Living with ghosts

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Perturbation theory for gravity in dimensions greater than two requires higher derivatives in the free action. Higher derivatives seem to lead to ghosts, states with negative norm. We consider a fourth order scalar field theory and show that the problem with ghosts arises because, in the canonical treatment, ϕ and $\Box\phi$ are regarded as two independent variables. Instead, we base quantum theory on a path integral, evaluated in Euclidean space and then Wick rotated to Lorentzian space. The path integral requires that quantum states be specified by the values of ϕ and $\phi_{,\tau}$. To calculate probabilities for observations, one has to trace out over $\phi_{,\tau}$ on the final surface. Hence one loses unitarity, but one can never produce a negative norm state or get a negative probability. It is shown that transition probabilities tend toward those of the second order theory, as the coefficient of the fourth order term in the action tends to zero. Hence unitarity is restored at the low energies that now occur in the universe.

I. INTRODUCTION

In standard, second order theory the Lagrangian is a function of the fields and their first derivatives. The path integral is calculated by perturbation theory, with the part of the action that contains quadratic terms in the fields and their first derivatives regarded as the free field action, and the remaining terms as interactions. One then calculates Feynman diagrams, using the interactions as vertices, and the propagator defined by the free part of the action. This is equivalent to calculating the expectation value of the interactions in the Gaussian measure defined by the free action. One would therefore expect perturbation theory to make sense, when and only when the interaction action is bounded by the free action.

This is born out by the examples we know. In two dimensions, the free action of a scalar field ϕ,

$$S = \int dx^2 [\phi\Box\phi + m^2\phi^2], \qquad (1)$$

is the first Sobolev norm[1] $\|\phi\|_{2,1}$ of the field ϕ. In two dimensions, the first Sobolev norm bounds the pointwise value of ϕ; thus, it also bounds the volume integral of any entire function of ϕ. This means that the free action bounds any interaction action, so perturbation theory should work. Indeed one finds that in two dimensions, any quantum field theory is renormalizable.

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[1]For a function $f \in C^\infty(M)$, $1 \leq p < \infty$, and an integer $k \geq 0$, the Sobolev norm is defined [1] as

$$\|f\|_{p,k} = \left[\int_M \sum_{0 \leq j \leq k} |D^j f|^p \mu_g\right]^{1/p}, \qquad (2)$$

where $|D^j f|$ is the pointwise norm of the jth covariant derivative and μ_g is the Riemannian volume element.

In four dimensions on the other hand, the first Sobolev norm does not bound the pointwise value of ϕ, but only the volume integral of ϕ^4. This means that the free action bounds the interactions only for theories with quartic interactions, like $\lambda\phi^4$, or Yang-Mills theory. Indeed, these are the quantum field theories that are renormalizable in four dimensions. Note that even Yang-Mills theory is not renormalizable in dimensions higher than four, because the interactions are not bounded by the free action. Similarly, Born-Infeld theory is not renormalizable in dimensions higher than two.

When one does perturbation theory for gravity, one writes the metric as $g_0 + \delta g$, where g_0 is a background metric that is a solution of the field equations. The terms quadratic in δg are again regarded as the free action, and the higher order terms are the interactions. The latter include terms like $(\nabla \delta g)^2$, multiplied by powers of δg. The volume integral of such an interaction is not bounded by the free action and perturbation theory breaks down for gravity, which is not renormalizable [2]. Even if all the higher loop divergences canceled by some miracle in a supergravity theory, one could not trust the results, because one is using perturbation theory beyond its limit of validity; δg can be much larger than g_0 locally for only a small free action. In other words, there are large metric fluctuations below the Planck scale.

The situation is different however if one adds curvature squared terms to the Einstein-Hilbert action. The action is now quadratic in second derivatives of δg, so one takes the free action to be the quadratic terms in δg, and its first and second derivatives. This means that it is the second Sobolev norm $\|\delta g\|_{2,2}$ of δg, which bounds the pointwise value of δg. Hence the free action bounds the interactions, and perturbation theory works. This is reflected in the fact that the $R + R^2$ theory is renormalizable [3–5], and in fact asymptotically free [6]. However, higher derivatives seem to lead to ghosts, states with negative norm, which have been thought to be a fatal flaw in any quantum field theory (see e.g. [7]).

In the next section we review why higher derivatives appear to give rise to ghosts. The existence of ghosts would mean that the set of all states would not form a Hilbert space with a positive definite metric. There would not be a unitary

S matrix, and there would apparently be states with negative probabilities. These seemed sufficient reasons to dismiss any quantum field theory, such as Einstein gravity, that had higher derivative quantum corrections and ghosts. However, we shall show that one can still make sense of higher derivative theories, as a set of rules for calculating probabilities for observations. But one cannot prepare a system in a state with a negative norm, nor can one resolve a state into its positive and negative norm components. So there are no negative probabilities, and no nonunitary S matrix.

Although gravity is the physically interesting case, in this paper we consider a fourth order scalar field theory, which has the same ghostly behavior, but does not have the complications of indices or gauge invariance. We show explicitly that the higher derivative theory tends toward the second order theory, as the coefficient of the fourth order term in the action tends to zero. Hence the departures from unitarity for higher derivative gravity are very small at the low energies that now occur in the universe.

