Maximum bounds on the surface redshift of anisotropic stars

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It is shown that for realistic anisotropic star models the surface redshift cannot exceed the values 3.842 or 5.211 when the tangential pressure satisfies the strong or the dominant energy condition, respectively. Both values are higher than 2, the bound in the perfect fluid case.

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I. INTRODUCTION

It is well known that the surface redshift s for a static perfect fluid sphere, whose density is positive and not increasing outwards, is not larger than $s_m=2$ [1]. The bound holds for the interior Schwarzschild solution with a constant fluid density ρ and infinite central pressure. It occurs before the appearance of a horizon.

Over the years, different arguments have been put forth for the existence of anisotropy in star models. It may be due to the presence of a solid core, phase transitions, a mixture of two fluids, or slow rotation [2]. The idea that the tangential pressure t may be different from the radial pressure p was suggested first by Lemaitre in 1933 [3]. He discussed a model sustained solely by tangential pressures and with constant density. This model was generalized for variable density by Florides [4] and was recently examined in Ref. [5]. The application of anisotropic fluid models to neutron stars began with the pioneering work of Bowers and Liang [6] and was done both analytically and numerically [7–15]. Some recent work may be found in Refs. [16–20].

In many papers it is stressed that arbitrarily big redshifts are obtained when t grows to infinity [5–7,10,12,16]. However, in realistic models t should be finite, positive, and should satisfy the dominant energy condition (DEC) $t\leqslant\rho$ or even the strong energy condition (SEC) $2t+p\leqslant\rho$. These may be written together as $t\leqslant\varepsilon\rho$ where $\varepsilon=1$ for DEC and $\varepsilon=1/2$ for SEC (if the realistic condition for positive p in the interior is accepted).

The bounds on s in anisotropic models were studied in Ref. [21] in a more general setting which incorporates soap bubbles, monopoles and wormholes and the focus was that a horizon does not form. It generalized this result in the perfect fluid case [22] to anisotropic fluids. Similar conclusions were reached in Ref. [23].

In this paper we elaborate further on the method used in Ref. [21] and give concrete values for the maximum surface redshift when the tangential pressure satisfies either DEC or SEC.

In Sec. II the field equations are given in a convenient form and the bound on t is implemented to derive an inequality for the mass-radius ratio.

In Sec. III this inequality is used to obtain maximum bounds on the surface redshift when SEC, DEC or a pseudo-isotropy condition is satisfied by t. Several realistic models are discussed and it is shown that the bounds cannot be saturated.

Section IV contains a short discussion.

II. FIELD EQUATIONS AND THE MAIN INEQUALITY

The metric element in curvature coordinates is

$$ds^2=e^{\nu}d\tau^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{1}$$

where ν,λ depend only on the radial coordinate r. The Einstein equations read [8]

$$\kappa\rho=\frac{e^{-\lambda}}{r}\left(\lambda'-\frac{1}{r}\right)+\frac{1}{r^2}, \tag{2}$$

$$\kappa p=\frac{e^{-\lambda}}{r}\left(\nu'+\frac{1}{r}\right)-\frac{1}{r^2}, \tag{3}$$

$$\kappa t=\frac{1}{2}e^{-\lambda}\left(\nu''-\frac{\lambda'\nu'}{2}+\frac{\nu'^2}{2}+\frac{\nu'-\lambda'}{r}\right), \tag{4}$$

where $'$ means derivative with respect to r, $\kappa=8\pi G/c^4$ and we use units with $G=c=1$. Equation (2) integrates to

$$z\equiv e^{-\lambda}=1-\frac{2m}{r},\quad m=\frac{\kappa}{2}\int_0^r\rho r^2dr. \tag{5}$$

Here m is the mass function, $m\geqslant 0$ and consequently $z\leqslant 1$. Equations (3),(4) give a linear second-order equation for $y\equiv e^{\nu/2}$:

$$2r^2zy''+(r^2z'-2rz)y'+[2(1-z)+rz'-2\kappa\Delta r^2]y=0. \tag{6}$$

Here $\Delta=t-p$ is the anisotropy factor. Finally, Eq. (3) may be written as

$$y'=\frac{1-z+\kappa pr^2}{2rz}y. \tag{7}$$

For stability reasons p must be positive and hence $y'>0$. Equation (6) has a more compact form

$$z^{1/2}\left(\frac{z^{1/2}y'}{r}\right)'=Dy,\quad D=\left(\frac{m}{r^3}\right)'+\frac{\kappa\Delta}{r}. \tag{8}$$

