Geometry of an $N=4$ twisted string

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We compare $N=2$ strings and $N=4$ topological strings within the framework of the sigma model approach. Being classically equivalent on a flat background, the theories are shown to lead to different geometries when put in a curved space. In contrast with the well studied Kähler geometry characterizing the former case, in the latter case a manifold has to admit a covariantly constant holomorphic two-form in order to support an $N=4$ twisted supersymmetry. This restricts the holonomy group to be a subgroup of SU(1,1) and leads to a Ricci-flat manifold. We speculate that the $N=4$ topological formalism is an appropriate framework to smooth down ultraviolet divergences intrinsic to the $N=2$ theory.

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I. INTRODUCTION

String theory and nonlinear sigma models are intimately linked to each other. Having originated from different sources, namely, the study of dual models of hadrons on the one hand and the search for renormalizable field theories in $d \geqslant 2$ on the other, they were soon shown to be connected, as the latter provides deep insight into the former (see, e.g., [1–3]). To mention only the most significant points, the gauging of global (super)symmetries of the nonlinear sigma model typically results in a string theory (in the Neveu-Schwarz-Ramond formalism) coupled to the background massless modes, while the one-loop finiteness fixes the effective low energy dynamics of the string partner. It should be remembered that, since the gauging brings extra constraints into the formalism (normally forming an N-extended superconformal algebra) the resulting string theory is not necessarily critical and that point is to be examined on its own.

Parallel to the progress of string theory on the supersymmetry route, the nonlinear sigma models revealed a number of striking properties in the supersymmetric area. With the number of global supersymmetries growing, the background geometry becomes severely restricted. Admitting an automatic $N=1$ supersymmetric generalization [4], the model was shown to require a Kähler geometry in order to support an $N=2$ global supersymmetry [5]. The $N=4$ case, which also corresponds to a maximally extended supersymmetry,[1] appeals to a hyper-Kähler space, the latter being automatically Ricci flat [5]. Interestingly enough, the $N \leqslant 4$ bound correlates well with that known for an N-extended superconformal algebra (SCA) admitting a central extension [6], the latter typically underlying a string theory with an N-extended local supersymmetry on the world sheet.

The gauged $N=2$ nonlinear sigma model, or $N=2$ string (coupled to background), has attracted considerable interest over the past decade (for a comprehensive list of references see, for example, Ref. [7]). The theory is critical in two spatial and two temporal dimensions (or a four-dimensional Euclidean space) and contains the only physical state in the quantum spectrum. Being a massless scalar, the latter can be associated with either the Kähler potential (closed string) or the Yang scalar (open string) [8]. Notice, however, that although the $N=2$ model does provide a satisfactory stringy description of self-dual gauge theory or self-dual gravity, the manifest Lorentz invariance is missing and, in spite of being the theory of an $N=2$, $d=2$ supergravity coupled to matter, the model fails to produce fermions in the quantum spectrum (see, however, the recent work [9]).

At the classical level the former drawback has been overcome recently [10,11] based on an earlier $N=4$ topological formalism by Berkovits and Vafa [12]. According to the $N=4$ topological prescription, one adds to the theory two more fermionic currents (of conformal spin 3/2) and two more bosonic ones (of conformal spin 1) which on the one hand extend the $N=2$ SCA to a small $N=4$ SCA, but on the other hand do not change the physical content of the model as they prove to be functionally dependent. The key point, however, is that this extension brings an extra $\mathrm{U}(1,1)_{outer}$ symmetry to the formalism (see [10,11] for an explicit real-

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[1] $N=3$ automatically implies $N=4$, as a product of two complex structures yields a third one [5].

ization), which thus raises the global symmetry group to that including the full Lorentz group [recall $SO(2,2) \simeq SU(1,1) \times SU(1,1)'$]. Quantum equivalence of the two approaches has been established by explicit evaluation of scattering amplitudes[2] in Ref. [12]. Based on the symmetry argument, the $N=4$ topological string action has been constructed in [10] just by installing the $U(1,1)_{outer}$ into the action of the $N=2$ string. Curiously enough, to a great extent the situation resembles what happens for the Green-Schwarz superstring, where extracting an independent set of fermionic first class constraints is known to be in conflict with manifest Lorentz covariance.[3]

Given the $N=4$ topological string action, the natural question to ask is, what is the geometry of a nonlinear sigma model associated with it? In the present paper we address this issue and show that (i) the global supersymmetry in question is actually $N=4$ twisted supersymmetry (hence the title of this paper) and (ii) apart from being a Kähler space, the target manifold has to admit a covariantly constant holomorphic two-form, in order to support the latter.

As is well known (see, e.g., [14]), the last point restricts the holonomy group to be a subgroup of SU(1,1), which is equivalent to the Ricci-flatness condition for the Riemann tensor (some constructive examples of such manifolds can be found, e.g., in Ref. [8]). Alternatively, working in real coordinates, in addition to the covariantly constant complex structure characterising the Kähler geometry, one reveals two covariantly constant ''real'' structures (almost product structures), forming altogether a pseudo quaternionic algebra. Geometry of this type is known as pseudo hyper-Kähler geometry [15,16] and has recently been discussed in the sigma model context by Hull [17] and by Abou Zeid and Hull [18].

Finally, as the one-loop calculation proceeds along the same lines both for the $N=2$ nonlinear sigma model and for the $N=4$ twisted generalization (requiring a Ricci-flat manifold for the one-loop ultraviolet finiteness [19,20]), one concludes that in the latter case the ultraviolet finiteness of the quantum theory[4] is guaranteed by the presence of a higher symmetry at the classical level.

The organization of the work is as follows. In the next section we briefly review the nonlinear sigma models with N-extended global supersymmetry and discuss the origin of the twisted supersymmetry. In Sec. III the $N=2$ nonlinear sigma model and the $N=4$ twisted generalization are compared in the complex coordinates common to their string partners. In the former case we reproduce the well known condition that a manifold must be Kähler, while in the latter case it is shown that in addition it has to admit a covariantly constant holomorphic two-form in order to support the $N=4$ twisted supersymmetry. We turn to string theory in Secs. IV and V and consider gauging of an $N=2$ global supersymmetry and an $N=4$ twisted global supersymmetry. We explicitly check that in both cases the gauging does not impose any new restrictions on the background geometry as compared to those implied already by the global supersymmetry and hence is valid to describe a consistent coupling of the $N=2,4$ strings to the external curved backgrounds. Calculation of the one-loop β function is outlined in Sec. VI. We summarize our results and discuss some further problems in the concluding Sec. VII. Two Appendixes contain our spinor notations in $d=2$ and some elementary notions from complex geometry on Kähler manifolds, these being relevant for the gauging procedure implemented in Secs. IV and V. Throughout the paper we work in components. The superfield technique is avoided in order to keep the connection between geometry and strings at a more transparent level.

II. NONLINEAR SIGMA MODELS AND N-EXTENDED TWISTED SUPERSYMMETRY

To elucidate the structure of an N-extended twisted supersymmetry in the sigma model context, it is worth recalling the original argument by Alvarez-Gaumé and Freedman [5] valid for ordinary supersymmetry transformations. Given the free field action on a flat background[5]

$$S=\frac{1}{2}\int d^2x\{\partial^\alpha\phi^n\partial_\alpha\phi_n-i\bar{\psi}^n\gamma^\alpha\partial_\alpha\psi_n\}, \tag{1}$$

where the target space index n runs from 0 to $N-1$ and γ are the Dirac matrices in two dimensions (for our $d=2$ spinor conventions see Appendix A), one can easily guess an ansatz for the global supersymmetry transformation:

$$\delta\phi^n=f^n{}_m(\bar{\epsilon}\psi^m), \quad \delta\psi^n=\lambda i f^n{}_m\partial_\alpha\phi^m(\gamma^\alpha\epsilon). \tag{2}$$

The bosonic fields ϕ^n and the fermionic partners ψ^n are taken to be real, λ is an arbitrary real constant, and f^n_m is an arbitrary real nondegenerate (constant) matrix, both to be determined below.

The invariance of the action under the ansatz leads f^n_m to obey (the target space flat metric is denoted by η_{nm})

$$\eta_{pn}f^n{}_m+\lambda f^k{}_p\eta_{km}=0, \tag{3}$$

where one has to take into account the identity $\gamma_a\gamma_b=-\eta_{ab}-\epsilon_{ab}\gamma_3$ and integrate by parts. Making use of the $d=2$ Fiertz identity (see also Appendix A)

[2]The $N=4$ topological prescription appeals to a specific topological twist which does not treat all the currents on equal footing and breaks the Lorentz group to U(1,1).

[3]For a detailed discussion on covariant quantization of the Green-Schwarz superstring and kappa symmetry, see, e.g., [13].

[4]According to the analysis of Ref. [21], for the case of two- and four-dimensional targets the Ricci-flatness condition one reveals at the one-loop level persists to higher orders in perturbation theory.

