S^2 also embedded in the 3-dimensional Euclidean space that has the same unit normal \mathbf{n} . This map is known as the Gaussian normal map of the convex surface M^2 into S^2 . The Gaussian normal map is discussed in more detail in Appendix A, where it is shown that

$$K dA = da, \quad (2.7)$$

where dA is the area element on M^2 and da is the corresponding area element on S^2 (with the same corresponding set of unit normal vectors \mathbf{n} in the embedding Euclidean 3-space).

From the divergence theorem applied to the interior of M^2 , one can easily prove

$$\int_{M^2} \mathbf{n} dA = 0, \quad (2.8)$$

and from the divergence theorem applied to the interior of S^2 , one can similarly prove

$$\int_{M^2} \mathbf{n} K dA = \int_{S^2} \mathbf{n} da = 0. \quad (2.9)$$

The unit normal \mathbf{n} to the embedding M^2 of a thorny sphere is not well defined at each conical singularity. As one goes around the k th conical singularity infinitesimally close to it, the unit normal \mathbf{n} to the embedding (well defined everywhere except at the conical singularities themselves, and hence defining a smooth Gaussian normal map from that smooth part of M^2 into the round S^2 of unit-normal directions) sweeps out a topological circle on the round sphere S^2 of possible directions for the unit normal, with the area of the disk D_k within this topological circle on the round S^2 being the deficit angle μ_k . The set of these n disks is the part of the round S^2 that is not mapped into from the smooth part of the M^2 but instead represents the conical singularities. Thus in the case that μ_k is not infinitesimal, the direction of the unit normal at that conical singularity is spread out over this cone (the disk D_k on the S^2) and has an angular uncertainty of the order of $\mu_k^{1/2}$.

C. Constraint equation

The integral of Eq. (2.8) is the same as what it would be if one excluded the zero-area conical singularities and inserted the unit curvature K for $\text{reg}(M^2)$, the smooth part of M^2 :

$$\int_{\text{reg}(M^2)} \mathbf{n} K dA = \int_{S^2 - \sum D_k} \mathbf{n} da = 0. \quad (2.10)$$

If one subtracts the second integral here from the second integral of Eq. (2.9), one gets

$$\int_{\sum D_k} \mathbf{n} da = 0, \quad (2.11)$$

which is one version of what we shall call the constraint equation for the thorny sphere, restricting the orientation of its conical singularities in the embedding Euclidean 3-space.

One can regard this constraint as arising from the fact that we have restricted the thorny sphere to have constant Gaussian curvature everywhere except at the conical singularities, and this restricts three combinations of the strengths and positions of these singularities. A more physical interpretation of the constraint is as a force-balance equation, to which we now turn.

One can define

$$\mathbf{F}_k \equiv \int_{D_k} \mathbf{n} da, \quad (2.12)$$

which can be interpreted as the force, defined as a vector in the embedding Euclidean 3-space that can trivially be parallel transported over it and added to other such forces, exerted by a string that produces the angle deficit angle μ_k at the k th conical singularity.

Since the area of the disk D_k is

$$\mu_k = \int_{D_k} da \quad (2.13)$$

(see the proof in Appendix A), one can write

$$\mathbf{F}_k = \mu_k \hat{\mathbf{n}}_k, \quad (2.14)$$

where $\hat{\mathbf{n}}_k$ is the normal \mathbf{n} averaged over the area of the disk D_k :

$$\hat{\mathbf{n}}_k \equiv \frac{\int_{D_k} \mathbf{n} da}{\int_{D_k} da}. \quad (2.15)$$

For $\sum_{k=1}^n \mu_k \ll 4\pi$, each disk D_k is nearly round, and the averaged normal will have the length

$$|\hat{\mathbf{n}}_k| \approx 1 - \frac{\mu_k}{4\pi}, \quad (2.16)$$

with this relation being exact when the disk D_k is precisely round. Thus the length is nearly unity when μ_k is small, but it decreases below that to a minimum value of $1/2$ when μ_k is increased to its maximum value of 2π (assuming a round disk D_k , which one indeed gets in the case of just two conical singularities, a case in which the geometry is not determined by the geodesic distances between conical singularities, simply π in this case, but which involves an arbitrary deficit angle $\mu < 2\pi$ and is discussed in Appendix B).

For an opposite extreme case, in which $\sum_{k=1}^n \mu_k = 4\pi$ and in which the embedding of the thorny sphere gives the two sides of a convex polygon, the deficit angle at a vertex is 2π minus twice the corresponding interior angle of the polygon (since the surface M^2 corresponds to both sides), and the disk D_k is one interior lune-shaped region between two great circles that intersect at angle μ_k . Then one can show that in this extreme case,

$$|\hat{\mathbf{n}}_k| = \frac{\pi}{\mu_k} \sin \frac{\mu_k}{4}, \quad (2.17)$$

which for small μ_k has the limit $\pi/4$ rather than the limit of unity that Eq. (2.16) has. Thus for Eq. (2.16) to be valid, it is required that the disk D_k be nearly round, for which it is not sufficient merely that μ_k be small, but also that the effects of all the other conical singularities on the embedding also be small.

In terms of the precisely defined forces \mathbf{F}_k , the constraint equation (2.11) becomes the force-balance equation that the sum of these forces vanishes:

$$\mathbf{F} \equiv \sum_{k=1}^n \mathbf{F}_k = \sum_{k=1}^n \mu_k \hat{\mathbf{n}}_k = \sum_{k=1}^n \int_{D_k} \mathbf{n} da = 0. \quad (2.18)$$

This is one precise version of the constraint equation for arbitrary possible positive deficit angles.

One way to visualize this constraint equation is to imagine that one covers the disks D_k of the round S^2 with some material with constant mass per unit area. Then the constraint equation is the condition that the center of mass of the n disks be at the center of the round sphere.

Although this form of the constraint equation is easily visualizable and is precisely valid for general deficit angles (so long as they allow the thorny sphere to be rigidly embedded in Euclidean 3-space, which will be the case for positive deficit angles but need not be so for negative deficit angles), it is not very convenient for calculations, since for $n > 2$ it is a rather difficult procedure to construct the embedding of a thorny sphere into Euclidean 3-space. Therefore, it is also useful to look at other ways of representing thorny spheres, which we shall do in Sec. IV.

Let us emphasize that the above results can be generalized. Instead of a thorny sphere one may consider a *thornifold*, that is a closed 2-dimensional surface with conical singularities and arbitrary smooth metric outside them. We assume that this metric has positive Gaussian curvature K so that it can be isometrically embedded in Euclidean 3-space as a closed convex surface. As is shown in Appendix A, the constraint equation (2.18) is modified and takes the form

$$\mathbf{F} \equiv \sum_{k=1}^n \mu_k \hat{\mathbf{n}}_k = \int_{\text{reg}(M^2)} \mathbf{n} (1 - K) dA, \quad (2.19)$$

where K is the Gaussian curvature of M^2 . We call this relation a *generalized constraint equation*. The presence of non-constant Gaussian curvature in the right-hand side of this equation makes possible the existence of new configurations, e.g. with a single conical singularity. See the discussion of C metrics in Appendix A for some interesting physical applications of the generalized constraint equation.

III. SPHERICAL TRIANGULATIONS OF A THORNY SPHERE

A. Elementary deformation of a sphere and another count of the degrees of freedom

In this section we describe how to construct a thorny sphere starting with a triangulation of a regular unit sphere S^2 (see also [7]). In the next section we describe the mapping of thorny spheres into round spheres embedded in three-dimensional Euclidean space.

We start construction of a thorny sphere by taking a regular unit sphere with an arbitrary given triangulation of it, using spherical triangles. As above, let $N_v = n \geq 3$, $N_e = 3n - 6$, and $N_f = 2n - 4$ be the number of vertices, edges, and triangles for the triangulation, with Eqs. (2.3), (2.4), and (2.5) applying for the geometry of each triangle.

Now let us cut from the triangulation a spherical quadrangle Q which consists of two triangles with a common

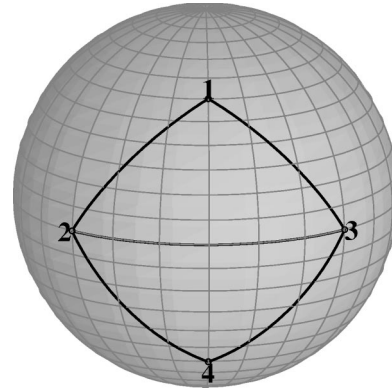


FIG. 1. Spherical quadrangle composed of two spherical triangles. The edges are the lines of the large circles of the sphere.

edge between vertices 2 and 3; see Fig. 1. We denote the length of this common edge by l_1 . The other edges have lengths l_2, l_3 and \bar{l}_2, \bar{l}_3 for the first and second triangles, respectively. Denote by Q' a new spherical quadrangle for which the length of the common edge is l'_1 , while the other lengths are the same. This change of the length l_1 results in a change of angles of each of two triangles which can be found by using Eq. (2.4). In this procedure the lengths $l_2, l_3, \bar{l}_2, \bar{l}_3$ of the edges of Q' remain the same as for Q . That is why the modified spherical quadrangle Q' can be glued back into a cut sphere. The resulting surface will be smooth everywhere except at 4 vertices where angle deficits will appear. We call this procedure an *elementary deformation* of the sphere.

For a given triangulation, one can use $N_e = 3n - 6$ elementary deformations independently to fix all of the edge lengths (and hence also all the deficit angles) and thus to determine the $(3n - 6)$ -parameter metric on a generic thorny sphere with n conical singularities. However, if we have the goal of fixing required deficit angles at the $N_v = n$ vertices, not all of the elementary deformations have independent effects upon them. Some combinations of these deformations do not generate angle deficits, but simply move vertices of the triangulation along the sphere. The number of such “degrees of freedom” that do not affect the deficit angles is $2N_v - 3 = 2n - 3$ (two “degrees of freedom” per vertex minus 3 “degrees of freedom” corresponding to rigid rotations of the sphere which preserve the lengths of each edge unchanged). Thus the total number of “real degrees of freedom” which generate deformations in the angle deficits is $N_e - 2N_v + 3 = n - 3$. These deformations are sufficient to create the required angle deficits at all the n vertices except 3. This is exactly what one can expect in a general case, since there exist exactly 3 consistency conditions in the vector constraint equations (1.1) and (1.2) relating angle deficits and positions of the singular points on the thorny sphere.

The above counting of the “degrees of freedom” gives us also the following useful information. Let us fix the values of all n angle deficits μ_k . Then $2n$ “degrees of freedom” characterizing the positions of the vertices must obey 3 constraint equations. We can also use 3 “degrees of freedom” of the rigid rotation of the sphere to put, say, the first point to the north pole of the sphere, and the second one on the meridian

$\phi=0$. After this there remain $2n-6$ free parameters. By adding to them the n parameters μ_k we return to the $3n-6$ parameters of a generic thorny sphere with n conical singularities.