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It is convenient to introduce the average density

$$\langle\rho\rangle=\frac{3}{r^3}\int_0^r \rho r^2 dr=\frac{6m}{\kappa r^3}. \tag{9}$$

Then Eqs. (5), (8), (9) give

$$D=\frac{\kappa}{2r}(\rho-\langle\rho\rangle+2\Delta). \tag{10}$$

A realistic requirement is that ρ should be finite and positive. It must decrease monotonically or stay constant for stability reasons, $\rho'\leqslant 0$. Then it is easily shown that $\langle\rho\rangle'\leqslant 0$ and $\rho \leqslant\langle\rho\rangle$. Written in another way

$$\frac{d\ln m}{d\ln r}\leqslant 3. \tag{11}$$

Now, let us divide Eqs. (7) and (8):

$$r\left(\ln\frac{z^{1/2}y'}{r}\right)'=\frac{\rho-\langle\rho\rangle+2\Delta}{p+\langle\rho\rangle/3}. \tag{12}$$

There are two cases: $D\leqslant 0$ everywhere and $D>0$ somewhere. In the first case the right-hand side (RHS) of Eq. (12) is bounded from above by zero. Perfect fluids $(\Delta=0)$ form a subcase of this case. Anisotropic fluids with $\Delta\leqslant 0$, which have radially dominated pressure, form another subcase. Even when $\Delta>0$ in some regions there is a third subcase with $D\leqslant 0$. As was mentioned in the Introduction, a realistic t satisfies the inequality $t\leqslant\varepsilon\rho$. Then $\Delta\leqslant t\leqslant\varepsilon\rho$ and a sufficient condition for nonpositive D is

$$\frac{d\ln m}{d\ln r}\leqslant\frac{3}{1+2\varepsilon}. \tag{13}$$

When $\varepsilon=0$ we return to the isotropic case and Eq. (11) although $t=p$ instead of $t\leqslant 0$. Equation (13) shows that when the slope of m in a logarithmic scale is not steep enough, no positive Δ is able to ensure $D>0$. We shall show in the following section that models with $D\leqslant 0$ satisfy the Buchdahl bound on the redshift, while models with $D>0$ can develop in principle bigger redshifts.

Anisotropic models are subjected to three field equations but possess five characteristics z,y,t,p,ρ. Therefore, two of them or their combinations must be given explicitly. Equation (13) is important for models with a given density profile. This profile should break the criterion at least once in order to possibly achieve redshifts bigger than 2. Another possibility is to satisfy directly the condition $D>0$, which is more general, but can be checked without solving the field equations only when Δ is the second given function.

When $D>0$ in some region, the following chain of inequalities holds:

$$\frac{\rho-\langle\rho\rangle+2\Delta}{p+\langle\rho\rangle/3}\leqslant\frac{2(t-p)}{p+\langle\rho\rangle/3}\leqslant\frac{6t}{\langle\rho\rangle}\leqslant 6\varepsilon. \tag{14}$$

The isotropic case is regained again when $\varepsilon=0$. Inserting the bound from Eq. (14) into Eq. (12) and integrating from r to the boundary of the fluid sphere at $r=R$ yields

$$y'\geqslant A(R)\left(\frac{r}{R}\right)^{6\varepsilon}\frac{r}{z^{1/2}},\quad A(r)=\frac{z^{1/2}y'}{r}. \tag{15}$$

At R the interior fluid solution should be matched to the exterior Schwarzschild solution, which requires $z(R)=1-2M/R$. Here $M\equiv m(R)$ is the total mass of the fluid sphere. Equation (9) provides the inequality

$$\frac{2m}{r}\geqslant\frac{2M}{R^3}r^2. \tag{16}$$

After these remarks, let us integrate Eq. (15) from the center to the boundary and take into account that $y(0)\geqslant 0$. The result reads

$$\left(1-\frac{2M}{R}\right)^{1/2}\geqslant\frac{M}{2R^{3+6\varepsilon}}\int_0^{R^2}\frac{x^{3\varepsilon}dx}{\left(1-\frac{2M}{R^3}x\right)^{1/2}}. \tag{17}$$