[5]In what follows, we denote flat $d=2$ vector indices by the letters from the beginning of the Latin alphabet. Those from the end are reserved for the target space indices. On a flat background $e_a{}^\alpha=\delta_a^\alpha$ and $\gamma^a e_a{}^\alpha\equiv\gamma^\alpha$.

$$(\bar{\epsilon}_1\partial_\alpha\psi^n)\gamma^\alpha\epsilon_2=-(\bar{\epsilon}_1\gamma^\alpha\epsilon_2)\partial_\alpha\psi^n-\tfrac{1}{2}(\bar{\epsilon}_1\epsilon_2)\gamma^\alpha\partial_\alpha\psi^n$$
$$-\tfrac{1}{2}(\bar{\epsilon}_1\gamma^\alpha\partial_\alpha\psi^n)\epsilon_2$$
$$+\tfrac{1}{2}(\bar{\epsilon}_1\gamma_3\epsilon_2)\gamma_3\gamma^\alpha\partial_\alpha\psi^n$$
$$+\tfrac{1}{2}(\bar{\epsilon}_1\gamma_3\gamma^\alpha\partial_\alpha\psi^n)\gamma_3\epsilon_2, \tag{4}$$

the algebra of the transformations can be readily evaluated (on-shell relations)

$$[\delta_1,\delta_2]\phi^n=i\lambda f^n{}_m f^m{}_k(\bar{\epsilon}_2\gamma^\alpha\epsilon_1-\bar{\epsilon}_1\gamma^\alpha\epsilon_2)\partial_\alpha\phi^k, \tag{5}$$

and the same for the field ψ^n. Thus, for the transformation (2) to be the standard supersymmetry transformation one has to set

$$\lambda f^n{}_m f^m{}_k=-\delta^n{}_k. \tag{6}$$

If one wishes to realize an extended supersymmetry then each f must satisfy the conditions (3),(6) with the corresponding λ involved and also the vanishing of cross brackets for two different transformations implies

$$\lambda_1(f_2f_1)^n{}_k+\lambda_2(f_1f_2)^n{}_k=0,$$
$$\lambda_2(f_2f_1)^n{}_k+\lambda_1(f_1f_2)^n{}_k=0. \tag{7}$$

Observing further that one of the transformations can always be generated by the unit matrix

$$f^n_{0m}=\delta^n{}_m\rightarrow\lambda_0=-1, \tag{8}$$

one ends up with the algebraic conditions

$$f_i^2=-1,\quad \lambda_i=1,\quad i\neq 0, \tag{9}$$

while Eq. (7) reduces to the Clifford algebra (for $i\neq j$). Notice that on a flat background the number of supersymmetries does not seem to be bounded from above. It suffices to raise the value of N and find an appropriate representation of the Clifford algebra.

The same arguments apply also to the more complicated case of a curved manifold or the nonlinear sigma model with an N-extended global supersymmetry. A minimal coupling of the toy system (1) to a curved background metric $g_{nm}(\phi)$ turns out to be not enough to respect the supersymmetry and some new terms need to be added [4,5]:

$$S=\frac{1}{2}\int d^2x\Big\{\partial^\alpha\phi^n\partial_\alpha\phi^m g_{nm}(\phi)-i\bar{\psi^n}\gamma^\alpha\partial_\alpha\psi^m g_{nm}(\phi)$$
$$-g_{nm}(\phi)\Gamma^m{}_{pk}(\phi)\partial_\alpha\phi^p i\bar{\psi^n}\gamma^\alpha\psi^k$$
$$-\frac{1}{6}R_{nmpk}(\phi)(\bar{\psi^n}\psi^p)(\bar{\psi^m}\psi^k)\Big\}, \tag{10}$$

where $\Gamma^m{}_{pk}$ is the Levi-Civita connection and R_{nmpk} is the Riemann tensor. In addition, the transformations themselves need to be slightly modified and f^n_m acquires the ϕ dependence

$$\delta\phi^n=f^n{}_m(\phi)(\bar{\epsilon}\psi^m),$$
$$\delta\psi^n=\lambda i f^n{}_m(\phi)\partial_\alpha\phi^m(\gamma^\alpha\epsilon)-\Gamma^n{}_{mk}(\phi)f^m{}_p(\phi)(\bar{\epsilon}\psi^p)\psi^k. \tag{11}$$

Varying the action and verifying the algebra of the transformations one again comes to the algebraic conditions (3), (6), (7), (8), (9) where the flat metric is to be exchanged with the curved one. In addition, there appears a new restriction that f^n_m must be covariantly constant

$$\nabla_k f^n{}_m=0. \tag{12}$$

Notice that the latter is trivially satisfied for $f^n_{0m}=\delta^n{}_m$ and hence at least one supersymmetry can always be realized without any restrictions on the background geometry [4]. In checking the symmetry, the integrability condition (Ricci identity)

$$R_{nmlp}f^l{}_k-\lambda R_{nmlk}f^l{}_p=0, \tag{13}$$

and the standard properties of the Riemann tensor

$$R^n{}_{mpk}+\text{cycle}(mpk)=0,$$
$$\nabla_l R^n{}_{mpk}+\text{cycle}(lpk)=0 \tag{14}$$

are much used, while the analysis of the (on-shell) algebra appeals to the fermionic equation of motion

$$i\gamma^\alpha\mathcal{D}_\alpha\psi^k+\tfrac{1}{3}R^k{}_{mpl}\psi^p(\bar{\psi^m}\psi^l)=0, \tag{15}$$

with $\mathcal{D}_\alpha\psi^k=\partial_\alpha\psi^k+\Gamma^k{}_{nm}\partial_\alpha\phi^n\psi^m$. It is worth noting that the easiest way to deal with the last term in the action is to reduce the $d=2$ spinors involved in it to their irreducible components, as we do in Sec. IV below (see also Appendix A). This allows one to exploit the symmetry properties of the Riemann tensor more efficiently and liberates one from the necessity to use Fiertz identities.

In contrast to the flat case, the fact that a target manifold has to admit a covariantly constant tensor or, in other words, a tensor commuting with all elements from the holonomy group of a manifold, severely restricts the number of possible supersymmetries. The $N\leqslant 4$ bound has been revealed [5] for an irreducible manifold, this based on the Schur lemma applied to real representations of the holonomy group. Since a product of two complex structures necessarily yields a third one (a quaternionic structure), the Kähler geometry corresponding to $N=2$ and the hyper-Kähler geometry associated with $N=4$ seem to be the only options available.[6]

An additional interesting possibility has been brought into focus quite recently [15–18]. The observation made [17] is

[6]In Ref. [22] geometric models of $N=4$ supersymmetric mechanics have been proposed, which can be viewed as one-dimensional counterparts of the two-dimensional $N=2$ supersymmetric sigma-models of [5].

that for one of the four transformations,[7] say f_2, one can give up the conventional sign in the commutation relation (5), thus leading to a twisted supersymmetry algebra and a ''real'' structure (or an almost product structure [23])

$$f^n_{2m} f^m_{2\,k} = \delta^n{}_k, \quad \nabla_k f^n_{2m} = 0. \tag{16}$$

A product of the latter with the complex structure f_1 then yields another real structure while a product of the two real structures gives back the complex structure, altogether inducing a pseudo quaternionic structure [16,15]. The full algebra is then an $N=4$ twisted supersymmetry algebra and the corresponding geometry is known as a pseudo hyper-Kähler geometry. A detailed discussion of the latter point in the sigma model context can be found in Refs. [17,18]. Notice that the possibility of twisting the $N=4$ supersymmetry algebra is consistent with Jacobi identities since for the $d=2$ (twisted) super Poincaré algebra

$$\{Q_A, Q_B\} = 2\sigma(\gamma_0\gamma^\alpha)_{AB} P_\alpha,$$

$$[M, Q_A] = \tfrac{1}{2}\gamma_{3AB} Q_B, \quad [M, P_\alpha] = \epsilon_{\alpha\beta} P^\beta, \tag{17}$$

those hold for both $\sigma = 1$ and $\sigma = -1$.

III. KÄHLER AND PSEUDO HYPER-KÄHLER GEOMETRIES

As was mentioned in the Introduction, the prime concern of this work is to compare the geometry of the $N=2$ string and its $N=4$ topological extension. Since gauging global (super)symmetries of a sigma model results in a string theory coupled to background, it is worth considering the sigma model conformed to the complex frame intrinsic to the $N=2$ string. The Lagrangian to start with is that of the $N=2$ string in the superconformal gauge,

$$S = -\frac{1}{2\pi}\int d\tau d\sigma \{\partial^\alpha z^a \partial_\alpha \bar{z}^{\bar{a}} \eta_{a\bar{a}} - i\bar{\psi}^{\bar{a}}\gamma^\alpha \partial_\alpha \psi^a \eta_{a\bar{a}} + i\partial_\alpha \bar{\psi}^{\bar{a}}\gamma^\alpha \psi^a \eta_{a\bar{a}}\}. \tag{18}$$

Here z^a, with $a=0,1$ in the critical dimension, is a $d=2$ complex scalar and ψ^a_A is a $d=2$ complex (Dirac) spinor. Since the target space is essentially complex we shall distinguish between the index carried by a field and its complex conjugate $(z^a)^* = \bar{z}^{\bar{a}}$, $(\psi^a)^* = \bar{\psi}^{\bar{a}}$.

It is straightforward to check that the action exhibits the invariance with respect to an $N=2$ global supersymmetry (the parameter is a complex spinor)

$$\delta z^a = \bar{\epsilon}\psi^a, \quad \delta\psi^a = -\frac{i}{2}\partial_\alpha z^a (\gamma^\alpha \epsilon), \tag{19}$$

thus forming the conventional algebra

$$[\delta_1, \delta_2] = \tfrac{1}{2}(i\bar{\epsilon}_1\gamma^\alpha\epsilon_2 - i\bar{\epsilon}_2\gamma^\alpha\epsilon_1)\partial_\alpha. \tag{20}$$

In addition, a closer look reveals the invariance under an $N=2$ twisted supersymmetry

$$\delta z^a = \epsilon_{\bar{a}\bar{b}}\eta^{\bar{a}a}(\bar{\psi}^{\bar{b}}\epsilon), \quad \delta\psi^a = \tfrac{1}{2}\epsilon_{\bar{a}\bar{b}}\eta^{\bar{a}a} i\partial_\alpha z^{\bar{b}}(\gamma^\alpha\epsilon), \tag{21}$$

with ϵ_{ab} the Levi-Civita totally antisymmetric tensor [$\epsilon_{01} = -1$ and $(\epsilon_{ab})^* = \epsilon_{\bar{a}\bar{b}}$]. The corresponding algebra differs from the standard one (20) by the sign on the right hand side:

$$[\delta_1^{twist}, \delta_2^{twist}] = -\tfrac{1}{2}(i\bar{\epsilon}_1\gamma^\alpha\epsilon_2 - i\bar{\epsilon}_2\gamma^\alpha\epsilon_1)\partial_\alpha. \tag{22}$$

In verification of Eq. (22) the trivial identity

$$\eta^{\bar{a}a}\eta^{\bar{b}c}\epsilon_{\bar{a}\bar{b}} = \epsilon^{ac}, \tag{23}$$

with $\eta^{\bar{a}a} = \mathrm{diag}(-,+)$, is helpful. The cross commutator $[\delta, \delta^{twist}]$ proves to vanish on shell.

