Life in an energy eigenstate: Decoherent histories analysis of a model timeless universe

J. J. Halliwell and J. Thorwart
Theory Group, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
(Received 23 January 2002; published 29 April 2002)

Inspired by quantum cosmology, in which the wave function of the universe is annihilated by the total Hamiltonian, we consider the internal dynamics of a simple particle system in an energy eigenstate. Such a system does not possess a uniquely defined time parameter, and all physical questions about it must be posed without reference to time. We consider in particular the following question: what is the probability that the system's trajectory passes through a set of regions of configuration space without reference to time? We first consider the classical case, where the answer has a variety of forms in terms of a phase-space probability distribution function. We then consider the quantum case, and we analyze this question using the decoherent histories approach to quantum theory, adapted to questions which do not involve time. When the histories are decoherent, the probabilities approximately coincide with the classical case, with the phase-space probability distribution replaced by the Wigner function of the quantum state. For some initial states, decoherence requires an environment, and we compute the required influence functional and examine some of its properties. Special attention is given to the inner product used in the construction (the induced or Rieffel inner product), the construction of class operators describing the histories, and the extent to which reparametrization invariance is respected. Our results indicate that simple systems without an explicit time parameter may be quantized using the decoherent histories approach, with the expected classical limit extracted. The results support, for simple models, the usual heuristic proposals for the probability distribution function associated with a semiclassical wave function satisfying the Wheeler-DeWitt equation.

DOI: 10.1103/PhysRevD.65.104009 PACS number(s): 04.60.Gw, 04.60.Kz, 98.80.Hw

I. INTRODUCTION

There are a variety of interesting physical situations in which our classical view of the world inspires us to ask questions that quantum theory does not easily answer. For example, classical mechanics concerns simultaneously specified values of coordinates and momenta while quantum theory has to go through some contortions to say what this means operationally. Another important class of problems of this type are those that involve time in a nontrivial way. The arrival time problem and tunneling time problem, for example, have been the subject of considerable recent interest [1].

Perhaps more intriguing than these are questions in quantum mechanics that do not involve time at all. Consider, for example, the following situation. Suppose we have a system of particles in a state of fixed total energy. It could, for example, be a light particle orbiting a massive particle. Then classically, we can ask whether the light particle passes through a certain region of configuration space at any stage during its orbit. Or we can ask which of the possible classical orbits the light particle follows. The important point here is that these sorts of questions do not involve time explicitly. Much experimental and observation data are in fact of this type. For example, astronomical observations yield planetary orbits, and particle physics experiments often yield a photograph of a track in a bubble chamber. We would like to ask the same question in quantum theory: given that the system is in an energy eigenstate, what is the probability that it is found in a certain region of configuration space, irrespective of time?

The primary motive for considering this question is quantum cosmology [2]. There, in simple cosmological models, the wave function of the system obeys the Wheeler-DeWitt equation,

$$H\Psi(\mathbf{x})=0, \tag{1.1}$$

where H is the total Hamiltonian of the gravitational field plus all matter sources in the universe. The most significant feature of this equation is that it contains absolutely no reference to time whatsoever. It is usually argued that ''time,'' or more precisely, the physical systems that we use to measure time, are contained already in the gravitational and matter fields [3–5]. While this is very plausible, it leaves us with the question as to how to extract interesting physical predictions from this wave function, given the absence of the time coordinate that plays such a central role in standard quantum theory. In quantum cosmology, heuristic methods (mainly the ''WKB interpretation'') have been used to extract useful predictions [6], but what we are concerned with here is how such heuristic ideas may be incorporated in a properly defined interpretational framework for the quantum theory of timeless models. Furthermore, although simple (minisuperspace) models are unlikely to be reasonable physical approximations to a full quantum gravity theory, it does seem likely that a quantum theory of gravity will center around a timeless equation of the form (1.1) (the loop variable approach, for example, involves such an equation [7]), hence it remains important to understand the quantization and interpretation of such systems at an elementary level.

Quantum cosmology, then, is our main motivation for studying the internal dynamics of a closed system in an energy eigenstate. The question we shall focus on is this: given an energy eigenstate, what is the probability that the system enters a region Δ of configuration space? Or similarly, sup-

posing the coordinates of the system are $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, what is the probability distribution of, say, $x_2,\ldots,x_n$, given the value of x_1? Classically, these questions may be answered reasonably easily. We look for all the phase-space initial data points whose classical trajectories pass through Δ, and then the desired probability for entering Δ is the probability measure on this subset of phase space. In quantum theory, the question is considerably more complicated, and, like the question of phase-space samplings, a variety of different (generally inequivalent) approaches may be employed.

It is perhaps of interest to spell out in more detail how the absence of a time parameter affects these considerations. In a standard nonrelativistic particle theory, the variables of interest are, for example, positions at a fixed moment of time $\mathbf{x}(t)$. Here, the time t is regarded as an observable physical parameter. In a theory without time, by contrast, the quantities of interest are curves in configuration space $\mathbf{x}(s)$ (or more generally, in phase space). Here, s is *not* a physically measurable time, but is simply a parameter labeling the points along the curve, and the curves are parametrized in this way for mathematical convenience. Furthermore, one of the characteristic features of genuinely timeless theories is that none of the components of $\mathbf{x}$ are monotonic in s. This means that it is not possible to use one of the components of $\mathbf{x}$ as a ''time'' parameter, except for local sections of the curve. Despite these features, such classical theories are well-defined and predictive. Indeed, many classical cosmological models are of this type. For example, in the massive scalar field cosmological model [for a positive curvature Friedmann-Robertson-Walker (FRW) metric] [8], the classical solutions go backwards and forwards in both the scale factor a and the scalar field ϕ. Therefore, the most general such classical theory is one in which there is a probability distribution on the set of classical trajectories, and this is also well-defined (subject to careful normalization) as we shall see. Such probability distributions are of particular interest for predicting, for example, the likelihood of the initial conditions for inflation [9].

A closely related issue is the fact that a vanishing Hamiltonian [or in the quantum theory, Eq. (1.1)] is associated with the symmetry of reparametrization invariance, which is essentially the freedom to redefine the parameter s labeling points along a trajectory. Individual phase-space points are not reparametrization-invariant, since they are moved along the classical trajectories by a reparametrization. But a useful invariant quantity is the entire classical trajectory, as we shall see later in more detail. This simple observation turns out to be a useful focal point for what we do in this paper.

Turning now to the quantum theory, there are, as stated, many possible approaches to these systems, all of which are complicated by the absence of a time parameter. However, the aspects of the classical theory outlined above suggest that a particularly useful approach to quantization is the decoherent histories approach [10–12]. This is because it deals directly with entire trajectories and obviously does not require a time coordinate. The aim of this paper is to show that the decoherent histories approach can be used to calculate probabilities for histories in configuration space in simple timeless models. In particular, we shall show that in the classical limit, the decoherent histories approach produces a set of classical trajectories with a probability measure on that set.

The other significant approach to this problem that is currently being pursued involves constructing operators corresponding to the questions posed above, and which commute with the Hamiltonian, thus they are ''observables'' [13–18]. For a free particle in two dimensions, for example, the classical trajectories have the form

$$x_1(t)=x_1+\frac{p_1 t}{m}, \quad x_2(t)=x_2+\frac{p_2 t}{m} \tag{1.2}$$

and we may eliminate t between them to write

$$x_1(t)=x_1+\frac{p_1}{p_2}[x_2(t)-x_2]. \tag{1.3}$$

This is the classical answer to the question, what is the value of x_1 at a given value of x_2? It commutes with the free-particle Hamiltonian,

$$H=\tfrac{1}{2}(p_1^2+p_2^2), \tag{1.4}$$

thus it is an observable and we may find states that are eigenstates of both Eqs. (1.3) and (1.4), from which one may begin to address the questions set out above. We will not say much about this approach here, but it is important to mention because any other approach ought to make some kind of contact with it at some stage, and in particular, the decoherent histories approach must contain some notion corresponding to the notion of an observable in the operator approach. (See also Refs. [19,20] for some much earlier approaches to these issues.)

We begin Sec. II by describing the classical result. We introduce a classical phase-space distribution function $w(\mathbf{p},\mathbf{x})$ and compute the probability that a trajectory in configuration space passes through a region Δ. Most of the key ideas for this paper are in fact contained in this classical result. In particular, we discuss the reparametrization invariance of the system, and introduce observables corresponding to entire classical trajectories. We write the classical result in a number of different forms, including a form in terms of a flux across a hypersurface, closely related to the heuristic WKB interpretation of quantum cosmology.

We begin the quantum case in Sec. III with the construction of the decoherence functional for timeless models. This has two important aspects. The first concerns the choice of inner product in the construction, since solutions to equations of the form Eq. (1.1) are typically not normalizable in the simple Schrödinger inner product. The appropriate choice is the induced (or Rieffel) inner product, which we describe. We also show how this inner product for the quantum case implies a useful normalization for the classical phase-space distribution function. The second aspect is the construction of the class operators, which, in this case, are propagators describing coarse-grained sets of histories passing through restricted regions of configuration space.

In Sec. IV, we discuss the semiclassical limit of the decoherence functional. A key step is the construction of class operators corresponding to restricted sets of histories enter-

ing the region Δ. The obvious candidates for these class operators are not in fact compatible with the constraint equation, and we therefore show how they may be appropriately modified. This turns out in fact to be the crucial step in the construction of the decoherence functional. We then show that, for the special initial states for which the histories are decoherent, the probabilities for the histories approximately coincide with the classical case, with the phase-space distribution function $w(\mathbf{p},\mathbf{x})$ replaced by the Wigner function $W(\mathbf{p},\mathbf{x})$ of the quantum system.

In Sec. V, we consider the special case in which the system is a collection of harmonic oscillators in a fixed energy eigenstate. For this system it is possible to introduce a special class of eigenstates of the Hamiltonian sometimes called ''timeless coherent states,'' which have the property that they are concentrated about an entire classical phase-space trajectory. We discuss the decoherence and probabilities associated with these states and obtain the intuitively expected physical results for the probabilities of entering a region Δ.

Since decoherence is only obtained for special initial states, we consider, in Sec. VI, the addition of an environment to produce decoherence for a wide variety of initial states. We repeat the calculation of decoherence and probabilities with intuitively expected results, in agreement with classical expectations.

The calculations of Secs. IV and VI used initial states consisting of single WKB wave packets. In Sec. VII, we therefore extend to the case of superpositions of such wave packets. This turns out in fact to be straightforward, and very similar to earlier calculations performed with the reduced density matrix. We easily find that the interference terms between different WKB wave packets are very small. We summarize and conclude in Sec. VIII.

Finally, we outline the relationship of the present paper to other works in the field. This paper is part of a program to apply the decoherent histories approach to quantum cosmology, or models of the type used in quantum cosmology, and ultimately to quantum gravity more generally. The decoherent histories approach (generally) and its applications to quantum cosmology have been set out at length by Hartle in his 1992 Les Houches Lectures [21], and more recent relevant aspects of this were discussed by Hartle and Marolf [22]. This formalism has been applied to particular models in three places, most recently by the present authors [23], who used it to construct in detail the decoherence functional for the relativistic particle and compute probabilities for crossing spacelike surfaces. The present paper is in some ways an extension of that work. Whelan [24] has used the formalism to compute probabilities on timelike surfaces for the relativistic particle. Also, Craig and Hartle [25] have applied the formalism to a Bianchi IX quantum cosmological model. The last two papers use the Klein-Gordon inner product, whereas here we use the positive-definite induced inner product to construct the decoherence functional. There is also some connection with the work on probabilities for nontrivial spacetime coarse grainings in nonrelativistic quantum mechanics [26].

It is also perhaps worth mentioning that this paper is very much in the spirit of Ref. [27], which attempts to interpret the Wheeler-DeWitt equation in terms of emergent trajectories by introducing model detectors into the Hamiltonian. This was inspired in turn by Barbour's observations [28] on the similarity between the Wheeler-DeWitt equation and Mott's calculation showing the emergence of a straight line track from a spherical wave in alpha decay [29], together with some of Barbour's more general observations about the Wheeler-DeWitt equation and timeless theories [28,30,31].