II. HIGHER DERIVATIVE GHOSTS

We consider a scalar field ϕ with a fourth-order Lagrangian in Lorentzian signature:

$$L = -\frac{1}{2}\phi(\Box - m_1^2)(\Box - m_2^2)\phi - \lambda\phi^4 \qquad (3)$$

where $m_2 > m_1$. Defining

$$\psi_1 = \frac{(\Box - m_2^2)\phi}{[2(m_2^2 - m_1^2)]^{1/2}}, \quad \psi_2 = \frac{(\Box - m_1^2)\phi}{[2(m_2^2 - m_1^2)]^{1/2}} \qquad (4)$$

the Lagrangian can be rewritten as

$$L = \frac{1}{2}\psi_1(\Box - m_1^2)\psi_1 - \frac{1}{2}\psi_2(\Box - m_2^2)\psi_2$$

$$- \frac{4\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^4. \qquad (5)$$

The action of ψ_2 has the wrong sign. Classically it means that the energy of the ψ_2 field is negative, while that of ψ_1 is positive. If there were no interaction term, this negative energy would not matter because each of the fields, ψ_1 and ψ_2, would live in its own world and the two worlds would not communicate with each other. However, if there is an interaction term, like ϕ^4, it will couple ψ_1 and ψ_2 together. Energy can then flow from one to the other, and one can have runaway solutions, with the positive energy of ψ_1 and the negative energy of ψ_2 both increasing exponentially.

In quantum theory, on the other hand, one is in trouble even in the absence of interactions, as can be seen by looking at the free field propagator for ϕ. In momentum space, this is the inverse of a fourth order expression in p, which can be expanded as

$$G(p) = \frac{1}{(m_2^2 - m_1^2)}\left(\frac{1}{(p^2 + m_1^2)} - \frac{1}{(p^2 + m_2^2)}\right). \qquad (6)$$

This is just the difference of the propagators for ψ_1 and ψ_2. The important point is that the propagator for ψ_2 appears with a negative sign. This would mean that states with an odd number of ψ_2 particles would have a negative norm. In other words, ψ_2 particles are ghosts. There would not be a positive definite Hilbert space metric, nor a unitary S matrix.

If there were not any interactions, the situation would not be too serious. The state space would be the direct sum of two Hilbert spaces, one with positive definite metric and the other negative. There would not be any physically realized operators that connected the two Hilbert spaces, so ghost number would be conserved by a superselection rule. A ϕ^4 interaction, however, would allow ψ_2 particles to be created or destroyed. As in the classical theory, there will be instabilities, with runaway production of ψ_1 and ψ_2 particles. These instabilities show up in the fact that interactions tend to shift the ghost poles in the two point function for ϕ into the complex p-plane, where they represent exponentially growing and decaying modes [8,9].

It seems to add up to a pretty damning indictment of higher derivative theories in general, and quantum gravity and quantum supergravity in particular. However, the problem with ghosts arises because in the canonical treatment, ϕ and $\Box\phi$ are regarded as two independent variables, although they are both determined by ϕ. We shall show that, by basing quantum theory on a path integral over the field, evaluated in Euclidean space and then Wick rotated to Lorentzian space, one can obtain a sensible set of rules for calculating probabilities for observations in higher derivative theories.

III. EUCLIDEAN PATH INTEGRAL

According to the canonical approach, one would perform the path integral over all ψ_1 and ψ_2. The path integral over ψ_1 will converge, but the path integral over ψ_2 is divergent, because the free action for ψ_2 is negative definite. However, one should not do the path integrals over ψ_1 and ψ_2 separately because they are not independent fields; they are both determined by ϕ. The fourth order free action for ϕ is positive definite; thus the path integral over all ϕ in Euclidean space should converge, and should define a well determined Euclidean quantum field theory.

One way to compute the path integral for a fourth order theory is to expand ϕ in eigenfunctions of the differential operator $\hat{O}$ in the action. One then integrates over the coefficients in the harmonic expansion, which gives $(\det\hat{O})^{-1/2}$. Another way is to use time slicing, by dividing the period into a number of short time steps ϵ and approximating the derivatives by

$$\phi_{,\tau} \sim \frac{(\phi_{n+1} - \phi_n)}{\epsilon}, \quad \phi_{,\tau\tau} \sim \frac{(\phi_{n+2} - 2\phi_{n+1} + \phi_n)}{\epsilon^2}. \qquad (7)$$

One then integrates over the values of ϕ on each time slice. In a second order theory, where the action depends on ϕ and

$\phi_{,\tau}$ but not on $\phi_{,\tau\tau}$, the path integral will depend on the values of ϕ on the initial and final surfaces. However, in a fourth order theory, the use of three neighbor differences means that one has to specify $\phi_{,\tau}$ on the initial and final surfaces as well.

One can also see what needs to be specified on the initial and final surfaces as follows. In classical second order theory, a state can be defined by its Cauchy data on a space-like surface, i.e. the values of ϕ and $\phi_{,\tau}$ on the surface. In a canonical $3+1$ treatment, these are regarded as the position of the field and its conjugate momentum. In quantum theory, position and momentum do not commute, so instead one describes a state by a wave function in either position space or momentum space. In ordinary quantum mechanics, the position and momentum representations are regarded as equivalent: one is just the Fourier transform of the other. However, with path integrals, one has to use wave functions in the position representation. This can be seen as follows. Imagine using the path integral to go from a state at τ_1 to a state at τ_2, and then to a state at τ_3. In the position representation, the amplitude to go from a field ϕ_1 on τ_1, to ϕ_2 at τ_2, is given by a path integral over all fields ϕ with the given boundary values. Similarly, the amplitude to go from ϕ_2 at τ_2, to ϕ_3 at τ_3, is given by another path integral. These amplitudes obey a composition law:

