Quantum mechanics of a superparticle with 1/4 supersymmetry breaking

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We study quantum mechanics of a massive superparticle in $d=4$ which preserves 1/4 of the target space supersymmetry with eight supercharges, and so corresponds to the partial breaking $N=8\rightarrow N=2$. Its worldline action contains a Wess-Zumino term, explicitly breaks $d=4$ Lorentz symmetry and exhibits one complex fermionic κ symmetry. We perform the Hamiltonian analysis of the model and quantize it in two different ways: with gauge-fixed κ symmetry and in the Gupta-Bleuler formalism. Both approaches give rise to the same supermultiplet structure of the space of states. It contains three irreducible $N=2$ multiplets with the total number of $(4+4)$ complex on-shell components. These states prove to be in one-to-one correspondence with the de Rham complex of p-forms on a three-dimensional subspace of the target x manifold. We analyze the vacuum structure of the model and find that the nontrivial vacua are given by the exact harmonic one- and two-forms. Despite the explicit breaking of $d=4$ Lorentz symmetry in the fermionic sector, the $d=4$ mass-shell condition is still valid in the model.

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I. INTRODUCTION

Nowadays, partial breaking of global supersymmetry (PBGS) [1,2] is widely understood to be an inborn feature of supersymmetric extended objects (for a recent review, see Ref. [3]). Exhibiting local kappa invariance, conventional p-brane models enjoy the feature of breaking half of the target space global supersymmetry. Viewed differently, the PBGS concept can be exploited to construct superbrane actions in a static gauge [4], the technical tool here being the method of nonlinear realizations [5].

Recently, there has been growing interest in PBGS options other than the 1/2 breaking [6–15]. This is essentially due to the discovery of the $d=11$ supergravity solutions preserving 1/4 or 1/8 of the $d=11$ supersymmetry [7] and their subsequent interpretation in terms of intersecting branes. Since branelike world volume effective actions which would be capable of describing those solutions are still unknown, it seems interesting to study pointlike models that mimic the exotic supersymmetry breaking options inherent in the intersecting branes. Such models could share some characteristic features of the systems of intersecting branes, much like the ordinary superparticle bears a similarity to the Green-Schwarz superstring.

In a series of recent papers [16–19] superparticle models exhibiting 3/4 or 1/4 PBGS have been constructed. In contrast with the conventional superparticle which, like a single superbrane, preserves half of the target space supersymmetry, these models reveal some new interesting peculiarities. Following the argument of Refs. [16,17,19], in order to realize 3/4, 1/4 or some further fractional PBGS options one has to extend the standard $R^{4|N}$ superspace by new central charge bosonic coordinates. In one of the 1/4 PBGS massive superparticle models of Ref. [18] the target superspace is $R^{7|8}$. The 1/4 breaking of the original $N=8$ supersymmetry[1] down to $N=2$ manifests itself in the presence of only one complex κ symmetry in the corresponding worldline action. This is achieved at cost of the explicit breaking of the target space Lorentz symmetry down to $SO(3)$ symmetry (in the fermionic sector).

In the present paper we continue the analysis of Ref. [18] and study quantum aspects of this particular $N=8\rightarrow N=2$ model as a typical example of massive superparticles with 1/4 PBGS. We first simplify the Lagrangian of Ref. [18] by taking a real slice in the sector of bosonic variables. This does not change the structure of global and local symmetries, while still provides us with an example of 1/4 PBGS in an ordinary four-dimensional Minkowski space-time (with $R^{4|8}$ as the target superspace and explicitly broken Lorentz symmetry). Prior to quantization, Hamiltonian analysis is accomplished in full detail. A subspace of physical variables is specified. The supersymmetry algebra is shown to acquire an extra constant central term which appears differently in the commutation relations of the broken and unbroken supersymmetry generators. This is typical of the superbranes in the PBGS approach and allows one to evade the no-go argument of [20] in the line of the general reasoning of [2]. We quantize the model in two different ways: in a fixed gauge and using the Gupta-Bleuler method requiring no gauge fixing. Both approaches perfectly match. We obtain a spectrum of eight complex on-shell states, four bosons and four fermions, which prove to be in one-to-one correspondence with the space of differential zero-, one-, two- and three-forms on

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[1]Throughout the paper, N denotes the number of independent *real* supersymmetries from the one-dimensional worldline perspective.

the x manifold. It is worth mentioning that a similar correspondence is known to hold in one of the versions of Witten supersymmetric quantum mechanics [21]. The vacuum structure of the theory is elucidated and shown to be provided by exact harmonic one- and two-forms on the manifold. Finally, we elaborate on the structure of the representations of the unbroken $N=2$ supersymmetry acting in a space of the excited states. This space is shown to contain two $SO(3)$ scalar and one $SO(3)$ vector supermultiplets.

II. CLASSICAL HAMILTONIAN ANALYSIS

According to the original formulation of Ref. [18], a superparticle realizing the $N=8\to N=2$ PBGS mechanism propagates in $R^{7|8}$ superspace. The even part of the supermanifold is parametrized by seven bosonic coordinates $x^0, x^i, \overline{x^i}$, $i=1,2,3$. The model exhibits an $N=8$ rigid space-time supersymmetry, as well as a local κ invariance with one complex parameter. It is noteworthy that, without spoiling the symmetry structure, one can reduce the model to the real subspace $x^i=\overline{x^i}$, after which the bosonic coordinates can be regarded to parametrize the usual four-dimensional flat Minkowski space. Since this reduction does not invalidate the basic features of the problem we are dealing with, but considerably simplifies the analysis, in the rest of the paper we shall concentrate just on this "real slice" of the original model. Its dynamics is governed by the action functional with a Wess-Zumino term $\sim m$

$$S=\int d\tau\left\{\frac{1}{2e}(-\Pi^0\Pi^0+\Pi^i\Pi^i)-\frac{1}{2}e\,m^2+im(\theta\dot{\overline{\theta}}-\psi^i\dot{\overline{\psi^i}})\right\}, \tag{2.1}$$

where

$$\Pi^0=\dot{x}^0+\frac{i}{2}\theta\dot{\overline{\theta}}+\frac{i}{2}\overline{\theta}\dot{\theta}+\frac{i}{2}\psi^i\dot{\overline{\psi}}^i+\frac{i}{2}\overline{\psi^i}\dot{\psi}^i,$$

$$\Pi^i=\dot{x}^i+i\psi^i\dot{\theta}+i\overline{\psi^i}\dot{\overline{\theta}} \tag{2.2}$$

and θ,ψ^i are four complex fermions parametrizing the odd sector of the model.