B. Constraints for elementary deformations of a sphere

Now to illustrate the nature of these constraints we discuss a case when all angle deficits are infinitesimally small. Consider an infinitesimal elementary deformation of a spherical quadrangle Q shown on Fig. 1. To produce conical singularities one can deform the length of the common edge (2,3) while keeping the lengths of the other edges unchanged. This deformation changes internal angles at vertices 1,2,3,4 and yields conical singularities. Let us introduce the following notations:

$$\begin{aligned} w_3 &= (\mathbf{n}_2 \cdot \mathbf{n}_1), & w_2 &= (\mathbf{n}_3 \cdot \mathbf{n}_1), & \bar{w}_3 &= (\mathbf{n}_2 \cdot \mathbf{n}_4), \\ \bar{w}_2 &= (\mathbf{n}_3 \cdot \mathbf{n}_4), & w_1 &= (\mathbf{n}_2 \cdot \mathbf{n}_3), \end{aligned} \quad (3.1)$$

where \mathbf{n}_k are the unit vectors (with the beginning at the center of the sphere) which define the positions of the corresponding vertices. If l_k is the length of the edge opposite to the k th vertex, then $w_k = \cos l_k$, which is true for triangles whose edges are arcs of great circles. For triangles on S^2 , Eq. (2.4) gives

$$w_1 = \frac{c_2 c_3 + c_1}{s_2 s_3}, \quad w_2 = \frac{c_1 c_3 + c_2}{s_1 s_3}, \quad w_3 = \frac{c_2 c_1 + c_3}{s_2 s_1}, \quad (3.2)$$

where $c_k = \cos \gamma_k, s_k = \sin \gamma_k$. Analogous relations for \bar{w}_k can be obtained from Eq. (3.2) by replacing γ_k by $\bar{\gamma}_k$. Variations of w_1 produce changes of the angles $\delta \gamma_k \equiv -x_k, \delta \bar{\gamma}_k \equiv -\bar{x}_k$. The condition that these variations do not change w_2 can be written in linear order as

$$\begin{aligned} \delta w_2 &= \frac{1}{s_1 s_3} [(c_1 s_3 w_2 + c_3 s_1) x_1 \\ &\quad + (c_3 s_1 w_2 + c_1 s_3) x_3 + x_2 s_2] \\ &= \frac{s_2}{s_1 s_3} (w_3 x_1 + w_1 x_3 + x_2) = 0. \end{aligned} \quad (3.3)$$

To get the last line we used Eq. (3.2). A similar relation follows from the variation of \bar{w}_2 . Their combination yields

$$\begin{aligned} (w_3 x_1 + w_1 x_3 + x_2) + (\bar{w}_3 \bar{x}_1 + \bar{w}_1 \bar{x}_3 + \bar{x}_2) \\ = w_3 x_1 + (x_2 + \bar{x}_2) + w_1 (x_3 + \bar{x}_3) + \bar{w}_3 \bar{x}_1 \\ = w_3 \mu_1 + \mu_2 + \mu_3 w_1 + \mu_4 \bar{w}_3 = 0, \end{aligned} \quad (3.4)$$

where we took into account that $w_1 = \bar{w}_1$. It is easy to see that $\mu_1 = x_1, \mu_2 = x_2 + \bar{x}_2, \mu_3 = x_3 + \bar{x}_3$, and $\mu_4 = \bar{x}_1$ are the conical angle deficits produced by the deformation at the

vertices 1,2,3,4. Consider the particular form of Eq. (1.2) when only these four vertices have nonzero deficit angles:

$$\mathbf{n}_1 \mu_1 + \mathbf{n}_2 \mu_2 + \mathbf{n}_3 \mu_3 + \mathbf{n}_4 \mu_4 = 0. \quad (3.5)$$

On projecting it on the vector \mathbf{n}_2 we get exactly Eq. (3.4).

To get Eq. (3.5) directly, and not just its projection on \mathbf{n}_2 , we can do some further algebra and find that for each of the two spherical triangles with only the length l_1 perturbed,

$$x_1 \mathbf{n}_1 + x_2 \mathbf{n}_2 + x_3 \mathbf{n}_3 = \frac{\delta w_1}{1 - w_1^2} \mathbf{n}_3 \times \mathbf{n}_2, \quad (3.6)$$

$$\bar{x}_1 \mathbf{n}_4 + \bar{x}_2 \mathbf{n}_2 + \bar{x}_3 \mathbf{n}_3 = \frac{\delta w_1}{1 - w_1^2} \mathbf{n}_2 \times \mathbf{n}_3. \quad (3.7)$$

When these two equations are added, the right hand sides cancel, and one gets Eq. (3.5).

A generalization of this linearized vector condition to finite deformations will be given in the next sections.

IV. MAPPING A THORNY SPHERE ONTO A ROUND SPHERE WITH CUTS

As we already mentioned, there exists an embedding of a thorny sphere in flat space (at least for positive deficit angles). But in practice it is very difficult to obtain this embedding explicitly and get the precisely defined forces \mathbf{F}_k for the force balance equation (2.18) unless the angle deficits are small. An exception for large deficits is a simple case when a thorny sphere has two conical singularities (see Appendix B). (This case is not covered by Sec. II since it cannot be triangulated using as vertices only the two conical singularities.) Therefore, in this section we present another description of a thorny sphere by mapping it onto a round sphere with cuts. This approach allows us to formulate the constraint equation in an *explicit* algebraic form. There are several ways to do this. We describe here two simple methods.

A. Method A

One method of representing the thorny sphere, \bar{S}^2 , with $n = N_v$ conical singularities $A_k, 1 \leq k \leq n$, is the following, applicable for $n \geq 3$: Let the singularities be labeled so that the sequence of the shortest geodesic from A_1 to A_2 (say geodesic segment γ_1 with beginning at A_1 and end at A_2), that from A_2 to A_3 (say γ_2), \dots , that from $A_{(n-1)}$ to A_n (say $\gamma_{(n-1)}$), and that from A_n to A_1 (say γ_n) forms a closed path $\bar{\mathcal{P}}$ that does not intersect itself. For example, one can choose some regular point, find the shortest geodesic from that point to each conical singularity, arbitrarily choose one of the singular points to be A_1 , and then label the remaining conical singularities in the same order as the angles, at the regular point, of the tangent vectors of these geodesics from the regular point to those singularities. It is not obvious whether or not the resulting sequence of geodesics between the conical singularities chosen in this order (or in any other

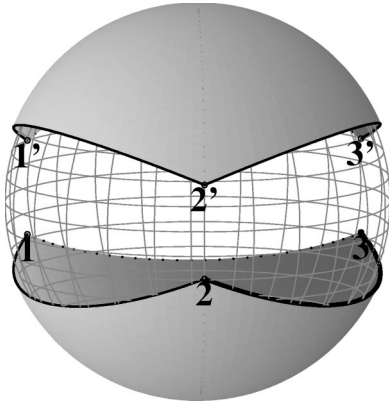


FIG. 2. A unit sphere with cuts. By gluing the cuts one obtains a sphere with three conical singularities.

possible specification of the order) will necessarily form a closed path $\tilde{\mathcal{P}}$ that does not intersect itself, but here we shall assume that it does.

Now the closed path \mathcal{P} divides the thorny sphere, \tilde{S}^2 , into two parts, say $\tilde{\mathcal{M}}_1$, which is encircled clockwise by \mathcal{P} , and $\tilde{\mathcal{M}}_2$, which is encircled counterclockwise by \mathcal{P} . Because the interiors of both contain no conical singularities, they can be isometrically mapped to corresponding regions, \mathcal{M}_1 and \mathcal{M}_2 , on the round unit S^2 of the same unit curvature as the part of the thorny sphere \tilde{S}^2 away from the conical singularities. The boundaries of these two regions on S^2 are the geodesic polygons that are the images \mathcal{B}_1 and \mathcal{B}_2 of $\tilde{\mathcal{P}}$ in these maps from the thorny sphere \tilde{S}^2 into the round sphere S^2 . In the simplest case of three conical singularities the regions $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ are spherical triangles. Their map on the sphere is shown on Fig. 2.

The two maps preserve the lengths of the geodesic edges γ_k of the polygon, and they preserve the angles $\phi_{(1,k)}$ and $\phi_{(2,k)}$ between the two successive geodesics that meet at the conical singularity A_k . Let us define $\phi_{(1,k)}$ to be the angle between the tangent vector of the geodesic ending at A_k and that of the geodesic beginning at A_k , measured in the region $\tilde{\mathcal{M}}_1$ and taken to be positive if clockwise, so that the interior angle at that vertex of the polygon is $\pi - \phi_{(1,k)}$. Similarly, define $\phi_{(2,k)}$ to be the angle between the tangent vector of the geodesic ending at A_k and that of the geodesic beginning at A_k , measured in the region $\tilde{\mathcal{M}}_2$ and also taken to be positive if clockwise, so that the interior angle at that vertex of the polygon is $\pi + \phi_{(2,k)}$ (now with a plus sign since with the ordering given for the geodesic edges, the polygon encircles $\tilde{\mathcal{M}}_2$ in the counterclockwise orientation rather than in the clockwise orientation as it does $\tilde{\mathcal{M}}_1$). Then the conical deficit angle at A_k is $\mu_k = \phi_{(1,k)} - \phi_{(2,k)}$.

The maps from $\tilde{\mathcal{M}}_1$ on the thorny sphere \tilde{S}^2 to \mathcal{M}_1 on the round sphere S^2 , and from $\tilde{\mathcal{M}}_2$ on \tilde{S}^2 to \mathcal{M}_2 on S^2 , give points $A_{(1,k)}$ on S^2 that are the vertices of \mathcal{M}_1 , and points $A_{(2,k)}$ on S^2 that are the vertices of \mathcal{M}_2 . Then the locations of the $2n$ points $A_{(1,k)}$ and $A_{(2,k)}$ on the round sphere S^2 uniquely determine the geometry of the thorny sphere \tilde{S}^2 ,

since they determine the polygon boundaries \mathcal{B}_1 and \mathcal{B}_2 of the regions \mathcal{M}_1 and \mathcal{M}_2 whose interiors have the unit-curvature metric inherited from the unit sphere S^2 . Because the boundary segments are geodesics, the successive ones from \mathcal{M}_1 and \mathcal{M}_2 can be identified so that the union of the two regions with this identification forms the thorny sphere \tilde{S}^2 with no singularities except for the conical singularities with deficit angles μ_k at the n vertices.

Let us see how we get the right parameter count from this construction. Naively one has two parameters for locating each of the $2n$ points $A_{(1,k)}$ and $A_{(2,k)}$ on the round sphere S^2 , or $4n$ total. However, there is the constraint that the geodesic segments from the successive $A_{(1,k)}$'s must match those from the successive $A_{(2,k)}$'s, which gives n constraints, leaving only $3n$ parameters arbitrary. Then there is an arbitrary 3-parameter rotation that one can separately apply to both \mathcal{M}_1 and \mathcal{M}_2 , so of the $3n$ arbitrary parameters, only $3n - 6$ are physically significant in determining the geometry of \tilde{S}^2 . This is precisely equal to the number of continuous parameters of a thorny sphere with n conical singularities of arbitrary strength, the number of edge lengths N_e of a triangulation of it, as discussed above (under the assumption that the triangulation exists, although the number of continuous parameters would not be expected to depend on this assumption).

Given the location of \mathcal{M}_1 on the round sphere S^2 (which has three Euler-angle parameters of arbitrariness), one could fix the three arbitrary rotation angles for the location of \mathcal{M}_2 on the S^2 so that two successive vertices of \mathcal{M}_1 (say $A_{(1,n)}$ and $A_{(1,1)}$ in order to refer to them explicitly below) coincide with the corresponding ones of \mathcal{M}_2 (i.e., $A_{(2,n)}$ and $A_{(2,1)}$ respectively), with these two regions being on opposite sides of the geodesic segment γ_n joining these two successive vertices and providing the common boundary between \mathcal{M}_1 and \mathcal{M}_2 . The union of these two regions with their common boundary, γ_n , no longer a boundary, then gives one single simply connected region on the round S^2 that represents the thorny sphere \tilde{S}^2 . We will denote this union region as \mathcal{M}_C , and its boundary as C . Note, however, that it is not ensured that the two regions that have been joined, \mathcal{M}_1 and \mathcal{M}_2 , will not overlap somewhere other than where they have been joined, so the map from \tilde{S}^2 to its image in S^2 is not necessarily one to one but can in some regions be two to one, and in general C may have self-intersections.

In Fig. 3 we demonstrate the regions for $n=6$ conical singularities. Regions \mathcal{M}_1 and \mathcal{M}_2 are shown on pictures (a) and (b), respectively. Points 1 and 6 have the same coordinates for both regions. In Fig. 4 \mathcal{M}_1 and \mathcal{M}_2 are united in the region \mathcal{M}_C by gluing them along the edge between points 1 and 6. The closed curve on Fig. 4 is the boundary C of \mathcal{M}_C , and in the considered example it does not have self-intersections. Hence, one can cut the region inside C and then glue the corresponding points (2 with 2', 3 with 3', 4 with 4' and 5 with 5'). This yields a thorny sphere with some configuration of 6 conical singularities with deficit angle $\mu = \pi/12$.

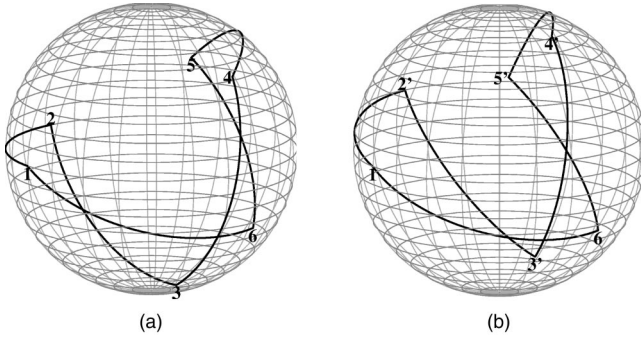


FIG. 3. (a) demonstrates the region \mathcal{M}_1 for a the thorny sphere with 6 singularities. \mathcal{M}_1 lies on the right when one goes from point 1 to point 2. (b) shows region \mathcal{M}_2 for the same sphere. It lies on the left hand side when one goes from point 1 to point 2'. On both pictures points 1,5,5' and 6 lie on the back side of the sphere.

