Contribution to muon $g-2$ from $\pi^0\gamma$ and $\eta\gamma$ intermediate states in the vacuum polarization

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(Received 14 February 2002; published 9 May 2002)

Using new experimental data, we calculate the contribution to the anomalous magnetic moment of the muon from the $\pi^0\gamma$ and $\eta\gamma$ intermediate states in vacuum polarization with high precision: $a_\mu(\pi^0\gamma)+a_\mu(\eta\gamma)=(54.7\pm1.5)\times10^{-11}$. We also find a small contribution from $e^+e^-\pi^0$, $e^+e^-\eta$ and $\mu^+\mu^-\pi^0$ intermediate states equal to 0.5×10^{-11}.

DOI: 10.1103/PhysRevD.65.097302 PACS number(s): 13.40.Em, 11.55.Fv, 14.60.Ef

New experimental data [1–3] allow us to calculate the contribution to the anomalous magnetic moment of the muon $a_\mu\equiv(g_\mu-2)/2$ from the $\pi^0\gamma$ and $\eta\gamma$ intermediate states in vacuum polarization with high precision. We have also found the contribution from $e^+e^-\pi^0$, $e^+e^-\eta$, and $\mu^+\mu^-\pi^0$ intermediate states.

The contribution to a_μ from the arbitrary intermediate state X (hadrons, hadrons$+\gamma$, etc.) in the vacuum polarization can be obtained via the dispersion integral

$$a_\mu=\left(\frac{\alpha m_\mu}{3\pi}\right)^2\int\frac{ds}{s^2}K(s)R(s), \qquad (1)$$

$$R(s)\equiv\frac{\sigma(e^+e^-\to X)}{\sigma(e^+e^-\to\mu^+\mu^-)},$$

$$\sigma(e^+e^-\to\mu^+\mu^-)\equiv\frac{4\pi\alpha^2}{3s},$$

$$K(s>4m_\mu^2)=\frac{3s}{m_\mu^2}\left\{x^2\left(1-\frac{x^2}{2}\right)+(1+x)^2\left(1+\frac{1}{x^2}\right)\times\left[\ln(1+x)-x+\frac{x^2}{2}\right]+\frac{1+x}{1-x}x^2\ln(x)\right\}$$

$$=\frac{3}{a^3}\left(16(a-2)\ln\frac{a}{4}-2a(8-a)-8(a^2-8a+8)\frac{\operatorname{arctanh}(\sqrt{1-a})}{\sqrt{1-a}}\right),$$

$$x=\frac{1-\sqrt{1-\frac{4m_\mu^2}{s}}}{1+\sqrt{1-\frac{4m_\mu^2}{s}}},\quad a=\frac{4m_\mu^2}{s},$$

$$K(s<4m_\mu^2)=\frac{3}{a^3}\left(16(a-2)\ln\frac{a}{4}-2a(8-a)-8(a^2-8a+8)\frac{\arctan(\sqrt{a-1})}{\sqrt{a-1}}\right).$$

Evaluating integral (1) with the trapezoidal rule for the experimental data from SND [1,2], see Fig. 1(a), we found the contribution of $\pi^0\gamma$:

$$a_\mu(\pi^0\gamma)=(46.2\pm.6\pm1.3)\times10^{-11},\quad 600\ \text{MeV}<\sqrt{s}<1039\ \text{MeV}. \qquad (2)$$

The first error is statistical, the second is systematic. For the energy region $\sqrt{s}<600$ MeV we used the theoretical formula for the cross section:

$$\sigma(e^+e^-\to\pi^0\gamma)=\frac{8\alpha f^2}{3}\left(1-\frac{m_{\pi^0}^2}{s}\right)^3\frac{1}{\left(1-\frac{s}{m_\omega^2}\right)^2}, \qquad (3)$$

where $f^2=(\pi/m_{\pi^0}^3)\Gamma_{\pi^0\to\gamma\gamma}\cong10^{-11}/\ \text{MeV}^2$ according to [4]. Equation (3) has been written in the approximation

$$\Gamma_\rho=\Gamma_\omega=0,\quad m_\rho-m_\omega=0. \qquad (4)$$

The $\gamma^*\to\pi^0\gamma$ amplitude is normalized on the $\pi^0\to\gamma\gamma$ one at $s=0$. The result is

$$a_\mu(\pi^0\gamma)=1.3\times10^{-11},\quad \sqrt{s}<600\ \text{MeV}. \qquad (5)$$

Note that the region $\sqrt{s}<2m_\mu$ gives the negligible contribution 2×10^{-13}. We neglect the small errors dealing with the experimental error in the width $\Gamma_{\pi^0\to\gamma\gamma}$ (7%) and the approximation (4) (1.5%).