II. THE CLASSICAL CASE

We are interested in the following question: ''What is the probability associated with a given region of configuration space when the system is in an energy eigenstate?'' We begin by analyzing the classical problem.

We will consider a classical system described by a $2n$-dimensional phase space, with coordinates and momenta $(\mathbf{x},\mathbf{p})=(x_k,p_k)$, and Hamiltonian

$$H=\sum_{k=1}^{n}\frac{p_k^2}{2M}+V(\mathbf{x}). \tag{2.1}$$

More generally, we are interested in a system for which the kinetic part of the Hamiltonian has the form $g^{kj}(\mathbf{x})p_kp_j$, where $g^{kj}(\mathbf{x})$ is an inverse metric of hyperbolic signature. Most minisuperspace models in quantum cosmology have a Hamiltonian of this form. However, the focus of this paper is the timelessness of the system, and the form of the configuration space metric turns out to be unimportant. So for simplicity, we will concentrate on the form Eq. (2.1).

We assume that there is a classical phase-space distribution function $w(\mathbf{p},\mathbf{x})$, which is normalized according to

$$\int d^np\,d^nx\,w(\mathbf{p},\mathbf{x})=1 \tag{2.2}$$

and obeys the evolution equation

$$\frac{\partial w}{\partial t}=\sum_k\left(-\frac{p_k}{M}\frac{\partial w}{\partial x_k}+\frac{\partial V}{\partial x_k}\frac{\partial w}{\partial p_k}\right)=\{H,w\}, \tag{2.3}$$

where $\{\}$ denotes the Poisson brackets. The interesting case is that in which w is the classical analogue of an energy eigenstate, in which case $\partial w/\partial t=0$, so the evolution equation is simply

$$\{H,w\}=0. \tag{2.4}$$

It follows that

$$w(\mathbf{p}^{\rm cl}(t),\mathbf{x}^{\rm cl}(t))=w(\mathbf{p}(0),\mathbf{x}(0)), \tag{2.5}$$

where $\mathbf{p}^{\rm cl}(t),\mathbf{x}^{\rm cl}(t)$ are the classical solutions with initial data $\mathbf{p}(0),\mathbf{x}(0)$, so w is constant along the classical orbits. (The normalization of w then becomes an issue if the classical orbits are infinite, but we will return to this in the quantum case discussed below.)

Given a set of classical solutions $(\mathbf{p}^{\rm cl}(t),\mathbf{x}^{\rm cl}(t))$ and a phase-space distribution function w, we are interested in the probability that a classical solution will pass through a region

Δ of configuration space. We construct this as follows. First of all, we introduce the characteristic function of the region Δ,

$$f_\Delta(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Delta \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

To see whether the classical trajectory $\mathbf{x}^{\rm cl}(t)$ intersects this region, consider the phase-space function

$$A(\mathbf{x},\mathbf{p}_0,\mathbf{x}_0) = \int_{-\infty}^{\infty} dt\ \delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t)) \tag{2.7}$$

(in the case of periodic classical orbits, the range of t is taken to be equal to the period). This function is positive for points $\mathbf{x}$ on the classical trajectory labeled by $\mathbf{p}_0,\mathbf{x}_0$ and zero otherwise. Hence intersection of the classical trajectory with the region Δ means

$$\int d^n\mathbf{x}\, f_\Delta(\mathbf{x}) \int_{-\infty}^{\infty} dt\ \delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t)) > 0, \tag{2.8}$$

or, equivalently, that

$$\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{x}(t)) > 0. \tag{2.9}$$

This quantity is essentially the amount of parameter time the trajectory spends in the region Δ. We may now write down the probability for a classical trajectory entering the region Δ. It is

$$p_\Delta = \int d^n p_0\ d^n x_0\ w(\mathbf{p}_0,\mathbf{x}_0)\,\theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{x}^{\rm cl}(t)) - \epsilon \right). \tag{2.10}$$

In this construction, ϵ is a small positive number that is eventually set to zero, and is included to avoid possible ambiguities in the θ function at zero argument. The θ function ensures that the phase-space integral is over all initial data whose corresponding classical trajectories spend a time greater than ϵ in the region Δ.

The classical solution $\mathbf{x}^{\rm cl}(t)$ depends on some fiducial initial coordinates and momenta, $\mathbf{x}_0$ and $\mathbf{p}_0$, say. In the case of a free particle, for example,

$$\mathbf{x}^{\rm cl}(t) = \mathbf{x}_0 + \frac{\mathbf{p}_0 t}{M}. \tag{2.11}$$

The construction is independent of the choice of fiducial initial points. If we shift $\mathbf{x}_0,\mathbf{p}_0$ along the classical trajectories, the measure, phase-space distribution function w, and the θ function are all invariant. Hence the integral over $\mathbf{x}_0,\mathbf{p}_0$ is effectively a sum over classical trajectories. The shift along the classical trajectories may also be thought of as a reparametrization, and the quantity (2.10) is in fact a reparametrization-invariant expression of the notion of a classical trajectory. This means that the probability (2.10) has the form of a phase-space overlap of the ''state'' with a reparametrization-invariant operator.

It is useful also to write this result in a different form, which will be more relevant to the results we get in the quantum theory case. In the quantum theory, we generally deal with propagation between fixed points in configuration space, rather than with the phase-space point. Therefore, in the free-particle case, consider the change of variables from $\mathbf{x}_0,\mathbf{p}_0$ to $\mathbf{x}_0,\mathbf{x}_f$, where

$$\mathbf{x}_f = \mathbf{x}_0 + \frac{\mathbf{p}_0}{M}\tau. \tag{2.12}$$

Hence $\mathbf{x}_f$ is the position after evolution for starting from $\mathbf{x}_0$ for parameter time τ. The probability then becomes

$$p_\Delta = \frac{M}{\tau} \int d^n x_f\ d^n x_0\ w(\mathbf{p}_0,\mathbf{x}_0)\,\theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{x}_0^f(t)) - \epsilon \right), \tag{2.13}$$

where $\mathbf{p}_0 = M(\mathbf{x}_f-\mathbf{x}_0)/\tau$ and

$$\mathbf{x}_0^f(t) = \mathbf{x}_0 + \frac{(\mathbf{x}_f-\mathbf{x}_0)}{\tau} t. \tag{2.14}$$

The parameter τ may in fact be scaled out of the whole expression, hence the probability is independent of it.

The result now has the form of an integral over ''initial'' and ''final'' points, analogous to similar results in quantum theory. The result is again essentially a sum over classical trajectories with the trajectories now labeled by any pair of points $\mathbf{x}_0,\mathbf{x}_f$ along the trajectories, and is invariant under shifting $\mathbf{x}_0$ or $\mathbf{x}_f$ along those trajectories. Naively, one might have thought that the restriction to paths that pass through Δ is imposed by summing over all finite length classical paths which intersect Δ as they go from the ''initial'' point $\mathbf{x}_0$ to the ''final'' point $\mathbf{x}_f$, that is, Δ lies *between* the initial and final points. This is also what one might naively expect in the quantum theory version. However, one can see from the above construction that the correct answer is in fact to sum over *all* classical paths (which can be of infinite length) passing through $\mathbf{x}_0$ and $\mathbf{x}_f$ that intersect Δ at *any* point along the entire trajectory, even if Δ does not lie between the two points (see Fig. 1). This feature is related to the reparametrization invariance of the system.

The above point turns out to be quite crucial to what follows in the rest of this paper, so it is worth saying it in an alternative form. Loosely speaking, the statement is that only the entire classical path respects the reparametrization invariance associated with the constraint equation. A section of the classical path does not. This may be expressed more precisely in terms of the function $A(\mathbf{x},\mathbf{p}_0,\mathbf{x}_0)$ introduced in Eq. (2.7). This function is concentrated on the entire classical trajectory, and is zero when $\mathbf{x}$ is not on the trajectory. It is easy to see that it has vanishing Poisson bracket with the Hamiltonian $H = H(\mathbf{p}_0,\mathbf{x}_0)$, since we have

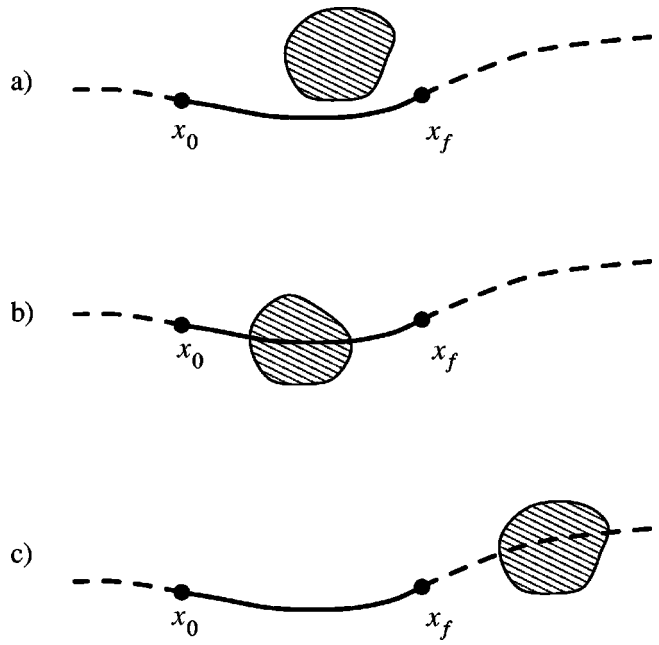

FIG. 1. The rewritten classical probability Eq. (2.13) in terms of a sum over initial and final points $\mathbf{x}_0$ and $\mathbf{x}_f$. The probability for not entering Δ is a sum over paths as in case (a). The probability for entering Δ includes a sum over classical paths in which Δ lies between the initial and final points, as in case (b). But, to agree with the phase-space form of the result Eq. (2.10), it must also include a sum over initial and final points for which Δ does not lie between them, as in case (c). This figure also applies to the semiclassical propagator Eq. (4.8) in the quantum case.

$$\{H,A(\mathbf{x},\mathbf{p}_0,\mathbf{x}_0)\}=\int_{-\infty}^{\infty}dt\{H,\delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t))\}$$

$$=-\int_{-\infty}^{\infty}dt\,\frac{d}{dt}\,\delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t))$$

$$=0. \tag{2.15}$$

This is the precise sense in which the entire trajectory is reparametrization-invariant, and the phase-space function A may be regarded as an observable—a quantity which commutes with the constraint H [16,17]. By way of comparison, consider a second phase-space function similarly defined, but on only a finite section of trajectory,

$$B(\mathbf{x},\mathbf{p}_0,\mathbf{x}_0)=\int_0^{\tau}dt\;\delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t)). \tag{2.16}$$

It is easily seen that

$$\{H,B(\mathbf{x},\mathbf{p}_0,\mathbf{x}_0)\}=-\delta(\mathbf{x}-\mathbf{x}^{\rm cl}(\tau))+\delta(\mathbf{x}-\mathbf{x}^{\rm cl}(0)). \tag{2.17}$$

Hence B ''almost'' commutes with H, failing only at the end points, and it is in this sense that a finite section of trajectory does not fully respect reparametrization invariance.