Apart from conventional τ reparametrizations, the action (2.1) is invariant under the local κ transformations

$$\delta\theta=\kappa,\quad \delta x^i=-i\psi^i\delta\theta-i\overline{\psi^i}\delta\overline{\theta},\quad \delta\psi^i=\frac{\Pi^i\delta\overline{\theta}}{\Pi^0+me},$$

$$\delta x^0=-\frac{i}{2}\theta\delta\overline{\theta}-\frac{i}{2}\overline{\theta}\delta\theta-\frac{i}{2}\psi^i\delta\overline{\psi^i}-\frac{i}{2}\overline{\psi^i}\delta\psi^i, \tag{2.3}$$

$$\delta e=\frac{2ie(\delta\theta\dot{\overline{\theta}}+\delta\overline{\theta}\dot{\theta})}{\Pi^0+me}.$$

Here, $\kappa(\tau)$ is a complex Grassmann parameter. The action is also invariant under the rigid x^0, x^i translations extended by the supertranslations with eight real parameters (or four complex ones ϵ^i, ϵ):

$$\delta\psi^i=\epsilon^i,\quad \delta x^0=-\frac{i}{2}\epsilon^i\overline{\psi^i}-\frac{i}{2}\overline{\epsilon^i}\psi^i,$$

$$\delta x^i=-i\epsilon^i\theta-i\overline{\epsilon^i\theta}, \tag{2.4}$$

$$\delta\theta=\epsilon,\quad \delta x^0=-\frac{i}{2}\epsilon\overline{\theta}-\frac{i}{2}\overline{\epsilon}\theta. \tag{2.5}$$

The algebra of the corresponding quantum Noether generators is given below in Eq. (2.29). Besides, the action (2.1) enjoys a global $SO(3)$ symmetry acting as rotations in the vector index i. As distinct from the standard massive superparticles with a Wess-Zumino term [22–24] corresponding to 1/2 PBGS, the full $d=4$ Lorentz symmetry is explicitly broken in Eq. (2.1) and is restored only in the limit of vanishing fermions. One more distinction is that the fermionic variables are split into a singlet and triplet of the group $SO(3)$, like x^0, x^i, while in the case of 1/2 superparticles they are in a spinor representation of the space-time group. In this respect the considered model resembles a spinning particle where both fermionic and bosonic fields are space-time vectors. This analogy, however, is rather far-fetched, since no manifest space-time supersymmetry is present in the spinning particle.[2] It is also worth noting that the algebra of the global supersymmetry (2.4),(2.5) is a truncation of the most general extension of the standard $N=2,d=4$ ($N=8,d=1$) superalgebra by tensorial "central charges" [25,14], with P_0, P_i being combinations of the standard $d=4$ translation generators and the central charge ones [18]. One more symmetry of Eq. (2.1) is the invariance under phase $U(1)$ transformations of the fermionic variables [θ and ψ^i have opposite $U(1)$ charges].

It has to be stressed that, although the manifest Lorentz covariance is missing in the model under consideration, we can still treat the variable x^0 as a time coordinate in the target space. The corresponding momentum then specifies the energy

$$p_0=-p^0=-E, \tag{2.6}$$

with $\eta_{nm}=\mathrm{diag}\ (-,+,+,+)$. In support of this assertion, the mass shell condition still holds in the model [18] [see Eq. (2.10) below]. The Lorentz invariance gets broken in the sector of Fermi variables only. Curiously enough, the situation resembles what happens in the $N=2$ string theory, where the $U(1)$ current of the $N=2$ superconformal algebra

[2] An interplay between a $d=4$ spinning particle and one of the 1/4 PBGS models of Ref. [18] was studied in [19].

is constructed out of fermionic fields, which is known to break the full Lorentz group $SO(2,2)$ down to the subgroup $U(1,1)$ [26].

As is well known, the presence of local symmetries is characteristic of a constrained dynamical system which requires special care in quantization. Following Dirac's recipe, in the Hamiltonian framework one finds five primary constraints

$$\begin{aligned} &A \equiv p_\theta - \frac{i}{2}(p_0+m)\bar{\theta} - i p^i \psi^i = 0, \\ &\bar{A} = -p_{\bar{\theta}} + \frac{i}{2}(p_0+m)\theta + i p^i \bar{\psi^i} = 0, \\ &A_i \equiv p_{\psi i} - \frac{i}{2}(p_0-m)\bar{\psi_i} = 0, \\ &\bar{A_i} = -p_{\bar{\psi i}} + \frac{i}{2}(p_0-m)\psi_i = 0, \quad p_e = 0, \end{aligned} \tag{2.7}$$

while the complete canonical Hamiltonian reads

$$\begin{aligned} H = p_e \lambda_e + A\lambda_\theta - \bar{A}\lambda_{\bar{\theta}} + A^i \lambda^i_\psi - \bar{A^i}\lambda^i_{\bar{\psi}} + \frac{1}{2} e (m^2 - p_0 p_0 \\ + p^i p^i). \end{aligned} \tag{2.8}$$

Here $(p_\theta, p_0, p_i, p_{\psi^i}, p_e)$ stand for the momenta canonically conjugate to the variables $(\theta, x^0, x^i, \psi^i, e)$ and λ_e, etc. are the Lagrange multipliers. The canonical brackets read

$$\begin{aligned} &\{x^0, p_0\} = \{e, p_e\} = 1, \quad \{x^i, p^k\} = \delta^{ik}, \\ &\{p_\theta, \theta\} = \{p_{\bar{\theta}}, \bar{\theta}\} = 1, \\ &\{p^i_\psi, \psi^k\} = \{p^i_{\bar{\psi}}, \bar{\psi^k}\} = \delta^{ik}. \end{aligned} \tag{2.9}$$

Given the Hamiltonian (2.8), conservation of the primary constraints implies one secondary constraint

$$M \equiv m^2 - p_0 p_0 + p^i p^i = 0, \tag{2.10}$$

and specifies some of the Lagrange multipliers

$$\begin{aligned} &(p_0-m)\lambda_{\psi i} + p_i \lambda_{\bar{\theta}} = 0, \quad (p_0-m)\lambda_{\bar{\psi i}} + p_i \lambda_\theta = 0, \\ &(p_0+m)\lambda_\theta + p^i \lambda^i_{\bar{\psi}} = 0, \quad (p_0+m)\lambda_{\bar{\theta}} + p^i \lambda^i_\psi = 0. \end{aligned} \tag{2.11}$$

Hereafter, we eliminate the canonically conjugate pair of nondynamical variables p_e and e from the consideration in a standard way (by fixing the gauge $e = \text{const}$, after which the constraint p_e becomes second class and p_e can be removed altogether by passing to the appropriate Dirac brackets).

Aiming at the construction of a quantum mechanical description of the system at hand, in the following we shall restrict ourselves to the upper shell of the massive hyperboloid (2.10):

$$p^0 = E \geqslant m, \quad \text{or} \quad p_0 \leqslant -m, \Rightarrow p_0 - m \neq 0, \tag{2.12}$$

thus omitting configurations with negative energy. Under this assumption, Eqs. (2.11) determine the value of λ_ψ and $\lambda_{\bar{\psi}}$, while still leaving $\lambda_\theta, \lambda_{\bar{\theta}}$ arbitrary. The latter fact signals the presence of two first-class fermionic constraints in the formalism. The separation of the constraints into the first- and second-class ones becomes more transparent after the simple redefinition

$$A \to A' = A + \frac{1}{(p_0-m)} p^i \bar{A^i}. \tag{2.13}$$

In the new basis the full set of the canonical Poisson brackets between the basic constraints is as follows:

$$\begin{aligned} &\{A', \bar{A'}\} = -i\frac{1}{(p_0-m)}(m^2 - p_0 p_0 + p^i p^i) \approx 0, \\ &\{A', A^i\} = \{A', \bar{A^i}\} = \{\bar{A'}, A^i\} = \{\bar{A'}, \bar{A^i}\} = 0, \\ &\{A^i, \bar{A^k}\} = i(p_0-m)\delta^{ik}, \quad \{A^i, A^k\} = \{\bar{A^i}, \bar{A^k}\} = 0. \end{aligned} \tag{2.14}$$

We see that $A', \bar{A'}, M$ are first class, while $A_i, \bar{A_i}$ are second class:

$$\underline{\text{First class}}: \quad A' \approx 0, \quad \bar{A'} \approx 0, \quad M \approx 0, \tag{2.15}$$

$$\underline{\text{Second class}}: \quad A^i \approx 0, \quad \bar{A^i} \approx 0. \tag{2.16}$$

With respect to the corresponding Dirac bracket the constraints $A', \bar{A'}, M$ generate, respectively, the complex κ symmetry and τ reparametrizations. Such a bracket is easy to construct, but we postpone giving its explicit form until fixing a gauge with respect to the κ symmetry.