The boundary C has $2n-2$ vertices $A_{(1,1)}=A_{(2,1)}, A_{(1,n)}=A_{(2,n)}, A_{(1,k)}$ for $2 \leq k \leq n-1$, and $A_{(2,k)}$ for $2 \leq k \leq n-1$, which can be at arbitrary locations except for the constraints that the $n-1$ geodesic distances between the successive $A_{(1,k)}$'s must be the same as those between the corresponding successive $A_{(2,k)}$'s ($n-1$ constraints, since we have already imposed the fact that the distance between $A_{(1,1)}$ and $A_{(1,n)}$ is the same as that between $A_{(2,1)}$ and $A_{(2,n)}$ by putting $A_{(1,1)}$ at the same location as $A_{(2,1)}$ and $A_{(1,n)}$ at the same location as $A_{(2,n)}$). Therefore, the number of free parameters done this way is twice the $2n-2$ vertices, minus the $n-1$ constraints, or $3n-3$, all of which are arbitrary, but of which three merely determine the orientation of the entire region on the S^2 , so that the number of true physical parameters is $3n-6$. (One could arbitrarily remove these three remaining gauge parameters of the freedom to rotate the coordinates by putting, say, $A_{(1,1)}$ at the ‘‘north pole’’ of the S^2 , at polar angle $\theta=0$, and then putting $A_{(1,n)}$ along the ‘‘prime meridian,’’ $\phi=0$.)

As we shall see below, the $3n-6$ coordinate-independent parameters that are in one-to-one correspondence with the geometries on a unit-curvature thorny sphere with n conical singularities (up to the discrete choice of the ordering of the

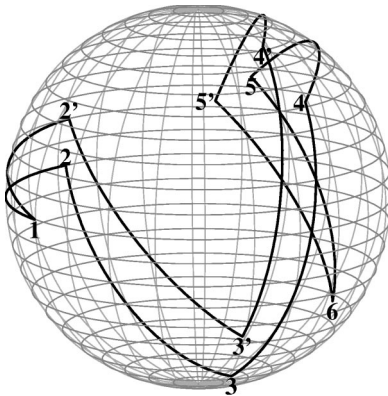


FIG. 4. This figure demonstrates the region \mathcal{M}_C constructed by method A. Points 1, 5, 5' and 6 lie on the back side of the sphere. Region \mathcal{M}_C is the unification of regions \mathcal{M}_1 and \mathcal{M}_2 shown on Fig. 3. The thorny sphere is obtained by gluing points k and k' .

singularities in the construction above, or of the choice of the triangulation when one takes its edge lengths as the $3n-6$ parameters) can be interpreted as $2n-3$ parameters for the relative locations of the n conical singularities on some S^2 (i.e., after taking out an overall rotation), plus $n-3$ conical deficit angles that can be freely specified once the relative locations are fixed. There is then a constraint fixing three of the conical deficit angles, which, at least in the case of small deficit angles, becomes the force-balance Eq. (1.2) for the strings that produce the deficit angles.

One way to see this constraint on the deficit angles is to consider how much freedom one has to specify the deficit angles μ_k after the points $A_{(1,k)}$ have been specified. Specifying the points $A_{(1,k)}$ determines the angles $\phi_{(1,k)}$ between the successive geodesics joining those points, but the deficit angles are $\mu_k = \phi_{(1,k)} - \phi_{(2,k)}$, and the angles $\phi_{(2,k)}$ are determined by the location of the points $A_{(2,k)}$. As noted above, without loss of generality we can orient \mathcal{M}_2 relative to \mathcal{M}_1 so that $A_{(1,1)}$ coincides with $A_{(2,1)}$ and $A_{(1,n)}$ coincides with $A_{(2,n)}$. Then the successive $A_{(2,k+1)}$'s must be at the same distances from the $A_{(2,k)}$'s as the $A_{(1,k+1)}$'s are from the $A_{(1,k)}$'s, but for $k < n-2$, the direction from $A_{(2,k)}$ to $A_{(2,k+1)}$ (at angle $\phi_{(2,k)}$ clockwise from the direction of the geodesic coming from $A_{(2,k-1)}$ to $A_{(2,k)}$) is a free parameter, whose choice fixes the deficit angle $\mu_k = \phi_{(1,k)} - \phi_{(2,k)}$ at that vertex. However, when one gets to $A_{(2,n-2)}$, the angle $\phi_{(2,n-2)}$ is fixed (up to a twofold degeneracy) so that the vertex $A_{(2,n-1)}$ is at the same distance from $A_{(2,n)}$ as $A_{(1,n-1)}$ is from $A_{(1,n)}$. Then when $A_{(2,n-1)}$ is thus fixed, the angles $\phi_{(2,n-2)}$, $\phi_{(2,n-1)}$, and $\phi_{(2,n)}$ are fixed, and hence these final three deficit angles, $\mu_{(n-2)}$, $\mu_{(n-1)}$, and μ_n , are determined (up to the twofold degeneracy of the two possible locations for $A_{(2,n-1)}$ at the fixed distances from $A_{(2,n-2)}$ and from $A_{(2,n)}$) and are not free parameters. Therefore, once the $2n-3$ parameters of the relative locations $A_{(1,k)}$ of the n conical singularities are determined (as seen from within \mathcal{M}_1), one is free to specify only $n-3$ of the deficit angles, giving a total of $3n-6$ parameters.

Another way to express the three constraints on the deficit angles once their $A_{(1,k)}$ locations are fixed is from the constraint on the holonomy from going successively around all of the conical singularities in a three-dimensional space. This will be discussed in Sec. V.

B. Method B

Another possible method to cut the thorny sphere with conical singularities into a piece that can be fitted onto the round sphere is the following: Let us choose one of the singular points, say point A_n , and connect it by shortest geodesics with the rest of the points, A_k , $k=1, \dots, n-1$. Since the thorny sphere is a complete Riemannian manifold, the shortest geodesics which connect A_n with different A_k do not intersect. If there are more than one geodesic with the same length, we choose one of them. Let us also make the following convention for ordering of points A_k , $k=1, \dots, n-1$. Choose one of these points and denote it by A_1 . We can go clockwise around A_n starting from the geodesic between A_n and A_1 . The convention is that the next geodesic we hit cor-

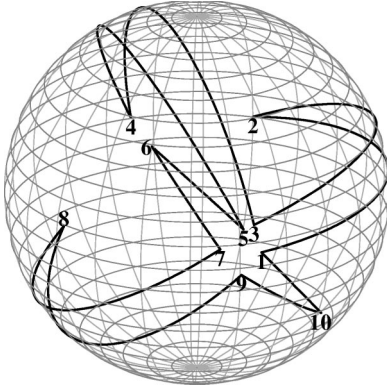


FIG. 5. This figure demonstrates the method B. Points 2,4,8 on the figure lie on the back side of the sphere. After cutting the star-like region and gluing the rest one gets a sphere with six conical singularities with deficit angle $\mu = \pi/12$. After gluing points 1, 3, 5, 7 and 9 are identified.

responds to the geodesic between A_n and A_2 , the next geodesic after that corresponds to point A_3 , etc.

Let us now make cuts of \tilde{S}^2 from A_n to points A_k along the geodesics. This procedure yields a spherical polygon \mathcal{M}_C with a boundary C which is a closed curve. In general, this polygon will have a shape different from the polygon obtained by method A, although the number of edges and vertices is the same, $2n - 2$. The boundary now consists of $n - 1$ pairs of geodesics with equal lengths. The geodesics in the given pair are connected at a vertex which corresponds to point A_n on \tilde{S}^2 , while different pairs are connected at vertices A_k , $k = 1, \dots, n - 1$. Let denote by B_{2k} the vertex which corresponds to A_k , $k = 1, \dots, n - 1$. Then a vertex between $B_{2(k-1)}$ and B_{2k} is the image of A_n . We denote it by B_{2k-1} .

As in the previous case \mathcal{M}_C can be glued on a regular sphere S^2 , and the boundary C of \mathcal{M}_C will be mapped on a closed contour on S^2 . Each vertex B_k on C has a uniquely defined coordinate \mathbf{x}_k on S^2 . We can now define coordinates of the singular points on \tilde{S}^2 as follows: points A_k ($k = 1, \dots, n - 1$) have coordinates $\mathbf{n}_k = \mathbf{x}_{2k}$ (coordinates of B_{2k}), and the coordinate \mathbf{n}_n of A_n can be chosen as the coordinate of one of B_{2k+1} . It is convenient to put $\mathbf{n}_n = \mathbf{x}_1$.

The internal angles at points B_{2k} are $\beta_{2k} = \alpha_k = 2\pi - \mu_k$, which are polar angles around A_k , $k = 1, \dots, n - 1$, and μ_k are conical angle deficits at A_k . If β_{2k+1} are internal angles at points B_{2k+1} , then they are related to the polar angle around A_n as

$$\alpha_n = \sum_{k=0}^{n-2} \beta_{2k+1}. \quad (4.1)$$

By assumption, all $\mu_k > 0$ and hence $0 < \alpha_k < 2\pi$, $k = 1, \dots, n$. The remarkable property of the contour C is that the edge between B_{2k+1} and B_{2k} can be obtained by rotating the edge between B_{2k-1} and B_{2k} counterclockwise around B_{2k} by angle β_{2k} . An example of \mathcal{M}_C obtained by cutting the sphere by the given method is shown on Fig. 5.

V. CONSTRAINT EQUATION

A. Derivation of the constraint equation

The procedure to construct a sphere with conical singularities by either method A or B requires that the contour C be closed; see Figs. 4 and 5. This imposes a constraint on the positions of the vertices on C . We describe how this restriction can be found by using method A of cutting the sphere. One can show that the constraint required for method B is the same.

Consider a region \mathcal{M}_C which appears after cutting \tilde{S}^2 by method A. Denote the coordinates of points $A_{(1,k)}$ on S^2 by \mathbf{n}_k , $k = 1, \dots, n$ and the coordinates of points $A_{(2,k)}$ by \mathbf{n}'_k , $k = 2, \dots, n - 1$. Let us also introduce matrices $O(\mathbf{n}, \alpha)$ which belong to the group $SO(3)$ and describe rotations by angle α around the axes defined by the unit vector \mathbf{n} normal to the S^2 . Our convention is that positive α corresponds to counterclockwise rotation around \mathbf{n} (as seen by looking down upon the sphere, with \mathbf{n} up). Define matrices $O_k = O(\mathbf{n}_k, \mu_k)$, $k = 1, \dots, n$, where μ_k is the conical angle deficit at the corresponding singular point on \tilde{S}^2 . Given matrices O_k and coordinates of the vertices $A_{(1,k)}$ on the boundary C , coordinates of the rest of the vertices $A_{(2,k)}$ on C can be found as follows.

The angle at the vertex $A_{(1,1)}$ between geodesics connecting $A_{(1,1)}$ with $A_{(1,2)}$ and $A_{(2,2)}$ is μ_1 . Therefore, $A_{(2,2)}$ is the image of $A_{(1,2)}$ obtained by rotating $A_{(1,2)}$ around $A_{(1,1)}$ by angle μ_1 , and by using the matrices one can write

$$\mathbf{n}'_2 = O_1 \mathbf{n}_2. \quad (5.1)$$

Consider now a rotation of $A_{(1,3)}$ around $A_{(1,2)}$ by angle μ_2 . This gives a point A'_3 which could be obtained by making a cut on the thorny sphere which goes through $A_{(1,2)}$ and $A_{(1,3)}$. To get $A_{(2,3)}$ one has to do an additional rotation of A'_3 around $A_{(1,1)}$ by angle μ_1 . This second rotation takes into account that point $A_{(1,2)}$ itself has to be rotated around $A_{(1,1)}$. Thus, for the coordinates of $A_{(2,3)}$ one gets

$$\mathbf{n}'_3 = O_1 O_2 \mathbf{n}_3. \quad (5.2)$$

This procedure can be continued further to get the coordinates of the other points $A_{(2,k)}$:

$$\mathbf{n}'_k = O_1 O_2 \cdots O_{k-1} \mathbf{n}_k. \quad (5.3)$$

If $k = n$ we come to the final point $A_{(1,n)}$. This point and its image coincide,

$$\mathbf{n}'_n = \mathbf{n}_n. \quad (5.4)$$

This means that the vector \mathbf{n}_n is on the axis of rotation defined by the matrix

$$M = O_1 O_2 \cdots O_{n-1}. \quad (5.5)$$

So for M we can write $M = O(\mathbf{n}_n, \alpha)$ where α is some angle.