A third version of the classical result is also useful. It is of interest to obtain an expression for the probability for intersecting an $(n-1)$-dimensional surface Σ. Since the result (2.10) involves the parameter time spent in a finite volume region Δ it does not apply immediately. However, suppose that the set of trajectories contained in the probability distribution w intersect the $(n-1)$-dimensional surface Σ only once. Then we may consider a finite volume region Δ obtained by thickening Σ along the direction of the classical flow. If this thickening is by a small (positive) parameter time Δt, then the quantity appearing in the θ function in Eq. (2.10) is

$$\int dt\int_{\Delta}d^nx\;\delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t))$$

$$=\Delta t\int dt\int_{\Sigma}d^{n-1}x\;\mathbf{n}\cdot\frac{d\mathbf{x}^{\rm cl}(t)}{dt}\,\delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t))$$

$$=\Delta t I[\Sigma,\mathbf{x}^{\rm cl}(t)], \tag{2.18}$$

where $\mathbf{n}$ is the normal to Σ, and we suppose that the normal is chosen so that $\mathbf{n}\cdot d\mathbf{x}^{\rm cl}/dt$ is positive. The quantity $I[\Sigma,\mathbf{x}^{\rm cl}(t)]$, in a more general context, is the intersection number of the curve $\mathbf{x}^{\rm cl}(t)$ with the surface Σ, and takes the value 0 for no intersections, or ±1 (depending on whether there is an even or odd number of intersections). In this case we have assumed that the trajectories intersect at most once, hence $I=0$ or 1. We then have

$$\theta(\Delta t I-\epsilon)=\theta(I-\epsilon')=I \tag{2.19}$$

(where $\epsilon=\Delta t\epsilon'$) and the probability for intersecting Σ may be written

$$p_\Sigma=\int dt\int d^np_0\;d^nx_0\;w(\mathbf{p}_0,\mathbf{x}_0)$$

$$\times\int_{\Sigma}d^{n-1}x\;\mathbf{n}\cdot\frac{d\mathbf{x}^{\rm cl}(t)}{dt}\,\delta^{(n)}(\mathbf{x}-\mathbf{x}^{\rm cl}(t)). \tag{2.20}$$

At each t, we may perform a change of variables from $\mathbf{p},\mathbf{x}$ to new variables $\mathbf{p}'=\mathbf{p}^{\rm cl}(t)$, $\mathbf{x}'=\mathbf{x}^{\rm cl}(t)$, and using Eq. (2.5), we obtain the result

$$p_\Sigma=\frac{1}{M}\int dt\int_{\Sigma}d^np'\;d^{n-1}x'\;\mathbf{n}\cdot\mathbf{p}'w(\mathbf{p}',\mathbf{x}'). \tag{2.21}$$

Finally, the integrand is now in fact independent of t, so the t integral leads to an overall factor. (This might be infinite but is regularized as discussed below.) We therefore drop the t integral.

This result is relevant for the following reason. In the heuristic ''WKB interpretation'' of quantum cosmology, one considers WKB solutions to the Wheeler-DeWitt equation of the form

$$\Psi=Ce^{iS}. \tag{2.22}$$

It is usually asserted that this corresponds to a set of classical trajectories with momentum $\mathbf{p}=\nabla S$, and with a probability of intersecting a surface Σ given in terms of the flux of the

wave function across the surface [6,9]. As we shall show, from the decoherent histories analysis, the quantum theory gives a probability for crossing a surface Σ proportional to Eq. (2.21) with w replaced by the Wigner function of the quantum theory. The Wigner function of the WKB wave function is, approximately [32],

$$W(\mathbf{p},\mathbf{x}) = |C(\mathbf{x})|^2 \delta(\mathbf{p} - \nabla S). \tag{2.23}$$

Inserting in Eq. (2.21), we therefore obtain, up to overall factors, the probability distribution

$$p_\Sigma = \int_\Sigma d^{n-1}x \; \mathbf{n}\cdot\nabla S |C(\mathbf{x})|^2. \tag{2.24}$$

We therefore have agreement with the usual heuristic analysis.

III. THE QUANTUM CASE

A. Decoherent histories approach

The decoherent histories approach to quantum theory is described at length elsewhere [10–12], so only the briefest review will be given here. The central object of interest is the decoherence functional,

$$D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}(C_{\underline{\alpha}} \rho C^\dagger_{\underline{\alpha}'}), \tag{3.1}$$

where the histories are characterized by the class operators $C_{\underline{\alpha}}$, which satisfy

$$\sum_{\underline{\alpha}} C_{\underline{\alpha}} = 1 \tag{3.2}$$

and therefore

$$\sum_{\underline{\alpha},\underline{\alpha}'} D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}\; \rho = 1. \tag{3.3}$$

In nonrelativistic quantum mechanics, the class operators are given by time-ordered sequences of projection operators

$$C_{\underline{\alpha}} = P_{\alpha_n}(t_n)\cdots P_{\alpha_1}(t_1) \tag{3.4}$$

(and by sums of terms of this form), where $\underline{\alpha}$ denotes the string of alternatives α_1, $\alpha_2 \cdots \alpha_n$. The theory is, however, more general than this and we will exploit this generality here.

Intuitively, the decoherence functional is a measure of the interference between pairs of histories $\underline{\alpha}$, $\underline{\alpha}'$. When its real part is zero for $\underline{\alpha} \neq \underline{\alpha}'$, we say that the histories are consistent and probabilities

$$p(\underline{\alpha}) = D(\underline{\alpha},\underline{\alpha}) \tag{3.5}$$

obeying the usual probability sum rules may be assigned to them. Typical physical mechanisms which produce this situation usually cause both the real and imaginary part of $D(\underline{\alpha},\underline{\alpha}')$ to vanish. This condition is usually called decoherence of histories, and is related to the existence of so-called generalized records [10,33].

In the nonrelativistic case, for histories characterized by projections onto configuration space, a path integral version of the decoherence functional is available, and can be very useful. It has the form

$$D(\underline{\alpha},\underline{\alpha}') = \int_{\underline{\alpha}} \mathcal{D}x \int_{\underline{\alpha}'} \mathcal{D}y \exp\left(\frac{i}{\hbar} S[x(t)] - \frac{i}{\hbar} S[y(t)] \right) \times \rho(x_0, y_0), \tag{3.6}$$

where the sum is over pairs of paths $x(t), y(t)$ passing through the pairs of regions $\underline{\alpha}$, $\underline{\alpha}'$. This is equivalent to the form (3.1) and (3.4), when the histories are strings of projections onto ranges of positions. Equation (3.6) is a useful starting point for the generalization to timeless theories.

We would like to construct the decoherence functional for the situation in question, in which we have a system in an energy eigenstate, and we ask questions which do not refer in any way to time. Two new issues arise in this case. The first concerns the inner product through which the various pieces of the decoherence functional are put together. The second concerns the construction of the class operators. We take each in turn.

B. The induced inner product

For many situations, and especially for the analogous situation in quantum cosmology, the Hamiltonian has a continuous spectrum so the energy eigenstates are not normalizable in the usual inner product,

$$\langle \Psi_1 | \Psi_2 \rangle = \int d^n x \; \Psi_1^*(\mathbf{x}) \Psi_2(\mathbf{x}). \tag{3.7}$$

A way to deal with this has been developed, and goes by the name of the induced inner product, or Rieffel induction [34,22]. Consider the eigenvalue equation

$$H|\Psi_{E\lambda}\rangle = E|\Psi_{E\lambda}\rangle, \tag{3.8}$$

where λ denotes the degeneracy. These eigenstates will typically satisfy

$$\langle \Psi_{E'\lambda'} | \Psi_{E\lambda} \rangle = \delta(E - E')\, \delta_{\lambda\lambda'} \tag{3.9}$$

from which it is clear that the inner product diverges when $E = E'$. The induced inner product on a set of eigenstates of fixed E is defined, loosely speaking, by discarding the δ function $\delta(E - E')$. That is, the induced or physical inner product is then defined by

$$\langle \Psi_{E\lambda'} | \Psi_{E\lambda} \rangle_{\mathrm{phys}} = \delta_{\lambda\lambda'}. \tag{3.10}$$

This procedure can be defined quite rigorously, and has been discussed at some length in Refs. [34,22]. We will use it here to construct the decoherence functional. A simple prescription for using it in the decoherence functional is to regularize each propagator and energy eigenstate by using a different energy for each. The final answer will then involve a number of δ functions in energy, as in Eq. (3.9), which are simply dropped.

The induced inner product normalization for the wave functions does in fact suggest a normalization scheme for the corresponding classical phase-space distribution function, which, recall, is not normalizable in the case in which the classical trajectories are infinite (since w is constant along those trajectories). The idea is to consider the normalization of the Wigner function in the quantum case. This is defined by

$$W(\mathbf{p},\mathbf{X})=\frac{1}{(2\pi)^n}\int d^n v\, e^{-i\mathbf{p}\cdot\mathbf{v}}\rho(\mathbf{X}+\tfrac{1}{2}\mathbf{v},\mathbf{X}-\tfrac{1}{2}\mathbf{v}) \tag{3.11}$$

with inverse

$$\rho(\mathbf{x},\mathbf{y})=\int d^n p\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}W\left(\mathbf{p},\frac{\mathbf{x}+\mathbf{y}}{2}\right) \tag{3.12}$$

(see Refs. [35,36] for properties of the Wigner function). For an energy eigenstate $|\Psi_E\rangle$, we first of all construct a regularized density operator,

$$\rho_{EE'}=|\Psi_E\rangle\langle\Psi_{E'}| \tag{3.13}$$

which is normalized by

$$\mathrm{Tr}(\rho_{EE'})=\delta(E-E') \tag{3.14}$$

hence the corresponding Wigner function is normalized according to

$$\int d^n p\, d^n x\, W_{EE'}(\mathbf{p},\mathbf{x})=\delta(E-E'). \tag{3.15}$$

Now notice that the density operator obeys the equation

$$[H,\rho_{EE'}]=(E-E')\rho_{EE'}. \tag{3.16}$$

Taking the Wigner transform of this equation, we obtain

$$\mathcal{L}W_{EE'}=i(E-E')W_{EE'}, \tag{3.17}$$

where $\mathcal{L}$ is the phase-space operator

$$\mathcal{L}=\sum_k\left(-\frac{p_k}{M}\frac{\partial}{\partial x_k}+\frac{\partial V}{\partial x_k}\frac{\partial}{\partial p_k}\right)+\mathcal{L}_q. \tag{3.18}$$

It is a sum of the classical Liouville operator, plus a term $\mathcal{L}_q$ describing quantum modifications.

We may now see how to normalize the classical case. We take the classical distribution function to be described by Eq. (3.17) with the quantum term $\mathcal{L}_q$ set to zero. The Liouville operator may then be written

$$\mathcal{L}=-\frac{d}{ds} \tag{3.19}$$

for some parameter s, and Eq. (3.17) may be solved, with the result

$$w_{EE'}=e^{-is(E-E')}w_{EE}. \tag{3.20}$$

The exponential factor now effectively regularizes the phase-space distribution function. w_{EE} is constant along the classical trajectories, but $w_{EE'}$ is not, and, in the normalization (3.15), the part of the integral along the trajectories is an integral over s which produces the δ function.

C. Construction of the decoherence functional

The next part of the construction of the decoherence functional is the class operators, C_α. These are to describe histories of fixed energy which do or do not pass through the region Δ without regard to time, denoted $\alpha=\Delta$ and $\alpha=\bar{\Delta}$, respectively. We follow Refs. [21,22], in part.

Consider first the amplitude to go from $\mathbf{x}_0$ at time $t=0$ to $\mathbf{x}_f$ at time $t=\tau$ passing through the region Δ, or not, at any time in between. This is given by

$$g_\alpha(\mathbf{x}_f,\tau|\mathbf{x},0)=\int_\alpha \mathcal{D}\mathbf{x}(t)\exp\left(\frac{i}{\hbar}S[\mathbf{x}(t)]\right), \tag{3.21}$$

where the sum is over all paths $x(t)$ with t in the range $[0,\tau]$ which pass through Δ, or never pass through Δ. It therefore satisfies

$$\sum_\alpha g_\alpha=g_\Delta+g_{\bar{\Delta}}=g, \tag{3.22}$$

where $g=g(\mathbf{x}_f,\tau|\mathbf{x}_0,0)$ is the unrestricted propagator. There are many ways of constructing this sort of object more explicitly (see Refs. [23,37,38] for example), but here it is useful to exploit the construction used in the classical case. The amplitude to pass through Δ is therefore given by

$$g_\Delta(\mathbf{x}_f,\tau|\mathbf{x}_0,0)=\int \mathcal{D}\mathbf{x}(t)\exp(iS[\mathbf{x}(t)])\times\theta\left(\int_0^\tau dt\, f_\Delta(\mathbf{x}(t))-\epsilon\right). \tag{3.23}$$

Here, the θ function ensures that only paths $x(t)$ that spend a time in excess of ϵ in Δ contribute to the sum.

