General theory of quantum field mixing

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(Received 12 December 2001; published 20 May 2002)

We present a general theory of mixing for an arbitrary number of fields with integer or half-integer spin. The time dynamics of the interacting fields is solved and the Fock space for interacting fields is explicitly constructed. The unitary inequivalence of the Fock space of base (unmixed) eigenstates and the physical mixed eigenstates is shown by a straightforward algebraic method for any number of flavors in boson or fermion statistics. The oscillation formulas based on the nonperturbative vacuum are derived in a unified general formulation and then applied to both two- and three-flavor cases. Especially, the mixing of spin-1 (vector) mesons and the Cabibbo-Kobayashi-Maskawa mixing phenomena in the standard model are discussed, emphasizing the nonperturbative vacuum effect in quantum field theory.

DOI: 10.1103/PhysRevD.65.096015 PACS number(s): 11.10.−z, 11.15.Tk, 12.15.Ff, 14.60.Pq

I. INTRODUCTION

The mixing of quantum fields plays an important role in the phenomenology of high-energy physics [1–4]. Mixings of both $K^0\overline{K}^0$ and $B^0\overline{B}^0$ bosons provide the evidence of CP violation in the weak interaction [5], and $\eta\eta'$ boson mixing [6] in the SU(3) flavor group provides a unique opportunity to investigate the nontrivial QCD vacuum and fill the gap between QCD and the constituent quark model. In the fermion sector, neutrino mixing and oscillations are the likely resolution of the famous solar neutrino puzzle [7–10]. In addition, the standard model incorporates the mixing of fermion fields through the Cabibbo-Kobayashi-Maskawa (CKM) mixing of three quark flavors, a generalization of the original Cabibbo mixing matrix between the d and s quarks [11–14]. Therefore, careful theoretical analyses of the mixing problem in quantum field theory are an important step toward understanding the many-body aspects of high-energy phenomena and their relationship to other areas of physics involving phase transitions.

Moreover, the theory of mixing fields touches important, yet not fully answered, fundamental questions about the quantization of the interacting fields. The mixing transformation introduces very nontrivial relationships between the interacting and noninteracting (free) fields, which lead to a unitary inequivalence between the two Fock spaces [15,17] of the interacting fields and the free fields. This is different from the perturbation theories where the vacuum state of interacting fields is equal to the vacuum of free fields up to a less essential phase factor e^{iS_0} [18–20]. The mixing of quantum fields is one of the cases that can be solved nonperturbatively in the quantum field theory. Thus, it also allows us to investigate the accuracy of perturbation theory. For instance, the dynamics of a mixed-field Hamiltonian can be used for a partial summation of regular perturbation series as well as an improvement of the accuracy in perturbation theory.

Recently, the importance of the mixing transformations has prompted a fundamental examination of them from a quantum field theoretic perspective. The investigation of two-field unitary mixing in the fermion case demonstrated a rich structure of the interacting-field vacuum as an SU(2) coherent state and altered the oscillation formula including the antiparticle degrees of freedom. Momentum dependence of mixing, the existence of correlated antiparticle beam, and additional high-frequency oscillation terms have been found and at the same time the vacuum condensates have been analyzed for fermions [16,21–25]. Subsequent analyses for the boson case revealed similar features but much more complicated vacuum structure for interacting fields [15,26,27]. Especially, the pole structure in the inner product between the mass vacuum and the flavor vacuum was found and related to the convergence limit of perturbation series [27]. Attempts to look at the mixings of the three-fermion case have also been carried out [16,22,25].

In this paper, we extend the previous analyses of mixing phenomena and work out a unified theoretical framework for an arbitrary number of flavors with any integer (bosons) or half-integer (fermions) spin statistics. We build the representation of mixing transformation in the Fock space of quantum fields and demonstrate how this can be used to obtain exact oscillation effects. We then use the developed framework to carry out calculations of two-field and three-field unitary mixings for the typical spin (i.e., 0, 1/2, and 1) cases. We also comment on the use of a mixed-field solution to improve the perturbation series of mixing effects.