In the next sections we shall quantize the theory in two different ways, either eventually leading to the same spectrum of physical states. One of them is the Gupta-Bleuler quantization which can be performed with all local symmetries being kept manifest. Another one involves removing, prior to quantization, some irrelevant unphysical variables by fixing proper gauges with respect to the local symmetries. In this case one should necessarily deal with Dirac brackets.[3] In the rest of this section we describe the Hamiltonian formalism along the lines of the second approach.

We impose the gauge conditions

[3] A quantization of the massless superparticle with the gauge-fixed κ symmetry was accomplished in [27].

$$\theta=0, \quad \bar{\theta}=0. \tag{2.17}$$

Conservation of the gauge fully specifies the value of the remaining independent Lagrange multipliers in Eqs. (2.8), (2.11)

$$\lambda_\theta=\lambda_{\bar{\theta}}=0. \tag{2.18}$$

As usual, the gauge-fixing conditions, together with the former first-class constraints $A', \bar{A}'$, should now be treated as second-class constraints extending the set (2.16). The Dirac bracket then has to be used for the remaining variables. For the case at hand it is defined by

$$\begin{aligned}\{B,C\}_D=\{B,C\}&+\frac{i}{(p_0-m)}\{B,\theta\}M\{\bar{\theta},C\}\\&+\frac{i}{(p_0-m)}\{B,\bar{\theta}\}M\{\theta,C\}-\{B,\theta\}\{A',C\}+\{B,\bar{\theta}\}\\&\times\{\bar{A}',C\}-\{B,A'\}\{\theta,C\}+\{B,\bar{A}'\}\{\bar{\theta},C\}\\&+\frac{i}{(p_0-m)}\{B,A_i\}\{\bar{A}_i,C\}+\frac{i}{(p_0-m)}\{B,\bar{A}_i\}\\&\times\{A_i,C\},\end{aligned} \tag{2.19}$$

where M is defined in Eq. (2.10). Being evaluated in the coordinate sectors, Eq. (2.19) gives

$$\begin{aligned}&\{x^0,\psi^i\}_D=-\frac{1}{2(p_0-m)}\psi^i, \quad \{x^0,\bar{\psi}^i\}_D=-\frac{1}{2(p_0-m)}\bar{\psi}^i,\\&\{x^0,p_0\}_D=1, \quad \{x^i,p^j\}_D=\delta^{ij}, \quad \{\psi^i,\bar{\psi}^j\}_D=-\frac{i}{(p_0-m)}\delta^{ij},\end{aligned} \tag{2.20}$$

with all other brackets vanishing.

Let us dwell on the issue of global supersymmetry in the reduced phase space which might give us an idea of what type of symmetries one has to expect at the quantum level. In the chosen gauge, the equations of motion take their free form

$$\dot{x}^0=-p_0, \quad \dot{x}^i=p^i, \quad \dot{p}_0=0, \quad \dot{p}^i=0, \quad \dot{\psi}^i=0. \tag{2.21}$$

Then, recalling the original transformation laws (2.3)–(2.5), one finds that six of the global supersymmetries are now realized as

$$\delta\psi^i=\epsilon^i, \quad \delta x^0=-\frac{i}{2}\epsilon^i\bar{\psi}^i-\frac{i}{2}\bar{\epsilon}^i\psi^i, \quad \delta x^i=0. \tag{2.22}$$

Two remaining supersymmetries (2.5) are now modified by a compensating κ transformation (2.3) chosen so as to preserve the gauge (2.17)

$$\delta\psi^i=\frac{1}{(p_0-m)}\bar{\epsilon}p^i, \quad \delta x^i=i\psi^i\epsilon+i\bar{\psi}^i\bar{\epsilon},$$

$$\delta x^0=-\frac{i}{2(p_0-m)}p^i(\psi^i\epsilon+\bar{\psi}^i\bar{\epsilon}). \tag{2.23}$$

As a typical feature of the canonical formalism, the action of some symmetry generator Q^i is defined via the Dirac brackets as follows:

$$\delta B=i\{B,Q^i\}_D\epsilon^i+i\{B,\bar{Q}^i\}_D\bar{\epsilon}^i. \tag{2.24}$$

Aiming at quantization of the system as the eventual goal, we first diagonalize the brackets (2.20) by redefining the fermionic fields

$$\begin{aligned}&\psi^i\to\psi'^i=\psi^i\sqrt{p_0-m}, \quad \bar{\psi}^i\to\bar{\psi}'^i=\bar{\psi}^i\sqrt{p_0-m},\\&\{x^0,p_0\}_D=1, \quad \{x^i,p_j\}_D=\delta^i{}_j, \quad \{\psi'^i,\bar{\psi}'^j\}_D=-i\delta^{ij},\end{aligned} \tag{2.25}$$

which makes the passage to a quantum description straightforward. In the new basis the supersymmetry generators take the form (hereafter, we omit primes on the new fields)

$$\begin{aligned}&Q^i=\bar{\psi}^i\sqrt{p_0-m}, \quad \bar{Q}^i=\psi^i\sqrt{p_0-m},\\&Q=\frac{1}{\sqrt{p_0-m}}\psi^ip^i, \quad \bar{Q}=\frac{1}{\sqrt{p_0-m}}\bar{\psi}^ip^i.\end{aligned} \tag{2.26}$$

One should add to these generators also the generators of $SO(3)$ rotations

$$J_i=\epsilon_{ijk}x^jp^k-i\epsilon_{ijk}\bar{\psi}^j\psi^k. \tag{2.27}$$

We can now write the full closed superalgebra at once in the quantum case, making the standard replacement of the Dirac brackets by the graded commutator

$$\{\quad\}_D\Rightarrow-i(\{\quad\}, \quad [\quad]), \tag{2.28}$$

where the anticommutator is chosen for the bracket between two fermionic operators.

The full quantum algebra including Eq. (2.26) together with the translation generators $p_0=-i\partial/\partial x^0, p_i=-i\partial/\partial x^i$ and the generators of $SO(3)$ rotations (2.27) then reads

$$\begin{aligned}&\{Q^i,\bar{Q}^j\}=(p_0-m)\delta^{ij},\\&\{Q,\bar{Q}\}=(p_0+m)+\frac{1}{(p_0-m)}(m^2-p_0p_0+p^ip^i),\\&\{Q,Q^i\}=p^i, \quad \{\bar{Q},\bar{Q}^i\}=p^i,\\&[J_i,J_j]=i\epsilon_{ijk}J^k, \quad [J_i,p_j]=i\epsilon_{ijk}p^k,\\&[J_i,Q_j]=i\epsilon_{ijk}Q^k, \quad [J_i,\bar{Q}_j]=i\epsilon_{ijk}\bar{Q}^k.\end{aligned} \tag{2.29}$$

Other (anti)commutators prove to vanish. One can directly check that these generators, by the general rule (2.24) [with the replacement (2.28)], yield for the target superspace coordinates just the supersymmetry transformations (2.22), (2.23), translations and standard $SO(3)$ rotations.