$$G(\phi_3,\phi_1)=\int d\phi_2 G(\phi_3,\phi_2)G(\phi_2,\phi_1). \quad (8)$$

The composition law holds, only because one can join a field from ϕ_1 to ϕ_2 to a field from ϕ_2 to ϕ_3, to obtain a field from ϕ_1 to ϕ_3. Although in general $\phi_{,\tau}$ will be discontinuous at t_2, the field will still have a well defined action:

$$S(\phi_3,\phi_1)=S(\phi_3,\phi_2)+S(\phi_2,\phi_3). \quad (9)$$

On the other hand, if one would use the momentum representation and wave functions in terms of $\phi_{,\tau}$, the composition law would no longer hold, because the discontinuity of ϕ at τ_2 would make the action infinite. Thus in second order theories, one should use wave functions in terms of ϕ rather than $\phi_{,\tau}$.

In a fourth order theory, a classical state is determined by the values of ϕ and its first three time derivatives on a space-like surface. In a canonical treatment, ϕ and $\phi_{,\tau\tau}$ are usually taken to be independent coordinates. For the scalar field theory (3) we then have the conjugate momenta

$$\Pi_\phi=-\phi_{,\tau\tau\tau}+(m_1^2+m_2^2-2\vec{\nabla}^2)\phi_{,\tau}, \quad \Pi_{\phi_{,\tau\tau}}=-\phi_{,\tau}. \quad (10)$$

This suggests that in quantum theory, one should describe a state by a wave functional $\Psi(\phi,\phi_{,\tau\tau})$ on a surface. Indeed, this is closely related to using the fields ψ_1 and ψ_2 that we introduced earlier. These were linear combinations of ϕ and $\Box\phi$; thus taking the wave function to depend on ψ_1 and ψ_2, is equivalent to it depending on ϕ and $\phi_{,\tau\tau}$. However, if one does the path integral between fixed values of ϕ and $\phi_{,\tau\tau}$, one gets in trouble with the composition law, because the

values of $\phi_{,\tau}$ on the intermediate surface at τ_2 are not constrained, Hence $\phi_{,\tau}$ will be in general discontinuous at τ_2, which implies that $\phi_{,\tau\tau}$ will have a delta function when one joins the fields above and below τ_2. In a second order action $\phi_{,\tau\tau}$ appears linearly; thus the delta function can be integrated by parts and the action of the combined field is finite. But in a fourth order action $(\phi_{,\tau\tau})^2$ appears, rendering the action of the combined field infinite if $\phi_{,\tau\tau}$ is a delta function.

Therefore, the path integral requires that quantum states be specified by ϕ and $\phi_{,\tau}$ in order to get the composition law for amplitudes in a fourth order theory. In the next section we show how one can obtain transition probabilities for observations from the Euclidean path integral over ϕ.

IV. HIGHER DERIVATIVE HARMONIC OSCILLATOR

A. Ground state wave function

To illustrate how probabilities can be calculated, we consider a higher derivative harmonic oscillator, for which in Euclidean signature we take the action

$$S=\int d\tau\left[\frac{\alpha^2}{2}\phi_{,\tau\tau}^2+\frac{1}{2}\phi_{,\tau}^2+\frac{1}{2}m^2\phi^2\right]. \quad (11)$$

The general solution to the equation of motion is given by

$$\phi(\tau)=A\sinh\lambda_1\tau+B\cosh\lambda_1\tau+C\sinh\lambda_2\tau+D\cosh\lambda_2\tau, \quad (12)$$

where λ_1 and λ_2 are given by Eq. (A2). For small α, $\lambda_1\sim m$ and $\lambda_2\sim 1/\alpha$.

For $\alpha^2>0$, this is similar to our scalar field model, since in the latter we can take Fourier components so that spatial derivatives behave like masses. Indeed, even for $\lambda\to 0$ in the quantum scalar field theory (3), ghost instabilities arise. This is because of the lack of any lower-energy bound mentioned above, allowing unlimited creation of pairs of ghost particles and normal particles. This process is obviously forbidden in the zero-dimensional case of the oscillator considered here, but any perturbation with a finite amplitude leads to similar instabilities. This can be seen from the general solution (12) of the equation of motion. Although in Lorentzian signature, this just corresponds to oscillatory behavior, the latter mode has negative energy. Therefore, for small α, even a small amplitude for the negative mode leads to enormously negative energies, rendering the system unstable. A $\lambda\phi^4$ interaction in the field theory (3) would make the situation worse, by introducing exponentially growing runaway solutions even at the classical level. For our purposes here, in which we consider the former problem, it is sufficient to deal with the higher derivative harmonic oscillator (11). In Sec. V we shall show that our proposal on how to deal with the former problem also solves the problems with ghosts in interacting field theories.