The paper is organized as follows. In Sec. II, we define the ladder operators for flavor fields and explicitly show the unitary inequivalence between the flavor Fock space and the Fock space of mass eigenstates. In Sec. III, we find the time dynamics of the flavor ladder operators and derive general expressions for the particle condensations and the number operators as functions of time. We also present some remarks on the Green-function method in the mixing problem. We then specifically consider, in Sec. IV, the mixing of two spin-1 fields (vector mesons) along with the mixing of spin-$\frac{1}{2}$ fields and show the consistency with previously known results. A summary and conclusion follow in Sec. V. In Appendix A, the mixing parameters are shown explicitly for spin 0, $\frac{1}{2}$, and 1. In Appendix B, we present a derivation of the flavor vacuum state by solving an infinite system of coupled equations which appears as a condition of the vacuum annihilation. In Appendix C, we summarize our results of the three-field mixing for spin 0, $\frac{1}{2}$, and 1 using the SU(3) Wolfenstein parametrization.

II. THE THEORY OF QUANTUM FIELD MIXINGS

In this section, we consider the mixing problem for N fields of fermions or bosons. To discuss the dynamics of the flavor (mixed) fields, we define a flavor field ϕ_μ as a mixture of the free fields φ_j $(j=1,2,\ldots,N)$; i.e.,

$$\phi_\mu=\sum_j U_{\mu j}\varphi_j, \tag{2.1}$$

where $U_{\mu j}$ is a unitary mixing matrix element. We use the latin indices $i,j,k,\ldots$ to label the fields of mass eigenstates and the greek indices $\mu,\nu,\xi,\ldots$ to label the flavor fields. We also denote $\overline{\phi}$ and $\overline{\varphi}$ as the entire columns $\overline{\phi}=(\phi_1,\phi_2,\ldots,\phi_N)^\top$ and $\overline{\varphi}=(\varphi_1,\varphi_2,\ldots,\varphi_N)^\top$, respectively. The evolution of the fields ϕ_μ is generated by the Hamiltonian of the form[1]

$$H(\overline{\phi})=H_{\text{free}}(\overline{\varphi})=H_{\text{free}}(U^\dagger\overline{\phi})=H_0(\overline{\phi})+\overline{\phi}^\dagger\mathcal{M}\overline{\phi}, \tag{2.2}$$

where $H_{\text{free}}(\overline{\varphi})$ is the free field Hamiltonian for φ_i with the corresponding mass eigenvalues m_i, $H_0(\overline{\phi})$ is the free flavor field Hamiltonian, and $\mathcal{M}$ is a mixing matrix.

The existence of the explicit relationship between free (φ) and flavor (ϕ) fields, given by Eq. (2.1), allows us to work out the quantum-field theoretical solution to the problem given by

$$\frac{d}{dt}\phi_\mu=i[H(\overline{\phi}),\phi_\mu]. \tag{2.3}$$

In fact, the solution of Eq. (2.3) is contained in Eq. (2.1) with the free field (φ_i) given by

$$\varphi_i=\sum_\sigma\int\frac{d\vec{k}}{\sqrt{2\epsilon_{i\vec{k}}}}[u^i_{\vec{k}\sigma}a_{i\vec{k}\sigma}(t)e^{i\vec{k}\vec{x}}+v^i_{\vec{k}\sigma}b^\dagger_{i\vec{k}\sigma}(t)e^{-i\vec{k}\vec{x}}], \tag{2.4}$$

where $a_{i\vec{k}\sigma}(t)=e^{-i\epsilon_{i\vec{k}}t}a_{i\vec{k}\sigma}$ and $b_{i\vec{k}\sigma}(t)=e^{-i\epsilon_{i\vec{k}}t}b_{i\vec{k}\sigma}$ with the standard equal time commutation/anticommutation relationships for bosons/fermions, i.e.,

$$[a_\alpha(t),a^\dagger_{\alpha'}(t)]_\pm=[b_\alpha(t),b^\dagger_{\alpha'}(t)]_\pm=\delta_{\alpha,\alpha'}.$$