Turning to a curved background, it seems natural to assume that the complex structure intrinsic to the flat model persists in a curved space. A target manifold is thus taken to be Hermitian with a Hermitian metric $g_{n\bar{m}}(z,\bar{z})$ [we denote the inverse by $g^{\bar{m}n}$ and conjugate as $(g_{n\bar{m}})^* = g_{m\bar{n}}$] while the action functional of the nonlinear sigma model in this framework acquires the form

$$\begin{aligned} S = -\frac{1}{2\pi}\int d\tau d\sigma \{ & \partial^\alpha z^n \partial_\alpha \bar{z}^{\bar{m}} g_{n\bar{m}} - i\bar{\psi}^{\bar{m}}\gamma^\alpha\partial_\alpha\psi^n g_{n\bar{m}} \\ & - i\bar{\psi}^{\bar{m}}\gamma^\alpha\psi^k \ \Gamma^n{}_{pk}\partial_\alpha z^p g_{n\bar{m}} \\ & + i\partial_\alpha\bar{\psi}^{\bar{m}}\gamma^\alpha\psi^n g_{n\bar{m}} + i\bar{\psi}^{\bar{k}}\gamma^\alpha\psi^n \Gamma^{\bar{m}}{}_{\bar{p}\bar{k}}\partial_\alpha \bar{z}^{\bar{p}} g_{n\bar{m}} \\ & + 2R_{\bar{m}n\bar{p}k}(\bar{\psi}^{\bar{m}}\psi^n)(\bar{\psi}^{\bar{p}}\psi^k)\}. \end{aligned} \tag{24}$$

Variation of all but the last terms in the action under an $N=2$ global supersymmetry transformation

$$\delta z^n = (\bar{\epsilon}\psi^n),$$

$$\delta\psi^n = -\frac{i}{2}\partial_\alpha z^n(\gamma^\alpha\epsilon) - \Gamma^n{}_{mp}(\bar{\epsilon}\psi^m)\psi^p, \tag{25}$$

yields

$$\begin{aligned} & (\bar{\epsilon}\psi^k)\partial^\alpha z^n \partial_\alpha \bar{z}^{\bar{m}}(\partial_k g_{n\bar{m}} - \partial_n g_{k\bar{m}}) \\ & + (\bar{\epsilon}\gamma_3\psi^n)\partial_\beta \bar{z}^{\bar{m}}\epsilon^{\beta\alpha}\partial_\alpha z^{\bar{k}}\partial_{\bar{k}} g_{n\bar{m}} \\ & - i(\bar{\psi}^{\bar{m}}\gamma^\alpha\psi^p)(\bar{\psi}^{\bar{k}}\epsilon)\partial_\alpha z^n R_{\bar{m}p\bar{k}n} \\ & - i(\bar{\psi}^{\bar{m}}\gamma^\alpha\psi^p)(\bar{\psi}^{\bar{n}}\epsilon)\partial_\alpha z^k R_{p\bar{m}k\bar{n}} + \text{c.c.} \end{aligned} \tag{26}$$

Thus for the invariance to hold in a curved action one has to demand

$$\partial_k g_{n\bar{m}} - \partial_n g_{k\bar{m}} = 0, \quad \partial_{\bar{k}} g_{n\bar{m}} - \partial_{\bar{m}} g_{n\bar{k}} = 0, \tag{27}$$

which means that the target Hermitian manifold must be torsion free or, equivalently, Kähler. On a Kähler manifold the

[7] In what follows we assume that the first transformation is generated by the unit matrix and the second one by the ordinary complex structure.

Riemann tensor acquires extra symmetries (for completeness we list them in Appendix B) which then can be used to show that the variation of the last term in Eq. (24) exactly cancels the remnant in Eq. (26). Making use of the fermionic equation of motion one can verify also that the algebra (20) persists on a Kähler space.

Let us now proceed to the $N=2$ twisted transformation. Because the metric carries indices of different types, the naive guess like $\epsilon_{ab}\to\epsilon_{ab}/\sqrt{-\det g}$ that one could try to implement in passing to a curved space does not yield a tensor field. Hence one is forced to introduce an arbitrary two-form B_{nm}, ϵ_{nm} being the flat limit, and consider the ansatz

$$\delta z^n = B_{\bar{k}\bar{p}} g^{\bar{k}n}(\bar{\psi}^{\bar{p}}\epsilon),$$
$$\delta\psi^n = \tfrac{1}{2} B_{\bar{k}\bar{p}} g^{\bar{k}n} i\partial_\alpha z^{\bar{p}}(\gamma^\alpha\epsilon) - \Gamma^n{}_{pm} g^{\bar{k}p} B_{\bar{k}\bar{p}}(\bar{\psi}^{\bar{p}}\epsilon)\psi^m. \tag{28}$$

With respect to this generalization the nonlinear sigma model action holds invariant provided

$$\nabla_k B_{nm}=0, \quad \partial_{\bar{k}} B_{nm}=0,$$
$$\nabla_{\bar{k}} B_{\bar{n}\bar{m}}=0, \quad \partial_k B_{\bar{n}\bar{m}}=0. \tag{29}$$

In checking the invariance, the integrability conditions

$$R^k{}_{nm\bar{s}} B_{kp} - R^k{}_{nm\bar{p}} B_{ks}=0,$$
$$R^{\bar{k}}{}_{nm\bar{s}} B_{\bar{k}\bar{p}} - R^{\bar{k}}{}_{nm\bar{p}} B_{\bar{k}\bar{s}}=0 \tag{30}$$

prove to be helpful. Evaluating the algebra further, one encounters one more condition

$$B_{\bar{n}\bar{m}} B_{sp} g^{\bar{m}s} = g_{p\bar{n}}; \tag{31}$$

this, however, does not seem to be an extra restriction, since for an irreducible manifold the right hand side must be proportional to the metric with a constant real coefficient (recall that both g and B are covariantly constant), the latter can be obsorbed into a redefinition of B [23]. Finally, taking into account the conditions (29), (30) and the Fiertz identity (4) one can verify that the cross commutator $[\delta,\delta^{twist}]$ vanishes on shell which leads to the full $N=4$ twisted algebra.

Thus, for the nonlinear sigma model on a Kähler space to admit an extra $N=2$ twisted global supersymmetry, the manifold must admit a covariantly constant holomorphic two-form. This means that the holonomy group of a manifold, which is generally a subgroup of U(1,1), reduces to a subgroup of SU(1,1) (see, for example, [14]), the latter point implies a Ricci-flat space. Actually, contracting the first of the integrability conditions (30) with the tensor $g^{\bar{l}p} B_{\bar{l}r} g^{\bar{r}s}$ one immediately arrives at

$$R^k{}_{k\bar{n}\bar{m}}=0, \tag{32}$$

which is the familiar Ricci-flatness condition.

As is well known, for the $N=2$ model the latter condition appears at the quantum level as a requirement of the one-loop ultraviolet finiteness [19] of the theory. Curiously enough, as we have seen above, the $N=4$ topological prescription implies it already in the classical area where it proves to be encoded into a higher symmetry of the formalism.

IV. GAUGING $N=2$ GLOBAL SUPERSYMMETRY

We now turn to string theory and consider gauging of the extended global supersymmetry discussed above. Given a global symmetry transformation, a conventional way to convert it to a local one consists in applying the Noether procedure. In general, extra fields are needed and for the case of local supersymmetry these usually fill up one or another $d=2$, N-extended conformal supergravity multiplet. Since gauging the $N=2$ global supersymmetry in the sigma model (24) should result in an $N=2$ string consistently coupled to the curved background, the structure of the Noether couplings is prompted by the $N=2$ string itself. For the chiral half, the analysis has been done in Ref. [24] while for the ordinary (untwisted) $N=4$ string the question has been addressed in Ref. [25] (for some related work, see [26–31]).

In order to avoid cumbersome calculations caused by the $d=2$ Fiertz rearrangement rules one has to use in checking the local symmetries, as in Ref. [10] we choose to get rid of $d=2$ γ matrices and work directly in terms of irreducible components on the world sheet [for example $\psi^a_A=\binom{\psi^a_{(+)}}{\psi^a_{(-)}}$; for our $d=2$ spinor conventions, see Appendix A]. The action functional of the $N=2$ gauged nonlinear sigma model is then a sum of Eq. (24), which we rewrite here in terms of the irreducible components

$$\begin{aligned} S_{N=2} = -\frac{1}{2\pi}\int d\tau d\sigma\sqrt{-g}\{ & g^{\alpha\beta}\partial_\alpha z^n \partial_\beta z^{\bar{m}} g_{n\bar{m}} + i\sqrt{2}(\psi^n_{(+)}\partial_\alpha\psi^{\bar{m}}_{(+)} + \psi^{\bar{m}}_{(+)}\partial_\alpha\psi^n_{(+)}) e_-{}^\alpha g_{n\bar{m}} + i\sqrt{2}(\psi^n_{(-)}\partial_\alpha\psi^{\bar{m}}_{(-)} \\ & + \psi^{\bar{m}}_{(-)}\partial_\alpha\psi^n_{(-)}) e_+{}^\alpha g_{n\bar{m}} + i\sqrt{2}\psi^{\bar{m}}_{(+)}\psi^s_{(+)}\Gamma^n{}_{ps}\partial_\alpha z^p e_-{}^\alpha g_{n\bar{m}} - i\sqrt{2}\psi^{\bar{s}}_{(+)}\psi^n_{(+)}\Gamma^{\bar{m}}{}_{\bar{p}\bar{s}}\partial_\alpha z^{\bar{p}} e_-{}^\alpha g_{n\bar{m}} \\ & + i\sqrt{2}\psi^{\bar{m}}_{(-)}\psi^s_{(-)}\Gamma^n{}_{ps}\partial_\alpha z^p e_+{}^\alpha g_{n\bar{m}} - i\sqrt{2}\psi^{\bar{s}}_{(-)}\psi^n_{(-)}\Gamma^{\bar{m}}{}_{\bar{p}\bar{s}}\partial_\alpha z^{\bar{p}} e_+{}^\alpha g_{n\bar{m}} + 4R_{\bar{n}p\bar{m}k}\psi^{\bar{n}}_{(+)}\psi^p_{(-)}\psi^{\bar{m}}_{(-)}\psi^k_{(+)}\}, \end{aligned} \tag{33}$$