The class operator C_α is a propagator at fixed energy, E, say, so this is given by

$$\langle\mathbf{x}_f|C_\alpha|\mathbf{x}_0\rangle=\int_{-\infty}^{\infty}d\tau\, e^{-iE\tau}g_\alpha(\mathbf{x}_f,\tau|\mathbf{x}_0,0). \tag{3.24}$$

When g_α is replaced with an unrestricted propagator, we require that Eq. (3.24) is annihilated by $H-E$, and this is why we choose an infinite range for τ, rather than a half-infinite one [39–41]. (As we shall see below, when g_α is a restricted propagator, we encounter some difficulties here, although the correct range for τ is still the infinite one.) The total (regularized) decoherence functional is therefore given by

$$D(\alpha,\alpha')=\int d^n x_f\, d^n x_0\, d^n x_0' \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau'\; e^{-iE\tau} e^{iE'\tau'} \times g_\alpha(\mathbf{x}_f,\tau|\mathbf{x}_0,0) g^*_{\alpha'}(\mathbf{x}_f,\tau'|\mathbf{x}_0',0) \times \Psi_{E_0}(\mathbf{x}_0)\Psi^*_{E_0'}(\mathbf{x}_0'). \tag{3.25}$$

This may also be written as

$$D(\alpha,\alpha')=\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau'\; e^{-iE\tau} e^{iE'\tau'} \times \int_{\alpha} \mathcal{D}\mathbf{x}(t) \int_{\alpha'} \mathcal{D}\mathbf{x}'(t) \exp\{iS_0^{\tau}[\mathbf{x}(t)] - iS_0^{\tau'}[\mathbf{x}'(t)]\} \Psi_{E_0}(\mathbf{x}_0)\Psi^*_{E_0'}(\mathbf{x}_0') \tag{3.26}$$

where note that the two actions in the path integral are over different ranges of time. It is straightforward to show that

$$\sum_{\alpha,\alpha'} D(\alpha,\alpha') = \delta(E-E_0)\,\delta(E'-E_0')\,\delta(E_0-E_0'). \tag{3.27}$$

In the induced inner product scheme, we therefore replace the right-hand side by 1, verifying that the construction is correctly normalized.

D. Modified class operators

The basic scheme described above runs into an interesting difficulty in that the class operators defined by Eq. (3.24) do not satisfy the constraint equation. We have, for example, for the class operator for paths that enter the region Δ,

$$C_\Delta(\mathbf{x}_f,\mathbf{x}_0)=\int_{-\infty}^{\infty} d\tau\, e^{-iE\tau} \int \mathcal{D}\mathbf{x}(t) \exp(iS[\mathbf{x}(t)]) \times \theta\left(\int_0^{\tau} dt\, f_\Delta(\mathbf{x}(t)) - \epsilon \right). \tag{3.28}$$

It may be shown that this satisfies the constraint everywhere except on the boundaries of the region Δ. That is, there is a discontinuity as one of the end points crosses the boundary of Δ. This is an issue because a basic rule of the game of constructing the decoherence functional for these systems is that we only work with objects which satisfy the constraint equation (or operators which commute with it). Mathematically, this is to ensure that the underlying symmetry, reparametrization invariance, is fully respected (the induced inner product, note, is defined only between objects which satisfy the constraint). Physically, it is related to the fact that the universe is a closed system, and measurements of it (the class operators are generalizations of the notions of measurement) must not displace its wave function.

Because of this difficulty, it is necessary to define a modified class operator C'_Δ which is as close as possible to the path integral one, but satisfies the constraint equation everywhere [22]. A way to do this is to first compute the class operator C_Δ when one of the end points is inside Δ. This defines a solution to the constraints for one end point inside and the other outside. We then *define* C'_Δ to be the object which satisfies the constraint everywhere and matches this expression for one end point inside and the other outside.

An example of this difficulty was encountered and solved in Ref. [23], which concerned the decoherent histories analysis of the relativistic particle. Suppose one considers the class operator for propagating between spacetime points x_0^μ and x_f^μ, with the restriction that the paths never cross the spacelike surface $x^0=$const. This object, denoted $C_r(x_f,x_0)$, is readily constructed using the free propagator together with the method of images. However, it does not satisfy the constraint everywhere: the constraint operating on C_r gives a δ function on the spacelike surface. One can also see physically why there might be problems here. Suppose for simplicity that x_f and x_0 are timelike separated and consider the trajectory of a classical particle passing through these two points. Such a trajectory, if extended beyond these points, must cross *every* surface of constant x^0. Similarly, in the quantum theory, there are no nontrivial solutions to the Klein-Gordon equation that are zero on one side of a spacelike surface. Hence the only sensible possible answer for the class operator for this situation is $C_r=0$. This is indeed the solution used in Ref. [23] and led to physically expected results.

This issue concerning the replacement of the original class operators with modified ones is related to the fact that in the original class operator (3.28), the functions restricting the paths to enter the region Δ involve a time integral over a finite interval $[0,\ \tau]$, whereas the classical result (2.13) involves a similar restricting function but with a time integral over an infinite range. If a finite range is used in the classical result, it is no longer reparametrization-invariant, as we saw in Sec. II. This is why we discussed the classical result at such length.

The construction of these modified class operators is crucial to the construction of the decoherence functional for reparametrization-invariant theories. It is probably fair to say that the method suggested above for constructing them has not at this stage been fully explored. We will show below that a physically plausible modified class operator is readily constructed in the semiclassical approximation, but a more thorough investigation of this issue will be deferred to a later publication. It should also be noted that reparametrization invariance can be rather subtle. For example, the path-integral constructed class operators Eq. (3.28) certainly *appear* to be reparametrization-invariant at the level of symmetry transformations at the Lagrangian level, yet do not quite satisfy the constraint. Here, we have used the expression ''reparametrization-invariant'' to mean satisfying the constraint everywhere (or having a zero Poisson bracket with the Hamiltonian everywhere, in the classical case). This issue is related to the connections between Lagrangian and Hamiltonian symmetries, and between the path integral and Dirac quantizations. See Refs. [21,41] for further discussion.

IV. THE SEMICLASSICAL APPROXIMATION TO THE DECOHERENCE FUNCTIONAL AND PROBABILITIES

We may now compute the decoherence functional. It is

$$D(\alpha,\alpha') = \int d^n x_f\, d^n x_0\, d^n y_0\; C'_\alpha(\mathbf{x}_f,\mathbf{x}_0) \times C'^{*}_{\alpha}(\mathbf{x}_f,\mathbf{y}_0)\Psi(\mathbf{x}_0)\Psi^*(\mathbf{y}_0). \tag{4.1}$$

(For convenience, here and in what follows we will in fact drop the notation involving different values of E to regularize the expressions, unless necessary.) To see how the semiclassical approximation works out, in this section we will *assume* decoherence (for example, by restricting to special initial states) and concentrate on the construction of the probabilities that the system passes through the region Δ. We will return to the question of decoherence for general initial states in Sec. VI.

We begin by computing the semiclassical approximation to the modified class operator $C'_\Delta(\mathbf{x}_f,\mathbf{x}_0)$. Recall that it is given by a suitable modification of the path integral expression (3.28) with one end point inside Δ and the other outside. In the absence of any restrictions on the paths, the path integral will be dominated by the classical paths connecting the initial and final points. The classical paths will be the solutions to the equations of motion

$$M\ddot{\mathbf{x}} + \boldsymbol{\nabla} V(\mathbf{x}) = \mathbf{0} \tag{4.2}$$

which satisfy the boundary conditions

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(\tau) = \mathbf{x}_f. \tag{4.3}$$

In addition, these paths must satisfy the constraint equation

$$\tfrac{1}{2} M\dot{\mathbf{x}}^2 + V(\mathbf{x}) = E. \tag{4.4}$$

This equation determines the time τ in terms of $\mathbf{x}_0$, $\mathbf{x}_f$, and E, hence the final form of the extremizing paths has no reference to time. It is also useful to introduce $A(\mathbf{x}_f,\mathbf{x}_0)$, the classical action from $\mathbf{x}_0$ to $\mathbf{x}_f$. It obeys the time-independent Hamilton-Jacobi equation with respect to each end point,

$$\frac{1}{2M}(\boldsymbol{\nabla} A)^2 + V = E. \tag{4.5}$$

The initial and final momenta are given by derivatives of the classical action,

$$\mathbf{p}_f = \boldsymbol{\nabla}_f A(\mathbf{x}_f,\mathbf{x}_0), \quad \mathbf{p}_0 = -\boldsymbol{\nabla}_0 A(\mathbf{x}_f,\mathbf{x}_0). \tag{4.6}$$

The semiclassical approximation to the unrestricted path integral is given by a sum of terms each of the form

$$G(\mathbf{x}_f,\mathbf{x}_0) = P(\mathbf{x}_f,\mathbf{x}_0) e^{iA(\mathbf{x}_f,\mathbf{x}_0)}. \tag{4.7}$$

The quantity P is a prefactor, whose specific form is lengthy to calculate, but will not in fact be required. [For the case of the time-dependent propagator, the prefactor would be given in terms of the determinant of the matrix of second derivatives of $A(\mathbf{x}_f,\mathbf{x}_0)$. Here, because of the constraint equation (4.5), this matrix is in fact singular and the expression for the prefactor is more complicated [42,43].]

The semiclassical form (4.7) satisfies the constraint equation (i.e., is annihilated by $H-E$) in the WKB approximation, as it should. If the classical trajectory from $\mathbf{x}_0$ to $\mathbf{x}_f$ is unique, there will be just one term in the semiclassical approximation. If there is more than one, there will be a sum of similar terms, one for each trajectory. For the moment we will assume that there is just one. We will also assume that the extremizing classical solution is real, and thus the action $A(\mathbf{x}_f,\mathbf{x}_0)$ is real.

With the restriction that the paths must pass through Δ, we expect that the class operator will be given again by Eq. (4.7) when the classical path passes through Δ, and will be zero when the classical path does not pass through Δ. It is then not difficult to see that the modified class operator for this case may therefore be written

$$C'_\Delta(\mathbf{x}_f,\mathbf{x}_0) = \theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{x}_0^f(t)) - \epsilon \right) P(\mathbf{x}_f,\mathbf{x}_0) e^{iA(\mathbf{x}_f,\mathbf{x}_0)}. \tag{4.8}$$

The θ function here is the same as in the (rewritten) classical case Eq. (2.13) in terms of ''initial'' and ''final'' points, where $\mathbf{x}_0^f(t)$ denotes the classical path from $\mathbf{x}_0$ to $\mathbf{x}_f$. (This is exactly as in the classical case depicted in Fig. 1.) Note also that

$$\boldsymbol{\nabla} A \cdot \boldsymbol{\nabla}\, \theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{x}_0^f(t)) - \epsilon \right) = 0 \tag{4.9}$$

as may be shown by shifting the t integration. It follows that the modified class operator is a WKB solution to the constraint equation, as required. We have therefore succeeded in computing, in the semiclassical approximation, a modified class operator satisfying the constraint equation everywhere, corresponding to the restriction to paths passing through the region Δ. This is a very simple result but turns out to be crucial to the rest of the derivation.

It is important that t is integrated over an infinite range in the quantity inside the θ function, otherwise the modified class operator would not in fact satisfy the constraint. Recall that the originally defined class operator Eq. (3.24) contained a similar θ function, with a finite range of time integration, which one might have been tempted to use in the semiclassical approximation, but this class operator does not in fact satisfy the constraint.

Hence we see that the difference between the modified and original class operators in the semiclassical approximation is the difference between using the entire classical trajectory or using finite segments of it in the θ functions. We also see that these modified class operators are the correct ones to use in order to be consistent with the discussion of the classical case and Eq. (2.13). There, we saw that it is appropriate to sum over classical paths intersecting Δ even if Δ does not lie on the segment of classical trajectory *between*

$\mathbf{x}_0$ and $\mathbf{x}_f$. This feature therefore appears to be necessary for the particular type of reparametrization invariance used here. Only the entire trajectory is reparametrization-invariant notion. A finite section of trajectory is not. (See Ref. [41] for a further discussion of reparametrizations in this sort of context.)