We can also construct the coordinates of the images in a different way by starting with the point $A_{(2,n-1)}$ which is obtained by the rotation of $A_{(1,n-1)}$ around $A_{(1,n)}$ by the

angle μ_n . According to our convention, this rotation should be in the opposite direction, so for the coordinates of the points we get

$$\mathbf{n}'_{n-1} = O_n^{-1} \mathbf{n}_{n-1}. \quad (5.6)$$

By proceeding as earlier we get for the coordinates of the k th point

$$\mathbf{n}'_k = O_n^{-1} O_{n-1}^{-1} \cdots O_{k+1}^{-1} \mathbf{n}_k. \quad (5.7)$$

Because \mathbf{n}_1 and its image coincide the matrix

$$N = O_n^{-1} O_{n-1}^{-1} \cdots O_2^{-1} \quad (5.8)$$

corresponds to a rotation around \mathbf{n}_1 by some angle β , $N = O(\mathbf{n}_1, \beta)$. By using Eqs. (5.5) and (5.8) we can write

$$\begin{aligned} MO_n &= O(\mathbf{n}_n, \mu + \alpha), \\ O_1 N^{-1} &= O(\mathbf{n}_1, \mu - \beta). \end{aligned} \quad (5.9)$$

This relation shows that matrices $O(\mathbf{n}_n, \mu + \alpha)$ and $O(\mathbf{n}_1, \mu - \beta)$ are identical. In the general case, if points $A_{(1,1)}$, $A_{(1,n)}$ are not on the same axis, this implies that the matrices are the unit matrices. Thus, we come to the following constraint equation which follows from Eq. (5.9):

$$O(\mathbf{n}_1, \mu_1) O(\mathbf{n}_2, \mu_2) \cdots O(\mathbf{n}_n, \mu_n) = I_3, \quad (5.10)$$

where I_3 is the unit 3×3 matrix.

Figure 3 gives an example in which the coordinates of the points on (a) obey the condition (5.10) for $\mu = \pi/12$. The method of rotations described above was used to produce images $2', 3', 4', 5'$ of points 2,3,4,5, respectively. The constraint equation is also obeyed for the 6 points 2,4,6,8,10,1 in Fig. 5 with the same angle deficit $\mu = \pi/12$.

B. Alternate derivation of the constraint equation using holonomies

Let the 2-metric on the thorny sphere be $d\tilde{\Omega}^2$ (with unit Gaussian curvature away from the conical singularities), and consider the following three dimensional metric that is flat everywhere away from the conical singularities, which form strings in the radial directions:

$$ds^2 = dr^2 + r^2 d\tilde{\Omega}^2. \quad (5.11)$$

One can then calculate the holonomy of going around various closed curves and parallel transporting one's frame. Since the space is flat except at the strings, the holonomy will be trivial when the closed curve can be shrunk to a point without crossing any strings, but it will generally be non-trivial when the closed curve encircles one or more strings.

Choose a regular point, say A_0 in $\tilde{\mathcal{M}}_1$, and take a curve that starts at A_0 and stays in $\tilde{\mathcal{M}}_1$ until it nears the conical singularity at $A_{(1,k)}$. Then have the curve encircle that singularity (but no other one), in the counterclockwise direction

(by going briefly into $\tilde{\mathcal{M}}_2$), and then return in $\tilde{\mathcal{M}}_1$ to A_0 . Let the element of the holonomy group generated by that curve be labeled H_k .

All of the elements of the holonomy group can be obtained by products of these elements and their inverses. For example, the curve that first goes out from A_0 to encircle $A_{(1,k)}$ clockwise and then goes across $\tilde{\mathcal{M}}_1$ to encircle $A_{(1,j)}$ clockwise before returning in $\tilde{\mathcal{M}}_1$ to A_0 is $H_j^{-1} H_k^{-1}$. To take a slightly more complicated example, the curve that goes out from A_0 to leave $\tilde{\mathcal{M}}_1$ between $A_{(1,1)}$ and $A_{(1,2)}$ and then goes across $\tilde{\mathcal{M}}_2$ to encircle $A_{(2,4)}$ clockwise and then return back along its previous path to A_0 generates the holonomy group element $\tilde{H}_4 = H_2 H_3 H_4^{-1} H_3^{-1} H_2^{-1}$.

Now consider the curve that starts at A_0 , goes out in $\tilde{\mathcal{M}}_1$ and encircles $A_{(1,n)}$ counterclockwise, returns directly in $\tilde{\mathcal{M}}_1$ to A_0 , and then in turn goes out and encircles $A_{(1,n-1)}$ counterclockwise and returns, and then encircles $A_{(1,n-2)}$ counterclockwise, etc., until finally it encircles $A_{(1,1)}$ counterclockwise and returns to A_0 . Thus it encircles each of the conical singularities clockwise in order, staying in $\tilde{\mathcal{M}}_1$ except for each time it encircles a singularity. The total holonomy generated by this curve is $H_1 H_2 H_3 \cdots H_{n-1} H_n$.

But since this curve encircles all of the singularities, it can be deformed so that all of it lies in $\tilde{\mathcal{M}}_2$ except for the initial part leaving A_0 and the final part returning to A_0 . This curve can then be shrunk to zero without crossing any singularities, so for consistency it must represent the trivial holonomy element (the identity). Therefore, we get the constraint equation

$$H_1 H_2 H_3 \cdots H_{n-1} H_n = I. \quad (5.12)$$

In the $SO(3)$ representation this constraint coincides with Eq. (5.10) which was found by the alternate computation, with $H_k = O(\mathbf{n}_k, \mu_k)$.

If we use an $SU(2)$ spinor representation of the holonomy, then the conical singularity at A_1 with deficit angle μ_k generates the holonomy element

$$\begin{aligned} H_k &= U_k = U(\mathbf{n}_k, \mu_k) \equiv e^{(i/2)\mathbf{h}_k \mu_k} \\ &= I \cos \frac{\mu_k}{2} + i \mathbf{h}_k \sin \frac{\mu_k}{2}, \end{aligned} \quad (5.13)$$

where

$$\mathbf{h}_k \equiv \mathbf{n}_k \cdot \boldsymbol{\sigma} \equiv n_k^i \sigma_i = \begin{pmatrix} n_k^3 & n_k^1 + i n_k^2 \\ n_k^1 - i n_k^2 & -n_k^3 \end{pmatrix} \quad (5.14)$$

with n_k^i being the i th Cartesian coordinates of the unit normal \mathbf{n}_k to the round S^2 embedded in flat three-space, at the point $A_{(1,k)}$ of \mathcal{M}_1 that represents A_k on the thorny sphere \tilde{S}^2 . The Pauli matrices σ_i are chosen such that $i\sigma_1\sigma_2\sigma_3 = 1$ which guarantees that U_k corresponds to counterclockwise rotation around \mathbf{n}_k . For generic assumed deficit angles, without imposing the constraint (5.12), the product of all the U_k 's will also be a holonomy element of the form

$$U_1 U_2 U_3 \cdots U_{n-1} U_n = e^{(i/2)\mathcal{N}}, \quad (5.15)$$

where

$$\mathcal{N} \equiv \mathbf{N} \cdot \boldsymbol{\sigma} \equiv N^i \sigma_i = \begin{pmatrix} N^3 & N^1 + iN^2 \\ N^1 - iN^2 & -N^3 \end{pmatrix} \quad (5.16)$$

for some vector \mathbf{N} with Cartesian coordinates N^i (and which without loss of generality can be taken to have length, which represents the total angle of rotation, $\leq 2\pi$). Then the constraint (5.12) is the condition that the total rotation vector is zero,

$$\mathbf{N} = 0, \quad \text{or} \quad N^i = 0 \text{ for each } i, \quad (5.17)$$

which gives the three conditions on the deficit angles.

The constraint Eq. (5.12) has two immediate consequences: (1) there cannot exist a thorny sphere with a single conical singularity with a deficit angle $0 < |\mu| < 2\pi$, and (2) on a sphere with a pair of conical singularities, the singular points lie on the same axis.

We can note that Eq. (5.12) in the $SU(2)$ spinor representation follows from the single equation

$$\text{Tr}[U_1 U_2 U_3 \cdots U_{n-1} U_n] = 2. \quad (5.18)$$

Indeed, the product of matrices in the left hand side of Eq. (5.15) is a unitary matrix which corresponds to a rotation by an angle φ around some axis. If the trace of this matrix is 2, then the angle is $\varphi = 4\pi m$, where m is an integer, and the unitary matrix is simply the unit matrix. Conversely, when the trace of a 2×2 unitary matrix is 2, it must be the unit matrix.

C. Constraint equations for small angle deficits

In the case of small deficit angles, so that all of the sums of the products of different matrices \mathcal{h}_k are small, then the constraint Eq. (5.12) or Eq. (5.17) becomes

$$0 = \mathbf{N} \approx \sum_{k=1}^n \mu_k \mathbf{n}_k, \quad (5.19)$$

which is Eq. (1.2) of the Introduction. Since in the three-dimensional space with small deficit angles, the conical singularities correspond to strings with tension μ_k at directions given by \mathbf{n}_k , the constraint equation becomes the equilibrium condition for the forces exerted by the strings, a force-balance equation. For an even number of conical singularities Eq. (5.19) is satisfied when for each conical singularity whose position is defined by the vector \mathbf{n}_k , there is another singularity with the same deficit angle whose position is $-\mathbf{n}_k$. This situation is realized for polyhedral configurations of singularities discussed in [2].

VI. CONSTRUCTION OF THORNY SPHERES WITH LARGE DEFICITS: EXAMPLES

A. Three conical singularities

1. Case of equal angle deficits

We now discuss some examples of spheres with large deficit angles. To construct them it is enough to solve the constraint equation (5.12). Let us consider first the case in which there are three deficit angles that are all the same and equal to μ , and the conical singularities are at points $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$. The constraint Eq. (5.12) for this configuration yields the holonomy around one point in terms of holonomies of two other points. We will write this in the following form, using the $SU(2)$ representation of the holonomy given by Eq. (5.13):

$$U(\mathbf{n}_1, \mu) U(\mathbf{n}_2, \mu) = U^{-1}(\mathbf{n}_3, \mu) = U(-\mathbf{n}_3, \mu). \quad (6.1)$$

Suppose that the angle between vectors \mathbf{n}_2 and \mathbf{n}_1 is a ($0 < a < \pi$). We can choose \mathbf{n}_1 and \mathbf{n}_2 lying in the (xy) plane such that matrices for the corresponding points are

$$\mathcal{h}_1 = \cos a \sigma_1 + \sin a \sigma_2, \quad \mathcal{h}_2 = \sigma_1. \quad (6.2)$$

We get

$$\begin{aligned} U(\mathbf{n}_1, \mu) U(\mathbf{n}_2, \mu) &= \left(\cos^2 \frac{\mu}{2} - \cos a \sin^2 \frac{\mu}{2} \right) I \\ &+ i \sin \frac{\mu}{2} \left[\cos \frac{\mu}{2} (1 + \cos a) \sigma_1 \right. \\ &+ \cos \frac{\mu}{2} \sin a \sigma_2 \\ &\left. + \sin \frac{\mu}{2} \sin a \sigma_3 \right]. \end{aligned} \quad (6.3)$$

To satisfy Eq. (6.1) one has to choose a such that

$$\cos a = - \frac{\cos(\mu/2)}{1 + \cos(\mu/2)}. \quad (6.4)$$

In the limit that $\mu = 0$, one gets $a = 4\pi/3$, so each spherical triangle fills a hemisphere. As μ is increased, the length a of the sides decreases and reaches 0 when $\mu = 4\pi/3$.