and a chain of the Noether couplings involving an $N=2$, $d=2$ world-sheet supergravity multiplet $(e_a{}^\beta, \chi_{A\beta}, A_\beta)$ (a stands for a flat index)

$$S^{Noether}_{N=2}=-\frac{1}{2\pi}\int d\tau d\sigma\sqrt{-g}\{-\sqrt{2}\,\bar{\psi}^m_{(+)}\psi^n_{(+)}A_\alpha e^\alpha_- g_{n\bar{m}}-\sqrt{2}\,\bar{\psi}^m_{(-)}\psi^n_{(-)}A_\alpha e_+{}^\alpha g_{n\bar{m}}+2i\partial_\alpha z^n\bar{\psi}^m_{(-)}\chi_{\beta(+)}e_-{}^\alpha e_+{}^\beta g_{n\bar{m}}$$
$$-2i\partial_\alpha z^n\bar{\psi}^m_{(+)}\chi_{\beta(-)}e_+{}^\alpha e_-{}^\beta g_{n\bar{m}}-2i\partial_\alpha z^{\bar{m}}\psi^n_{(+)}\bar{\chi}_{\beta(-)}e_+{}^\alpha e_-{}^\beta g_{n\bar{m}}+2i\partial_\alpha z^{\bar{m}}\psi^n_{(-)}\bar{\chi}_{\beta(+)}e_-{}^\alpha e_+{}^\beta g_{n\bar{m}}$$
$$-2\bar{\psi}^m_{(+)}\psi^n_{(-)}\bar{\chi}_{\alpha(+)}\chi_{\beta(-)}e_+{}^\alpha e_-{}^\beta g_{n\bar{m}}-2\bar{\psi}^m_{(-)}\psi^n_{(+)}\bar{\chi}_{\alpha(-)}\chi_{\beta(+)}e_+{}^\beta e_-{}^\alpha g_{n\bar{m}}$$
$$+\bar{\psi}^m_{(+)}\psi^n_{(+)}\bar{\chi}_{\beta(-)}\chi_{\alpha(-)}(e_+{}^\alpha e_-{}^\beta+e_+{}^\beta e^\alpha_-)g_{n\bar{m}}+\bar{\psi}^m_{(-)}\psi^n_{(-)}\bar{\chi}_{\beta(+)}\chi_{\alpha(+)}(e_+{}^\alpha e_-{}^\beta+e_+{}^\beta e_-{}^\alpha)g_{n\bar{m}}\}. \tag{34}$$

Because all the terms in a variation of the action which are proportional either to $\bar{\psi}^m_{(+)}\psi^n_{(+)}g_{n\bar{m}}$ or to $\bar{\psi}^m_{(-)}\psi^n_{(-)}g_{n\bar{m}}$ can be compensated by an appropriate variation of the gauge field A_α, it suffices to check the invariance modulo those terms. For an $N=2$ local world-sheet supersymmetry one finds

$$\delta z^n=i\bar{\epsilon}_{(-)}\psi^n_{(+)},\quad \delta\psi^n_{(-)}=-i\bar{\epsilon}_{(-)}\Gamma^n{}_{pk}\psi^p_{(+)}\psi^k_{(-)},$$

$$\delta\chi_{\alpha(+)}=0,\quad \delta e^\alpha_+=0,$$

$$\delta\psi_{(+)}=\frac{1}{\sqrt{2}}\epsilon_{(-)}\partial_\alpha z e^\alpha_+-\frac{i}{\sqrt{2}}\epsilon_{(-)}\psi_{(-)}\bar{\chi}_{\gamma(+)}e_+{}^\gamma+\frac{i}{2\sqrt{2}}\psi_{(+)}(\bar{\epsilon}_{(-)}\chi_{\gamma(-)}-\epsilon_{(-)}\bar{\chi}_{\gamma(-)})e_+{}^\gamma,$$

$$\delta e^\alpha_-=-\frac{i}{\sqrt{2}}e_+{}^\alpha(\bar{\epsilon}_{(-)}\chi_{\gamma(-)}+\epsilon_{(-)}\bar{\chi}_{\gamma(-)})e_-{}^\gamma,$$

$$\delta\chi_{\alpha(-)}=\nabla_\alpha\epsilon_{(-)}+\frac{i}{2}\epsilon_{(-)}A_\alpha+\frac{i}{2\sqrt{2}}(\bar{\epsilon}_{(-)}\chi_{\alpha(-)}+\epsilon_{(-)}\bar{\chi}_{\alpha(-)})\chi_{\gamma(-)}e_+{}^\gamma+\frac{i}{\sqrt{2}}\epsilon_{(-)}\chi_{\alpha(-)}\bar{\chi}_{\gamma(-)}e_+{}^\gamma, \tag{35}$$

and

$$\delta z^n=i\bar{\epsilon}_{(+)}\psi^n_{(-)},\quad \delta\psi^n_{(+)}=-i\bar{\epsilon}_{(+)}\Gamma^n{}_{pk}\psi^p_{(-)}\psi^k_{(+)},$$

$$\delta\chi_{\alpha(-)}=0,\quad \delta e^\alpha_-=0,$$

$$\delta\psi_{(-)}=\frac{1}{\sqrt{2}}\epsilon_{(+)}\partial_\alpha z e_-{}^\alpha+\frac{i}{\sqrt{2}}\epsilon_{(+)}\psi_{(+)}\bar{\chi}_{\gamma(-)}e_-{}^\gamma-\frac{i}{2\sqrt{2}}\psi_{(-)}(\bar{\epsilon}_{(+)}\chi_{\gamma(+)}-\epsilon_{(+)}\bar{\chi}_{\gamma(+)})e_-{}^\gamma,$$

$$\delta e^\alpha_+=\frac{i}{\sqrt{2}}e_-{}^\alpha(\bar{\epsilon}_{(+)}\chi_{\gamma(+)}+\epsilon_{(+)}\bar{\chi}_{\gamma(+)})e_+{}^\gamma,$$

$$\delta\chi_{\alpha(+)}=-\nabla_\alpha\epsilon_{(+)}-\frac{i}{2}\epsilon_{(+)}A_\alpha-\frac{i}{2\sqrt{2}}(\bar{\epsilon}_{(+)}\chi_{\alpha(+)}+\epsilon_{(+)}\bar{\chi}_{\alpha(+)})\chi_{\gamma(+)}e_-{}^\gamma-\frac{i}{\sqrt{2}}\epsilon_{(+)}\chi_{\alpha(+)}\bar{\chi}_{\gamma(+)}e_-{}^\gamma. \tag{36}$$

In addition, apart from the usual reparametrization invariance, local Lorentz transformations, and Weyl symmetry, the model exhibits invariance under two extra bosonic transformations:

$$\delta A_\alpha=\partial_\alpha a,\quad \delta\psi_{(\pm)}=-\frac{i}{2}a\psi_{(\pm)},$$

$$\delta\chi_{(\pm)}=-\frac{i}{2}a\chi_{(\pm)}, \tag{37}$$

$$\delta A_\alpha=e^{-1}\epsilon_{\alpha\beta}g^{\beta\gamma}\partial_\gamma b,\quad \delta\psi_{(+)}=-\frac{i}{2}b\psi_{(+)},$$

$$\delta\psi_{(-)}=\frac{i}{2}b\psi_{(-)},$$

$$\delta\chi_{(+)}=\frac{i}{2}b\chi_{(+)},\quad \delta\chi_{(-)}=-\frac{i}{2}b\chi_{(-)}, \tag{38}$$

where $e^{-1}=[\det(e_n{}^\alpha)]^{-1}=\sqrt{-g}$, and the super Weyl transformation

$$\delta A_\alpha=\frac{1}{\sqrt{2}}g_{\alpha\beta}e_+{}^\beta e_-{}^\gamma(\bar{\nu}_{(+)}\chi_{\gamma(-)}+\bar{\chi}_{\gamma(-)}\nu_{(+)})+\frac{1}{\sqrt{2}}g_{\alpha\beta}e^\beta_-e^\gamma_+(\bar{\nu}_{(-)}\chi_{\gamma(+)}+\bar{\chi}_{\gamma(+)}\nu_{(-)}),$$

$$\delta\chi_{\alpha(+)}=g_{\alpha\beta}e_+{}^\beta\nu_{(-)},\quad \delta\chi_{\alpha(-)}=g_{\alpha\beta}e_-{}^\beta\nu_{(+)}, \tag{39}$$

these just preserving their flat form. In checking the local symmetries one has to use essentially the fact that the target manifold is Kähler and the metric is covariantly constant. In addition, special care is to be taken of the terms requiring

integration by parts. When integrating by parts, the derivative ∂_α will hit the background metric

$$\partial_\alpha g_{n\bar{m}} = \partial_\alpha z^k \partial_k g_{n\bar{m}} + \partial_\alpha z^{\bar{k}} \partial_{\bar{k}} g_{n\bar{m}} = \partial_\alpha z^k \Gamma^l{}_{kn} g_{l\bar{m}} + \partial_\alpha z^{\bar{k}} \Gamma^{\bar{l}}{}_{\bar{k}\bar{m}} g_{n\bar{l}}, \tag{40}$$

thus inducing extra terms as compared to the flat case.

Notice that omitting the world-sheet supergravity fields in the transformation laws above and taking the parameter to be a constant, one is left precisely with the sigma model global supersymmetry transformations, thus supporting the consistency of the gauging done.