The off-diagonal terms of the decoherence functional are now given in the semiclassical approximation by

$$D(\Delta,\overline{\Delta}) = \int d^n x_f\, d^n x_0\, d^n y_0\; \theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{x}_0^f(t)) - \epsilon\right)\left[1-\theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{y}_0^f(t)) - \epsilon\right)\right]$$
$$\times P(\mathbf{x}_f,\mathbf{x}_0)P^*(\mathbf{x}_f,\mathbf{y}_0)\exp[iA(\mathbf{x}_f,\mathbf{x}_0) - iA(\mathbf{x}_f,\mathbf{y}_0)]\rho(\mathbf{x}_0,\mathbf{y}_0). \tag{4.10}$$

It is now essentially a sum over pairs of classical paths, and the θ functions restrict the paths to either pass or not pass through the region Δ. We see also from the invariance property of the θ functions, Eq. (4.9), that they are invariant under shifting the regions Δ_α along their classical trajectories. This is the expression of the idea that the decoherence functional, in the semiclassical approximation, knows only about entire trajectories.

It is now convenient to introduce the variables

$$\mathbf{X}_0 = \tfrac{1}{2}(\mathbf{x}_0+\mathbf{y}_0), \quad \mathbf{v} = \mathbf{x}_0 - \mathbf{y}_0 \tag{4.11}$$

and thus

$$\mathbf{x}_0 = \mathbf{X}_0 + \tfrac{1}{2}\mathbf{v}, \quad \mathbf{y}_0 = \mathbf{X}_0 - \tfrac{1}{2}\mathbf{v} \tag{4.12}$$

and we also rewrite the density operator in terms of the Wigner function Eq. (3.11). We will discuss the detailed mechanism of decoherence in the next section. For the moment, we will simply assume decoherence, which essentially means assuming that $\mathbf{v}$ is concentrated around zero, and work out the form of the probabilities. Although we note that this assumption can be justified for initial states $\rho(\mathbf{x}_0,\mathbf{y}_0)$ which are approximately diagonal in position. We now set $\mathbf{v}=\mathbf{0}$ in the prefactors P and in the θ functions, and we obtain the probability

$$p_\Delta = \int d^n x_f\, d^n X_0 d^n v\;\; d^n p\; \theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{X}_0^f(t)) - \epsilon\right)|P(\mathbf{x}_f,\mathbf{X}_0)|^2 \exp[iA(\mathbf{x}_f,\mathbf{X}_0+\tfrac{1}{2}\mathbf{v}) - iA(\mathbf{x}_f,\mathbf{X}_0-\tfrac{1}{2}\mathbf{v}) + i\mathbf{p}\cdot\mathbf{v}]W(\mathbf{p},\mathbf{X}_0). \tag{4.13}$$

Expanding the action terms to linear order in $\mathbf{v}$ (decoherence again allows us to drop the higher-order terms), the $\mathbf{v}$ integral may be performed and we obtain

$$p_\Delta = \int d^n x_f d^n X_0 d^n p\; \theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{X}_0^f(t)) - \epsilon\right)|P(\mathbf{x}_f,\mathbf{X}_0)|^2 \delta^{(n)}(\mathbf{p}+\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0))W(\mathbf{p},\mathbf{X}_0), \tag{4.14}$$

where ∇_0 operates on the initial point $\mathbf{X}_0$. Finally, the integration over $\mathbf{x}_f$ may be performed. The δ-function constraint then means that the quantity $\mathbf{X}_0^f(t)$ (the classical path from $\mathbf{X}_0$ to $\mathbf{x}_f$) is replaced by $\mathbf{X}^{\rm cl}(t)$ (the classical path with initial data $\mathbf{X}_0$, $\mathbf{p}_0$). Although we have not worked out the explicit form of the prefactor P, we deduce that it must in fact drop out when the $\mathbf{x}_f$ integration is carried out, because the probability must equal 1 when the θ function is removed. [From this we deduce that $|P(\mathbf{x}_f,\mathbf{X}_0)|^2$ must be the Jacobian factor in the change of integration variables from $\mathbf{x}_f$ to $\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0)$.] We therefore obtain the final result

$$p_\Delta = \int d^n X_0\; d^n\, \theta\left(\int_{-\infty}^{\infty} dt\, f_\Delta(\mathbf{X}^{\rm cl}(t)) - \epsilon\right) W(\mathbf{p},\mathbf{X}_0). \tag{4.15}$$

As expected, this is the classical result Eq. (2.10) with the classical phase-space distribution function replaced by the Wigner function.

V. SYSTEMS OF HARMONIC OSCILLATORS

There is one simple system for which the discussion of decoherence and probabilities is particularly simple, and this is the case of a collection of harmonic oscillators. It also enjoys the property that its spectrum is discrete, hence the induced inner product is not required for normalization.

In nonrelativistic quantum mechanics, in the search for emergent quasiclassical histories, it is of interest to consider histories characterized by strings of phase-space quasiprojectors P_Γ. These are positive Hermitian operators concentrated on a region Γ of phase space, but are not quite projectors since they only have $P_\Gamma^2 \approx P_\Gamma$. Omnès has proved an important theorem about these projectors, which is essentially that they are approximately preserved in form under unitary evolution and moreover approximately follow classical evolution [12]. That is,

$$e^{-iHt} P_\Gamma e^{iHt} \approx P_{\Gamma_t}, \tag{5.1}$$

where Γ_t is the classical evolution of the phase-space cell Γ. The approximation holds when the phase-space cells are significantly larger than a quantum-sized cell, and for times not so long that wave-packet spreading becomes significant. In the special case of the harmonic oscillator, the approximation holds for all time. This result allows one to show that, first, histories of phase-space projectors are approximately decoherent for a wide variety of initial states, and second, that their probabilities are peaked about classical evolution. Differently put, on sufficiently coarse-grained scales, quantum systems have an approximate determinism that ensures decoherence and approximate correspondence with classical physics.

It seems reasonable to suppose that the timeless models considered here might have analogous properties. We will demonstrate this for a system of harmonic oscillators in an energy eigenstate. The Hamiltonian for a set of N identical harmonic oscillators is

$$H_0 = \tfrac{1}{2}(\mathbf{p}^2 + \mathbf{x}^2) - \tfrac{1}{2}N \tag{5.2}$$

(the factor of $\frac{1}{2}N$ is included to subtract the vacuum state energy and avoid certain phase factors). The standard coherent states (see Ref. [44] for example) are denoted $|\mathbf{p},\mathbf{x}\rangle$ and they have the important property that they are preserved in form under unitary evolution,

$$e^{-iH_0 t}|\mathbf{p},\mathbf{x}\rangle = |\mathbf{p}_t,\mathbf{x}_t\rangle, \tag{5.3}$$

where $\mathbf{p}_t$, $\mathbf{x}_t$ are the classical solutions matching $\mathbf{p}$, $\mathbf{x}$ at $t=0$, hence they are strongly peaked about the classical path. In Ref. [27], a set of states were introduced which are timeless analogues of the usual coherent states. They are

$$\begin{aligned} |\phi_{\mathbf{px}}\rangle &= \delta(H_0 - E)|\mathbf{p},\mathbf{x}\rangle \\ &= \int_0^{2\pi} \frac{dt}{2\pi} e^{-i(H_0-E)t}|\mathbf{p},\mathbf{x}\rangle \\ &= \int_0^{2\pi} \frac{dt}{2\pi} e^{iEt}|\mathbf{p}_t,\mathbf{x}_t\rangle. \end{aligned} \tag{5.4}$$

These states were referred to in Ref. [27] as ''timeless coherent states'' (see also [14,45]). They are exact eigenstates of the Hamiltonian,

$$H_0|\phi_{\mathbf{px}}\rangle = E|\phi_{\mathbf{px}}\rangle. \tag{5.5}$$

Furthermore, since the coherent states $|\mathbf{p}_t,\mathbf{x}_t\rangle$ are concentrated at a phase-space point for each t, integrating t over a whole period produces a state which is concentrated along the entire classical trajectory. They are therefore the natural analogues of the usual coherent states. Their properties are similar in many ways to the usual coherent states and are described in more detail in Ref. [27].

Each state is labeled by a fiducial phase-space point $\mathbf{p}$, $\mathbf{x}$ which determines the classical trajectory the state is peaked about. Under evolution of the fiducial point $\mathbf{p}$, $\mathbf{x}$ to another point, $\mathbf{p}_s$, $\mathbf{x}_s$, say, along the same classical trajectory, the state changes by a phase,

$$|\phi_{\mathbf{px}}\rangle \to |\phi_{\mathbf{p}_s\mathbf{x}_s}\rangle = e^{iEs}|\phi_{\mathbf{px}}\rangle. \tag{5.6}$$

Two timeless coherent states of different energy are exactly orthogonal. If they have the same energy, then they are approximately orthogonal if they correspond to sufficiently distinct classical solutions. They also obey a completeness relation,

$$\int \frac{d^N p\, d^N x}{(2\pi)^N} |\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}| = \delta(H_0 - E). \tag{5.7}$$

[Note that the notation $\delta(H_0-E)$ is a rather loose one. This object is really the projection operator onto the subspace of energy E, for which it is exactly true that $\delta(H_0-E)^2 = \delta(H_0-E)$.] Since $\delta(H_0-E)|\psi\rangle = |\psi\rangle$ on any solution to the eigenvalue equation $(H_0-E)|\psi\rangle = 0$, this is essentially a completeness relation on the set of solutions to the eigenvalue equation. We may therefore write any solution $|\psi\rangle$ as a superposition of timeless coherent states,

$$|\psi\rangle = \int \frac{d^N p\, d^N x}{(2\pi)^N} |\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|\psi\rangle. \tag{5.8}$$

Given these preliminaries, we may now discuss the decoherence functional. We will consider coarse grainings in which the paths in configuration space either pass or do not pass through a series of regions denoted $\Delta = \Delta_1, \Delta_2, \ldots$. We will take Δ_1, $\Delta_2, \ldots$ to lie along a classical path. Hence we need at least two such regions to fix a configuration space path.

The decoherence functional, in terms of the modified class operators, is, in the semiclassical approximation,

$$\begin{aligned} D(\Delta,\overline{\Delta}) = \int d^n x_f\, d^n x_0\, d^n y_0\; & C'_\Delta(\mathbf{x}_f,\mathbf{x}_0) \\ & \times C'^{*}_{\overline{\Delta}}(\mathbf{x}_f,\mathbf{y}_0)\Psi(\mathbf{x}_0)\Psi^*(\mathbf{y}_0). \end{aligned} \tag{5.9}$$

The modified class operator $C'_\Delta(\mathbf{x}_f,\mathbf{x}_0)$ is given by Eq. (4.8), so it is equal to the unrestricted semiclassical propagator G when $\mathbf{x}_f$ or $\mathbf{x}_0$ lie on the classical path specified by Δ and is zero otherwise. Also, $C'_{\overline{\Delta}} = \delta(H_0-E) - C'_\Delta$.

We first consider the case in which the initial state is a timeless coherent state, $|\phi_{\mathbf{px}}\rangle$. It is then straightforward to see that

$$C'_\Delta|\phi_{\mathbf{px}}\rangle \approx |\phi_{\mathbf{px}}\rangle \tag{5.10}$$

when the trajectory labeled by the fiducial points **p,x** passes through the regions Δ, and

$$C'_\Delta|\phi_{\mathbf{px}}\rangle \approx 0 \tag{5.11}$$

otherwise. Also, since

$$\delta(H_0-E)|\phi_{\mathbf{px}}\rangle = |\phi_{\mathbf{px}}\rangle, \tag{5.12}$$

it follows that

$$C'_{\overline{\Delta}}|\phi_{\mathbf{px}}\rangle \approx 0 \tag{5.13}$$

when the trajectory labeled by **p,x** passes through Δ. From these results it is easy to see that the decoherence functional is approximately diagonal. Furthermore, the probability for entering the regions Δ is then approximately 1 or 0, depending on whether the classical trajectory of the timeless coherent state passes through Δ.