The fourth order action (11) for ϕ is positive definite; thus it gives a well defined Euclidean quantum field theory. In this theory, one can calculate the amplitude to go from a state $(\phi_1,\phi_{1,\tau})$ at time τ_1, to a state $(\phi_2,\phi_{2,\tau})$ at time τ_2. In

particular, one can calculate the ground state wave function, the amplitude to go from zero field in the infinite Euclidean past, up to the given values $(\phi_0, \phi_{0,\tau})$ at $\tau = 0$. This yields (see the Appendix)

$$\Psi_0(\phi_0, \phi_{0,\tau}) = N' \exp\left[-F'\left(\phi_{0,\tau}^2 + \frac{m}{\alpha}\phi_0^2 \right) \right.$$
$$\left. + \frac{2m^2 - m/\alpha}{(\lambda_2 - \lambda_1)^2}\phi_0\phi_{0,\tau} \right] \qquad (13)$$

where

$$F' = \frac{(1 - 4m^2\alpha^2)}{2\alpha^2(\lambda_1 + \lambda_2)(\lambda_2 - \lambda_1)^2} \qquad (14)$$

and $N'(\alpha, m)$ is a normalization factor.

Similarly, one can calculate the Euclidean conjugate ground state wave function Ψ_0^*, the amplitude to go from the given values at $\tau = 0$, to zero field in the infinite Euclidean future. This conjugate wave function is equal to the original ground state wave function, with the opposite sign of $\phi_{0,\tau}$. The probability that a quantum fluctuation in the ground state gives the specified values ϕ_0 and $\phi_{0,\tau}$ on the surface $\tau = 0$ is then given by

$$P(\phi_0, \phi_{0,\tau}) = \Psi_0\Psi_0^* = N'^2 \exp\left[-2F'\left(\phi_{0,\tau}^2 + \frac{m}{\alpha}\phi_0^2 \right) \right]. \qquad (15)$$

The probability dies off at large values of ϕ and $\phi_{,\tau}$ and is normalizable; thus the probability distribution in the Euclidean theory is well defined. However if one Wick rotates to Minkowski space, $\phi_{,\tau}^2$ picks up a minus sign. The probability distribution becomes unbounded for large Lorentzian $\phi_{,\tau}$ and can no longer be normalized. This is another reflection of the same problem as the ghosts. You cannot fully determine a state on a spacelike surface, because that would involve specifying ϕ and Lorentzian $\phi_{,t}$, which does not have a physically reasonable probability distribution.

Although one cannot define a probability distribution for ϕ and Lorentzian $\phi_{,t}$ on a spacelike surface, one can calculate a probability distribution for ϕ alone, by integrating out over Euclidean $\phi_{,\tau}$. This integral converges because the probability distribution is damped at large values of Euclidean $\phi_{,\tau}$. This is just what one would calculate in a second order theory. So the moral is as follows: a fourth order theory can make sense in Lorentzian space, if you treat it like a second order theory. The normalized probability distribution that a ground state fluctuation gives the specified value ϕ_0 on a spacelike surface is then given by

$$P(\phi_0) = \left(\frac{2F'm}{\pi\alpha} \right)^{1/2} \exp\left[-\frac{2mF'}{\alpha}\phi_0^2 \right]. \qquad (16)$$

As the coefficient α of the fourth order term in the action tends to zero, this becomes

$$P(\phi_0) = \left(\frac{m}{\pi} \right)^{1/2}\left(1 + \frac{m\alpha}{2} \right)\exp[-m(1 + m\alpha)\phi_0^2], \qquad (17)$$

which tends toward the result for the second order theory.

B. Transition probabilities

In this section we compute the Euclidean transition probability to go from a specified value ϕ_1 at time τ_1, to ϕ_2 at time τ_2, for the higher derivative harmonic oscillator.

In a second order theory, a state can be described by a wave function that depends on the values of ϕ on a spacelike surface. Thus a transition amplitude is given by a path integral from an initial state ϕ_1 on τ_1, to a final state ϕ_2 on τ_2. To calculate the probability to go from the initial state to the final, one multiplies the amplitude by its Euclidean conjugate. This can be represented as the path integral from a third surface, at τ_3, back to τ_2. Because the path integral in a second order theory depends only on ϕ on the boundary, what happens above τ_3 and below τ_1 does not matter. Furthermore, the path integrals above and below τ_2 can be calculated independently, which implies the probability to go from initial to final can be factorized into the product of an S matrix and its adjoint. The S matrix is unitary, because probability is conserved.

Now let us calculate the probability to go from an initial to a final state in the fourth order theory (11). The path integral requires quantum states to be specified by ϕ and $\phi_{,\tau}$. The transition amplitude to go from a state $(\phi_1, \phi_{1,\tau})$ at time $\tau_1 = -T$, to a state $(\phi_2, \phi_{2,\tau})$ at time $\tau_2 = 0$, reads

$$\langle(\phi_2, \phi_{2,\tau}; 0)|(\phi_1, \phi_{1,\tau}; -T)\rangle$$
$$= \int_{(\phi_1, \phi_{1,\tau})}^{(\phi_2, \phi_{2,\tau})} d[\phi(\tau)]\exp[-S(\phi)]. \qquad (18)$$

This is evaluated in the Appendix, by writing $\phi = \phi_{cl} + \phi'$, where ϕ_{cl} obeys the equation of motion with the given boundary conditions on both surfaces. The result is

$$\langle(\phi_2, \phi_{2,\tau}; 0)|(\phi_1, \phi_{1,\tau}; -T)\rangle$$
$$= \left(-\frac{\alpha(1 + \alpha N)H}{2\pi^2} \right)^{1/2}\exp[-E(\phi_1^2 + \phi_2^2)$$
$$-F(\phi_{1,\tau}^2 + \phi_{2,\tau}^2) - G\phi_{1,\tau}\phi_{2,\tau} + H\phi_1\phi_2$$
$$-K(\phi_{2,\tau}\phi_2 - \phi_{1,\tau}\phi_1) - L(\phi_{2,\tau}\phi_1 - \phi_{1,\tau}\phi_2)]. \qquad (19)$$

The coefficient functions in the exponent are given by Eq. (A6), and N is a normalization factor.