Equation (3) agrees with the data in the energy region $\sqrt{s}<700$ MeV; at higher energies the approximation (4) does not work carefully, see Fig. 1(b). If we use the pointlike model, as in [5], we will get Eq. (3) without factor $[1-s/m_\omega^2]^{-2}$. This formula predicts the contribution from low energies several times less than Eq. (5); see also Fig. 1(b).

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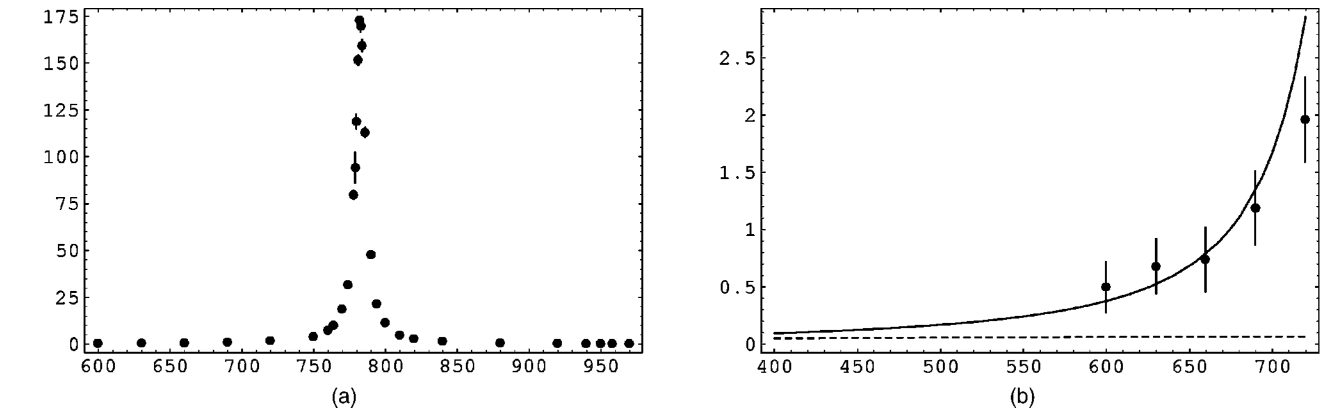

FIG. 1. (a) Plot of the dependence $\sigma(e^+e^-\to\pi^0\gamma)$, in nb upon $\sqrt{s}$ in MeV (SND experimental data). (b) Comparison of the theoretical formulas for $\sigma(e^+e^-\to\pi^0\gamma)$. Equation (3) is shown with the solid line; the pointlike model prediction is shown with the dashed line.

Treating the data from CMD-2 [3] in the same way, we get a contribution of $\eta\gamma$:

$$a_\mu(\eta\gamma)=(7.1\pm.2\pm.3)\times10^{-11},\quad 720\ \text{MeV}<\sqrt{s}<1040\ \text{MeV}. \tag{6}$$

According to the quark model (and the model of vector dominance also), the energy region $\sqrt{s}<720$ MeV is dominated by the ρ resonance, hence $\sigma(e^+e^-\to\eta\gamma)\cong\sigma(e^+e^-\to\rho\to\eta\gamma)$. So we change Eq. (3) according to this fact, take into account the ρ width, and get the small contribution:

$$a_\mu(\eta\gamma)=0.1\times10^{-11},\quad \sqrt{s}<720\ \text{MeV}. \tag{7}$$

Summing Eqs. (2), (5), (6), and (7), we can write

$$a_\mu(\pi^0\gamma)+a_\mu(\eta\gamma)=(54.7\pm0.6\pm1.4)\times10^{-11}, \tag{8}$$

where statistical and systematic errors are separately added in quadrature. In Table I we present our results with statistical and systematic errors added in quadrature. Comparing Eq. (8) with the analogous calculation in [5] (see Table I), one can see that our result is 27% more and the error is 2.5 times less. The contribution (8) accounts for 1.37 of the projected error of the E821 experiment at Brookhaven National Laboratory (40×10^{-11}) or 36% of the reached accuracy $(150\times10^{-11}$ [6]).