Worth noting is the appearance of the constant central charge $\pm m$ in the anticommutators $\{Q,\overline{Q}\}$ and $\{Q_i,\overline{Q}_j\}$ and the weakly vanishing term in the first anticommutator. The latter property is typical for gauge-fixed theories. Recall that the equation

$$M=m^2-p_0p_0+p^ip^i=0 \tag{2.30}$$

is the only first-class constraint remaining in the formalism. Following Dirac's method one should require it to vanish on physical states. When restricted to the physical subspace, the algebra (2.29) thus acquires its rigorous form. From the structure of the algebra one can also infer that in the realization on the states the generators $Q^i,\overline{Q^i}$ should correspond to the spontaneously broken symmetries (recall that by assumption $p_0-m\neq 0$), while $Q,\overline{Q}$ can be chosen to be unbroken and so annihilating the vacuum. The appearance of the constant central charge m with opposite signs in the anticommutators of broken and unbroken supersymmetries ensures evading the arguments of [20] against the possibility of partial breaking, in accord with the generic reasoning of Ref. [2] (it is applicable to any superbrane theory).

Finally, it is important to stress that it is the mass-shell condition (2.30) that allowed one to construct an $N=8$ supersymmetry algebra out of the operators at hand, with six supersymmetries being broken. Similar to other superbrane-like models, the partial breaking thus holds at the free theory level, without need to introduce a potential of a specific shape, as it takes place in the standard nonrelativistic supersymmetric quantum mechanics [20,28,29].

III. QUANTIZATION IN A FIXED GAUGE AND THE VACUUM STRUCTURE

Let us proceed to the more detailed exposition of the quantization procedure. After replacing the Dirac brackets by (anti)commutators according to the rule (2.28) we represent the fermionic coordinates by means of conventional creation-annihilation operators

$$\psi^i\rightarrow a^i,\quad \overline{\psi^i}\rightarrow a^{i+},\quad \{a^i,a^{j+}\}=\delta^{ij}. \tag{3.1}$$

For the bosonic operators we keep the ordinary coordinate representation, with

$$p_0=-i\frac{\partial}{\partial x^0},\quad p_i=-i\frac{\partial}{\partial x^i}. \tag{3.2}$$

Given a single pair of fermionic operators, a convenient matrix representation is [28,30]

$$a=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad a^+=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \{a,a^+\}=1. \tag{3.3}$$

A representation space is trivially constructed and consists of a vacuum state and a single filled state

$$|0\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |\uparrow\rangle=a^+|0\rangle=\begin{pmatrix}0\\1\end{pmatrix}. \tag{3.4}$$

To construct a representation for the triplet (3.1), it suffices to find a matrix which anticommutes with both a and a^+. Such a matrix is readily constructed

$$\tau=[a,a^+]=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \tau^2=1, \tag{3.5}$$

and the sought representation is then given by 8×8 matrices

$$a_1=a\times 1_2\times 1_2,\quad a_2=\tau\times a\times 1_2,\quad a_3=\tau\times\tau\times a,$$

$$a_1^+=a^+\times 1_2\times 1_2,\quad a_2^+=\tau\times a^+\times 1_2,\quad a_3^+=\tau\times\tau\times a^+. \tag{3.6}$$

It is noteworthy that the properties of the τ operator allow one to identify it with a Z_2-grading operator (sometimes referred to as the Klein operator) acting in the Hilbert space (see, e.g. [31,32]). In particular, the eigenstates corresponding to the eigenvalue $+1$ of this operator are identified with bosonic states (for the simplest case of one pair of the creation-annihilation operators there is only one such state, $|0\rangle$), while those corresponding to the eigenvalue -1 are identified with fermions (in the simplest case the only fermionic state is $|\uparrow\rangle$).

In accord with the realization (3.6), the representation space of the full algebra is eight-dimensional[4]

[4]The direct products of the states $|0\rangle$ and $|\uparrow\rangle$ amount to the usual eight component columns

$$|0\rangle\times|0\rangle\times|0\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix},\quad |\uparrow\rangle\times|0\rangle\times|0\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix},\quad \cdots,$$

$$|\uparrow\rangle\times|\uparrow\rangle\times|\uparrow\rangle=\begin{pmatrix}\vdots\\0\\0\\1\end{pmatrix}.$$

$$
\begin{array}{ccc}
 & |0\rangle\times|0\rangle\times|0\rangle\otimes\Phi(x), & \\
|\uparrow\rangle\times|0\rangle\times|0\rangle\otimes\Psi_1(x), & |0\rangle\times|\uparrow\rangle\times|0\rangle\otimes\Psi_2(x), & |0\rangle\times|0\rangle\times|\uparrow\rangle\otimes\Psi_3(x), \\
|0\rangle\times|\uparrow\rangle\times|\uparrow\rangle\otimes\Phi_1(x), & |\uparrow\rangle\times|0\rangle\times|\uparrow\rangle\otimes\Phi_2(x), & |\uparrow\rangle\times|\uparrow\rangle\times|0\rangle\otimes\Phi_3(x), \\
 & |\uparrow\rangle\times|\uparrow\rangle\times|\uparrow\rangle\otimes\Psi(x), &
\end{array}
\tag{3.7}
$$

with $x=(x^0,x^i)$, and it is the direct sum of two complex $SU(2)$ singlets Φ, Ψ and two complex $SU(2)$ triplets Φ_i, Ψ_i (total of $8+8$ states). It should be stressed that we do not assign the Fermi statistics to any of the x-dependent functions appearing above. The statistics of the states is defined entirely with respect to the Z_2-grading operator $\tau\times\tau\times\tau$. Thus, we have a single boson in the first line of Eq. (3.7), a triplet of fermions in the second line, a triplet of bosons in the third line and a single fermion in the last line. Note that this decomposition into fermionic and bosonic states is to some extent conventional. As the Z_2-grading operator one could equally take $-\tau$, with respect to which the bosonic states become fermionic and vice versa. Similarly, the vacuum and filled states in Eq. (3.3), as well as the creation and annihilation operators, alternate their status. Without loss of generality, in what follows we shall stick to the first grading.

Of frequent use in the literature are also alternative representations which deal with either superfields or a more abstract Fock space (see, e.g, [28,29]). In the next section we shall present the superfield Gupta-Bleuler quantization of the same system and show that it yields an equivalent spectrum of states.

At first glance, it seems somewhat surprising that a pithy part of the states (3.7) is described by purely bosonic functions. Observe, however, that the four levels in Eq. (3.7) are in one-to-one correspondence with the space of differential zero-, one-, two-, and three-forms on a manifold (the components of a 2-form are defined as $\Phi_{ij}=\epsilon_{ijk}\Phi^k$ while those of a 3-form as $\Psi_{ijk}=\epsilon_{ijk}\Psi$). At this stage it seems relevant to mention that the *de Rham* complex of a (curved) manifold, the space of all p-forms, can be described within the framework of supersymmetric quantum mechanics [21]. This correspondence between fermionic states and p-forms is also reminiscent of Kähler's geometric reformulation of spinors and Dirac equation in terms of differential forms (for a comprehensive review and further references see Ref. [33]).