Given Eq. (6.4), the position of the third point, \mathbf{n}_3 , is fixed by the matrix which follows from Eq. (6.1),

$$\mathcal{h}_3 = \cos a \sigma_1 - \cos \frac{\mu}{2} \sin a \sigma_2 - \sin \frac{\mu}{2} \sin a \sigma_3. \quad (6.5)$$

Equations (6.2) and (6.5) give coordinates of the conical singularities with deficit angle μ . The corresponding sphere with conical singularities can be constructed from a spherical polygon \mathcal{M}_C with a boundary C on regular sphere S^2 . This can be done by either method A or B. Consider, for instance, method A. The contour C consists of two parts, C_1 and C_2 . The contour C_1 consists of two shortest geodesics connecting points \mathbf{n}_1 with \mathbf{n}_2 and \mathbf{n}_2 with \mathbf{n}_3 . The contour C_2 con-

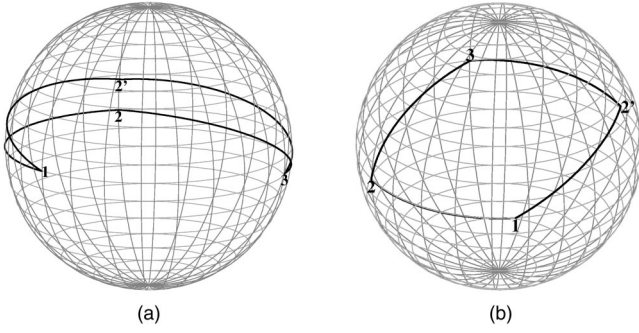


FIG. 6. (a) and (b) demonstrate cuts which after gluing yield thorny spheres with three conical singularities with deficits $\mu = \pi/12$ and $7\pi/6$, respectively. On (a) points 1 and 3 lie on the back side of the sphere.

sists of geodesics connecting \mathbf{n}_1 with \mathbf{n}'_2 and \mathbf{n}'_2 with \mathbf{n}_3 where \mathbf{n}'_2 is the image of \mathbf{n}_2 obtained by rotation around \mathbf{n}_1 by angle μ :

$$\mathbf{n}'_2 = U(\mathbf{n}_1, \mu) \mathbf{n}_2 U^{-1}(\mathbf{n}_1, \mu). \quad (6.6)$$

For some values of μ and a the corresponding contours are presented in Fig. 6.

2. Arbitrary angle deficits μ_k

In the general case of $n=3$ conical singularities with arbitrary deficit angles μ_k , let $\mathbf{A} = \mathbf{n}_1 \tan(\mu_1/2) = \pm A \mathbf{n}_1$, $\mathbf{B} = \mathbf{n}_2 \tan(\mu_2/2) = \pm B \mathbf{n}_2$, and $\mathbf{C} = \mathbf{n}_3 \tan(\mu_3/2) = \pm C \mathbf{n}_3$, where A , B , and C are chosen to be positive. Then

$$U_1 = U(\mathbf{n}_1, \mu_1) = \cos \frac{\mu_1}{2} (I + i\mathbf{A}) = \frac{I + i\mathbf{A}}{\sqrt{1 + A^2}}, \quad (6.7)$$

$$U_2 = U(\mathbf{n}_2, \mu_2) = \cos \frac{\mu_2}{2} (I + i\mathbf{B}) = \frac{I + i\mathbf{B}}{\sqrt{1 + B^2}}, \quad (6.8)$$

$$U_3 = U(\mathbf{n}_3, \mu_3) = \cos \frac{\mu_3}{2} (I + i\mathbf{C}) = \frac{I + i\mathbf{C}}{\sqrt{1 + C^2}}, \quad (6.9)$$

where the signs of the square roots in the denominators are chosen to be the same as those of the respective $\cos(\mu_k/2)$'s (positive if $|\mu_k| < \pi$). Then for $U_1 U_2 U_3$ to be the unit matrix that represents the trivial holonomy, one uses the Pauli-matrix identity (for $i\sigma_1 \sigma_2 \sigma_3 = 1$)

$$\mathbf{A}\mathbf{B} \equiv (\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) = (\mathbf{A} \cdot \mathbf{B})I - i(\mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma} \quad (6.10)$$

and gets the constraint

$$\begin{aligned} & \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \\ & - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} + (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} = 0. \end{aligned} \quad (6.11)$$

One can see that the linear part of this is simply that the sum of the three vectors is zero, $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$.

In the nonlinear case, if, say, \mathbf{A} and \mathbf{B} are specified, then the solution for \mathbf{C} is

$$\mathbf{C} = -\frac{\mathbf{A} + \mathbf{B} + \mathbf{A} \times \mathbf{B}}{1 - \mathbf{A} \cdot \mathbf{B}}. \quad (6.12)$$

Again one can see that if \mathbf{A} and \mathbf{B} are small, to linear order in those vectors, $\mathbf{C} = -\mathbf{A} - \mathbf{B}$. Alternatively, if the three unit normals \mathbf{n}_k are specified, then the solution for the deficit angles is

$$\tan \frac{\mu_1}{2} = \frac{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_3)}, \quad (6.13)$$

$$\tan \frac{\mu_2}{2} = \frac{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}{(\mathbf{n}_2 \times \mathbf{n}_1) \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}, \quad (6.14)$$

$$\tan \frac{\mu_3}{2} = \frac{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}{(\mathbf{n}_3 \times \mathbf{n}_1) \cdot (\mathbf{n}_3 \times \mathbf{n}_2)}. \quad (6.15)$$

A third specification would be to fix the three deficit angles and thereby to fix the three lengths A , B , and C of the three vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} respectively. Then the constraint determines the relative directions of \mathbf{A} , \mathbf{B} , and \mathbf{C} . Suppose that these are given by the cosines of the angles between them, say

$$\alpha = \cos a = \mathbf{n}_2 \cdot \mathbf{n}_3 = \mathbf{B} \cdot \mathbf{C} / (BC), \quad (6.16)$$

$$\beta = \cos b = \mathbf{n}_3 \cdot \mathbf{n}_1 = \mathbf{C} \cdot \mathbf{A} / (CA), \quad (6.17)$$

$$\gamma = \cos c = \mathbf{n}_1 \cdot \mathbf{n}_2 = \mathbf{A} \cdot \mathbf{B} / (AB). \quad (6.18)$$

Then by equating the squared magnitudes of the two sides of Eq. (6.12), one can solve for

$$\gamma = \frac{1}{AB} \left[1 \mp \sqrt{\frac{1 + A^2 + B^2 + A^2 B^2}{1 + C^2}} \right]. \quad (6.19)$$

Similarly, by cyclic permutations one gets

$$\alpha = \frac{1}{BC} \left[1 \mp \sqrt{\frac{1 + B^2 + C^2 + B^2 C^2}{1 + A^2}} \right], \quad (6.20)$$

$$\beta = \frac{1}{CA} \left[1 \mp \sqrt{\frac{1 + C^2 + A^2 + C^2 A^2}{1 + B^2}} \right]. \quad (6.21)$$

One must choose the signs so that the cosines are between -1 and 1 . (For small A , B , and C , one must choose the upper signs.) In order that the cosines can be between -1 and 1 , the absolute magnitudes of the three deficit angles must obey the triangular inequalities (each larger than the absolute difference between the other two, and each smaller than the sum of the other two), which translates into the nonlinear inequalities for A , B , and C that

$$\frac{|A - B|}{|1 + AB|} \leq C \leq \frac{A + B}{|1 - AB|}, \quad (6.22)$$

$$\frac{|B - C|}{|1 + BC|} \leq A \leq \frac{B + C}{|1 - BC|}, \quad (6.23)$$

$$\frac{|C-A|}{|1+CA|} \leq B \leq \frac{C+A}{|1-CA|}. \quad (6.24)$$

In the case of three conical singularities that we are presently considering, the regions \mathcal{M}_1 and \mathcal{M}_2 are simply spherical triangles that are identical except for their orientation. The interior angles at the vertices **A**, **B**, and **C** are $\pi - \mu_1/2$, $\pi - \mu_2/2$, and $\pi - \mu_3/2$ respectively, and the cosines of the angular lengths of the opposite sides are $\alpha = \cos a$, $\beta = \cos b$, and $\gamma = \cos c$ respectively. Then, as an alternative to the constraint equations given above, one can use the standard formulas (2.3) and (2.4) for spherical triangles. For example, from Eq. (2.3),

$$\alpha = \cos a = \frac{\cos(\mu_2/2)\cos(\mu_3/2) - \cos(\mu_1/2)}{\sin(\mu_2/2)\sin(\mu_3/2)} \quad (6.25)$$

and cyclically for $\beta = \cos b$ and $\gamma = \cos c$. This reduces to Eq. (6.4) in the special case in which all of the three deficit angles are equal to μ .

B. Four conical singularities

1. Case of equal angle deficits

Consider now a sphere with four conical singularities with deficits μ_k at points \mathbf{n}_k , $k=1,2,3,4$. The constraint (5.12) on the holonomies is

$$U(\mathbf{n}_1, \mu_1)U(\mathbf{n}_2, \mu_2)U(\mathbf{n}_3, \mu_3)U(\mathbf{n}_4, \mu_4) = I. \quad (6.26)$$

Given the coordinates and angular deficits of three points, we can find from Eq. (6.26) the coordinates and the deficit of the fourth point. We first present here a particular solution of Eq. (6.26) for the case in which all the deficit angles coincide. Then the constraint (6.26) can be rewritten as

$$\begin{aligned} U(\mathbf{n}, \beta) &= U(\mathbf{n}_1, \mu)U(\mathbf{n}_2, \mu) \\ &= U(-\mathbf{n}_4, \mu)U(-\mathbf{n}_3, \mu), \end{aligned} \quad (6.27)$$

where \mathbf{n} is a unit vector in the Euclidean 3-space. The parameters β and \mathbf{n} are uniquely defined by Eq. (6.27) if μ and the coordinates of the pair of points $\mathbf{n}_1, \mathbf{n}_2$ are known. For $\mathbf{n}_1, \mathbf{n}_2$ defined as in Eq. (6.2),

$$\begin{aligned} \hbar &= C^{-1/2} \left(\cos \frac{\mu}{2} (1 + \cos a) \sigma_1 + \cos \frac{\mu}{2} \sin a \sigma_2 \right. \\ &\quad \left. - \sin \frac{\mu}{2} \sin a \sigma_3 \right), \end{aligned} \quad (6.28)$$

$$C = \sin^2 a + \cos^2 \frac{\mu}{2} (1 + \cos a)^2. \quad (6.29)$$

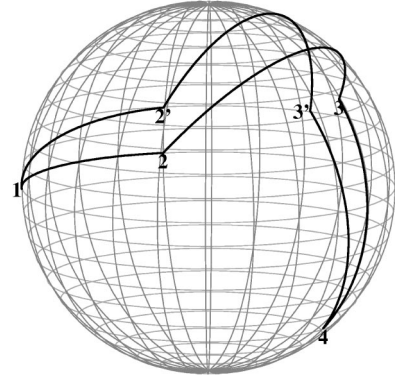


FIG. 7. This figure shows the cut for 4 conical singularities with deficits $\mu = \pi/12$ and parameters $\cos a = 0.2$ and $\theta = \pi/2$. Points 3 and 3' lie on the back side of the sphere.

Given coordinates \mathbf{n}_1 and \mathbf{n}_2 it is easy to see that Eq. (6.27) holds if \mathbf{n}_3 and \mathbf{n}_4 are obtained by a rotation of $-\mathbf{n}_2, -\mathbf{n}_1$ around \mathbf{n} by some angle θ , i.e.,

$$\begin{aligned} \hbar_3 &= -U(\mathbf{n}, \theta) \hbar_2 U^{-1}(\mathbf{n}, \theta), \\ \hbar_4 &= -U(\mathbf{n}, \theta) \hbar_1 U^{-1}(\mathbf{n}, \theta). \end{aligned} \quad (6.30)$$

This procedure yields a three-parameter family of spheres with four conical singularities. The parameters are μ, θ and the angle a between \mathbf{n}_1 and \mathbf{n}_2 . The internal region \mathcal{M}_C of an example of such a sphere obtained by method B is shown in Fig. 7. Points \mathbf{n}'_2 and \mathbf{n}'_3 are the images of \mathbf{n}_2 and \mathbf{n}_3 , respectively. The same method was applied to get configurations with 6 conical singularities on Figs. 4 and 5.

2. Arbitrary angle deficits μ_k

Now let us turn to the case when the $n=4$ conical singularities have arbitrary conical deficit angles μ_k . Similar to what was done for $n=3$, let $\mathbf{A} = \mathbf{n}_1 \tan(\mu_1/2) = \pm A \mathbf{n}_1$, $\mathbf{B} = \mathbf{n}_2 \tan(\mu_2/2) = \pm B \mathbf{n}_2$, $\mathbf{D} = \mathbf{n}_3 \tan(\mu_3/2) = \pm D \mathbf{n}_3$, $\mathbf{E} = \mathbf{n}_4 \tan(\mu_4/2) = \pm E \mathbf{n}_4$, where A, B, D , and E are chosen to be positive. Now there are various ways to proceed with solving the constraint $U_1 U_2 U_3 U_4 = I$, depending on what is specified and what is to be solved for.