We can also take into account the intermediate state $\pi^0e^+e^-$, using the obvious relation

$$\sigma(e^+e^-\to\pi^0e^+e^-,s)=\frac{2}{\pi}\int_{2m_e}^{\sqrt{s}-m_{\pi^0}}\frac{dm}{m^2}\Gamma_{\gamma*\to e^+e^-}(m)\times\sigma(e^+e^-\to\pi^0\gamma*,s,m), \tag{9}$$

where m is the invariant mass of the e^+e^- system, $\Gamma_{\gamma*\to e^+e^-}(m)=(1/2)\alpha\beta_e m(1-\beta_e^2/3)$, $\beta_e=\sqrt{1-4m_e^2/m^2}$, $\sigma(e^+e^-\to\pi^0\gamma*,s,m)=[p(m)/p(0)]^3\sigma(e^+e^-\to\pi^0\gamma,s)$, and $p(m)=(\sqrt{s}/2)\sqrt{[1-(m_{\pi^0}+m)^2/s][1-(m_{\pi^0}-m)^2/s]}$ is the momentum of $\gamma*$ in s.c.m.

In the same way we can calculate $a_\mu(\mu^+\mu^-\pi^0)$ and $a_\mu(e^+e^-\eta)$. The result is

$$a_\mu(e^+e^-\pi^0)+a_\mu(\mu^+\mu^-\pi^0)+a_\mu(e^+e^-\eta)=(0.4+0.026+0.057)\times10^{-11}=0.5\times10^{-11}. \tag{10}$$

Note that if $m\gtrsim m_\rho$ we have the effect of the excitation of resonances in the reaction $e^+e^-\to\pi^0(\rho,\omega)\to\pi^0e^+e^-$. However, this effect increases the final result (10) less than by 10% because of the factor $[p(m)/p(0)]^3$, which suppresses the high m. So we ignore this correction. We also neglect $a_\mu(\mu^+\mu^-\eta)=2\times10^{-14}$.

As it was noted in [5] and [7], it is necessary to take into account also

$$a_\mu(\text{hadrons}+\gamma,\text{rest})=a_\mu(\pi^+\pi^-\gamma)+a_\mu(\pi^0\pi^0\gamma)+a_\mu(\text{hadrons}+\gamma,s>1.2\ \text{GeV}^2).$$

We take $a_\mu(\pi^+\pi^-\gamma)=(38.6\pm1.0)\times10^{-11}$ from [7] (see also [5]), $a_\mu(\pi^0\pi^0\gamma)+a_\mu(\text{hadrons}+\gamma,s>1.2\ \text{GeV}^2)=(4\pm1)\times10^{-11}$ from [5]. Adding this to Eq. (8), we get

$$a_\mu(\text{hadrons}+\gamma,\text{total})=(97.3\pm2.1)\times10^{-11}. \tag{11}$$

The contribution (11) accounts for 2.43 of the projected error of the E821 experiment or 65% of the reached accuracy. In fact, the errors in Eqs. (8) and (11) are negligible for any imaginable $(g-2)_\mu$ measurement in the near future.

This work was supported in part by RFBR, Grant No. 02-02-16061.

TABLE I. Contribution to $a_\mu\times10^{11}$.

State	Our value	Ref. [5]
$\pi^0\gamma$	47.5 ± 1.4	37 ± 3
$\eta\gamma$	$7.2\pm.4$	6.1 ± 1.4
$\pi^0\gamma+\eta\gamma$	54.7 ± 1.5	43 ± 4
hadrons$+\gamma$, total	97.3 ± 2.1	93 ± 11

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