Armed with these remarks, we now proceed to analyze the vacuum structure of the theory. Most elegantly this can be done again in terms of differential forms and our discussion here parallels that of Ref. [21]. Because of the algebra (2.29), the vacuum state of the unbroken supersymmetry defined by the conditions

$$
Q|\mathrm{vac}\rangle=\overline{Q}|\mathrm{vac}\rangle=0, \tag{3.8}
$$

necessarily has minimal energy

$$
p_0+m=0,\Rightarrow M=p^ip^i-p^0p^0+m^2=p^ip^i\equiv-\Delta. \tag{3.9}
$$

Then the supersymmetry charges Q and $\overline{Q}$ can be given a natural geometric interpretation. When acting on a vacuum state they coincide with the exterior differentiation d and the adjoint exterior differentiation δ, respectively

$$
\overline{Q}\sim d, \quad Q\sim\delta, \quad d\delta+\delta d=\frac{1}{2m}\Delta. \tag{3.10}
$$

An immediate consequence of Eqs. (3.8)–(3.10) is that the vacuum state of the unbroken supersymmetry necessarily involves a harmonic form. Since d increases the order of a form by one unit while δ decreases it by one unit, it suffices to apply the operators Q and $\overline{Q}$ directly to each level in Eq. (3.7) (to be more precise, one has to consider a linear combination of states at a given level) without need to consider any linear combination of states belonging to different levels.

On a manifold of trivial topology, which we assume in this work, one finds the following solution to Eq. (3.8) in terms of 0-, or 3-forms [the first and the fourth levels in Eq. (3.7)]:

$$
\begin{aligned}
|\mathrm{vac}\rangle^{(0)}_{(B)}&=|0\rangle\times|0\rangle\times|0\rangle\otimes e^{-imx^0}\alpha, \\
|\mathrm{vac}\rangle^{(0)}_{(F)}&=|\uparrow\rangle\times|\uparrow\rangle\times|\uparrow\rangle\otimes e^{-imx^0}\beta,
\end{aligned}
\tag{3.11}
$$

where α,β are some constants. These vacua are not too interesting. Indeed, on them

$$
p^i|\mathrm{vac}\rangle^{(0)}_{(B)}=p^i|\mathrm{vac}\rangle^{(0)}_{(F)}=0, \tag{3.12}
$$

and, as follows from the (anti)commutation relations (2.29), the $(Q,\overline{Q},p_0+m)$ and $(Q^i,\overline{Q^i},p_0-m)$ supersymmetries decouple from each other. The first supersymmetry is unbroken, while the second one is totally broken. The only Goldstone excitations are expected to be the complex Volkov-Akulov [34] Goldstone fermions associated with the generators Q^i, or $\overline{Q^i}$. The action of the latter on Eq. (3.11) produces a ring of ground states, every state possessing the minimal energy (3.9) and being a singlet of the $Q,\overline{Q}$ supersymmetry. The holomorphic set Q^i annihilates $|\mathrm{vac}\rangle^{(0)}_{(F)}$, while the conjugated set vanishes on $|\mathrm{vac}\rangle^{(0)}_{(B)}$.

Thus these vacua and the related sector of the full space of states do not correspond to the symmetry structure of the 1/4 PBGS superparticle of Ref. [18]. Indeed, in the latter case the translations p^i should also be necessarily broken, with the associated Goldstone excitations as the transverse superparticle coordinates.

The vacua with the desirable properties arise as solutions of Eqs. (3.8) for the second and third levels in Eq. (3.7). For the 1-forms [the second level in Eq. (3.7)] Eqs. (3.8) amount to

$$p^i\Psi^i=0, \quad \partial^{[i}\Psi^{j]}=0\rightarrow\Psi^i=e^{-imx^0}p^i \;\; \Sigma(\vec{x}), \quad \Delta\Sigma=0, \tag{3.13}$$

and the general structure of the corresponding fermionic vacuum state is

$$|\mathrm{vac}\rangle_{(F)}=a^{i+}|0\rangle\times|0\rangle\times|0\rangle\otimes p^i\, e^{-imx^0} \;\; \Sigma(\vec{x}), \quad \Delta\Sigma=0. \tag{3.14}$$

For the 2-forms [the third level in Eq. (3.7)] Eqs. (3.8) can be analyzed in the same spirit, yielding a bosonic vacuum state

$$|\mathrm{vac}\rangle_{(B)}=a^{i+}a^{j+}\epsilon_{ijk}|0\rangle\times|0\rangle\times|0\rangle \otimes p^k\, e^{-imx^0} \;\; \Omega(\vec{x}), \quad \Delta\Omega=0. \tag{3.15}$$

We are led to neglect in $\Sigma(\vec{x})$, $\Omega(\vec{x})$ zero modes $\sim x^i$, since the corresponding pieces belong to the ring of ''trivial'' vacua (3.11).

It is straightforward to check that none of the generators Q^i and $\overline{Q^i}$ annihilate the vacuum states defined in this way; these generators rather produce one or another multiplet of the unbroken $N=2$ supersymmetry. The resulting states certainly do not belong to the ring of vacua [i.e., do not obey Eqs. (3.8)] in view of the anticommutation relations (2.29) and the important property

$$p^i|\mathrm{vac}\rangle_{(F,B)}\neq 0. \tag{3.16}$$

In full agreement with the classical consideration [18], one concludes that these six supersymmetries are spontaneously broken together with three transverse translations, i.e., this vacuum structure and the associated sector of the space of quantum states precisely match the ''real slice'' of the 1/4 PBGS superparticle of [18] with which we started in Sec. II.

It remains to discuss the generators of the $SO(3)$ rotations. Making use of the explicit representation (2.27) one can readily verify the relations

$$\begin{aligned} J_i|\mathrm{vac}\rangle_{(F)}&=a^{j+}|0\rangle\times|0\rangle\times|0\rangle\otimes e^{-imx^0}p^j\Sigma_i(\vec{x}),\\ \Sigma_i(\vec{x})&=\epsilon_{ijk}x^jp^k\Sigma(\vec{x}),\\ J_i|\mathrm{vac}\rangle_{(B)}&=a^{j+}a^{k+}\epsilon_{jkl}|0\rangle\times|0\rangle\times|0\rangle\\ &\quad\otimes e^{-imx^0}p^l\Omega_i(\vec{x}),\\ \Omega_i(\vec{x})&=\epsilon_{ijk}x^jp^k\Omega(\vec{x}). \end{aligned} \tag{3.17}$$

Since the operators J_i do not annihilate these vacuum states, but rather produce new vacua of the same sort, generically they are spontaneously broken. Note that they are vanishing on the ''trivial'' vacua (3.11), indicating that $SO(3)$ is unbroken in the sector corresponding to two decoupled supersymmetries. However, it can be chosen unbroken in the considered ''1/4 PBGS superparticle'' sector as well, provided one selects some subclass in the set of vacua (3.14),(3.15). Indeed, the relations

$$\Sigma_i=\Omega_i=0 \tag{3.18}$$

hold on the spherically symmetric solutions of the Laplace equation:

$$\Sigma(\vec{x})=\mathrm{const}_1+\frac{1}{|\vec{x}|}, \quad \Omega(\vec{x})=\mathrm{const}_2+\frac{1}{|\vec{x}|} \tag{3.19}$$

[actually, const_1 and const_2 drop out from the corresponding subset of the vacua (3.14) and (3.15)].