Again, one can construct a three layer "sandwich" to calculate the probability to go from the initial state to the final. However, in contrast with the second order theory the path integral now depends on both ϕ and $\phi_{,\tau}$ on the boundaries. This has two important implications for the calculation of the transition probability. First, as we just showed, one cannot observe Lorentzian $\phi_{,\tau}$ because it has an unbounded Lorentzian probability distribution. Therefore one should take $\phi_{,\tau}$

to be continuous on the surfaces and integrate over all values, fixing only the values of ϕ on the surfaces. Because the path integrals above and below $\tau_2 = 0$ both depend on $\phi_{2,\tau}$, the probability $P(\phi_2, \phi_1)$ to observe the initial and final specified values of ϕ does not factorize into an S matrix and its adjoint. Instead, there is loss of quantum coherence, because one cannot observe all the information that characterizes the final state.

After multiplying by the Euclidean conjugate amplitude and integrating out over $\phi_{2,\tau}$ we obtain

$$
-\frac{\alpha(1+\alpha N)H}{2\pi^2}\left(\frac{\pi}{2F}\right)^{1/2}\exp\left[-2E(\phi_1^2+\phi_2^2)-2F\phi_{1,\tau}^2\right.
$$

$$
\left. +2H\phi_1\phi_2+\frac{G^2}{2F}\phi_{1,\tau}^2\right]. \tag{20}
$$

Another consequence of the dependence of the path integral on $\phi_{,\tau}$ is that what goes on outside the sandwich now affects the result. The most natural choice would be the vacuum state above $\tau_3 = T$ and below $\tau_1 = -T$. In other words, one takes the path integral to be over all fields that have the given values on the three surfaces, and that go to zero in the infinite Euclidean future and past. This means that to obtain the transition probability we also ought to multiply by the appropriately normalized ground state wave function $\Psi_0(\phi_1, \phi_{1,\tau})$ and its Euclidean conjugate. The probability $P(\phi_2, \phi_1)$ is then given by

$$
P(\phi_2, \phi_1) = \int d[\phi_{1,\tau}]\Psi_0\Psi_0^*
$$

$$
\times \int d[\phi_{2,\tau}\langle(\phi_1, \phi_{1,\tau})|(\phi_2, \phi_{2,\tau})\rangle
$$

$$
\times \langle(\phi_2, \phi_{2,\tau})|(\phi_1, \phi_{1,\tau})\rangle
$$

$$
= \left(\frac{\alpha^2(1+\alpha\tilde{N})^2H^2}{2\pi^2(4F(F'+F)-G^2)}\right)^{1/2}
$$

$$
\times \exp\left[-2E(\phi_1^2+\phi_2^2)-2\frac{mF'}{\alpha}\phi_1^2\right.
$$

$$
\left. +2H\phi_1\phi_2\right]. \tag{21}
$$

Here $F'(\alpha, m)$ is the coefficient in the exponent of the ground wave function (13) and $\tilde{N}$ is a normalization factor. In the limit $\alpha \to 0$, this reduces to

$$
P(\phi_2, \phi_1)
$$

$$
= \frac{m}{2\pi \sinh mT}
$$

$$
\times \exp\left[-\frac{m\cosh mT(\phi_1^2+\phi_2^2)-2m\phi_1\phi_2}{\sinh mT}-m\phi_1^2\right]. \tag{22}
$$

Hence the probability given by the sandwich tends toward that of the second order theory, as the coefficient of the fourth order term in the action tends to zero. This is important, because it means that fourth order corrections to graviton scattering can be neglected completely at the low energies that now occur in the universe. On the other hand, in the very early universe, when fourth order terms are important, we expect the Euclidean metric to be some instanton, like a four sphere. In such a situation, one cannot define scattering or ask about unitarity. The only quantities we have any chance of observing are the n-point functions of the metric perturbations, which determine the n-point functions of fluctuations in the microwave background. With Reall we have shown that Starobinsky's model of inflation [10], in which inflation is driven by the trace anomaly of a large number of conformally coupled matter fields, can give a sensible spectrum of microwave fluctuations, despite the fact it has fourth order terms and ghosts [11]. Moreover, the fourth order terms can play an important role in reducing the fluctuations to the level we observe.

Finally, in order to obtain the Minkowski space probability, one analytically continues τ_2 to future infinity in Minkowski space, and τ_1 and τ_3 to past infinity, keeping their Euclidean time values fixed. This gives the Minkowski space probability, to go from an initial value ϕ_1 to a final value ϕ_2.

V. RUNAWAYS AND CAUSALITY

Our discussion in Sec. II suggests that even the slightest amount of a fourth order term will lead to runaway production of positive and negative energies, or of real and ghost particles. The classical theory is certainly unstable, if one prescribes the initial value of ϕ and its first three time derivatives. However, in quantum theory every sensible question can be posed in terms of vacuum to vacuum amplitudes. These can be defined by Wick rotating to Euclidean space and doing a path integral over all fields that die off in the Euclidean future and past. Thus the Euclidean formulation of a quantum field theory implicitly imposes the final boundary condition that the fields remain bounded. This removes the instabilities and runaways, like a final boundary condition removes the runaway solution of the classical radiation reaction force.