Equation (17) is a particular case of Eq. (66) from Ref. [21]. In this paper polar Gaussian coordinates were used, the bound in Eq. (14) was assumed to hold for some positive ε, not connected in general with t, and the maximum of $2m/r$ was not obliged to be on the surface of the configuration. Equation (17) can be written as an inequality just for $\alpha\equiv 2M/R$:

$$4\alpha^{3\varepsilon}(1-\alpha)^{1/2}\geqslant\int_0^{\alpha}\frac{x^{3\varepsilon}dx}{(1-x)^{1/2}}. \tag{18}$$

This is the main inequality to be used for finding redshift bounds.

III. BOUNDS AND MODELS

Equation (18) provides maximum values for the mass-radius ratio α and the surface redshift

$$s=(1-\alpha)^{-1/2}-1, \tag{19}$$

both of which depend on ε. In the perfect fluid case formally $\varepsilon=0$ and Eq. (18) becomes

$$3(1-\alpha)^{1/2}\geqslant 1. \tag{20}$$

This gives the values found by Buchdahl: $\alpha_m=8/9$ and $s_m=2$. When anisotropy is present and t is untied from p, $\varepsilon>0$ and there is possibility for higher redshifts. In general, the integral on the RHS of Eq. (18) is expressed through the hypergeometric function. When 3ε is an integer it becomes a sum of different powers of $1-\alpha$. Thus, when DEC holds for $t(\varepsilon=1)$ we have

$$(1-\alpha)^{1/2}\left[\alpha+2\alpha^3-\frac{1}{7}(1-\alpha)^3+\frac{3}{5}(1-\alpha)^2\right]\geqslant\frac{16}{35} \tag{21}$$

and the maximum bounds are $\alpha_m=0.974$ and $s_m=5.211$. These values were found in Ref. [21].

A useful relation which follows from the field equations is the Tolman-Oppenheimer-Volkoff (TOV) equation [6]

$$p'=-(\rho+p)\frac{1-z+\kappa pr^2}{2rz}+\frac{2\Delta}{r}. \tag{22}$$

It shows that for realistic and, hence, finite at the center p,p' we have $\Delta(0)=0$. Furthermore, the radial pressure is obliged to vanish at the boundary. If the more imperative SEC holds for t, $\rho\geqslant p+2t$ gives $t\leqslant\frac{1}{2}\rho$ and even $t(0)\leqslant\frac{1}{3}\rho(0)$. Let us accept the stronger bound, which holds in the interior of perfect fluids, and call this a pseudoisotropic case, with t,p being of the same magnitude, although not equal. Then $\varepsilon=1/3$ and Eq. (18) becomes

$$(1-\alpha)^{1/2}\left(1+\frac{7}{2}\alpha\right)\geqslant1, \tag{23}$$

which yields $\alpha_m=0.946$ and $s_m=3.310$. These values are still above the Buchdahl ones.

The general value of ε for SEC, however, is 1/2 and this case is the most realistic one. Now 3ε is not an integer, but the integral in Eq. (18) is once more expressed in terms of elementary functions and gives

$$4(1-\alpha)^{1/2}\alpha^{3/2}\geqslant\frac{2\alpha^3+\alpha^2-3\alpha}{4\sqrt{\alpha-\alpha^2}}+\frac{3}{8}\arcsin(2\alpha-1)+\frac{3\pi}{16}. \tag{24}$$

Computer calculations show that $\alpha_m=0.957$ and $s_m=3.842$. The last number is the main result of this paper. It shows an almost double increase in the Buchdahl bound when anisotropy is allowed and t satisfies SEC. The other assumptions made were $\rho>0$, $p\geqslant0$, $\rho'\leqslant0$ and the inevitable $t\geqslant p$. Of course, $D>0$ should also be true, otherwise the bound collapses to Buchdahl's one. Thus, realistic anisotropic star models can possess higher redshifts than the isotropic ones but they are limited and never reach infinity.