In the end of the next section we shall briefly discuss how this vacuum PBGS structure is related to the standard treatment of the partial breaking of supersymmetry in the field theory models, and in which precise sense it implies the presence of the appropriate Goldstone excitations in the spectrum.

Finally, let us comment on the structure of the representation of the $N=2$ unbroken supersymmetry which acts in a space of the ''excited'' (i.e., with $E=-p_0>m$) states. Since in this case

$$\vec{p}^2=p^ip^i\neq 0, \tag{3.20}$$

for the fermionic states from the second line in Eq. (3.7) one can use the decomposition

$$\Psi^i=\left(\delta^{ij}-\frac{p^ip^j}{\vec{p}^2}\right)\Psi^j+\frac{p^ip^j}{\vec{p}^2}\Psi^j\equiv\Psi^i_\perp+p^i\Upsilon, \quad p^i\Psi^i_\perp=0. \tag{3.21}$$

Analogously, the bosonic states from the third line in Eq. (3.7) can be represented as

$$\Phi^i=\left(\delta^{ij}-\frac{p^ip^j}{\vec{p}^2}\right)\Phi^j+\frac{p^ip^j}{\vec{p}^2}\Phi^j\equiv\Phi^i_\perp+p^i\Xi, \quad p^i\Phi^i_\perp=0. \tag{3.22}$$

With a simple inspection one can further verify that at each level the states

$$|0\rangle\times|0\rangle\times|0\rangle\otimes\Phi(x), \quad a^{i+}|0\rangle\times|0\rangle\times|0\rangle\otimes p^i\Upsilon(x), \tag{3.23}$$

$$\begin{aligned} &a^{i+}a^{j+}\epsilon_{ijk}|0\rangle\times|0\rangle\times|0\rangle\otimes\Phi^k_\perp(x),\\ &a^{i+}|0\rangle\times|0\rangle\times|0\rangle\otimes\Psi^i_\perp(x), \end{aligned} \tag{3.24}$$

$$\begin{aligned} &a^{i+}a^{j+}\epsilon_{ijk}|0\rangle\times|0\rangle\times|0\rangle\otimes p^k\Xi(x),\\ &a^{i+}a^{j+}a^{k+}\epsilon_{ijk}|0\rangle\times|0\rangle\times|0\rangle\otimes\Psi(x), \end{aligned} \tag{3.25}$$

form irreducible multiplets of the unbroken $N=2$ supersymmetry. One thus concludes that the space of the excited states is a direct sum of these three on-shell representations of one-dimensional $N=2$ supersymmetry, involving, respectively, $(2+2)$, $(4+4)$ and $(2+2)$ independent real components.

The rest of $N=8$ supersymmetry generators, $Q^i, \overline{Q^i}$, mix these $N=2$ multiplets with each other, combining them into an irreducible on-shell multiplet of the full supersymmetry.

IV. GUPTA-BLEULER QUANTIZATION

In the GB quantization (see, e.g., [35]) one represents the wave function by a complex superfield φ,

$$\varphi=\varphi(x^0,x^i,\theta,\overline{\theta},\psi^i,\overline{\psi^i}), \tag{4.1}$$

and imposes on it all the first-class constraints (2.15) and half of the second-class constraints (2.16) (without passing to Dirac bracket).

We shall enforce these constraints in two steps:

(1) Off-shell constraints: $A^i\varphi=0$, $\overline{A}'\varphi=0$;

(2) On-shell constraints: $A'\varphi=0$, $(m^2-p_0^2+p^ip^i)\varphi=0$.

We replace the momenta by differential operators

$$p_\theta\to i\frac{\partial}{\partial\theta},\quad \overline{p}_\theta\to i\frac{\partial}{\partial\overline{\theta}},\quad p^i_\psi\to i\frac{\partial}{\partial\psi^i},\quad \overline{p}^i_\psi\to i\frac{\partial}{\partial\overline{\psi^i}}, \tag{4.2}$$

after which the off-shell constraints take the form

$$D_i\varphi=0,\quad (\overline{D}+p^i\overline{\psi^i})\varphi=0, \tag{4.3}$$

where

$$D_i=\frac{\partial}{\partial\psi^i}-\frac{1}{2}(p_0-m)\overline{\psi}_i,\quad \overline{D}_i=-\frac{\partial}{\partial\overline{\psi^i}}+\frac{1}{2}(p_0-m)\psi_i, \tag{4.4}$$

$$D=\frac{\partial}{\partial\theta}-\frac{1}{2}(p_0+m)\overline{\theta},\quad \overline{D}=-\frac{\partial}{\partial\overline{\theta}}+\frac{1}{2}(p_0+m)\theta.$$

The solution of Eq. (4.3) reads

$$\varphi=u+\overline{\psi^i}\overline{\rho^i}+\overline{\psi}^{2i}\overline{v^i}+\overline{\psi}^3\overline{\eta}+\frac{1}{2}(p_0-m)\psi^i(\overline{\psi^i}u-\epsilon^{ijk}\overline{\psi}^{2j}\overline{\rho^k}+\overline{\psi}^3\overline{v^i})-\frac{1}{4}(p_0-m)^2\psi^{2i}(\overline{\psi}^{2i}u+\overline{\psi}^3)-\frac{1}{8}(p_0-m)^3\psi^3\overline{\psi}^3u. \tag{4.5}$$

Here

$$\psi^{2i}\equiv\frac{1}{2}\epsilon^{ijk}\psi^j\psi^k,\quad \psi^3\equiv\frac{1}{6}\epsilon^{ijk}\psi^i\psi^j\psi^k, \tag{4.6}$$

and the $\overline{\psi}$ monomials are defined by the same formulas. The superfields $\{u,\overline{\rho^i},\overline{v^i},\overline{\eta}\}$ depend only on $\{x^0,x^i,\theta,\overline{\theta}\}$ and obey the constraints

$$\overline{D}u=0,\quad \overline{D\rho^i}=-p^iu,\quad \overline{Dv^i}=\epsilon^{ijk}p^j\overline{\rho^k},\quad \overline{D\eta}=-p^i\overline{v^i}. \tag{4.7}$$

In terms of the component fields, the solution of the off-shell constraints reads

$$u=u_0+\theta\overline{\xi}-\frac{1}{2}\theta\overline{\theta}(p_0+m)u_0,$$

$$\overline{\rho^i}=\overline{\rho^i_0}+\theta\overline{\phi^i}-\overline{\theta}(p^iu_0)+\theta\overline{\theta}\left(p^i\overline{\xi}-\frac{1}{2}(p_0+m)\overline{\rho^i_0}\right), \tag{4.8}$$

$$\overline{v^i}=\overline{v^i_0}+\theta\overline{\zeta^i}+\overline{\theta}\epsilon^{ijk}p^j\overline{\rho^k_0}+\theta\overline{\theta}\left(-\epsilon^{ijk}p^j\overline{\phi}^k-\frac{1}{2}(p_0+m)\overline{v^i_0}\right),$$

$$\overline{\eta}=\overline{\eta_0}+\theta\overline{\omega}-\overline{\theta}(p^i\overline{v^i_0})+\theta\overline{\theta}\left(p^i\overline{\zeta^i}-\frac{1}{2}(p_0+m)\overline{\eta_0}\right).$$