In practice, this is done as follows. The quantum instabilities produce extra poles in the Fourier transform of the n point functions, which are shifted off the real axis by the interactions. Therefore, taking the contour of integration along the real axis and closing it in the upper half plane would yield growing exponentials from the complex poles. However, our prescription is to Wick rotate from the Euclidean. In this case, one must take a different contour, namely one that can be deformed to the imaginary p_0 axis. This avoids the growing exponentials from the complex poles. This final boundary condition was used in [11] to compute the 2-point functions of metric perturbations in trace anomaly driven inflation.

It has also been argued that in order to avoid pathologies in semiclassical gravity, one should only consider solutions

that are perturbatively expandable in $\hbar$ [13,14]. However, it has subsequently been shown that physically meaningful solutions can be obtained that go well beyond the approximate, perturbative solutions, but which are not sensitive to higher order corrections to the equation of motion [15]. In particular, these solutions often describe phenomena where the deviations from classical gravity are locally small but where long term cumulative effects may result in very large global deviations from classical solutions. Further evidence for their validity is provided by justifying the semiclassical equations by the $1/N$ approximation, in which case it is natural to adopt the view that semiclassical gravity should in general be adequate to describe phenomena on scales larger than the Planck scale. This was done in [11], where we concluded that the Starobinsky solution of trace anomaly inflation falls into this category.

The price one pays for removing runaways with a final boundary condition is a slight violation of causality. For instance, with the classical radiation reaction force, a particle would start to accelerate before a wave hit it. This can be seen by considering a single electron which is acted upon by a delta-function pulse [12]. The equation of motion for the x component reduces to

$$x_{,tt} = \lambda x_{,ttt} + \delta(t), \tag{23}$$

with $\lambda = 2e^2/3mc^3$. This has the solution

$$x(t) = \int \frac{d\omega}{2\pi} \exp[-i\omega t] \frac{1}{-\omega^2 - i\lambda\omega^3}. \tag{24}$$

The integrand has two singularities, at $\omega = 0$ and $\omega = i\lambda^{-1}$. The final boundary condition that $x_{,t}$ should tend to a finite limit implies one must choose an integration contour that stays close to the real axis, going below the second singularity. This yields

$$x(t) = \lambda \exp[t/\lambda], \quad t < 0$$

$$= t + \lambda, \quad t > 0 \tag{25}$$

which is without runaways, but acausal. However, this pre-acceleration is appreciable only for a period of time comparable with the time for light to travel the classical radius of the electron, and thus practically unobservable.

Similarly, if we would add an interaction term to the higher derivative scalar field theory (3), the imposition of a final boundary condition to eliminate the runaway solutions would lead to acausal behavior on the scale of m_2^{-1}, where m_2 is the mass of the ghost particle. However, in the context of quantum gravity, one could again never detect a violation of causality, because the presence of a mass introduces a logarithmic time delay $\Delta t \sim -m \log b$, where b is the impact parameter. Thus there is no standard arrival time, one can always arrive before any given light ray by taking a path which stays a sufficiently large distance from the mass.

VI. CONCLUDING REMARKS

We conclude that quantum gravity with fourth order corrections can make sense, despite apparently having negative energy solutions and ghosts. In doing this, we seem to go against the convictions of the last 25 years, that unitarity and causality are essential requirements of any viable theory of quantum gravity. Perturbative string theory has unitarity and causality, so it has been claimed as the only viable quantum theory of gravity. But the string perturbation expansion does not converge, and string theory has to be augmented by nonperturbative objects, like D-branes. One can have a worldsheet theory of strings without higher derivatives, only because two dimensional metrics are conformally flat, meaning perturbations do not change the light cone. Still, we live either on a 3-brane, or in the bulk of a higher dimensional compactified space. The world-sheet theory of D-branes with p greater than one has similar nonrenormalizability problems to Einstein gravity and supergravity. Thus string theory effectively has ghosts, though this awkward fact is quietly glided over.

To summarize, we first reviewed why perturbation theory for gravity in dimensions greater than two required higher derivatives in the free action. Higher derivatives seemed to lead to ghosts, states with negative norm. To analyze what was happening, we considered a fourth order scalar field theory. We showed that the problem with ghosts arises because in the canonical approach, ϕ and $\Box\phi$ are regarded as two independent coordinates. Instead, we based quantum theory on a path integral over ϕ, evaluated in Euclidean space and then Wick rotated to Lorentzian space. We showed the path integral required that quantum states be specified by the values of ϕ and $\phi_{,\tau}$ on a spacelike surface, rather than ϕ and $\phi_{,\tau\tau}$ as is usually done in a canonical treatment. The wave function in terms of ϕ and $\phi_{,\tau}$ is bounded in Euclidean space, but grows exponentially with Minkowski space $\phi_{,\tau}$. This means one cannot observe $\phi_{,\tau}$ but only ϕ. To calculate probabilities for observations one therefore has to trace out over $\phi_{,\tau}$ on the final surface, and lose information about the quantum state. One might worry that integrating out $\phi_{,\tau}$ would break Lorentz invariance. However, $\phi_{,\tau}$ is conjugate to $\phi_{,\tau\tau}$ so tracing over $\phi_{,\tau}$ is equivalent to not observing $\Box\phi$. Since, according to Eq. (4), ψ_1 and ψ_2 are linear combinations of ϕ and $\Box\phi$, this means that one only considers Feynman diagrams whose external legs are $\psi_1 - \psi_2$. You do not observe the other linear combination, $m_2^2\psi_1 - m_1^2\psi_2$.