V. GAUGING $N=4$ TWISTED GLOBAL SUPERSYMMETRY

An action functional for the $N=4$ twisted string in a flat space has been constructed in Refs. [10,11]. Just as we proceeded in the former case, in order to gauge the two remaining twisted global supersymmetries in the $N=2$ gauged nonlinear sigma model it suffices to mimic the structure of the terms entering the $N=4$ twisted string. To this end, on the world sheet there must be introduced two new real vectors $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$, and an extra complex fermion $\mu_{A\alpha}$ [10], these complementing an $N=2$, $d=2$ supergravity multiplet with an $N=4$, $d=2$ one and playing the role of the gauge fields for the extra local symmetry transformations. An amendment composed of the new fields reads

$$\begin{aligned} S^{Noether}_{N=4} = -\frac{1}{2\pi}\int d\tau d\sigma \sqrt{-g}\Big\{ & \sqrt{2}(\psi^n_{(-)}\psi^m_{(-)} e_+{}^\alpha + \psi^n_{(+)}\psi^m_{(+)} e_-{}^\alpha)(\mathcal{C}_\alpha + i\mathcal{D}_\alpha) B_{nm} + 2i\partial_\alpha z^n \psi^m_{(-)} \mu_{\beta(+)} e_-{}^\alpha e_+{}^\beta B_{nm} \\ & - 2i\partial_\alpha z^n \psi^m_{(+)} \mu_{\beta(-)} e_+{}^\alpha e_-^\beta B_{nm} - \tfrac{1}{2}\psi^n_{(+)}\psi^m_{(+)}\mu_{\alpha(-)}(\chi+\bar{\chi})_{\beta(-)} g^{\alpha\beta} B_{nm} - \tfrac{1}{2}\psi^n_{(-)}\psi^m_{(-)}\mu_{\alpha(+)}(\chi+\bar{\chi})_{\beta(+)} g^{\alpha\beta} B_{nm} \\ & + 2\psi^n_{(-)}\psi^m_{(+)}(\bar{\chi}_{\alpha(+)}\mu_{\beta(-)} + \bar{\chi}_{\beta(-)}\mu_{\alpha(+)}) e_+{}^\alpha e_-{}^\beta B_{nm} + 2\psi^n_{(+)}\bar{\psi}^{\bar{m}}_{(-)}\bar{\mu}_{\alpha(+)}\mu_{\beta(-)} e_+{}^\alpha e_-{}^\beta g_{n\bar{m}} + \text{c.c.}\Big\}. \end{aligned} \tag{41}$$

Before we display the local form of the twisted supersymmetry transformations, it is worth verifying that the addition of the further Noether couplings we gathered above in $S^{Noether}_{N=4}$ to the previous action $S_{N=2}+S^{Noether}_{N=2}$ does not destroy the local symmetries intrinsic to the $N=2$ gauged nonlinear sigma model. For this to be the case, the local $\epsilon_{(-)}$ transformation must include an extra piece in the variation of the field $\psi^n_{(+)}$ and additionally the fermionic gauge field $\mu_{\beta(-)}$ has to be involved too:

$$\begin{aligned} \delta\psi^n_{(+)} &= -\frac{i}{\sqrt{2}} B_{\bar{k}s} g^{\bar{k}n} \epsilon_{(-)} \bar{\psi}^s_{(-)} \bar{\mu}_{\gamma(+)} e_+{}^\gamma, \\ e_+{}^\beta \delta\mu_{\beta(+)} &= 0, \\ e_-{}^\beta \delta\mu_{\beta(-)} &= i\epsilon_{(-)}(\mathcal{C}+i\mathcal{D})_\beta e_-{}^\beta + \frac{i}{\sqrt{2}}\mu_{\beta(-)}(\bar{\epsilon}_{(-)}\chi_{\gamma(-)} + \epsilon_{(-)}\bar{\chi}_{\gamma(-)}) e_-{}^\gamma e_+{}^\beta \\ &\quad - \frac{i}{2\sqrt{2}}\epsilon_{(-)}\mu_{\gamma(-)}(\chi+\bar{\chi})_{\beta(-)} g^{\gamma\beta} - \frac{i}{2\sqrt{2}}\mu_{\beta(-)}(\bar{\epsilon}_{(-)}\chi_{\gamma(-)} - \epsilon_{(-)}\bar{\chi}_{\gamma(-)}) e_+{}^\gamma e^\beta_-. \end{aligned} \tag{42}$$

Similarly, for the $\epsilon_{(+)}$ transformation one finds

$$\begin{aligned} \delta\psi^n_{(-)} &= \frac{i}{\sqrt{2}} B_{\bar{k}s} g^{\bar{k}n} \epsilon_{(+)} \bar{\psi}^s_{(+)} \bar{\mu}_{\gamma(-)} e_-{}^\gamma, \\ e_-{}^\beta \delta\mu_{\beta(-)} &= 0, \\ e_+{}^\beta \delta\mu_{\beta(+)} &= -i\epsilon_{(+)}(\mathcal{C}+i\mathcal{D})_\beta e_+{}^\beta - \frac{i}{\sqrt{2}}\mu_{\beta(+)}(\bar{\epsilon}_{(+)}\chi_{\gamma(+)} + \epsilon_{(+)}\bar{\chi}_{\gamma(+)}) e_+{}^\gamma e_-{}^\beta \\ &\quad + \frac{i}{2\sqrt{2}}\epsilon_{(+)}\mu_{\gamma(+)}(\chi+\bar{\chi})_{\beta(+)} g^{\gamma\beta} + \frac{i}{2\sqrt{2}}\mu_{\beta(+)}(\bar{\epsilon}_{(+)}\chi_{\gamma(+)} - \epsilon_{(+)}\bar{\chi}_{\gamma(+)}) e_-{}^\gamma e_+{}^\beta. \end{aligned} \tag{43}$$

Because all the terms in the variation of the full action that are proportional either to $\psi^n_{(+)}\psi^m_{(+)}B_{nm}$ or to $\psi^n_{(-)}\psi^m_{(-)}B_{nm}$ can be compensated by an appropriate variation of the gauge field $\mathcal{C}_\alpha + i\mathcal{D}_\alpha$ and the same is obviously true for complex conjugates, it suffices to check the invariance modulo those terms. In addition, the transformation law of the field A_α which by the same reason we omitted in the previous section will be modified by new contributions involving $\mu_{(\pm)}$.

Turning to the transformations (37), (38), (39), one discovers that the following contributions from the extra gauge fields:

$$\delta_a\mu_{\beta(+)}=\frac{i}{2}a\mu_{\beta(+)},\quad \delta_a\mu_{\beta(-)}=\frac{i}{2}a\mu_{\beta(-)},$$

$$\delta_a(\mathcal{C}+i\mathcal{D})_\alpha=ia(\mathcal{C}+i\mathcal{D})_\alpha-a\frac{i}{2\sqrt{2}}\mu_{\gamma(-)}\chi_{\beta(-)}g^{\gamma\beta}e_\alpha{}^-$$
$$-a\frac{i}{2\sqrt{2}}\mu_{\gamma(+)}\chi_{\beta(+)}g^{\gamma\beta}e_\alpha{}^+; \tag{44}$$

$$\delta_b\mu_{\beta(+)}=-\frac{i}{2}b\mu_{\beta(+)},\quad \delta_b\mu_{\beta(-)}=\frac{i}{2}b\mu_{\beta(-)},$$

$$\delta_b(\mathcal{C}+i\mathcal{D})_\alpha=ib\left\{(\mathcal{C}+i\mathcal{D})_\gamma e_-{}^\gamma-\frac{1}{2\sqrt{2}}\mu_{\gamma(-)}\chi_{\beta(-)}g^{\gamma\beta}\right\}e_\alpha{}^-$$
$$-ib\left\{(\mathcal{C}+i\mathcal{D})_\gamma e_+{}^\gamma-\frac{1}{2\sqrt{2}}\mu_{\gamma(+)}\chi_{\beta(+)}g^{\gamma\beta}\right\}e_\alpha{}^+; \tag{45}$$

and

$$\delta_\nu(\mathcal{C}+i\mathcal{D})_\alpha=\frac{1}{2\sqrt{2}}\mu_{\gamma(-)}(\nu+\bar{\nu})_{(+)}e_-{}^\gamma e_\alpha{}^-$$
$$+\frac{1}{2\sqrt{2}}\mu_{\gamma(+)}(\nu+\bar{\nu})_{(-)}e_+{}^\gamma e_\alpha{}^+ \tag{46}$$

render the action invariant when combined with Eqs. (37), (38), and (39).

Having completed the consistency check, we now proceed to discuss the twisted local supersymmetry in the full action

$$S_{N=4}=S_{N=2}+S^{Noether}_{N=2}+S^{Noether}_{N=4}. \tag{47}$$

After an extremely tedious calculation with the extensive use of Eqs. (27), (29)–(31), one can verify that the action holds invariant under the $N=2$ twisted local supersymmetry with a fermionic parameter $\kappa_{(+)}$:

$$\delta z^n=B_{\bar{k}p}g^{\bar{k}n}i\bar{\psi}^{\bar{p}}_{(-)}\kappa_{(+)},\quad \delta\chi_{\alpha(-)}=0,\quad \delta\mu_{\alpha(-)}=0,\quad \delta e_-{}^\alpha=0,$$

$$\delta\psi^n_{(+)}=-\Gamma^n{}_{pk}g^{\bar{k}p}B_{\bar{k}p}i\bar{\psi}^{\bar{p}}_{(-)}\kappa_{(+)}\psi^k_{(+)},\quad \delta e_+{}^\alpha=\frac{i}{\sqrt{2}}(\kappa_{(+)}\mu_{\gamma(+)}+\bar{\kappa}_{(+)}\bar{\mu}_{\gamma(+)})e_+{}^\gamma e_-{}^\alpha,$$

$$e^\beta_+\delta\chi_{\beta(+)}=-i\bar{\kappa}_{(+)}(\mathcal{C}-i\mathcal{D})_\alpha e_+{}^\alpha-\frac{i}{\sqrt{2}}(\kappa_{(+)}\mu_{\gamma(+)}+\bar{\kappa}_{(+)}\bar{\mu}_{\gamma(+)})\chi_{\beta(+)}e_-{}^\beta e_+{}^\gamma-\frac{i}{2\sqrt{2}}\bar{\kappa}_{(+)}\bar{\mu}_{\gamma(+)}(\chi+\bar{\chi})_{\beta(+)}g^{\beta\gamma},$$