If \mathbf{A}, \mathbf{B} , and \mathbf{D} are specified and \mathbf{E} is to be solved for, one writes the constraint in the form

$$\begin{aligned} U_4 &\equiv \frac{I + i\mathbf{E}}{\sqrt{1 + E^2}} = U_3^{-1} U_2^{-1} U_1^{-1} \\ &= \frac{(I - i\mathbf{D})(I - i\mathbf{B})(I - i\mathbf{A})}{\sqrt{1 + D^2} \sqrt{1 + B^2} \sqrt{1 + A^2}}, \end{aligned} \quad (6.31)$$

and then one can explicitly solve for

$$\mathbf{E} = \frac{-\mathbf{A} - \mathbf{B} - \mathbf{D} - \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{D} - \mathbf{B} \times \mathbf{D} + (\mathbf{A} \cdot \mathbf{B}) \mathbf{D} - (\mathbf{A} \cdot \mathbf{D}) \mathbf{B} + (\mathbf{B} \cdot \mathbf{D}) \mathbf{A}}{1 - \mathbf{A} \cdot \mathbf{B} - \mathbf{A} \cdot \mathbf{D} - \mathbf{B} \cdot \mathbf{D} - (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D}}. \quad (6.32)$$

This obviously generalizes to arbitrary n : If the first $n-1$ positions and deficit angles are specified, so that U_k is given for $1 \leq i \leq n-1$, then one can readily solve for

$$U_n = U_{n-1}^{-1} U_{n-2}^{-1} \cdots U_2^{-1} U_1^{-1}. \quad (6.33)$$

If instead all of the deficit angles are given, and all but two consecutive positions on the sphere, then, assuming that the appropriate triangular inequalities are satisfied, one can solve for these two positions up to an overall rotation about an axis determined by what is given. (This rotation has non-trivial significance only for $n \geq 4$.) Let us illustrate this with the case $n=4$.

Suppose that the positions and deficit angles of the 3rd and 4th conical singularities are given, so that \mathbf{D} and \mathbf{E} are given, and the deficit angles but not the positions of the 1st and 2nd conical singularities are given, so that A and B are given, but not the direction \mathbf{n}_1 of \mathbf{A} and the direction \mathbf{n}_2 of \mathbf{B} . Then we can use the constraint in the form

$$U_2^{-1} U_1^{-1} = U_3 U_4, \quad (6.34)$$

or

$$\begin{aligned} \frac{I - \mathbf{A} \cdot \mathbf{B} - i(\mathbf{A} + \mathbf{B} + \mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma}}{\sqrt{1+A^2} \sqrt{1+B^2}} &= U_3 U_4 \\ &= \frac{I + i\mathcal{C}}{\sqrt{1+C^2}}, \end{aligned} \quad (6.35)$$

where for $n=4$

$$\mathbf{C} = \frac{\mathbf{D} + \mathbf{E} + \mathbf{D} \times \mathbf{E}}{1 - \mathbf{D} \cdot \mathbf{E}} \quad (6.36)$$

represents the combined effect of U_3 and U_4 .

Then we can proceed as we did for $n=3$ when the magnitudes A , B , and C were given, to solve for the angles between \mathbf{A} , \mathbf{B} , and \mathbf{C} . Since here \mathbf{C} is determined from the given \mathbf{D} and \mathbf{E} , the directions of \mathbf{A} and \mathbf{B} are then determined up to an overall rotation about the vector \mathbf{C} .

C. n conical singularities

Obviously, the procedure of the preceding section generalizes to higher n . If the U_k are specified for all $3 \leq i \leq n$, and if the deficit angles μ_1 and μ_2 are also specified (so that $A = |\tan \mu_1/2|$ and $B = |\tan \mu_2/2|$ are specified), then we define \mathbf{C} by

$$U_3 U_4 \cdots U_{n-1} U_n = \frac{I + i\mathcal{C}}{\sqrt{1+C^2}} \quad (6.37)$$

and solve for the angles between \mathbf{A} , \mathbf{B} , and \mathbf{C} . This determines \mathbf{A} and \mathbf{B} up to an overall rotation about the vector \mathbf{C} .

If all of the deficit angles are specified, and all but two non-consecutive positions, then one has to permute the positions and put in the appropriate commutators to solve for those positions (up to the arbitrary rotation). For example,

for $n=4$, suppose that the deficit angles but not the positions of the 1st and 3rd points are specified, and that both are specified for the 2nd and 4th points, so that U_2 and U_4 are given. Then if we define $\tilde{U}_3 = U_2 U_3 U_2^{-1}$, the constraint equation becomes $\tilde{U}_3^{-1} U_1^{-1} = U_2 U_4$, and one can use the previous procedure to solve for U_1 and \tilde{U}_3 , up to an overall rotation about the axis of the rotation $U_2 U_4$. Finally, one reconstructs $U_3 = U_2^{-1} \tilde{U}_3 U_2$. This procedure has a straightforward generalization for all higher n and for the unspecified positions to be further separated in the cyclic chain of rotations U_k that combine to produce the identity.

VII. TOPOLOGICAL ASPECTS OF SPHERE CUTTING

Using method A developed in Sec. III one gets a map

$$\Psi: \tilde{S}^2 \rightarrow \{\mathcal{M}_1, \mathcal{M}_2\} \quad (7.1)$$

of a thorny sphere \tilde{S}^2 onto a pair of simply connected regions on a unit round sphere. The boundaries $\mathcal{B}_1 = \partial \mathcal{M}_1$ and $\mathcal{B}_2 = \partial \mathcal{M}_2$ are isometric spherical polygons which are to be identified. Each of the polygons has n vertices, $A_{(1,k)}$ and $A_{(2,k)}$, where n is the number of conical singularities of the thorny sphere \tilde{S}^2 . It should be emphasized that the change of the reference point which is used to order the conical singularities may result in a change of order of the vertices $A_{(1,k)}$ and $A_{(2,k)}$, and as a result of this, one can get a different choice of regions $\mathcal{M}_1, \mathcal{M}_2$ representing the same thorny sphere. It is evident that the corresponding maps Ψ and Ψ' , formally different, are in fact equivalent.

When we construct regions \mathcal{M}_1 and \mathcal{M}_2 using the method described in Sec. IV A by solving the constraint equations we do not know in advance which ordering procedure of the conical singularities on the thorny sphere would correspond to the set of vertices obtained by gluing the boundaries \mathcal{B}_1 and \mathcal{B}_2 and identifying the vertices $A_{(1,k)}$ and $A_{(2,k)}$. In fact, the situation is even more complicated.

Let us note that the constraint equations (5.10) and (5.12) remain unchanged if one includes a unit operator between any two subsequent terms, say i and $i+1$ in the product of matrices. But a rotation along an axis \mathbf{n} by the angle $2\pi m$ is represented by a unit operator. Thus adding two new vertices with angle deficits $2\pi m$ (one for each of the regions \mathcal{M}_k) does not violate the constraint equations. We can choose the new angle deficits to be negative and put the new vertices at the north and south poles of a round sphere. This is equivalent to the usage of a covering space S_m^2 for a round sphere with a winding number $m > 0$. That is why by solving the constraint equations (5.10) and (5.12) one may end not with regular regions $\mathcal{M}_1, \mathcal{M}_2$ on a round sphere, but with regions on a covering space for S_m^2 . After identifying the points of m different leaves and projecting S_m^2 onto S^2 , one obtains boundaries \mathcal{B}_1 and \mathcal{B}_2 which are topologically circles S^1 , but which have intersections.

In order to exclude such cases one must be certain that after a solution of the constraint equations one does not have

undesirable extra vertices with $-2\pi m$ angle deficits. We describe now a procedure which allows one to do this.

Consider a map ψ_a of the region \mathcal{M}_a onto the region $\tilde{\mathcal{M}}_a$ of the thorny sphere:

$$\psi_a: \mathcal{M}_a \rightarrow \tilde{\mathcal{M}}_a. \quad (7.2)$$

Under this map the boundary \mathcal{B}_a of \mathcal{M}_a is transformed into the boundary $\tilde{\mathcal{B}}_a$ of $\tilde{\mathcal{M}}_a$. Take a regular point \tilde{p} inside $\tilde{\mathcal{M}}_a$ and let p_k be its images under ψ_a^{-1} . Let y^μ be coordinates near \tilde{p} and x_k^ν be coordinates near p_k . Since $\tilde{\mathcal{M}}_a$ and \mathcal{M}_a are orientable manifolds, we use local coordinate systems on both of them so that any transition from one coordinate system to another on the same manifold has the value of the Jacobian equal to $+1$. The *degree of a map* ψ_a is determined as

$$\text{deg } \psi_a = \sum_k \text{sgn det} \left(\frac{\partial y^\mu}{\partial x_k^\nu} \right)_{p_k}. \quad (7.3)$$

One can prove (see e.g. [8,9]) that the index of the map does not depend on the choice of the regular point \tilde{p} and is invariant under smooth homotopies. Moreover, the index of the map ψ_a is the same as the index of the map ψ_a restricted to the boundary \mathcal{B}_a . Since both of the boundaries are topologically circles S^1 , the degree of this map is just a winding number m . Note that \mathcal{M}_1 and \mathcal{M}_2 have the same degree of map because \mathcal{B}_2 can be obtained by a deformation of \mathcal{B}_1 (recall that we got vertices on \mathcal{B}_2 by rotating vertices on \mathcal{B}_1).

To calculate this winding number we consider a stereographic projection of \mathcal{B}_a into a plane. We take the origin of the stereographic projection to lie in the interior of the region \mathcal{M}_1 . For a resulting curve $\hat{\mathcal{B}}_a$ with intersections we define

$$m = \frac{1}{2\pi} \oint k dl, \quad (7.4)$$

where k is the Gaussian curvature of the curve $\hat{\mathcal{B}}_a$ and dl is the proper distance element. Since m is invariant under smooth homotopic transformations, it remains the same under a continuous change of the position of the ‘‘north’’ pole used for the stereographic projection until it crosses \mathcal{B}_a .

To summarize the above discussion we stress that using method A, starting with a given thorny sphere $\tilde{\mathcal{S}}^2$ one can define (not uniquely) regions $\mathcal{M}_{1,2}$ and the gluing procedure which recovers $\tilde{\mathcal{S}}^2$. In the inverse procedure, when one starts with a solution of the constraint equations, one must first check whether the winding number of the boundary is 1. Only in this case will the gluing procedure give a thorny sphere $\tilde{\mathcal{S}}^2$ without any additional angle deficits that are negative integer multiples of 2π .