Thus, off shell we have 8 complex bosonic fields

$$u_0,\ \overline{\phi^i},\ \overline{v^i_0},\ \overline{\omega} \tag{4.9}$$

and 8 complex fermions

$$\overline{\xi},\ \overline{\rho^i_0},\quad \overline{\zeta^i},\quad \overline{\eta_0}. \tag{4.10}$$

Now we turn to solving the on-shell constraints which have the form

$$(D-p^i\psi^i)\varphi=0,\quad (m^2-p_0^2+p^ip^i)\varphi=0. \tag{4.11}$$

Being rewritten in terms of $N=2$ superfields $\{u,\overline{\rho^i},\overline{v^i},\overline{\eta}\}$, they read

$$Du=\frac{1}{p_0-m}p^i\overline{\rho^i},\quad D\overline{v^i}=\frac{1}{p_0-m}p^i\overline{\eta},$$

$$D\overline{\rho^i}=\frac{1}{p_0-m}\epsilon^{ijk}p^j\overline{v^k},\quad D\overline{\eta}=0. \tag{4.12}$$

These conditions put all the fields on the mass shell

$$(m^2-p_0^2+p^ip^i)(\text{all bosons})=0,$$

$$(m^2-p_0^2+p^ip^i)(\text{all fermions})=0 \tag{4.13}$$

and add the following constraints:

$$\underline{\text{Bosons:}} \quad \overline{\phi^i}=\frac{1}{p_0-m}\epsilon^{ijk}p^j\overline{v_0^k}, \quad \overline{\omega}=0,$$
$$\underline{\text{Fermions:}} \quad \overline{\xi}=\frac{1}{p_0-m}p^i\overline{\rho_0^i}, \quad \overline{\zeta^i}=\frac{1}{p_0-m}p^i\overline{\eta_0}. \tag{4.14}$$

Therefore on shell we have 4 complex bosons $\{u_0,\overline{v_0^i}\}$ and 4 complex fermions $\{\overline{\rho_0^i},\overline{\eta_0}\}$. This is in a nice agreement with the on-shell content found in the end of the previous section. To see this in more detail, one should take into account that, as a consequence of Eqs. (4.3) and (4.11), the longitudinal ($\sim p^i$) parts of the $N=2$ superfields $\overline{\rho^i}$ and $\overline{v^i}$ are expressed as spinor derivatives of u and $\overline{\eta}$. Then the irreducible set of on-shell $N=2$ superfields at $p^ip^i\neq 0$ is as follows:

$$u=u_0+\theta\frac{1}{p_0-m}(p^i\overline{\rho_0^i})-\frac{1}{2}\theta\overline{\theta}(p_0+m)u_0,$$
$$\overline{\rho_\perp^i}=\overline{\rho_{0\perp}^i}+\theta\frac{1}{p_0-m}\epsilon^{ijk}p^j\overline{v_{0\perp}^k}-\frac{1}{2}\theta\overline{\theta}(p_0+m)\overline{\rho_{0\perp}^i},$$
$$\overline{v_\perp^i}=\overline{v_{0\perp}^i}+\overline{\theta}\epsilon^{ijk}p^j\overline{\rho_{0\perp}^k}+\frac{1}{2}\theta\overline{\theta}(p_0+m)\overline{v_{0\perp}^i}, \tag{4.15}$$
$$\overline{\eta}=\overline{\eta_0}-\overline{\theta}(p^i\overline{v_0^i})+\frac{1}{2}\theta\overline{\theta}(p_0+m)\overline{\eta_0}.$$

One can readily establish the correspondence with the wave functions (3.23)–(3.25) (up to factors containing p_0-m)

$$\Phi\sim u_0, \quad \Psi\sim\overline{\eta_0}, \quad \Psi_\perp^i\sim\overline{\rho_{0\perp}^i},$$
$$\Upsilon\sim(p^i\overline{\rho_0^i}),\Phi_\perp^i\sim\overline{v_{0\perp}^i}, \quad \Xi\sim(p^i\overline{v_0^i}). \tag{4.16}$$

Note that the superfields $\overline{\rho_\perp^i}$ and $\overline{v_\perp^i}$ are not independent: they describe the same on-shell $N=2$ supermultiplet $\overline{\rho_{0\perp}^i}(x)$, $\overline{v_{0\perp}^i}(x)$ and are related by

$$\overline{\rho_\perp^i}=\frac{1}{\vec{p}^2}\overline{D}(\epsilon^{ikl}p^k\overline{v_\perp^l}), \quad \overline{v_\perp^i}=-\frac{(p_0-m)}{\vec{p}^2}D(\epsilon^{ikl}p^k\overline{\rho_\perp^l}).$$

It is worth noting that the superfield wave function $\varphi(x,\theta,\psi)$ could be chosen fermionic rather than bosonic, with the corresponding exchange of Grassmann parities between the component wave functions. This freedom is of the same kind as a freedom of choosing either τ or $-\tau$ as the Z_2 grading operator in the fixed-gauge quantization. Also notice that one could put φ into some nontrivial representation of $SO(3)$ by attaching an extra $SO(3)$ index to it. In this way a reacher $SO(3)$ structure of the final wave functions can be achieved.

Finally, it is instructive to consider the ''vacuum'' solution within the GB quantization framework. It is singled out by the additional constraints

$$Q\varphi_{vac}=\overline{Q}\varphi_{vac}=0,$$

where

$$Q=\frac{\partial}{\partial\theta}+\frac{1}{2}(p_0+m)\overline{\theta}, \quad \overline{Q}=\frac{\partial}{\partial\overline{\theta}}+\frac{1}{2}(p_0+m)\theta,$$
$$\{Q,\overline{Q}\}=p_0+m$$

[cf. Eq. (3.8)]. It is straightforward to see that they imply, for all the component fields, the additional condition

$$(p_0+m)(\text{all components})=0,\Rightarrow\Delta(\text{all components})=0.$$

Besides, they require all components in the $\theta,\overline{\theta}$ expansions in Eq. (4.8), except for the first ones, to vanish. The latter requirement gives rise to the following relations and vacuum solutions:

$$\text{(a)}\ \begin{cases}p^iu_0=0\\ p^i\overline{\eta_0}=0\end{cases}\Rightarrow\begin{cases}u_0=\text{const}\times e^{-imx^0}\\ \overline{\eta_0}=\text{const}\times e^{-imx^0}\end{cases}$$
$$\text{(b)}\ \begin{cases}\epsilon^{ijk}p^j\overline{v_0^k}=0, \quad p^i\overline{v_0^i}=0\\ \epsilon^{ijk}p^j\overline{\rho_0^k}=0, \quad p^i\overline{\rho_0^i}=0\end{cases} \tag{4.17}$$
$$\Rightarrow\begin{cases}\overline{v_0^i}=e^{-imx^0}p^i\mathcal{B}(\vec{x}), & \Delta\mathcal{B}=0\\ \overline{\rho_0^i}=e^{-imx^0}p^i\mathcal{F}(\vec{x}), & \Delta\mathcal{F}=0.\end{cases}$$

The solutions (a) correspond to ''trivial'' vacua (3.11), while (b) to the vacua (3.15),(3.14).

Let us clarify the precise meaning of the PBGS phases associated with these vacuum solutions.