It is said sometimes for perfect fluids that the Buchdahl bound is optimal because there is a model which saturates it. Equation (12) shows that for such a model $\rho=\langle\rho\rangle$ which is possible only for constant ρ. Thus we arrive at the Schwarzschild incompressible sphere with the central pressure as a free parameter. Only the model with $p(0)=\infty$ saturates the bound. A possible explanation is that $y(0)\geqslant0$ and the equality is required for saturation. This leads to a singular metric and it is probably induced only by singular central pressure. When the pressure is required to satisfy SEC one has only $s_m=1/2$ [13]. We leave aside the problem that constant density leads to infinite speed of sound $v=(dp/d\rho)^{1/2}$, while a realistic speed of sound is bound by 1, the speed of light in our units. In conclusion, there is no realistic model saturating the Buchdahl bound.

What is the situation for anisotropic models? According to Eq. (12) a saturating model must satisfy

$$\rho-\langle\rho\rangle+2\Delta=6\varepsilon\left(p+\frac{1}{3}\langle\rho\rangle\right). \tag{25}$$

However, this is not possible at the center because there the LHS vanishes, while the RHS is positive. Again the explanation may be connected with the requirement $p(0)=\infty$ because of $y(0)=0$. Then $\Delta(0)<0$ and $D(0)<0$ since $t(0)$ is finite, but this contradicts the nature of the saturation ($D>0$).

One may ask for models which satisfy the limiting assumptions made during the derivation of Eq. (14). Namely, $\rho=\text{const}=\rho_0$, $p=0$ and $t=\varepsilon\rho_0$. These are three conditions while only two of the fluid's characteristics can be fixed beforehand. If we take the first two conditions, we arrive at the Lemaitre model [3–5]. Equation (22) then gives

$$t=\frac{1-z}{4z}\rho_0. \tag{26}$$

Obviously t is not as required. Using the bound on t gives

$$\alpha\leqslant\frac{4\varepsilon}{1+4\varepsilon}. \tag{27}$$

This inequality yields $\alpha_m=0.8$, $s_m=1.236$ for DEC and $\alpha_m=0.667$, $s_m=0.732$ for SEC. These values are higher than the ones of the realistic Schwarzschild interior solution, but far away from the absolute bounds derived above. Two models possessing the values when DEC holds were given recently [17] (models I and IV).

Most successful is the Bondi model [13] which also has a constant density, but the second given function is the constant ratio $Q=(p+2t)/\rho_0$. The region $Q\leqslant1$ is studied, which is equivalent to SEC. The redshift increases with Q and numerical simulation gives $s_m=1.352$ for $Q=1$. Close to this result stands example 2 from Ref. [18] with $s=1.2$. It utilizes a nonlocal equation of state and a density profile used in Refs. [4,9,14].

Finally, the conformally flat anisotropic models are worth mentioning [9]. The vanishing of the Weyl tensor implies

$$\frac{\kappa\Delta}{r}=-2\left(\frac{m}{r^3}\right)'. \tag{28}$$

The two terms in D are of the same nature, but do not compensate each other. All characteristics of the solution depend only on the density profile. D is definitely positive, however, already the Buchdahl bound is reached with infinite central radial pressure. This is analogous to the behavior of the interior Schwarzschild solution which is the only conformally flat solution in the perfect fluid case [24].

IV. DISCUSSION

The bounds found in this paper generalize the Buchdahl bound to anisotropic star models and show that claims for arbitrarily large redshifts are not realistic. The bounds are absolute, i.e., numbers depending on a few simple realistic requirements. They do not depend on the details of the mechanism generating anisotropy, equations of state, central density and so on. We have also not explored the consequences of the condition $0 \leqslant v \leqslant 1$, necessary for the causal behavior of the fluid. In the perfect fluid case a lower bound $s_m = 0.854$ was established heuristically when some of the above mentioned features are taken into account [25]. Probably, such a study can be performed in the anisotropic case too. It may concern not only the surface redshift but also the maximum masses and moments of inertia of neutron stars.

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