VIII. THORNY SPHERES AND SOLUTIONS OF EINSTEIN EQUATIONS WITH RADIAL STRINGS

Let us discuss now solutions of the Einstein equations for the thorny sphere configurations, with strings at the conical

singularities. The total action of the system is²

$$I = \frac{1}{16\pi G} \left[\int_{\mathcal{M}} \sqrt{-g} d^4x R + 2 \int_{\partial\mathcal{M}} K \sqrt{-h} d^3x \right] - \frac{1}{4\pi} \sum_k \hat{\mu}_k \int \sqrt{\sigma_k} d^2\zeta_k. \quad (8.1)$$

The last term in the right hand side of Eq. (8.1) is the Nambu-Goto action for the strings, where $(\sigma_k)_{\alpha\beta}$ is the metric induced on the world sheet of a particular string. We assume in Eq. (8.1) that the space-time \mathcal{M} has a timelike boundary. We take the metric in the form

$$ds^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta + e^{2\phi} a^2 d\tilde{\Omega}^2. \quad (8.2)$$

Here $\gamma_{\alpha\beta}$ is a 2D metric, $\phi = \phi(x)$ is a dilaton field which depends on coordinates x^α , and $d\tilde{\Omega}^2$ is the metric on the thorny sphere $\tilde{\mathcal{S}}^2$ with conical singularities. For a string located at fixed angles, the induced metric on a string world sheet coincides with $\gamma_{\alpha\beta}$. The parameter a in Eq. (8.2) has the dimensionality of length. Locally near each string the metric $d\tilde{\Omega}^2$ can be written as

$$d\tilde{\Omega}^2 = \sin^2 \theta d\varphi^2 + d\theta^2, \quad (8.3)$$

where $0 \leq \theta \leq \pi$, and φ is periodic with period $2\pi - \mu_k$. To proceed we have to take into account in Eq. (8.1) the presence of delta-function-like contributions due to the conical singularities (see, for instance [11])

$$\int_{\mathcal{M}} \sqrt{-g} d^4x R = \int_{\text{reg}(M^2)} \sqrt{-g} d^4x R + 2 \sum_k \mu_k \int \sqrt{\sigma_k} d^2\zeta_k, \quad (8.4)$$

where $\text{reg}(M^2)$ is the regular domain of \mathcal{M} . If we impose the on-shell condition $\mu_k = 8\pi G \hat{\mu}_k$, the contribution of the conical singularities in the curvature in Eq. (8.1) will cancel exactly the contribution from the string actions. There will remain only the bulk part of the action. On the metric (8.2) it will reduce to the 2D dilaton gravity action

$$I = \frac{1}{4G_2} \left[\int \sqrt{\gamma} d^2x \left(e^{2\phi} R_2 + 2e^{2\phi} (\nabla\phi)^2 + \frac{2}{a^2} \right) + 2 \int dy e^{2\varphi} (k - k_0) \right], \quad (8.5)$$

$$\frac{1}{G_2} = \frac{a^2 C}{G}, \quad (8.6)$$

²In this section we restore a normal value G for the Newton constant.

$$C = 1 - 2G \sum_k \hat{\mu}_k = 1 - \frac{1}{4\pi} \sum_k \mu_k. \quad (8.7)$$

The curvature R_2 in Eq. (8.5) is the 2D curvature determined by $\gamma_{\alpha\beta}$. As a result of the modification of the area of sphere due to the conical singularities, the gravitational action (including the boundary term) acquires an overall coefficient which depends on the μ_k 's. We included this coefficient in the definition of effective two dimensional gravitational coupling G_2 , Eq. (8.6). It is important that the action (8.5) has precisely the same form as the dilatonic action obtained under a spherical reduction of the gravitational action in the absence of cosmic strings. Therefore strings have no effect on the dynamical equations for the metric $\gamma_{\alpha\beta}$ and the dilaton ϕ . For these quantities one has standard solutions. In particular, the Birkhoff theorem can be applied in this case and guarantees that in the absence of other matter in the bulk, the solution is static and is a 2D black hole of mass M :

$$d\gamma^2 = -F dt^2 + F^{-1} dr^2, \quad F = 1 - \frac{2M}{r}. \quad (8.8)$$

The corresponding four-dimensional solution is a Schwarzschild black hole of the same mass parameter, but with strings in the radial direction. In a similar way, by using Eq. (8.5) one can construct non-static solutions in the presence of strings. Non-vacuum static spherically symmetric solutions, such as a charged black hole with strings, can be constructed as well by adding matter in the bulk.

For example, if we define $d^2\hat{\Omega} = d\tilde{\Omega}^2/C$ to give a rescaled thorny sphere with area 4π and a smooth part having Gaussian curvature no longer unity but

$$K = C = 1 - 2G \sum_k \hat{\mu}_k = 1 - \frac{1}{4\pi} \sum_k \mu_k, \quad (8.9)$$

then the Reissner-Nordstrom black hole generalizes to the following solution with strings:

$$ds^2 = - \left(K - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(K - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d^2\hat{\Omega}. \quad (8.10)$$

Here Q is precisely the charge, defined as $1/(4\pi)$ times the flux of electric field through each thorny sphere.

This form of the Reissner-Nordstrom metric remains valid (but is no longer asymptotically flat with a static timelike Killing vector $\partial/\partial t$) when the rescaled thorny sphere with positive Gaussian curvature, $K > 0$, on its smooth part, is replaced by a rescaled *thorny pseudosphere* with negative Gaussian curvature, $K < 0$, on its smooth part that is then locally isometric to a hyperbolic 2-space with constant negative curvature.

ACKNOWLEDGMENTS

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada, the Killam Trust and the NATO Collaborative Linkage Grant CLG.976417.

APPENDIX A: GAUSSIAN NORMAL MAPS AND GENERALIZED CONSTRAINT EQUATION

1. Gaussian map

Let M^2 be a closed 2-dimensional surface in a 3-dimensional Euclidean space

$$F(X^1, X^2, X^3) = 0, \quad \nabla F \neq 0, \quad (A1)$$

or locally $X^i = X^i(y^1, y^2)$. Using the rotational freedom in the choice of X^i , one can describe the surface M^2 locally as follows:

$$X^3 = f(y^1, y^2), \quad X^1 = y^1, \quad X^2 = y^2. \quad (A2)$$

The first quadratic form (induced metric) is

$$ds^2 = g_{ab} dy^a dy^b, \quad (A3)$$

$$g_{11} = 1 + f_{,1}^2, \quad g_{22} = 1 + f_{,2}^2,$$

$$g_{11} = f_{,1} f_{,2}, \quad \det g = 1 + f_{,1}^2 + f_{,2}^2, \quad (A4)$$

while the components of the second quadratic form are

$$b_{ab} = \frac{f_{,ab}}{\sqrt{\det g}}. \quad (A5)$$

The Gaussian curvature is

$$K = \frac{\det b}{\det g} = \frac{f_{,11} f_{,22} - f_{,12}^2}{(1 + f_{,1}^2 + f_{,2}^2)^2}. \quad (A6)$$

We also have

$$R = 2K, \quad (A7)$$

where R is the Ricci scalar for the induced metric g_{ab} .

Consider a unit sphere S^2 determined by the equation

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = 1. \quad (A8)$$

Let (ζ^1, ζ^2) be coordinates on the sphere, so

$$X^i = X^i(\zeta^a). \quad (A9)$$

Then the induced metric on the unit S^2 in these coordinates is

$$\gamma_{ab} = \delta_{ij} X^i_{,\zeta^a} X^j_{,\zeta^b}. \quad (A10)$$

The area element is $da = \sqrt{\det(\gamma)} d\zeta^1 d\zeta^2$.

We determine the *Gaussian normal map* $\varphi: M^2 \rightarrow S^2$ of M^2 into the sphere S^2 as follows (see e.g. [8,9]). Let $n^i(P)$ be a unit normal to M^2 at a point P . Then we put into cor-

respondence with P a point p on the unit sphere with coordinates $X^i = n^i(P)$ (which means that the normal vector to M^2 at point P coincides with the normal vector to S^2 at point p). This map determines the relation between coordinates y^a on M^2 and coordinates ζ^a on S^2 :

$$y^a = y^a(\zeta^b). \quad (\text{A11})$$

Let us now show that for the Gaussian normal map the following relation is valid [8,9]:

$$K dA = da, \quad (\text{A12})$$

where $dA = \sqrt{g} dy^1 dy^2$ is the surface area element on M^2 and da is the area element on S^2 . To prove this relation we choose X^3 to be orthogonal to M^2 at a given point p and $y^1 = X^1$ and $y^2 = X^2$ to be tangent to this surface. Then the surface M^2 is determined by the equation $X^3 = f(y^1, y^2)$, where $f_{,1} = f_{,2} = 0$ at the point p . Hence

$$K = \det \begin{vmatrix} f_{,11} & f_{,12} \\ f_{,21} & f_{,22} \end{vmatrix}, \quad g_{ab} = \delta_{ab}. \quad (\text{A13})$$

The Euclidean coordinates of a unit normal vector N^i in the vicinity of the point p are

$$n^i(y^1, y^2) = (1 + f_{,1}^2 + f_{,2}^2)^{-1/2} (f_{,1}, f_{,2}, -1). \quad (\text{A14})$$

We choose now the coordinates ζ^a on S^2 so that near a point $P = \varphi(p)$,

$$\bar{X}^1 = \zeta^1, \quad \bar{X}^2 = \zeta^2, \quad \bar{X}^3 = \sqrt{1 - (\zeta^1)^2 - (\zeta^2)^2}. \quad (\text{A15})$$

Here \bar{X}^i are such Cartesian coordinates that axis \bar{X}^3 coincides with the normal vector to S^2 at p , $N^i(p)$. In what follows we will denote the normal to S^2 the same as the normal to M^2 . In these coordinates $\gamma_{ab}(p) = \delta_{ab}$. Using Eq. (A14) we get

$$\zeta^a = \frac{f_{,a}}{\sqrt{1 + f_{,1}^2 + f_{,2}^2}}. \quad (\text{A16})$$

This relation establishes a relation between coordinates y^a on M^2 and coordinates ζ^a on S^2 . The canonical invariant element of area on a unit sphere at the point p written in the coordinates y^a is

$$\begin{aligned} da &= \det(\zeta^a_{,y^b})|_p dy^1 dy^2 \\ &= K dy^1 dy^2 = K \sqrt{g} dy^1 dy^2, \end{aligned} \quad (\text{A17})$$

which proves Eq. (A12). To obtain these equalities we use that $\det g|_P = \det \gamma|_p = 1$.

Suppose that M^2 is a compact 2 dimensional manifold diffeomorphic to the unit sphere S^2 and its embedding in R^3 is a closed convex surface, so that the Gaussian spherical map is a regular one-to-one map on S^2 . In this case one has

$$\int_{M^2} \mathbf{n} K dA = 0. \quad (\text{A18})$$

This relation directly follows from Eq. (A12) and the relation

$$\int_{S^2} \mathbf{n} da = 0. \quad (\text{A19})$$

2. Generalized constraint equation

Consider a closed 2D manifold M^2 with n conical singularities with positive deficit angles μ_k ($0 < \mu_k < 2\pi$), such that $\sum_k \mu_k < 4\pi$. We call such manifold a *thorny manifold*, or, for brevity, a *thornifold*. Assume that the Gaussian curvature K of M^2 is positive everywhere, so that one can isometrically embed the M^2 in Euclidean 3-space as a closed convex surface. Consider the Gaussian normal map of a regular domain $\text{reg}(M^2)$ of M^2 onto S^2 . To see what happens under the Gaussian map with conical singularities, consider a small region Σ_k around the k th conical singularity (but not including the singularity itself) with the boundary \mathcal{P}_k . The region Σ_k is mapped onto a region $\tilde{\Sigma}_k$ on S^2 with boundary $\tilde{\mathcal{P}}_k$. When \mathcal{P}_k shrinks to the conical singularity, the contour $\tilde{\mathcal{P}}_k$ shrinks to a contour C_k on S^2 , because the normal vector at a conical singularity does not have a unique direction. Let D_k be the region inside C_k . The remarkable property of D_k is that, although its form depends on the concrete thornifold M^2 , its surface area is μ_k where μ_k is the conical angle deficit at the corresponding singular point. To see this, apply the Gauss-Bonnet formula to the region Σ_k :

$$\int_{\Sigma_k} K dA + \int_{\mathcal{P}_k} k dl = 2\pi, \quad (\text{A20})$$

where k is the extrinsic curvature of \mathcal{P}_k embedded in M^2 . When \mathcal{P}_k shrinks to the conical singularity, the limit for integral $\int_{\mathcal{P}_k} k dl$ is $2\pi - \mu_k$. So we get from Eq. (A20) by using Eq. (A12)

$$A_k = \int_{D_k} da = \lim_{\Sigma_k \rightarrow 0} \int_{\Sigma_k} K dA = \mu_k. \quad (\text{A21})$$

To summarize, the Gaussian map of a thornifold M^2 with n conical singularities is a regular sphere with n disks D_k removed, each disk corresponding to a conical singularity. By using Eq. (A12) we can write

$$\int_{\text{reg}(M^2)} \mathbf{n} K dA = \int_{S^2} \mathbf{n} da - \sum_{k=1}^n \int_{D_k} \mathbf{n} da, \quad (\text{A22})$$

Because the first term in the left hand side of Eq. (A22) vanishes, we get the following identity:

$$\int_{\text{reg}(M^2)} \mathbf{n} K dA + \sum_{k=1}^n \int_{D_k} \mathbf{n} da = 0. \quad (\text{A23})$$

By taking into account Eq. (A21), this identity can also be rewritten as

$$\int_{\text{reg}(M^2)} \mathbf{n} K dA + \sum_{k=1}^n \mu_k \hat{\mathbf{n}}_k = 0, \quad (\text{A24})$$

where $\hat{\mathbf{n}}_k$ is the averaged normal \mathbf{n} over the area of the disk D_k [see Eq. (2.15)]. Now if we subtract Eq. (2.8) from Eq. (A24) and use the definition Eq. (2.12) of the force \mathbf{F}_k we get

$$\mathbf{F} \equiv \sum_{k=1}^n \mu_k \hat{\mathbf{n}}_k = \int_{\text{reg}(M^2)} \mathbf{n} (1-K) dA. \quad (\text{A25})$$

The constant 1 in this equation can be any constant, because of Eq. (2.8), but it is here chosen to be 1, the value of the Gaussian curvature on the unit thorny sphere, so that the right hand side is obviously zero for the thorny sphere. Equation (A25) can be considered as the generalized constraint equation for a closed thornifold whose Gaussian curvature K is not constant.