Because one is throwing away information, one gets a density matrix for the final state, and loses unitarity. However, one can never produce a negative norm state or get a negative probability. We illustrated with the example of a higher derivative harmonic oscillator that probabilities for observations tend toward those of the second order theory, as the coefficient of the fourth order term in the action tends to zero. This means that the departures from unitarity for higher derivative gravity will be very small at the low energies that now occur in the universe. On the other hand, the higher derivative terms will be important in the early universe, but there unitarity cannot be defined.

APPENDIX: TRANSITION AMPLITUDE

We compute the Euclidean transition amplitude, to go from an initial state $(\phi_1, \phi_{1,\tau})$ on a spacelike surface at $\tau = -T$, to a final state $(\phi_2, \phi_{2,\tau})$ at $\tau = 0$, for the higher derivative harmonic oscillator (11). The general solution to the equation of motion is given by

$$\phi(\tau) = A \sinh \lambda_1 \tau + B \cosh \lambda_1 \tau + C \sinh \lambda_2 \tau + D \cosh \lambda_2 \tau, \tag{A1}$$

where

$$\lambda_1 = \frac{1}{\sqrt{2\alpha^2}} \sqrt{(1 - \sqrt{1 - 4m^2\alpha^2})},$$

$$\lambda_2 = \frac{1}{\sqrt{2\alpha^2}} \sqrt{(1 + \sqrt{1 - 4m^2\alpha^2})}. \tag{A2}$$

The transition amplitude is given by a path integral,

$$\langle (\phi_2, \phi_{2,\tau}; 0) | (\phi_1, \phi_{1,\tau}; -T) \rangle$$

$$= \int_{(\phi_1, \phi_{1,\tau})}^{(\phi_2, \phi_{2,\tau})} d[\phi(\tau)] \exp[-S(\phi)]. \tag{A3}$$

This can be evaluated by separating out the "classical" part of ϕ. If we write $\phi = \phi_{cl} + \phi'$, where ϕ_{cl} obeys the equation of motion with the required boundary conditions on both surfaces $\tau = 0$ and $\tau = T$, then the amplitude becomes

$$\langle (\phi_2, \phi_{2,\tau}; 0) | (\phi_1, \phi_{1,\tau}; -T) \rangle$$

$$= \exp[-S_{cl}(\phi_1, \phi_{1,\tau}, \phi_2, \phi_{2,\tau})]$$

$$\times \int_{(0, -T)}^{(0,0)} d[\phi'(\tau)] \exp[-S(\phi')]. \tag{A4}$$

The classical action is

$$S_{cl} = \int_0^T d\tau \left[\frac{\alpha^2}{2} \phi_{cl,\tau\tau}^2 + \frac{1}{2} \phi_{cl,\tau}^2 + \frac{1}{2} m^2 \phi_{cl}^2 \right]$$

$$= E(\phi_1^2 + \phi_2^2) + F(\phi_{1,\tau}^2 + \phi_{2,\tau}^2)$$

$$+ G\phi_{1,\tau}\phi_{2,\tau} - H\phi_1\phi_2 + K(\phi_{2,\tau}\phi_2 - \phi_{1,\tau}\phi_1)$$

$$+ L(\phi_{2,\tau}\phi_1 - \phi_{1,\tau}\phi_2) \tag{A5}$$

where

$$E = \frac{-m(1 - 4m^2\alpha^2)}{2\alpha^3(\lambda_2^2 - \lambda_1^2)P^2} \left[\frac{2m}{\alpha} (\cosh \lambda_2 T - \cosh \lambda_1 T)(\lambda_1 \sinh \lambda_1 T + \lambda_2 \sinh \lambda_2 T) + \sinh \lambda_1 T \sinh \lambda_2 T \right.$$

$$\left. \times \left(\lambda_1 \left(2\lambda_2^2 + \frac{1}{\alpha^2} \right) \cosh \lambda_2 T \sinh \lambda_1 T - \lambda_2 \left(2\lambda_1^2 + \frac{1}{\alpha^2} \right) \sinh \lambda_2 T \cosh \lambda_1 T \right) \right]$$

$$F = \frac{(1 - 4m^2\alpha^2)}{2\alpha^2(\lambda_2^2 - \lambda_1^2)P^2} \left[\frac{2m}{\alpha} (\cosh \lambda_2 T - \cosh \lambda_1 T)(\lambda_2 \sinh \lambda_1 T + \lambda_1 \sinh \lambda_2 T) + \sinh \lambda_1 T \sinh \lambda_2 T \right.$$

$$\left. \times \left(\lambda_2 \left(2\lambda_1^2 + \frac{1}{\alpha^2} \right) \cosh \lambda_2 T \sinh \lambda_1 T - \lambda_1 \left(2\lambda_2^2 + \frac{1}{\alpha^2} \right) \sinh \lambda_2 T \cosh \lambda_1 T \right) \right]$$

$$G = \frac{(1 - 4m^2\alpha^2)}{\alpha^2(\lambda_2^2 - \lambda_1^2)P^2} \left(\frac{2m}{\alpha} (\cosh \lambda_2 T \cosh \lambda_1 T - 1) - \frac{1}{\alpha^2} \sinh \lambda_1 T \sinh \lambda_2 T \right) (\lambda_1 \sinh \lambda_2 T - \lambda_2 \sinh \lambda_1 T)$$