$$\delta\psi^n_{(-)}=\frac{1}{\sqrt{2}}B_{\bar{k}p}g^{\bar{k}n}\partial_\alpha z^{\bar{p}}\kappa_{(+)}e^\alpha_--\frac{i}{\sqrt{2}}\psi^n_{(+)}\kappa_{(+)}\mu_{\gamma(-)}e_-{}^\gamma-\frac{i}{\sqrt{2}}B_{\bar{k}p}g^{\bar{k}n}\bar{\psi}^{\bar{p}}_{(+)}\kappa_{(+)}\chi_{\gamma(-)}e_-{}^\gamma-\Gamma^n{}_{pk}g^{\bar{k}p}B_{\bar{k}p}i\bar{\psi}^{\bar{p}}_{(-)}\kappa_{(+)}\psi^k_{(-)},$$

$$e_+{}^\beta\delta\mu_{\beta(+)}=\nabla_\beta\bar{\kappa}_{(+)}e_+{}^\beta-\frac{i}{2}\bar{\kappa}_{(+)}A_\beta e_+{}^\beta-\frac{i}{2\sqrt{2}}\bar{\kappa}_{(+)}\bar{\chi}_{\beta(+)}\chi_{\gamma(+)}g^{\gamma\beta}-\frac{i}{\sqrt{2}}(\kappa_{(+)}\mu_{\gamma(+)}+\bar{\kappa}_{(+)}\bar{\mu}_{\gamma(+)})\mu_{\beta(+)}e_+{}^\gamma e_-{}^\beta, \tag{48}$$

and $\kappa_{(-)}$

$$\delta z^n=B_{\bar{k}p}g^{\bar{k}n}i\bar{\psi}^{\bar{p}}_{(+)}\kappa_{(-)},\quad \delta\chi_{\alpha(+)}=0,\quad \delta\mu_{\alpha(+)}=0,\quad \delta e_+{}^\alpha=0,$$

$$\delta\psi^n_{(-)}=-\Gamma^n{}_{pk}g^{\bar{k}p}B_{\bar{k}p}i\bar{\psi}^{\bar{p}}_{(+)}\kappa_{(-)}\psi^k_{(-)},\quad \delta e_-{}^\alpha=-\frac{i}{\sqrt{2}}(\kappa_{(-)}\mu_{\gamma(-)}+\bar{\kappa}_{(-)}\bar{\mu}_{\gamma(-)})e_-{}^\gamma e_+{}^\alpha,$$

$$e^\beta_-\delta\chi_{\beta(-)}=i\bar{\kappa}_{(-)}(\mathcal{C}-i\mathcal{D})_\alpha e_-{}^\alpha+\frac{i}{\sqrt{2}}(\kappa_{(-)}\mu_{\gamma(-)}+\bar{\kappa}_{(-)}\bar{\mu}_{\gamma(-)})\chi_{\beta(-)}e_+{}^\beta e_-{}^\gamma+\frac{i}{2\sqrt{2}}\bar{\kappa}_{(-)}\bar{\mu}_{\gamma(-)}(\chi+\bar{\chi})_{\beta(-)}g^{\beta\gamma},$$

$$\delta\psi^n_{(+)}=\frac{1}{\sqrt{2}}B_{\bar{k}p}g^{\bar{k}n}\partial_\alpha z^{\bar{p}}\kappa_{(-)}e_+{}^\alpha+\frac{i}{\sqrt{2}}\psi^n_{(-)}\kappa_{(-)}\mu_{\gamma(+)}e_+{}^\gamma+\frac{i}{\sqrt{2}}B_{\bar{k}p}g^{\bar{k}n}\bar{\psi}^{\bar{p}}_{(-)}\kappa_{(-)}\chi_{\gamma(+)}e_+{}^\gamma-\Gamma^n{}_{pk}g^{\bar{k}p}B_{\bar{k}p}i\bar{\psi}^{\bar{p}}_{(+)}\kappa_{(-)}\psi^k_{(+)},$$

$$e_-{}^\beta \delta\mu_{\beta(-)} = -\nabla_\beta \bar{\kappa}_{(-)} e_-{}^\beta + \frac{i}{2}\bar{\kappa}_{(-)} A_\beta e_-{}^\beta + \frac{i}{2\sqrt{2}}\bar{\kappa}_{(-)}\bar{\chi}_{\beta(-)}\chi_{\gamma(-)} g^{\gamma\beta} + \frac{i}{\sqrt{2}}(\kappa_{(-)}\mu_{\gamma(-)} + \bar{\kappa}_{(-)}\bar{\mu}_{\gamma(-)})\mu_{\beta(-)} e_-{}^\gamma e_+{}^\beta . \tag{49}$$

Here again we omitted rather lengthy expressions for the variations of the world-sheet vector fields A_α and $\mathcal{C}_\alpha + i\mathcal{D}_\alpha$, these being responsible for removing the terms proportional to $\psi^n_{(\pm)}\psi^{\bar{m}}_{(\pm)} g_{n\bar{m}}$ and $\psi^n_{(\pm)}\psi^m_{(\pm)} B_{nm}$. A relevant technical point to mention is that in verification of the $\kappa_{(\pm)}$ invariance it proves to be helpful to cancel the terms in the following sequence: first the terms involving $\nabla\psi$ and its complex conjugate, then those containing ∂z and $\partial\bar{z}$, then the terms quadratic in $\partial z, \partial\bar{z}$, and then the rest. This is because the integration by parts in the $\nabla\psi$ terms will contribute to ∂z, ∂z^2, and so on.

Apart from the transformations listed above, previous work on the structure of the $N=4$ topological string in a flat space [10,11] indicates the presence of further local symmetries. They prove to involve two complex bosonic parameters and two complex fermionic ones which together with $\kappa_{(\pm)}$ match perfectly the number of extra gauge fields on the world sheet. It is straightforward to check that the transformations persist in the curved space, just maintaining their flat form. Omitting the variations of A_α and $\mathcal{C}_\alpha + i\mathcal{D}_\alpha$, the bosonic pair can be represented as

$$\begin{aligned}
&\delta_c\psi^n_{(+)} = cB_{\bar{m}p} g^{\bar{m}n}\psi^p_{(+)}, \quad \delta_c\psi^n_{(-)} = cB_{\bar{m}p} g^{\bar{m}n}\psi^p_{(-)},\\
&\delta_c\chi_{\alpha(+)} = -c\mu_{\alpha(+)},\\
&\delta_c\chi_{\alpha(-)} = -c\mu_{\alpha(-)}, \quad \delta_c\mu_{\alpha(+)} = -\bar{c}\chi_{\alpha(+)},\\
&\delta_c\mu_{\alpha(-)} = -\bar{c}\chi_{\alpha(-)},
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&\delta_f\psi^n_{(+)} = fB_{\bar{m}p} g^{\bar{m}n}\psi^p_{(+)}, \quad \delta_f\psi^n_{(-)} = -fB_{\bar{m}p} g^{\bar{m}n}\psi^p_{(-)},\\
&\delta_f\chi_{\alpha(+)} = f\mu_{\alpha(+)},\\
&\delta_f\chi_{\alpha(-)} = -f\mu_{\alpha(-)}, \quad \delta_f\mu_{\alpha(+)} = \bar{f}\chi_{\alpha(+)},\\
&\delta_f\mu_{\alpha(-)} = -\bar{f}\chi_{\alpha(-)},
\end{aligned} \tag{51}$$

while for the fermionic transformation one finds

$$\delta\mu_{\alpha(-)} = \lambda_{(+)} g_{\alpha\beta} e_-{}^\beta, \quad \delta\mu_{\alpha(+)} = \lambda_{(-)} g_{\alpha\beta} e_+{}^\beta. \tag{52}$$

That the $N=4$ twisted global supersymmetry can be gauged without imposing further restrictions on the background geometry implies the consistency of the coupling and provides one with the action functional of the $N=4$ twisted string coupled to a Ricci-flat Kähler background.

VI. ONE-LOOP β FUNCTION

As we have seen above, the action functional to describe the $N=4$ twisted nonlinear sigma model is identical to that of the $N=2$ theory. Thus the structure of one-loop ultraviolet divergences in the $N=4$ model is immediately elucidated due to the analysis available for the $N=2$ case (see, e.g., [19]). For completeness of the presentation we mention here a few relevant points.

When analyzing the ultraviolet behavior of a theory, it is customary to use the background field method (for a review, see Ref. [32]). To maintain manifest covariance in perturbation theory, it is convenient to switch to normal coordinates (for the details and conventions see [32])

$$\begin{aligned}
S[\rho(x,s=1),\psi(x)] &= \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n}{ds^n} S[\rho(x,s),\psi(x)]\Big|_{s=0}\\
&= \sum \frac{1}{n!} D^n(s) S[\rho(x,s),\psi(x)]\Big|_{s=0}\\
&= S_0 + S_1 + S_2 + \cdots,
\end{aligned} \tag{53}$$

where S_0 is given by Eq. (24), the argument being the background field. Splitting the metric in the common way $g_{n\bar{m}} = V^a_n V^{\bar{b}}_{\bar{m}} \eta_{a\bar{b}}$ and redefining the quantum field $(d\rho^n/ds)|_{s=0} \to V^a_n (d\rho^n/ds)|_{s=0}$, one ends up with the usual framework of quantum field theory, the propagators and vertices being easy to define.

Turning to one-loop divergences, one reveals that a potentially divergent contribution from the fermionic fields involves the integral (in momentum space)

$$\int d^d p \frac{2p^\alpha p^\beta - \delta^{\alpha\beta} p^2}{[(p+k)^2 - m^2](p^2 - m^2)} + (\text{finite part as } d\to 2), \tag{54}$$

which proves to vanish as the divergent contributions from the first and second terms exactly cancel each other. Notice that this is in perfect agreement with the absence of an ultraviolet divergence in the self-energy of a minimally coupled vector potential in two dimensions [33]. The same result can be confirmed working in superfields [19,34].