Before quantization, the worldline action (2.1) in a ''static'' gauge $\tau=x^0$ and with the κ symmetry fully fixed by the gauge condition (2.17) (implemented at the classical level) can be considered as the minimal action of the Goldstone $N=2$ multiplet $x^i(\tau),\psi^i(\tau)$ corresponding to a nonlinear realization of the $d=1$ PBGS option $N=8\rightarrow N=2$ [18]. After quantization of the model associated with this action we obtained, as the space of quantum states, the above set of on-shell $N=2$ multiplets which are combined into a linear on-shell $N=8$ multiplet. Thus, proceeding from a nonlinear realization of $N=8$ supersymmetry in one dimension, we have finally arrived at a *linear* realization of this supersymmetry on a set of $N=2$ superfields bearing dependence on all four target space bosonic coordinates x^0,x^i.

An outcome of quantization of the 1/4 superparticle in question admits the standard interpretation as a first-quantized free supersymmetric field theory model in $d=4$. The ''1/4 Bogomol'nyi-Prasad-Sommerfield (BPS)'' conditions (4.17) extract those classical solutions of the free equations of motion which have a minimal energy and spontaneously break some of the involved symmetries. After shifting the superfields by the corresponding condensates, one can expect to find the relevant Goldstone excitations in the spectrum as collective coordinates related to the spontaneously broken generators. In particular, for the condensate (b) of Eq. (4.17) one can expect to recover the original worldline Goldstone multiplet in a new setting, within a *linear* realization of the original 1/4 PBGS option.

To see that this indeed occurs, let us restrict our attention to the bosonic condensate in (b) of Eq. (4.17) (it is unclear how to interpret the alternative Fermi condensate within the

field-theory framework; normally, the spontaneous supersymmetry breaking is induced just by bosonic condensates). We pass to the new "shifted" $N=2$ superfield $\hat{v}^i(x,\theta)$

$$\hat{v}^i \equiv \overline{v^i} - e^{-mx^0} p^i \mathcal{B}(\vec{x}) = \hat{v}_0^i + \theta \frac{1}{p_0 - m} p^i \overline{\eta}_0 + \overline{\theta} \epsilon^{ijk} p^j \overline{\rho}_0^k + \theta\overline{\theta}\left(\frac{1}{2}(p_0+m)\hat{v}_0^i - \frac{1}{p_0-m} p^i (p^k \hat{v}_0^k) \right), \tag{4.18}$$

and observe that under the broken p^i translations (with the parameters a^i) and $Q^i, \overline{Q^i}$ supertranslations the fields $\hat{v}_0^i(x)$ and $\overline{\eta}_0(x)$ are transformed as

$$\begin{aligned} \delta_a \hat{v}_0^i(x) &= i a^k p^k \hat{v}_0^i(x) + i a^k p^k p^i \mathcal{B}(\vec{x}) e^{-imx^0}, \\ \delta_\epsilon \overline{\eta}_0(x) &= 2m e^{-imx^0} \epsilon^k p^k \mathcal{B}(\vec{x}) + \cdots, \end{aligned} \tag{4.19}$$

where dots stand for terms which are linear in fields and vanish under restriction to the condensate $\mathcal{B}(\vec{x})$. These transformation laws directly stem from the $Q^i, \overline{Q^i}$ transformation of $\varphi(x,\theta,\psi)$,

$$\delta_\epsilon \varphi = (\epsilon^i Q_i + \overline{\epsilon^i}\,\overline{Q_i})\varphi,$$

$$Q_i = \frac{\partial}{\partial \psi^i} + \frac{1}{2}(p_0 - m)\overline{\psi_i} + \theta p_i,$$

$$\overline{Q_i} = \frac{\partial}{\partial \overline{\psi^i}} + \frac{1}{2}(p_0 - m)\psi_i + \overline{\theta} p_i,$$

rewritten in terms of $N=2$ superfields (4.8) while taking account of the on-shell relations (4.14). The inhomogeneous transformation laws (4.19) suggest the following decomposition:

$$\begin{aligned} \hat{v}_0^i(x) &= i y^k(x^0) e^{-imx^0} p^k p^i \mathcal{B} + \cdots, \\ \frac{1}{p_0 - m}\overline{\eta}_0(x) &= -\lambda^k(x^0) e^{-imx^0} p^k \mathcal{B} + \cdots, \end{aligned} \tag{4.20}$$

with

$$\delta_a y^i(x^0) = a^i, \quad \delta_\epsilon \lambda^i(x^0) = \epsilon^i + \cdots, \tag{4.21}$$

where the ellipses in Eq. (4.21) stand for terms vanishing upon restriction to the condensate and those in Eq. (4.20) for the homogeneously transforming parts of the fields. The bosonic and fermionic collective coordinates $y^i(x^0)$, $\lambda^i(x^0)$ form a closed multiplet of the unbroken $N=2$ supersymmetry,

$$\delta y^i = -\epsilon \lambda^i, \quad \delta \lambda^i = \frac{1}{2}\overline{\epsilon} p_0 y^i. \tag{4.22}$$

It is a linear realization counterpart of the above mentioned Goldstone multiplet $x^i(\tau), \psi^i(\tau)$ of the original 1/4 PBGS model.

It should be pointed out that this consideration is purely kinematical, since we deal with a free $d=4$ superfield theory. In realistic models of linear realizations of PBGS the vacuum condensate should arise dynamically as a sort of solitonic solution to self-interacting theory, with the Laplace equation in Eq. (4.17) being replaced by some nonlinear equation. In such models, the branelike Lagrangians of collective Goldstone modes appear as the leading low-energy approximation of the full nonlinear Lagrangian (see, e.g., [23,36]). In order to gain an interacting superfield theory as the result of quantization, we should start from a generalization of the worldline action (2.1) containing couplings to an external background and, perhaps, some potential terms.

V. CONCLUDING REMARKS

To summarize, in this paper we examined quantum mechanics of a massive superparticle model with 1/4 partial breaking of global supersymmetry which propagates in four-dimensional flat space-time. The spectrum was shown to contain a finite number of quantum states. This is in contrast with the massless twistor superparticle example realizing a 3/4 PBGS option [17] where infinitely many (massless) excitations are known to arise. Although the mass-shell condition is held in the model, the spectrum resembles very much the nonrelativistic supersymmetric quantum mechanics. In particular, we found a connection between the states and differential forms on a manifold, similar to that given in Ref. [21]. This connection implies a geometric interpretation for the generators of the unbroken supersymmetry as external differentials. The vacuum states for the case at hand proved to be related to the exact harmonic one- and two-forms on the x manifold.

It is worth noting that all the ingredients of our consideration here, in particular, the superalgebra (2.29), admit a straightforward extension to a supersymmetry containing $n+1$ complex supercharges Q, Q^i and $n+1$ real target bosonic translation generators P_0, P_i, $i=1,\ldots,n$, with $SO(n)$ being the only space-time symmetry group. In this generic case we still have one complex κ symmetry, and so it corresponds to the $1/(n+1)$ PBGS option. Another model which would be of interest to quantize along the lines of the present paper is the second $N=8 \to N=2$ model of Ref. [18]. As distinct from the system considered here, this alternative 1/4 PBGS model does not admit a straightforward generalization to higher-dimensional supersymmetry. As a first step, one has to construct the relevant worldline κ-invariant action which is still missing.

As for other possible developments, a generalization to manifolds of nontrivial topology and curved manifolds, as well as the construction of couplings to external background (super)fields would be natural next tasks. A generalization to the branes is also an obvious tempting point. In particular, there remains the problem of finding out explicit links with intersecting branes.

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