$$H = \frac{-m(1 - 4m^2\alpha^2)}{\alpha^3(\lambda_2^2 - \lambda_1^2)P^2} \left(\frac{2m}{\alpha} (\cosh \lambda_2 T \cosh \lambda_1 T - 1) - \frac{1}{\alpha^2} \sinh \lambda_1 T \sinh \lambda_2 T \right) (\lambda_1 \sinh \lambda_1 T - \lambda_2 \sinh \lambda_2 T)$$

$$K = \frac{1}{P^2} \left[\frac{m}{\alpha} \left(4m^2 + \frac{1}{\alpha^2} \right) \sinh \lambda_1 T \sinh \lambda_2 T (1 - \cosh \lambda_1 \cosh \lambda_2 T) + \frac{2m^2}{\alpha^2} (2 - 3(\cosh^2 \lambda_1 T + \cosh^2 \lambda_2 T)) \right]$$

$$L = \frac{-m(1-4m^2\alpha^2)}{\alpha^3(\lambda_2^2-\lambda_1^2)P^2}\left(\frac{2m}{\alpha}(\cosh\lambda_2 T\cosh\lambda_1 T-1)-\frac{1}{\alpha^2}\sinh\lambda_1 T\sinh\lambda_2 T\right)(\cosh\lambda_2 T-\cosh\lambda_1 T)$$

(A6)

with

$$P = (\lambda_1^2+\lambda_2^2)\sinh\lambda_1 T\sinh\lambda_2 T+2\lambda_1\lambda_2(1-\cosh\lambda_1 T\cosh\lambda_2 T).$$

(A7)

The preexponential factor in Eq. (A4) can be derived from the classical action alone [16]; it is basically the Jacobian of the change of variables $(\pi_1,\phi_1)\rightarrow(\phi_2,\phi_1)$. Because the Lagrangian is quadratic, the prefactor is independent of the values specifying the initial and final states, and the transition amplitude (A4) is exact. It is given by

$$\langle(\phi_2,\phi_{2,\tau};0)|(\phi_1,\phi_{1,\tau};-T)\rangle = \left(\frac{-\alpha(1+\alpha N)H}{2\pi^2}\right)^{1/2}\exp[-E(\phi_1^2+\phi_2^2)-F(\phi_{1,\tau}^2+\phi_{2,\tau}^2)-G\phi_{1,\tau}\phi_{2,\tau}+H\phi_1\phi_2$$

$$-K(\phi_{2,\tau}\phi_2-\phi_{1,\tau}\phi_1)-L(\phi_{2,\tau}\phi_1-\phi_{1,\tau}\phi_2)].$$

(A8)

The normalization factor N is independent of α to first order. It is determined by taking $T\rightarrow+\infty$ in Eq. (A8) and requiring that the amplitude tends toward the product of two normalized ground state wave functions $\Psi_0(\phi_1,\phi_{1,\tau})$ and $\Psi_0(\phi_2,\phi_{2,\tau})$.

For small α, $\lambda_1\sim m$ and $\lambda_2\sim 1/\alpha$, hence the transition amplitude becomes

$$\langle(\phi_2,\phi_{2,\tau};0)|(\phi_1,\phi_{1,\tau};-T)\rangle = \left(\frac{m\alpha}{2\pi^2\sinh mT}\right)^{1/2}\exp\left[-\frac{m\cosh mT(\phi_1^2+\phi_2^2)}{2\sinh mT}-\frac{\alpha}{2}(\phi_{1,\tau}^2+\phi_{2,\tau}^2)\right.$$

$$\left.+\frac{m\alpha\cosh mT(\phi_{2,\tau}\phi_2-\phi_{1,\tau}\phi_1)}{\sinh mT}-\frac{m(\alpha\phi_{2,\tau}+\phi_2)(\alpha\phi_{1,\tau}-\phi_1)}{\sinh mT}\right].$$

(A9)

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[1] A.L. Besse, *Einstein Manifolds* (Springer-Verlag, Berlin, 1987), p. 457.

[2] G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré, Sect. A **20**, 69 (1974).

[3] K.S. Stelle, Phys. Rev. D **16**, 953 (1977).

[4] R. Utiyama and B. DeWitt, J. Math. Phys. **3**, 608 (1962).

[5] E.S. Fradkin and G. Vilkovisky, Phys. Rev. D **8**, 4241 (1973).

[6] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. **B201**, 469 (1982).

[7] S.W. Hawking, in *Quantum Field Theory and Quantum Statistics: Essays in Honor of the 60th Birthday of E.S. Fradkin*, edited by A. Batalin, C. J. Isham, and C.A. Vilkovisky (Hilger, Bristol, UK, 1987).

[8] E. Tomboulis, Phys. Lett. **70B**, 361 (1977).

[9] D.A. Johnston, Nucl. Phys. **B297**, 721 (1987).

[10] A.A. Starobinsky, Phys. Lett. **91B**, 99 (1980).

[11] S.W. Hawking, T. Hertog, and H.S. Reall, Phys. Rev. D **63**, 083504 (2001).

[12] S. Coleman, in *Theory and Phenomenology in Particle Physics*, edited by A. Zichichi (Academic, New York, 1969).

[13] J. Simon, Phys. Rev. D **45**, 1953 (1992).

[14] L. Parker and J. Simon, Phys. Rev. D **47**, 1339 (1993).

[15] E.E. Flanagan and R.M. Wald, Phys. Rev. D **54**, 6233 (1996).

[16] M.S. Marinov, Phys. Rep. **60**, 1 (1980).