Now consider the case of a more general initial state. We expand it in timeless coherent states, as in Eq. (5.8). Using the above results, we therefore find

$$C'_\Delta|\psi\rangle \approx \int_D \frac{d^N p\, d^N x}{(2\pi)^N} |\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|\psi\rangle. \tag{5.14}$$

Here, D denotes the set of phase-space points **p,x** whose classical trajectories pass through the regions Δ in configuration space. Similarly,

$$C'_{\bar{\Delta}}|\psi\rangle \approx \int_{\bar{D}} \frac{d^N p\, d^N x}{(2\pi)^N} |\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|\psi\rangle, \tag{5.15}$$

where $\bar{D}$ denotes the set of phase-space points whose classical trajectories never pass through Δ. Clearly we again have approximate decoherence, because of the approximate determinism. We may therefore assign a probability for passing through Δ,

$$p_\Delta \approx \int_D \frac{d^N p\, d^N x}{(2\pi)^N} |\langle\phi_{\mathbf{px}}|\psi\rangle|^2. \tag{5.16}$$

It is the integral over the phase-space region Δ of the phase-space distribution function $|\langle\phi_{\mathbf{px}}|\psi\rangle|^2$. Because $|\psi\rangle$ is an eigenstate of the Hamiltonian, it is easy to see using the definition (5.4) of the timeless coherent states that

$$\langle\phi_{\mathbf{px}}|\psi\rangle = \langle\mathbf{p},\mathbf{x}|\psi\rangle \tag{5.17}$$

and so the probability now is

$$p_\Delta \approx \int_D \frac{d^N p\, d^N x}{(2\pi)^N} |\langle\mathbf{p},\mathbf{x}|\psi\rangle|^2. \tag{5.18}$$

It is then a standard result that the integrand is in fact a smeared Wigner function,

$$|\langle\mathbf{px}|\psi\rangle|^2 = \int d^N p'\, d^N x' \times e^{-1/2(\mathbf{p}-\mathbf{p}')^2 - 1/2(\mathbf{x}-\mathbf{x}')^2} W(\mathbf{p}',\mathbf{x}'). \tag{5.19}$$

This object is positive even though the original Wigner function W of $|\psi\rangle$ is not [44]. Hence we obtain a result which is essentially identical to the classically anticipated result (2.10), with a smeared Wigner function as the phase-space distribution function.

It is also of interest to note that the result for the probability may be written in the form

$$p_\Delta = \langle\psi|P_D|\psi\rangle, \tag{5.20}$$

where P_D is the approximate projection operator

$$P_D = \int_D \frac{d^N\mathbf{p}\, d^N\mathbf{x}}{(2\pi)^N} |\phi_{\mathbf{px}}\rangle\langle\phi_{\mathbf{px}}|. \tag{5.21}$$

Moreover, since the timeless coherent states $|\phi_{\mathbf{px}}\rangle$ are exact eigenstates of H_0, we have that

$$[P_D, H_0] = 0. \tag{5.22}$$

Hence, we see that the result may be written in the standard quantum-mechanical form for a probability, in terms of an operator which commutes with the constraint.

The result here, of approximate decoherence and simple expressions for the probabilities, like the corresponding nonrelativistic result is due to the approximate determinism contained in the quantum theory. It works only when we ask for the probabilities for approximately classical histories. To obtain probabilities for more complicated histories, and for systems which are not harmonic oscillators (where there is wave packet spreading), we need an environment to produce decoherence.

VI. DECOHERENCE THROUGH AN ENVIRONMENT

As stated above, the decoherence functional is typically not diagonal for most initial states, and a physical mechanism is required to produced decoherence. In this section, we therefore consider the addition of an environment to produce decoherence of histories. The results of this section therefore simply justify the assumed decoherence of Sec. IV, and little affect the final result of the probabilities, but it is important to see in detail how this works.

A. Semiclassical approximation to the decoherence functional with environment

For what we will do here, the specific form of the environment turns out not to be very important. But for definiteness, we take the environment to be a large collection of harmonic oscillators with coordinates denoted $\mathbf{q}_A$, where A runs over a large number of values, with a linear coupling to the system. For notational simplicity, we will assume that for each system coordinate **x** in the n-dimensional configuration space there is a set of n oscillators with coordinate **q** for the environment. The case of more oscillators is easily obtained from this. The total action of the system is

$$S = S_0[\mathbf{x}] + S_{\mathcal{E}}[\mathbf{x},\mathbf{q}] \tag{6.1}$$

and the corresponding Hamiltonian is

$$H = H_0(\mathbf{x}) + H_{\mathcal{E}}(\mathbf{x},\mathbf{q}). \tag{6.2}$$

We shall assume that the state of the whole system has the form

$$\Psi(\mathbf{x},\mathbf{q}) = \psi(\mathbf{x})\chi(\mathbf{x},\mathbf{q}). \tag{6.3}$$

This may be inserted into the Wheeler-DeWitt equation $(H-E)\Psi = 0$ to obtain a perturbative solution about the solution with no environment. We will concentrate on the case in

which the wave function is of oscillatory form, so the background solution is of WKB form,

$$\psi(\mathbf{x}) = C(\mathbf{x}) e^{iS(\mathbf{x})}, \tag{6.4}$$

where S obeys the Hamilton-Jacobi equation

$$\tfrac{1}{2}(\boldsymbol{\nabla} S)^2 + V(\mathbf{x}) = E \tag{6.5}$$

and C obeys the equation

$$\boldsymbol{\nabla}^2 C + 2\boldsymbol{\nabla} S \cdot \boldsymbol{\nabla} C = 0. \tag{6.6}$$

The environment wave functions obey the Schrödinger equation

$$i\boldsymbol{\nabla} S \cdot \boldsymbol{\nabla} \chi = H_{\mathcal{E}} \chi. \tag{6.7}$$

We will consider the case of a superposition of WKB states in Sec. VII.

The decoherence functional is now

$$D(\alpha,\alpha') = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-iE\tau} e^{iE'\tau'} \times \int_{\alpha} \mathcal{D}\mathbf{x}(t) \int_{\alpha'} \mathcal{D}\mathbf{y}(t) \exp\{iS_0^{\tau}[\mathbf{x}(t)] - iS_0^{\tau'}[\mathbf{y}(t)]\} \times F[\mathbf{x}(t),\mathbf{y}(t),\tau,\tau'] \psi_{E_0}(\mathbf{x}_0) \psi^*_{E_0'}(\mathbf{y}_0) \tag{6.8}$$

(suspending for the moment the necessity to use modified class operators). Here S_0^{τ} denotes the action over the fixed time range $[0, \tau]$ (and note that the time ranges are different on either side of the decoherence functional). The influence functional $F[\mathbf{x}(t),\mathbf{y}(t),\tau,\tau']$ is given by

$$F[\mathbf{x}(t),\mathbf{y}(t),\tau,\tau'] = \int \mathcal{D}\mathbf{q}(t) \mathcal{D}\mathbf{r}(t) \exp(iS_{\mathcal{E}}^{\tau}[\mathbf{x},\mathbf{q}] - iS_{\mathcal{E}}^{\tau'}[\mathbf{y},\mathbf{r}]) \chi(\mathbf{x}_0,\mathbf{q}_0) \times \chi^*(\mathbf{y}_0,\mathbf{r}_0). \tag{6.9}$$

It is different in form from the usual influence functional in two ways. First, the time ranges on either side are not the same, and secondly, the initial state χ depends on the system variables $\mathbf{x}$, with a different dependence on either side of the influence functional. The functional integral is over all pairs of paths $\mathbf{q}(t),\mathbf{r}(t)$ which meet at the final point,

$$\mathbf{q}(\tau) = \mathbf{r}(\tau'), \tag{6.10}$$

and this point is summed over. The paths also match the initial values $\mathbf{q}_0,\mathbf{r}_0$, which are then folded into the initial state.

It is useful to go now to the semiclassical approximation for the system variables. We also now recall that we must use modified class operators for the system variables (this does not affect the environment dynamics at this level of approximation). The off-diagonal terms of the decoherence functional are now given by

$$D(\Delta,\overline{\Delta}) = \int d^n x_f \, d^n x_0 \, d^n y_0 \; \theta\left(\int_{-\infty}^{\infty} dt \, f_{\Delta}(\mathbf{x}_0^f(t)) - \epsilon \right) \times \left[1 - \theta\left(\int_{-\infty}^{\infty} dt \, f_{\Delta}(\mathbf{y}_0^f(t)) - \epsilon \right) \right] \times P(\mathbf{x}_f,\mathbf{x}_0) P^*(\mathbf{x}_f,\mathbf{y}_0) \times \exp[iA(\mathbf{x}_f,\mathbf{x}_0) - iA(\mathbf{x}_f,\mathbf{y}_0)] \times F(\mathbf{x}_f,\mathbf{x}_0,\mathbf{y}_0) \psi(\mathbf{x}_0) \psi^*(\mathbf{y}_0). \tag{6.11}$$

Here, $F(\mathbf{x}_f,\mathbf{x}_0,\mathbf{y}_0)$ is the influence functional with the semiclassical approximation for the system variables inserted. That is, for $\mathbf{x}(t)$ we insert the classical trajectory from $\mathbf{x}_0$ to $\mathbf{x}_f$ in time τ, and the value of τ is then determined (in terms of $\mathbf{x}_0$ and $\mathbf{x}_f$) by the constraint equation, and similarly for $\mathbf{y}(t)$. The decoherence functional is again essentially a sum over pairs of classical paths for the system variables, the path from $\mathbf{x}_0$ to $\mathbf{x}_f$, and the path from $\mathbf{y}_0$ to $\mathbf{x}_f$.

B. Calculation of the influence functional

We may now calculate the influence functional with the semiclassical approximation for system variables inserted. The influence functional may be written

$$F(\mathbf{x}_f,\mathbf{x}_0,\mathbf{y}_0) = \int d^n q_f \; \phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{x}_0) \times \phi^*(\mathbf{q}_f,\mathbf{x}_f,\mathbf{y}_0), \tag{6.12}$$

where

$$\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{x}_0) = \int d^n q_0 \; g(\mathbf{q}_f,\mathbf{x}_f,\mathbf{q}_0,\mathbf{x}_0) \chi(\mathbf{q}_0,\mathbf{x}_0). \tag{6.13}$$

Here, we have introduced the propagator for the environment variables along the system classical trajectory from $\mathbf{x}_0$ to $\mathbf{x}_f$,

$$g(\mathbf{q}_f,\mathbf{x}_f|\mathbf{q}_0,\mathbf{x}_0) = \int \mathcal{D}\mathbf{q}(t) \exp(iS_{\mathcal{E}}^{\tau}[\mathbf{x}(t),\mathbf{q}(t)]). \tag{6.14}$$

The environment state χ may be normalized according to

$$\int d^n q |\chi(\mathbf{q},\mathbf{x})|^2 = 1 \tag{6.15}$$

for all $\mathbf{x}$. Since g propagates χ unitarily along a fixed system trajectory, it follows that

$$\int d^n q_f |\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{x}_0)|^2 = 1. \tag{6.16}$$

This means that the influence functional satisfies

$$|F(\mathbf{x}_f,\mathbf{x}_0,\mathbf{y}_0)|^2 \leqslant 1 \tag{6.17}$$

with equality when $\mathbf{x}_0 = \mathbf{y}_0$, indicating that the influence functional is peaked about $\mathbf{x}_0 = \mathbf{y}_0$, which is the decoherence effect we need.

The influence functional is difficult to evaluate in general. It can be evaluated exactly when both g and χ are Gaussian. It is also effectively Gaussian in form if we have a large number of oscillators in the environment, for then the many-oscillator influence functional is essentially the original one, Eq. (6.9), raised to a high power. This strongly enhances the peaking about $\mathbf{x}_0=\mathbf{y}_0$. A simple and reasonably general form for the influence functional may therefore be obtained by expanding about $\mathbf{x}_0=\mathbf{y}_0$, and assuming either a Gaussian form or a large number of oscillators (or both).