Then a detailed analysis shows [19] that the structure of one-loop divergences is specified completely by the S_2 vertex in a sector of the bosonic fields

$$-\frac{1}{2\pi}\int d^2\sigma R_{n\bar{m}k\bar{l}}\bar{\partial}^\alpha z^k \partial_\alpha \bar{z}^{\bar{l}} \langle \xi^n \bar{\xi}^{\bar{m}}\rangle, \tag{55}$$

the divergent part thus involving

$$-\frac{1}{4\pi\epsilon}R^{k}{}_{\bar{k}n\bar{m}}\partial^{\alpha}z^{m}\partial_{\alpha}\bar{z}^{\bar{n}}. \tag{56}$$

Beautifully enough, as the classical $N=4$ twisted model is formulated on a Ricci-flat Kähler manifold, one immediately concludes that the corresponding quantum theory is automatically free of ultraviolet divergences at the one-loop level.

VII. CONCLUSION

To summarize, in the present paper we have compared the nonlinear sigma model possessing an $N=2$ global supersymmetry with its $N=4$ twisted generalization. The extra twisted transformations were constructed with the use of a background two-form field. We argued that, in order to provide a symmetry of the $N=2$ action, the two-form must be covariantly constant and holomorphic. This is known to reduce the holonomy group to a subgroup of SU(1,1) and implies a Ricci-flat Kähler background. Gauging of both the $N=2$ and $N=4$ global (twisted) supersymmetries has been performed by appealing to the $N=2$, $d=2$ and $N=4$, $d=2$ supergravity multiplets on the world sheet, respectively. Recalling further the fact that the string partners of the sigma models are physically equivalent in a flat space, and that the $N=4$ extension has the advantage of being manifestly Lorentz invariant, it seems tempting to speculate that the latter point is responsible for the improved ultraviolet behavior of the $N=4$ twisted theory.

Turning to possible further developments, it would be interesting to derive the Kähler condition on the metric and those on the two-form field directly from the $N=2,4$ superconformal algebra in a curved space. The correct form of the superconformal currents is prompted by the gauged versions of the sigma models we constructed above. Since in the Hamiltonian framework the currents appear as secondary constraints, it is far from obvious that the information following from the closure of the algebra on a background will be as restrictive as that implied by the local Lagrangian symmetries, although we suspect they should match. Another interesting point is the superfield version of the analysis undertaken in this work.

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APPENDIX A

In this appendix we gather our $d=2$ spinor notations and discuss some technical points relevant for the verification of local symmetry transformations of the $N=2$ string and its $N=4$ twisted extension coupled to external curved background.

In order to describe fermions on the world sheet of a string, it is customary to use purely imaginary γ matrices:

$$\gamma_0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

$$\gamma_3=\gamma_0\gamma_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{A1}$$

These obey the algebraic properties

$$\{\gamma_a,\gamma_b\}=-2\eta_{ab},\quad \gamma_a\gamma_b=-\eta_{ab}-\epsilon_{ab}\gamma_3,$$
$$\epsilon^{ab}\gamma_b=\gamma_3\gamma^a, \tag{A2}$$

with $\eta_{ab}=\mathrm{diag}(-,+)$ and ϵ_{ab} the 2D Levi-Cività totally antisymmetric tensor, $\epsilon_{01}=-1$. The second and third identities are specific to the two-dimensional space and simplify dealing with the γ matrices considerably. Notice further that the charge conjugation matrix C, $\gamma_a^T=-C\gamma_aC^{-1}$, just coincides with γ_0 and furthermore $(\gamma_0\gamma_a)^+=\gamma_0\gamma_a$, $(\gamma_0\gamma_a)^T=\gamma_0\gamma_a$, where $(\cdots)^+$ stands for the Hermitian conjugation and $(\cdots)^T$ for the transposition.

Any 2×2 complex matrix M_{AB} can be decomposed with respect to the basis $\{1_2,\gamma_a,\gamma_3\}$:

$$M_{AB}=a\,\delta_{AB}+a_b\gamma^b_{AB}+b\,\gamma_{3AB}, \tag{A3}$$

the coefficients having the form

$$a=\tfrac{1}{2}\mathrm{Tr}(M),\quad a_b=-\tfrac{1}{2}\mathrm{Tr}(M\gamma_b),\quad b=\tfrac{1}{2}\mathrm{Tr}(M\gamma_3). \tag{A4}$$

Taking $M_{AB}=\psi_A\varphi_B$, with arbitrary spinors ψ_A, φ_B, and differentiating with respect to the latter one gets the basic Fiertz identity

$$\delta_{AK}\delta_{BN}=\tfrac{1}{2}\delta_{AB}\delta_{KN}-\tfrac{1}{2}\gamma^b_{AB}\gamma_{bNK}+\tfrac{1}{2}\gamma_{3AB}\gamma_{3NK}. \tag{A5}$$

This allows one to rearrange the order of spinors in various expressions involving world-sheet fermions. The key identity that proves to be of extensive use in checking the local supersymmetry of the $N=2$ string action is

$$(\bar{\psi}\varphi)(\bar{\chi}\gamma_3\lambda)-(\bar{\psi}\gamma_3\varphi)(\bar{\chi}\lambda)=(\bar{\chi}\gamma^b\varphi)(\bar{\psi}\gamma_b\gamma_3\lambda), \tag{A6}$$

while the one helpful in verification of global $\mathrm{U}(1,1)_{outer}$ invariance reads

$$(\bar{\psi}\varphi)(\bar{\chi}\gamma_3\lambda)+(\bar{\psi}\gamma_3\varphi)(\bar{\chi}\lambda)=-(\bar{\psi}\lambda)(\bar{\chi}\gamma_3\varphi)-(\bar{\psi}\gamma_3\lambda)\times(\bar{\chi}\varphi). \tag{A7}$$

Since in $d=2$ irreducible representations of the Lorentz group are one dimensional, it is sometimes convenient to use the light-cone notation for vectors and spinors:

$$A_{\pm}=\frac{1}{\sqrt{2}}(A_0\pm A_1),\quad A^nB_n=-A_+B_--A_-B_+,$$

$$\Psi_A=\begin{pmatrix}\psi_{(+)}\\ \psi_{(-)}\end{pmatrix}, \tag{A8}$$

$$\gamma_- = -i\sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_+ = i\sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A^+ = -A_-, \quad A^- = -A_+,$$

$$\gamma_3\gamma_+ = -\gamma_+, \quad \gamma_3\gamma_- = \gamma_-,$$

in which the Lorentz transformations simplify to

$$\delta A_\pm = \pm\Lambda A_\pm, \quad \delta\psi_{(\pm)} = \pm\tfrac{1}{2}\Lambda\psi_{(\pm)} = \tfrac{1}{2}\Lambda\gamma_3\psi. \quad \text{(A9)}$$

Obviously, the invariance is kept by contracting a "+" with a "−" [one could equally well contract a "(+)" with a "(−)" or a "+" with two "(−)"]. When a local version of the Lorentz transformations is considered, a spin connection ω_α is introduced, this allowing one to construct the covariant derivatives

$$\begin{aligned} \nabla_\alpha\psi_{(+)} &= \partial_\alpha\psi_{(+)} - \omega_\alpha\psi_{(+)}, \\ \nabla_\alpha\psi_{(-)} &= \partial_\alpha\psi_{(-)} + \omega_\alpha\psi_{(-)}, \\ \nabla_\alpha A_+ &= \partial_\alpha A_+ - 2\omega_\alpha A_+, \\ \nabla_\alpha A_- &= \partial_\alpha A_- + 2\omega_\alpha A_-, \end{aligned} \quad \text{(A10)}$$

with $\delta\omega_\alpha = \frac{1}{2}\partial_\alpha\Lambda$. The advantage of the light-cone notation is that it allows one to get rid of γ matrices and work explicitly in terms of the irreducible components of the tensors under consideration. Notice that taking into account the properties of the charge conjugation matrix it is easy to check that the object $\psi^a C\gamma^k\varphi^b\epsilon_{ab}$ transforms as a complex vector under the SO(1,1) Lorentz group, while $\psi^a C\varphi^b\epsilon_{ab}$ is a complex scalar.

Introducing the zweibein $e_b{}^\alpha$ on the world sheet $g^{\alpha\beta} = e_b{}^\alpha\eta^{bc}e_c{}^\beta$ and its inverse $e_\alpha{}^b$, $g_{\alpha\beta} = e_\alpha{}^b\eta_{bc}e_\beta{}^c$, where α stands for a curved index, one can finally verify the relations

$$\begin{aligned} e_n{}^\alpha e_k{}^\gamma - e_n{}^\gamma e_k{}^\alpha &= e\epsilon^{\alpha\gamma}\epsilon_{kn}, \\ g^{\alpha\gamma}e_k{}^\beta - g^{\alpha\beta}e_k{}^\gamma &= e\epsilon^{\gamma\beta}\epsilon_{kn}e^{n\alpha}. \end{aligned} \quad \text{(A11)}$$

These prove to be helpful in checking the local supersymmetry of the $N=2$ string. Other useful identities which we extensively use in the text are

$$\epsilon^{cp}\epsilon^{sk} = -\eta^{cs}\eta^{pk} + \eta^{ck}\eta^{ps}, \quad \epsilon^{mn}e_m{}^\alpha e_n{}^\beta = e\epsilon^{\alpha\beta},$$

$$\epsilon_{\alpha\beta}e_n{}^\alpha e_m{}^\beta = e\epsilon_{nm}, \quad \text{(A12)}$$

where $e = \det(e_n^\alpha)$ and $\epsilon_{\alpha\beta}$ is a totally antisymmetric matrix with $\epsilon_{01} = -1$.