3. C-metric example

To illustrate the action of the generalized constraint equation we discuss how it works for C metrics. The C metric is the following solution of the Einstein equations:

$$ds^2 = -H du^2 - 2 du dr - 2 a r^2 du dx + r^2 d\omega^2, \quad (\text{A26})$$

$$d\omega^2 = \frac{dx^2}{G(x)} + G(x) d\tilde{\phi}^2, \quad (\text{A27})$$

$$H = -a^2 r^2 G(x - 1/(ar)),$$

$$G(x) = 1 - x^2 - \alpha x^3. \quad (\text{A28})$$

This metric describes the gravitational field of a uniformly accelerated black hole (see e.g. [12]). The parameter a is the acceleration, and $\alpha = r_g a$, where $r_g = 2M$ is the Schwarzschild gravitational radius.

We focus our attention on the geometry of a 2-dimensional surface $u = \text{const}, r = \text{const}$. Its geometry is $r^2 d\omega^2$. It is easy to show that for $\alpha < 2/3\sqrt{3}$ the equation $G(x) = 0$ has 3 real roots. Two of them are negative, say $x_3 < x_2 < 0$, and one is positive $x_1 > 0$. From now on we assume that $x \in (x_2, x_1)$, so that the function G is positive. One also has $G'(x_2) > 0$ and $G'(x_1) < 0$. For an arbitrary period of $\tilde{\phi}$ the surface Σ with the metric $d\omega^2$ is a thornifold with two conical singularities at x_1 and x_2 . The singularity at x_1 vanishes when

$$\tilde{\phi} = b \phi \equiv \frac{2}{|G'(x_1)|} \phi, \quad \phi \in (0, 2\pi). \quad (\text{A29})$$

We fix this periodicity and write the metric (A27) as

$$d\omega^2 = h_{ab} dx^a dx^b = \frac{dx^2}{G(x)} + b^2 G(x) d\phi^2. \quad (\text{A30})$$

The Gaussian curvature of Σ is

$$K = \frac{1}{2} R = -\frac{1}{2} G'' = 1 + 3 \alpha x. \quad (\text{A31})$$

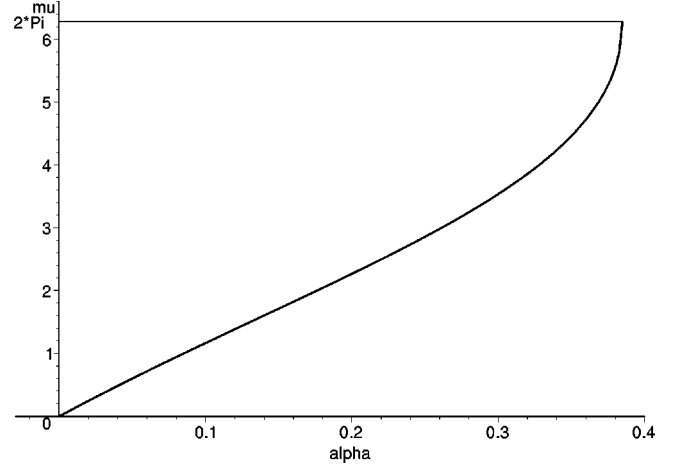


FIG. 8. The angle deficit μ as a function of α .

Integrating K over the regular part of Σ , we get

$$\int K \sqrt{h} d^2x = \frac{2\pi}{|G'(x_1)|} [G'(x_2) + |G'(x_1)|]. \quad (\text{A32})$$

In order to satisfy the Gauss-Bonnet equation, the conical singularity at x_1 must have the angle deficit

$$\mu = 2\pi \left(1 - \frac{G'(x_2)}{|G'(x_1)|} \right). \quad (\text{A33})$$

The dependence of μ on α can be presented in the following parametric form:

$$\mu = \frac{2\pi\epsilon(4-5\epsilon+2\epsilon^2)}{(1-\epsilon)(2+\epsilon-2\epsilon^2)}, \quad \alpha = \frac{\epsilon(1-\epsilon)}{(1-\epsilon+\epsilon^2)^{3/2}}. \quad (\text{A34})$$

The angle deficit μ as a function of α is shown in Fig. 8.

The surface Σ can be embedded into a 3-dimensional flat space as a surface of rotation. In cylindrical coordinates the equation of this surface is

$$\rho = b \sqrt{G(x)}, \quad z = - \int_{x_2}^x dx \sqrt{\frac{1-b^2 G'(x)^2/4}{G(x)}}. \quad (\text{A35})$$

Since $G'(0) = 0$, one has $\rho'(0) = 0$ and hence a normal vector to Σ at the line $x = 0$ is orthogonal to the z axis. Using Eq. (A31) we find that above this line $K < 1$ and below it $K > 1$.

When the string tension is small one can apply the generalized constraint equation to the case shown in Fig. 9. This equation shows that as a result of the action of the ‘‘inertial’’ force of acceleration, the form of the surface is changed so that extra positive curvature is located in the lower part of the surface, while the extra negative curvature is located in the upper part. As a result, the generalized constraint equation (A25) is obeyed.

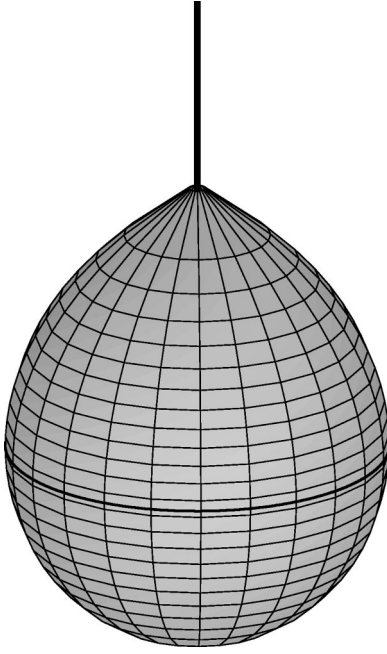


FIG. 9. Embedding of Σ into a 3 dimensional flat space ($\alpha = 0.1$). A “string” responsible for an angle deficit enters the north pole. A solid line on Σ separates it into two parts, the upper one where the Gaussian curvature is less than 1 and the lower one where it is greater than 1.

Another surface of interest is the event horizon. It is defined by the equation $H=0$, or $r=1/[a(x-x_3)]$. A surface of rotation in a 3-dimensional space which has the same internal geometry as the horizon is given by the equations

$$\rho = \frac{b \sqrt{G(x)}}{x-x_3},$$

$$z = - \int_{x_2}^x \frac{dx}{x-x_3} \sqrt{\frac{1-b^2 G'(x)^2/4}{G(x)}}. \quad (\text{A36})$$

The form of this surface is very similar to the one shown in Fig. 9.

APPENDIX B: SIMPLE EXAMPLES OF EMBEDDING OF A THORNY SPHERE INTO A EUCLIDEAN SPACE

When all the angle deficits are positive (which is the case of the most physical interest), a thorny sphere can be considered as a special limit of a 2-dimensional compact surface diffeomorphic to S^2 which has positive curvature and which can be embedded into a three-dimensional flat space. The curvature is everywhere constant except for N_v localized regions v_k where it is high. An angle deficit μ_k arises in the limit when the size of the region v_k tends to zero while the curvature inside it grows infinitely, so that the integral of the Gaussian curvature K (one-half the Ricci scalar curvature R) over this region remains finite and has the limit μ_k .

The simplest example of a thorny sphere is a sphere with two conical singularities located at its poles. This sphere can

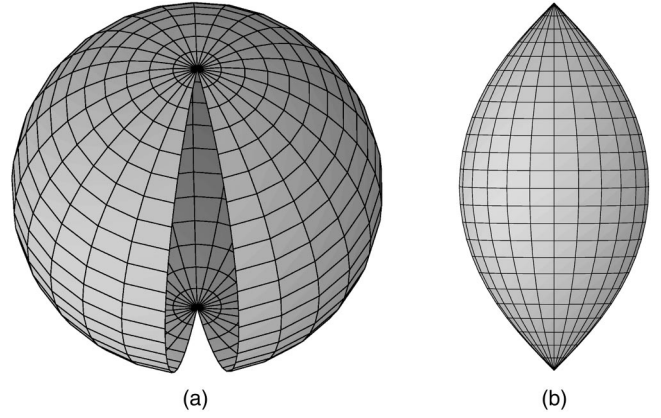


FIG. 10. A unit sphere with a cut (a), and embedding of a thorny sphere with two positive angle deficits into a 3-dimensional flat space (b).

be obtained by cutting a unit sphere by planes P_1 and P_2 at angles $\phi=0$ and $\phi=\mu$ and gluing the cuts together [see Fig. 10(a)]. It can also be obtained as the geometry on the surface of rotation embedded in a flat 3-space (see e.g. [10]). This surface is obtained by a rotation around the z axis of the following meridional curve:

$$x = a \cos \sigma, \quad y = 0,$$

$$z = \int_0^\sigma \sqrt{1 - a^2 \sin^2 \sigma} d\sigma, \quad |\sigma| \leq \pi/2. \quad (\text{B1})$$

Here $a \leq 1$. For $a=1$ the angle deficit vanishes. In the general case the angle deficit is $2\pi(1-a)$. This surface is shown in Fig. 10(b). A similar surface of rotation of constant curvature for a negative angle deficit is shown at Fig. 11. (Note that it is impossible to embed the entirety of this surface in 3-dimensional flat space in an axially symmetric way, so the embedding stops before one gets to the conical singularities with negative deficit angles.) The corresponding equations of the meridian curve which generate this rotational surface are

$$x = \sqrt{a^2 - \sin^2 \sigma}, \quad y = 0,$$

$$z = \int_0^\sigma \frac{\cos^2 \sigma}{\sqrt{a^2 - \sin^2 \sigma}} d\sigma. \quad (\text{B2})$$

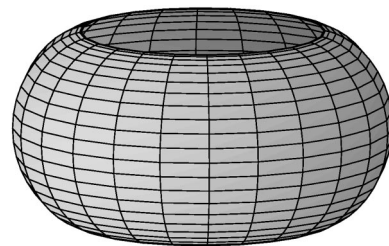


FIG. 11. Embedding of a thorny sphere with two negative angle deficits into a 3-dimensional flat space.

- [1] V.P. Frolov and D.V. Fursaev, Phys. Rev. D **63**, 124010 (2001).
- [2] V.P. Frolov and D.V. Fursaev, Class. Quantum Grav. **18**, 1535 (2001).
- [3] M. Aryal, L. Ford, and A. Vilenkin, Phys. Rev. D **34**, 2263 (1986).
- [4] J.S. Dowker and P. Chang, Phys. Rev. D **46**, 3458 (1992).
- [5] A.V. Pogorelov, *Extrinsic Geometry of Convex Surfaces*, Translations of Mathematical Monographs Vol. 35 (AMS, Providence, RI, 1973).
- [6] M. Spivak, *A Comprehensive Introduction to Differential Geometry* (Publish or Perish, Boston, 1975), Vol. 5.
- [7] M.G. Ivanov, hep-th/0111035.
- [8] B.A. Dubrovin, A.T. Fomenko, and S.A. Novikov, *Modern Geometry—Methods and Application: Part II, The Geometry and Topology of Manifolds*, Graduate Texts in Mathematics Vol. 104 (Springer-Verlag, Berlin, 1985).
- [9] T. Frankel, *The Geometry of Physics: An Introduction* (Cambridge University Press, Cambridge, England, 1999).
- [10] W. Blaschke, *Kreis und Kugel* (Walter de Gruyter und Co., Velt und Comp., Berlin, 1956).
- [11] D.V. Fursaev and S.N. Solodukhin, Phys. Rev. D **52**, 2133 (1995).
- [12] W. Kinnersley and M. Walker, Phys. Rev. D **2**, 1359 (1970).