We again use the coordinates $\mathbf{X}_0$, $\mathbf{v}$ defined in Eq. (4.11). We have

$$\begin{aligned}\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{x}_0)=&\,\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{X}_0)\\&+\tfrac{1}{2}v^a\partial_a\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{X}_0)\\&+\tfrac{1}{8}v^a v^b\partial_a\partial_b\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{X}_0)+\cdots,\end{aligned}\tag{6.18}$$

where $\partial_a=\partial/\partial X_0^a$. Inserting in the influence functional, and also introducing the notation $\phi_0=\phi(\mathbf{q}_f,\mathbf{x}_f,\mathbf{X}_0)$, we get

$$\begin{aligned}F=&\,1+\tfrac{1}{2}v^a\int d^nq_f(\phi_0^*\partial_a\phi_0-\phi_0\partial_a\phi_0^*)\\&+\tfrac{1}{8}v^av^b\int d^nq_f(\phi_0^*\partial_a\partial_b\phi_0+\phi_0\partial_a\partial_b\phi_0^*\\&-2\partial_a\phi_0\partial_b\phi_0^*)+\cdots.\end{aligned}\tag{6.19}$$

By differentiating the normalization of ϕ, Eq. (6.16), we see that

$$\int d^nq_f\ \phi_0\partial_a\phi_0^*=-\int d^nq_f\ \partial_a\phi_0\phi_0^*\tag{6.20}$$

and

$$\begin{aligned}&\int d^nq_f(\phi_0\partial_a\partial_b\phi_0^*+\phi_0^*\partial_a\partial_b\phi_0)\\&\quad=-\int d^nq_f(\partial_a\phi_0\partial_b\phi_0^*+\partial_b\phi_0\partial_a\phi_0^*).\end{aligned}\tag{6.21}$$

Using these relations, we may now write the influence functional as

$$\begin{aligned}F=&\,1+iv^a\Gamma_a-\tfrac{1}{2}v^av^b\Sigma_{ab}+\cdots\\&\approx\exp[iv^a\Gamma_a-\tfrac{1}{2}v^av^b(\Sigma_{ab}-\Gamma_a\Gamma_b)]+\cdots.\end{aligned}\tag{6.22}$$

The coefficients Γ_a and Σ_{ab} are given by

$$\begin{aligned}\Gamma_a(\mathbf{x}_f,\mathbf{X}_0)=&\,\frac{i}{2}\int d^nq_f(\phi_0\partial_a\phi_0^*-\phi_0^*\partial_a\phi_0),\\\Sigma_{ab}(\mathbf{x}_f,\mathbf{X}_0)=&\int d^nq_f\partial_a\phi_0\partial_b\phi_0^*.\end{aligned}\tag{6.23}$$

The approximation of writing F as a Gaussian becomes exact when g and χ are Gaussian, and is also true approximately when there are a large number of oscillators in the environment. We have therefore obtained the influence functional, as required.

C. Reparametrization invariance in the influence functional

The form of the influence functional indicates, as expected, that there is a suppression of interference for paths with $\mathbf{x}_0\neq\mathbf{y}_0$. This is the usual decoherence effect. However, the situation in the reparametrization-invariant theory considered here is not so simple. We expect that the decoherence functional depends, in some sense, only on reparametrization-invariant quantities, and in the semiclassical approximation used here, this means it depends on entire classical paths (rather than individual points). Differently put, we do not expect (or need) the destruction of interference for points $\mathbf{x}_0$, $\mathbf{y}_0$ lying on the *same* classical path (connecting each to $\mathbf{x}_f$), since these points are effectively equivalent. What we expect is that the influence functional will not be exponentially small when $\mathbf{x}_f$, $\mathbf{x}_0$, $\mathbf{y}_0$ lie along a single classical path. We must therefore see how reparametrization invariance is expressed in the influence functional.

We first note that

$$\phi_0=\int d^nq_0\ g(\mathbf{q}_f,\mathbf{x}_f|\mathbf{q}_0,\mathbf{X}_0)\chi(\mathbf{q}_0,\mathbf{X}_0).\tag{6.24}$$

Let us see how this quantity varies with $\mathbf{X}_0$. If this were the usual nonrelativistic quantum mechanics, with propagation from initial time t_0 to final time t_f, then ϕ_0 would in fact be independent of t_0. We expect a similar property here, that is, that ϕ_0 is constant (as a function of $\mathbf{X}_0$) along a certain vector field. The initial state $\chi(\mathbf{q}_0,\mathbf{X}_0)$ obeys the Schrödinger equation

$$i\boldsymbol{\nabla}_0S(\mathbf{X}_0)\cdot\boldsymbol{\nabla}_0\chi=H_\varepsilon(\mathbf{q}_0,\mathbf{x}_f,\mathbf{X}_0)\chi.\tag{6.25}$$

The propagator g, on the other hand, obeys Schrödinger equations with respect to both the final and initial points,

$$i\boldsymbol{\nabla}_fA(\mathbf{x}_f,\mathbf{X}_0)\cdot\boldsymbol{\nabla}_fg=H_\varepsilon(\mathbf{q}_f,\mathbf{x}_f,\mathbf{X}_f)g,\tag{6.26}$$

$$i\boldsymbol{\nabla}_0A(\mathbf{x}_f,\mathbf{X}_0)\cdot\boldsymbol{\nabla}_0g=H_\varepsilon(\mathbf{q}_0,\mathbf{x}_f,\mathbf{X}_0)g.\tag{6.27}$$

[Note that the expected minus sign in the Schrödinger equation with respect to the initial point is already contained through the fact that ∇_0A is *minus* the initial momentum, as in Eq. (4.6).] Now the point is here that g and χ obey different Schrödinger equations, so at this stage, ϕ_0 does not obviously have any constant directions in $\mathbf{X}_0$—neither $\boldsymbol{\nabla}_0S\cdot\boldsymbol{\nabla}_0\phi_0$ nor $\boldsymbol{\nabla}_0A\cdot\boldsymbol{\nabla}_0\phi_0$ are zero.

However, as we saw in Sec. IV (without environment), the path integral enforces the condition $\mathbf{p}=-\boldsymbol{\nabla}_0A$. We anticipate that this condition is approximately enforced with the environment in place. Furthermore, the initial Wigner function for a WKB state is of the approximate form

$$W(\mathbf{p},\mathbf{X}_0)=|C(\mathbf{X}_0)|^2\delta(p-\boldsymbol{\nabla}_0S(\mathbf{X}_0)).\tag{6.28}$$

It follows that the sum over paths is dominated by configurations for which

$$\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0)\approx -\nabla_0 S. \tag{6.29}$$

This means that the trajectories of S are the same as the classical trajectories from $\mathbf{X}_0$ to $\mathbf{x}_f$. From this it is then easy to show that

$$\nabla_0 S(\mathbf{X}_0)\cdot\nabla\phi_0\approx 0 \tag{6.30}$$

(essentially for the same reason that the analogous nonrelativistic version is independent of the initial time t_0). In the influence functional, two neighboring points $\mathbf{x}_0$, $\mathbf{y}_0$ on the same classical trajectory have $\mathbf{v}=\mathbf{x}_0-\mathbf{y}_0$ proportional to $\nabla_0 S$. It follows that

$$v^a\Gamma_a=0, \quad v^a\Sigma_{ab}=0, \tag{6.31}$$

which means that the influence functional does not suppress interference between points on the same trajectories, only between points on different trajectories. That is, when the condition $\mathbf{p}=-\nabla_0 A$ is true, we get the expected result that the influence functional is a function only of entire trajectories, and not of the individual points along those trajectories.

D. Decoherence and the evaluation of the v integral

The off-diagonal terms of the decoherence functional may now be written

$$D(\Delta,\overline{\Delta})=\int d^nx_f\, d^nX_0\, d^nv\, d^np\,\theta\left(\int_{-\infty}^{\infty}dt\, f_{\Delta}(\mathbf{x}_0^f(t))-\epsilon\right)\left[1-\theta\left(\int_{-\infty}^{\infty}dt\, f_{\Delta}(\mathbf{y}_0^f(t))-\epsilon\right)\right]P(\mathbf{x}_f,\mathbf{X}_0+\tfrac{1}{2}\mathbf{v})P^*(\mathbf{x}_f,\mathbf{X}_0-\tfrac{1}{2}\mathbf{v})$$
$$\times\exp\{i[\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0)+\mathbf{p}]\cdot\mathbf{v}+O(\mathbf{v}^3)\}W(\mathbf{p},\mathbf{X}_0)\exp(iv^a\Gamma_a-\tfrac{1}{2}v^av^b\sigma_{ab}), \tag{6.32}$$

where

$$\sigma_{ab}=\Sigma_{ab}-\Gamma_a\Gamma_b \tag{6.33}$$

and we have again introduced the variables $\mathbf{X}_0$, $\mathbf{v}$ in the exponential part. The decoherence functional is a sum over pairs of classical paths, one set of paths intersecting $\mathbf{x}_0$, $\mathbf{x}_f$ and passing through Δ at any stage along the path, the other set of paths intersecting $\mathbf{y}_0$, $\mathbf{x}_f$ and never passing through Δ at any stage along the path (see Fig. 1). It is easily seen that for this particular coarse graining, in which we are interested in paths that either pass or do not pass through the region Δ, we do not in fact encounter the situation discussed in Sec. VI C, in which $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{x}_f$ lie along the same classical trajectory. That is, the coarse graining is such that $\mathbf{v}$ is never proportional to $\nabla_0 A$, and the potentially singular situation (6.31) does not arise. The influence functional therefore does its job of suppressing the contribution from nonzero values of $\mathbf{v}$ to a decoherence width determined by the inverse of the nonzero eigenvalues of σ_{ab}. If the size of the coarse-graining region Δ is greater than this width, then the off-diagonal terms of the decoherence functional are approximately zero.

We therefore have approximate decoherence and we may examine the probability for passing through Δ, which is

$$p(\Delta)=\int d^nx_f\, d^nX_0\, d^nv\, d^np\;\theta\left(\int_{-\infty}^{\infty}dt\, f_{\Delta}(\mathbf{X}_0^f(t))-\epsilon\right)P(\mathbf{x}_f,\mathbf{X}_0+\tfrac{1}{2}\mathbf{v})P^*(\mathbf{x}_f,\mathbf{X}_0-\tfrac{1}{2}\mathbf{v})\exp\{i[\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0)+\mathbf{p}]\cdot\mathbf{v}$$
$$+O(\mathbf{v}^3)\}W(\mathbf{p},\mathbf{X}_0)\exp(iv^a\Gamma_a-\tfrac{1}{2}v^av^b\sigma_{ab}), \tag{6.34}$$

where $\mathbf{v}$ has been set to zero in the θ function. We are now summing over pairs of classical paths which *both* pass through the region Δ, so now we do have the possibility of $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{x}_f$ lying along the same path, and hence the matrix σ_{ab} is potentially singular, by Eq. (6.31). This means that some care is necessary in the $\mathbf{v}$ integral.

If we formally carry out the integral over $\mathbf{v}$, we get

$$p_{\Delta}=\int d^nx_f\, d^nX_0\, d^np\;\theta\left(\int_{-\infty}^{\infty}dt\, f_{\Delta}(\mathbf{X}_0^f(t))-\epsilon\right)|P(\mathbf{x}_f,\mathbf{X}_0)|^2W(\mathbf{p},\mathbf{X}_0)$$
$$\times\exp\{-\tfrac{1}{2}[\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0)+\mathbf{p}+\Gamma]^a\sigma_{ab}^{-1}[\nabla_0 A(\mathbf{x}_f,\mathbf{X}_0)+\mathbf{p}+\Gamma]^b\}. \tag{6.35}$$

Changing integration variables from $\mathbf{x}_f$ to $\mathbf{p}_0 = -\boldsymbol{\nabla}_0 A(\mathbf{x}_f, \mathbf{X}_0)$. This is conveniently written

$$p_\Delta = \int d^n p_0 \; d^n X_0 \; \theta\left(\int_{-\infty}^{\infty} dt \, f_\Delta(\mathbf{X}^{\rm cl}(t)) - \boldsymbol{\epsilon} \right) \widetilde{W}(\mathbf{p}_0, \mathbf{X}_0), \tag{6.36}$$

where we have defined the smeared Wigner function

$$\widetilde{W}(\mathbf{p}_0, \mathbf{X}_0) = \int d^n p \, \exp[-\tfrac{1}{2} (\mathbf{p}_0 - \mathbf{p} - \Gamma)^a \times \sigma^{-1}_{ab} (\mathbf{p}_0 - \mathbf{p} - \Gamma)^b] W(\mathbf{p}, \mathbf{X}_0). \tag{6.37}$$

Again the prefactors P drop out in the change of integration variables, as in Sec. IV. This smearing of the Wigner function represents environmentally induced fluctuations about the classical evolution, and the small additional term Γ in the exponent represents the backreaction of the environment on the classical equations of motion.