Turning to the light-cone framework, some identities relevant to this work are

$$\begin{aligned} g^{\alpha\beta} &= -e_+{}^\alpha e_-{}^\beta - e_-{}^\alpha e_+{}^\beta, \\ e\epsilon^{\alpha\beta} &= -e_+{}^\alpha e_-{}^\beta + e_-{}^\alpha e_+{}^\beta, \\ g_{\alpha\beta} &= -e_\alpha{}^+ e_\beta{}^- - e_\alpha{}^- e_\beta{}^+, \\ e^{-1}\epsilon_{\alpha\beta} &= e_\alpha{}^+ e_\beta{}^- - e_\alpha{}^- e_\beta{}^+, \\ e_-{}^\alpha e_\alpha{}^- &= 1, \quad e_+{}^\alpha e_\alpha{}^+ = 1, \\ e_-{}^\alpha e_\alpha{}^+ &= 0, \quad e_+{}^\alpha e_\alpha{}^- = 0, \\ \eta_{++} &= \eta_{--} = 0, \quad \eta_{+-} = \eta_{-+} = -1, \\ \epsilon_{+-} &= 1, \quad \epsilon_{-+} = -1. \end{aligned} \quad \text{(A13)}$$

APPENDIX B

In order to make the presentation self-contained, in this appendix we list symmetry properties of the Riemann tensor on a Kähler manifold. These prove to be of great use both in verification of an N-extended global supersymmetry for the sigma model under consideration and in establishing the local version of the latter.

Given a complex manifold with a Hermitian metric $g_{n\bar{m}}$ (we denote the inverse by $g^{\bar{m}n}$), one introduces covariant derivatives

$$\begin{aligned} &\nabla_n v_m = \partial_n v_m - \Gamma^k{}_{nm}v_k, \quad \nabla_{\bar{n}}v_m = \partial_{\bar{n}}v_m, \\ &\nabla_{\bar{n}}v_{\bar{m}} = \partial_{\bar{n}}v_{\bar{m}} - \Gamma^{\bar{k}}{}_{\bar{n}\bar{m}}v_{\bar{k}}, \quad \nabla_n v_{\bar{m}} = \partial_n v_{\bar{m}}. \end{aligned} \quad \text{(B1)}$$

Assuming the covariant constancy of the metric

$$\begin{aligned} \nabla_n g_{m\bar{k}} &= \partial_n g_{m\bar{k}} - \Gamma^p{}_{nm}g_{p\bar{k}} = 0, \\ \nabla_{\bar{n}}g_{m\bar{k}} &= \partial_{\bar{n}}g_{m\bar{k}} - \Gamma^{\bar{p}}{}_{\bar{n}\bar{k}}g_{m\bar{p}} = 0, \end{aligned} \quad \text{(B2)}$$

one readily finds the explicit form of the Levi-Civita connection

$$\Gamma^k{}_{nm} = g^{\bar{p}k}\partial_n g_{m\bar{p}}, \quad \Gamma^{\bar{p}}{}_{\bar{n}\bar{k}} = g^{\bar{p}m}\partial_{\bar{n}}g_{m\bar{k}}. \quad \text{(B3)}$$

The simplest way to define the curvature and the torsion tensors is to consider a commutator of two covariant derivatives. For example,

$$[\nabla_A, \nabla_B]v^p = R^p{}_{kAB}v^k - T^D_{AB}\nabla_D v^p, \quad \text{(B4)}$$

where A is a collective notation for a and $\bar{a}$, and the sum over D involves both d and $\bar{d}$. A simple inspection of the latter relation with the use of Eq. (B3) shows that the only nonvanishing components of the torsion tensor are

$$\begin{aligned} T^k_{nm} &= \Gamma^k{}_{nm} - \Gamma^k{}_{mn} = g^{\bar{k}k}(\partial_n g_{m\bar{k}} - \partial_m g_{n\bar{k}}), \\ T^{\bar{k}}_{nm} &= \Gamma^{\bar{k}}{}_{nm} - \Gamma^{\bar{k}}{}_{mn} = g^{\bar{k}k}(\partial_{\bar{n}}g_{k\bar{m}} - \partial_{\bar{m}}g_{k\bar{n}}), \end{aligned} \quad \text{(B5)}$$

while those of the curvature tensor are exhausted by

$$\begin{aligned} R^k{}_{n\bar{p}m} &= -R^k{}_{nm\bar{p}} = \partial_{\bar{p}}\Gamma^k{}_{mn}, \\ R^{\bar{k}}{}_{n\bar{p}\bar{m}} &= -R^{\bar{k}}{}_{\bar{n}\bar{m}p} = \partial_p\Gamma^{\bar{k}}{}_{\bar{m}\bar{n}}. \end{aligned} \quad \text{(B6)}$$

Introducing the notation

$$R_{\bar{k}n\bar{p}m} = g_{a\bar{k}} R^{a}{}_{n\bar{p}m}, \tag{B7}$$

and making use of the explicit form of the connection, one finds

$$R_{n\bar{m}p\bar{k}} = -R_{mn\bar{p}\bar{k}}. \tag{B8}$$

Assuming finally that the manifold at hand is a Kähler space

$$T^{k}_{nm} = 0 \to \Gamma^{k}{}_{nm} = \Gamma^{k}{}_{mn} \to \partial_n g_{m\bar{k}} - \partial_m g_{n\bar{k}} = 0, \tag{B9}$$

one immediately reveals an extra symmetry for the Riemann tensor

$$R^{k}{}_{nm\bar{p}} = R^{k}{}_{mn\bar{p}}. \tag{B10}$$

When combined with those valid for an arbitrary Hermitian manifold, the last equation yields a chain of relations

$$R_{n\bar{m}p\bar{k}} = R_{p\bar{m}n\bar{k}} = R_{n\bar{k}p\bar{m}} = R_{p\bar{k}n\bar{m}}, \tag{B11}$$

which are known as the cyclic property of the Riemann tensor on a Kähler manifold.

Finally, when dealing with the terms involving the Riemann tensor that enter the sigma model action the Bianchi identities

$$\nabla_k R_{n\bar{m}p\bar{l}} - \nabla_m R_{n\bar{k}p\bar{l}} = 0,$$

$$\nabla_{\bar{k}} R_{n\bar{m}p\bar{l}} - \nabla_{\bar{n}} R_{k\bar{m}p\bar{l}} = 0, \tag{B12}$$

which are valid for a Kähler space, prove to be helpful.

[1] E. Fradkin and A. Tseytlin, Nucl. Phys. **B261**, 1 (1985).
[2] C. Callan, D. Friedan, E. Martinec, and M. Perry, Nucl. Phys. **B262**, 593 (1985).
[3] A. Sen, Phys. Rev. D **32**, 2102 (1985).
[4] D. Freedman and P. Townsend, Nucl. Phys. **B177**, 282 (1981).
[5] L. Alvarez-Gaumé and D. Freedman, Commun. Math. Phys. **80**, 443 (1981).
[6] P. Ramond and J. Schwarz, Phys. Lett. **64B**, 75 (1976).
[7] G. Chalmers, O. Lechtenfeld, and B. Niemeyer, Nucl. Phys. **B591**, 39 (2000).
[8] H. Ooguri and C. Vafa, Nucl. Phys. **B361**, 469 (1991).
[9] S. Bellucci, A. Galajinsky, and O. Lechtenfeld, Nucl. Phys. **B609**, 410 (2001).
[10] S. Bellucci and A. Galajinsky, Nucl. Phys. **B606**, 119 (2001).
[11] S. Bellucci and A. Galajinsky, Phys. Rev. D **65**, 044013 (2002).
[12] N. Berkovits and C. Vafa, Nucl. Phys. **B433**, 123 (1995).
[13] S. Bellucci, Phys. Lett. B **227**, 61 (1989); Mod. Phys. Lett. A **5**, 2253 (1990); S. Bellucci and R.N. Oerter, Nucl. Phys. **B363**, 573 (1991).
[14] A. Lichnerowicz, *Global Theory of Connections and Holonomy Groups* (Noordhoff, Amsterdam, 1976), p. 207.
[15] N. Hitchin, Atti Accad. Sci. Torino, Cl. Sci. Fis., Mat. Nat. Suppl. 124 (1990).
[16] J. Barrett, G. Gibbons, M. Perry, C. Pope, and P. Ruback, Int. J. Mod. Phys. A **9**, 1457 (1994).
[17] C. Hull, Nucl. Phys. **B509**, 252 (1998).
[18] M. Abou Zeid and C. Hull, Nucl. Phys. **B561**, 293 (1999).
[19] L. Alvarez-Gaumé, D. Freedman, and S. Mukhi, Ann. Phys. (N.Y.) **134**, 85 (1981).
[20] L. Alvarez-Gaumé and P. Ginsparg, Commun. Math. Phys. **102**, 311 (1985).
[21] M. Grisaru, A. Van de Ven, and D. Zanon, Phys. Lett. B **173**, 423 (1986).
[22] S. Bellucci and A. Nersessian, Phys. Rev. D **64**, 021702(R) (2001); Nucl. Phys. B (Proc. Suppl.) **102**, 227 (2001).
[23] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces* (Pergamon, Oxford, 1965).
[24] E. Bergshoeff, E. Sezgin, and H. Nishino, Phys. Lett. **166B**, 141 (1986).
[25] M. Pernici and P. van Nieuwenhuizen, Phys. Lett. **169B**, 381 (1986).
[26] S. Bellucci, S. James Gates, Jr., B. Radak, and Sh. Vashakidze, Mod. Phys. Lett. A **4**, 1985 (1989).
[27] S. Bellucci, Z. Phys. C **36**, 229 (1987); **41**, 631 (1989); Prog. Theor. Phys. **79**, 1288 (1988).
[28] C. Hull, G. Papadopoulos, and B. Spence, Nucl. Phys. **B363**, 593 (1991).
[29] E. Ivanov, S. Krivonos, and V. Leviant, Int. J. Mod. Phys. A **7**, 287 (1992).
[30] E. Ivanov and A. Sutulin, Nucl. Phys. **B432**, 246 (1994); **B483**, 53(E) (1997).
[31] S. Bellucci and E. Ivanov, Nucl. Phys. **B587**, 445 (2000).
[32] E. Braaten, T. Curtright, and C. Zachos, Nucl. Phys. **B260**, 630 (1985).
[33] J. Honerkamp, Nucl. Phys. **B36**, 130 (1972).
[34] A.A. Deriglazov and S.V. Ketov, Nucl. Phys. **B359**, 498 (1991).