As stated, the results (6.31) suggest that the matrix σ_{ab} is singular, but it is easy to see the significance of this. When the matrix is nonsingular, the **v** integral produces a Gaussian peak about $\mathbf{p}_0 = \mathbf{p} + \Gamma$, which, as we have seen, represents fluctuations about classical evolution. If the matrix is singular in a certain direction, it is easy to see from Eq. (6.34) that the **v** integral in this direction will produce a δ function, instead of a Gaussian. It will still be peaked about the same configuration, but there are no fluctuations in that direction.

The result for the probabilities therefore approximately coincide with the classical result, Eq. (2.10). We have concentrated on the case in which Δ is a single region of configuration space, but the result straightforwardly generalizes to the case in which Δ consists of a series of regions $\Delta_1, \Delta_2, \ldots$. The above result then shows that the probability is peaked when the series of regions lies along a classical path (plus the environmental effects of a small backreaction and small fluctuations).

VII. SUPERPOSITION STATES

The calculations of Secs. IV and VI concerned only single WKB states of the form (6.4). It is therefore important to reconsider the decoherence calculation of Sec. VI for the more general case of a superposition of WKB states,

$$\Psi = \Psi_1 + \Psi_2 = C_1 e^{iS_1} \chi_1 + C_2 e^{iS_2} \chi_2 . \tag{7.1}$$

This turns out in fact to be quite straightforward, mainly because similar calculations (involving the reduced density matrix, not the decoherence functional) have already been done.

Inserting Eq. (7.1) in the decoherence functional, we obtain a result of the form

$$D = D_{11} + D_{12} + D_{21} + D_{22}, \tag{7.2}$$

where, in an obvious notation, D_{11} is the decoherence functional with initial density matrix $|\Psi_1\rangle\langle\Psi_1|$, D_{12} is the decoherence functional with the operator $|\Psi_1\rangle\langle\Psi_2|$ is the initial state slot, and so on. Clearly the analysis of D_{11} and D_{22} is identical to the case considered already—we get decoherence and probabilities given in terms of the Wigner functions of Ψ_1 and Ψ_2. Hence these two terms correspond to a statistical mixture of the two initial states.

The interesting terms are D_{12} and D_{21} $(= D_{12}^*)$, which correspond to interferences between different WKB branches. From Sec. VI, we see that

$$D_{12}(\alpha, \alpha') = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \; e^{-iE\tau} e^{iE'\tau'} \int_{\alpha} \mathcal{D}\mathbf{x}(t) \int_{\alpha'} \mathcal{D}\mathbf{y}(t) \exp\{ iS_0^\tau[\mathbf{x}(t)] - iS_0^\tau[\mathbf{y}(t)] \} \times F_{12}[\mathbf{x}(t), \mathbf{y}(t), \tau, \tau'] C_1(\mathbf{x}_0) e^{iS_1(\mathbf{x}_0)} C_2^*(\mathbf{y}_0) e^{-iS_2(\mathbf{y}_0)}, \tag{7.3}$$

where

$$F_{12}[\mathbf{x}(t), \mathbf{y}(t), \tau, \tau'] = \int \mathcal{D}\mathbf{q}(t) \mathcal{D}\mathbf{r}(t) \exp(iS_{\mathcal{E}}^{\tau}[\mathbf{x}, \mathbf{q}] - iS_{\mathcal{E}}^{\tau'}[\mathbf{y}, \mathbf{r}]) \chi_1(\mathbf{x}_0, \mathbf{q}_0) \chi_2^*(\mathbf{y}_0, \mathbf{r}_0) \tag{7.4}$$

(as in Sec. VI A, we should then replace the class operators with their modified version and go to the semiclassical limit). Now it is easy to see that the influence functional is the overlap of the two initial states, but with each unitarily evolved along two different trajectories. That is, in the semiclassical approximation for the system,

$$F_{12} = \langle \chi_2(\mathbf{y}_0) | U^\dagger(\mathbf{x}_f, \mathbf{y}_0) U(\mathbf{x}_f, \mathbf{x}_0) | \chi_1(\mathbf{x}_0) \rangle, \tag{7.5}$$

where $U(\mathbf{x}_f, \mathbf{x}_0)$ denotes the unitary evolution of the environment states along the system classical trajectory from $\mathbf{x}_0$ to $\mathbf{x}_f$.

Clearly $|F_{12}|^2 \leqslant 1$, and because F_{12} is an overlap between a pair of states it will typically be such that F_{12} is strictly less than 1. In this case, when raised to a high power, as happens when we take a large number of oscillators in the environment, we will get a very strong suppression of terms with $\mathbf{x}_0 \neq \mathbf{y}_0$. In particular, even when $\mathbf{x}_0 = \mathbf{y}_0$, we get

$$F_{12} = \langle \chi_2(\mathbf{x}_0) | \chi_1(\mathbf{x}_0) \rangle, \tag{7.6}$$

which will be less than 1, quite simply because χ_1 and χ_2 are different states. We therefore find that D_{12} and D_{21} are much

smaller than the diagonal terms D_{11}, D_{22}. This destruction of interference between WKB states therefore comes about for essentially the same reason that the corresponding off-diagonal terms of the density matrix are very small, as discussed previously [46–49].

It should be noted that in Eq. (7.5), it could in fact happen that $F_{12}=1$, as a result of a careful choice of $\chi_1 \neq \chi_2$ together with a suitable choice of $\mathbf{x}_0 \neq \mathbf{y}_0$, in particular, if

$$U(\mathbf{x}_f,\mathbf{x}_0)|\chi_1(\mathbf{x}_0)\rangle = U(\mathbf{x}_f,\mathbf{y}_0)|\chi_2(\mathbf{y}_0)\rangle. \tag{7.7}$$

The point here, however, is that this becomes very unlikely with a large environment. With a large collection of oscillators in the environment, the environment states are a tensor product over A of states $|\chi_1^A(\mathbf{x}_0)\rangle$, for example, and then

$$F_{12}=\prod_A \langle\chi_2^A(\mathbf{y}_0)|U^\dagger(\mathbf{x}_f,\mathbf{y}_0)U(\mathbf{x}_f,\mathbf{x}_0)|\chi_1^A(\mathbf{x}_0)\rangle. \tag{7.8}$$

As the size of the environment goes to infinity, the possibility of Eq. (7.8) being exactly 1 becomes negligible.

It is also of interest to look at the special case in which the wave function Eq. (7.1) is real (as is the case in the no-boundary wave function of Hartle and Hawking), so that $\Psi_2=\Psi_1^*$, and $\chi_2=\chi_1^*$. When $\mathbf{x}_0=\mathbf{y}_0$, we then have that

$$|F_{12}|^2=\left|\int d\mathbf{q}_0\ \chi_1^2(\mathbf{x}_0,\mathbf{q}_0)\right|^2. \tag{7.9}$$

Since χ_1 is generally complex [it obeys the complex Schrödinger equation (6.7)], the right-hand side of Eq. (7.9) will again be less than 1, so the argument still goes through [49]. The argument fails if χ_1 is real. But then it would have to be an eigenstate of the environment Hamiltonian for all values of $\mathbf{x}_0$, and this would not lead to decoherence, so we may disregard this case.

VIII. SUMMARY AND DISCUSSION

We have studied the quantization and interpretation of simple timeless models described by an equation of the type (1.1). In particular, we studied the following question: what is the probability that the system passes through a region Δ of configuration space without reference to time?

We obtained the classical answer to this problem, in three different forms, in terms of a classical phase-space distribution function $w(\mathbf{p},\mathbf{x})$, satisfying $\{H,w\}=0$, the analogue of Eq. (1.1). This function needs some care in normalization since it is constant along the (possibly infinite) classical orbits. A very useful step in the classical case was the introduction of the phase-space quantity Eq. (2.7), which is δ-function-peaked on the classical path and also has a vanishing Poisson bracket with the Hamiltonian, thus it is an observable. This quantity assists in understanding some aspects of the quantum theory.

We constructed the decoherence functional, following the general scheme of Refs. [21,22], using the induced inner product. Although the general scheme has been presented previously, a key part of our contribution to this area is the explicit identification (in the semiclassical approximation) of the class operator Eq. (4.8) *satisfying the constraint everywhere* describing histories restricted to pass through a region Δ of configuration space. Having made this identification, a major part of our work was to show that the decoherent histories approach then reduces, approximately, to the corresponding classical result, but with the classical phase-space distribution function $w(\mathbf{p},\mathbf{x})$ replaced by the Wigner function $W(\mathbf{p},\mathbf{x})$ of the quantum theory. We also explored the decoherence and probabilities for a system of harmonic oscillators using the timeless coherent states, in terms of which the analysis is particularly transparent and fully agrees with intuitive expectations.

In brief, therefore, we have shown that heuristic classically inspired notions of interpretation for simple timeless models may in fact be derived from the decoherent histories analysis of such models. This result is by no means unexpected, but the key aspects of the derivation are the elucidation of the role of the constraint and the related reparametrization invariance in the construction of both the classical and quantum results. Furthermore, the complete absence of a time parameter is not an obstruction to quantization.

There are a number of issues which the present work has generated and will be discussed in a later publication, but we mention them briefly below.

First of all, the main difficulty in computing the decoherence functional for our chosen coarse graining is the calculation of the modified class operators. Even before modification, class operators of the type Eq. (3.21) are difficult to calculate (typically they can only be obtained exactly in the very simple situations in which the method of images may be used). The suggested scheme for constructing modified operators obeying the constraint has not yet been explored fully. Here, we have constructed physically plausible modified class operators in the semiclassical approximation, obtaining full agreement with the classical results. Some exact modified class operators for simple coarse grainings of the relativistic particle have been constructed in Ref. [23], but it is not yet clear how general those results are. Hence a more detailed investigation of these modified class operators is called for. [We note in passing that, from the simple examples in which the modified class operators have been calculated, their calculation does in fact appear to be considerably easier than the original ones, Eq. (3.21).]

Second, we have assumed (except in Sec. V) that both the initial states and the propagators are in the oscillatory regime. This means in the propagator that we assume the dominant contribution comes from real configurations (rather than Euclidean or complex ones). Many interesting models in quantum cosmology have a Euclidean region, corresponding, for example, to ''tunneling from nothing.'' It is not immediately clear how the semiclassical calculation of Sec. IV is modified to include this case, the main difficulty being understanding what the class operators are. This case is therefore probably related to the question of a more general formula for the modified class operators.

Third, it is generally understood that decoherence of histories is related to the existence of ''records'' [10,33]. This means that it is possible to find a projection operator R_α which is perfectly correlated with the class operators C_α in

terms of which the probabilities may be written,

$$p(\underline{\alpha}) = \mathrm{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}}) = \mathrm{Tr}(R_{\underline{\alpha}} \rho). \tag{8.1}$$

In the case of a nonrelativistic model where decoherence is produced by an environment, it is possible to explicitly identify the environmental variables which store the information about the system [33]. It would be very desirable to do this in the timeless case considered here. It seems likely that the variables are very similar to the case of Ref. [33], but the interesting question is the role of reparametrization invariance in this situation, and whether the records are closely related to observables in the operator approach.

A fourth issue concerns the connection between the decoherent histories analysis considered here and the master equation for the reduced density operator (in the case in which decoherence is produced by an environment). It would be of interest to see if the discussion of decoherence and probabilities can be reexpressed in the simpler language of the density operator, as it sometimes can in nonrelativistic decoherence models. This is currently under investigation [50].

These and related issues will be taken up in future publications.

ACKNOWLEDGMENTS

We are grateful to Jim Hartle, John Klauder, and Don Marolf for useful conversations. J.T. was funded by Evangelisches Studienwerk.

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