In Eq. (2.4), $u^i_{\vec{k}\sigma}$ and $v^i_{\vec{k}\sigma}$ are the free particle and antiparticle amplitudes, respectively, and σ is the helicity quantum number given by

$$(\vec{n}\cdot\vec{s})u^i_{\vec{k}\sigma}=\sigma u^i_{\vec{k}\sigma},\quad(\vec{n}\cdot\vec{s})v^i_{\vec{k}\sigma}=\sigma v^i_{\vec{k}\sigma}, \tag{2.5}$$

where $\vec{s}$ is the spin operator and $\vec{n}=\vec{k}/|\vec{k}|$. We also define the following parameters that are useful in extracting the ladder operators from the field operators:

$$H^{\mu j}_{\vec{k}\sigma}\delta_{\sigma,\sigma'}=u^{\mu\dagger}_{\vec{k}\sigma}u^j_{\vec{k}\sigma'}=v^{\mu\dagger}_{-\vec{k}-\sigma}v^j_{-\vec{k}-\sigma'},$$
$$h^{\mu j}_{\vec{k}\sigma}\delta_{\sigma,\sigma'}=u^{\mu\dagger}_{\vec{k}\sigma}v^j_{-\vec{k}-\sigma'}. \tag{2.6}$$

For the analysis of arbitrary flavor mass parametrizations, we use the general notation given by Eq. (2.6) including both flavor and mass degrees of freedom. Although both indices μ and j are numbers running from 1 to N, the mass for the first index should be used as the flavor mass while the second index is for the mass eigenvalue m_j. One should note that H and h are both symmetric for bosons while H is symmetric and h is antisymmetric for fermions. The explicit representations of H and h are presented in the Appendix A for the spin-0, -$\frac{1}{2}$, and -1 cases.

Now, if $\Lambda(U,t)$ is the representation of the mixing transformation defined in the equal-time quantization, then

$$\overline{\phi}(t)=U\overline{\varphi}(t)=\Lambda(U,t)^\dagger\overline{\varphi}(t)\Lambda(U,t). \tag{2.7}$$

In the associate Fock space, this corresponds to

$$|\alpha,t\rangle_f=\Lambda(U,t)^\dagger|\alpha,t\rangle_m, \tag{2.8}$$

where the subscript $f(m)$ is used to denote the flavor (mass) Fock space. For the given time t, Eq. (2.2) can then be written as

$$H(\overline{\phi}(t))=\Lambda(U,t)^\dagger H(\overline{\varphi}(t))\Lambda(U,t). \tag{2.9}$$

As noticed from the two-field mixing analysis [15,16,26,27], $H(\overline{\phi}(t))$ and $H(\overline{\varphi}(t))$ cannot be in general related by the same operator at all times so that $\Lambda(U,t)$ is essentially time-dependent. The vacuum state of the flavor fields, defined as the state with the minimum energy, is $\Lambda(U,t)^\dagger|0\rangle_m$ and changes with time, satisfying

$$\begin{aligned}&{}_f\langle\alpha|H(\overline{\phi}(t))|\alpha\rangle_f\\&={}_m\langle\alpha|H(\overline{\varphi}(t))|\alpha\rangle_m\geqslant{}_m\langle0|H(\overline{\varphi}(t))|0\rangle_m\\&={}_f\langle0|H(\overline{\phi}(t))|0\rangle_f.\end{aligned} \tag{2.10}$$

We now define the ladder operators for the flavor fields as $\tilde{a}_{\mu=i,\vec{k}\sigma}(t)=\Lambda(U,t)^\dagger a_{i\vec{k}\sigma}(t)\Lambda(U,t)$. Using linearity of the mixing transformation, we then can solve the explicit structure of $\tilde{a}_{\mu\vec{k}\sigma}(t)$ without finding $\Lambda(U,t)$ itself.

Such an approach in fact has been known for some time for the fermion case [25], where it was noticed that fermion ladder operators for spin $\frac{1}{2}$ can be extracted from quantum fields by means of

$$a_{i\vec{k}\sigma}(t)=\frac{\sqrt{2\epsilon_{i\vec{k}}}}{H^{ii}_{\vec{k}\sigma}}u^{i\dagger}_{\vec{k}\sigma}\varphi_{i\vec{k}}(t),$$

[1]When there is an additional interaction Hamiltonian for $\overline{\phi}$ given by $H_I=\overline{\phi}^\dagger W\overline{\phi}$, the Hamiltonian of the system is, of course, extended to $\tilde{H}(\overline{\phi})=H(\overline{\phi})+H_I=H_0(\overline{\phi})+\overline{\phi}^\dagger\mathcal{M}\overline{\phi}+\overline{\phi}^\dagger W\overline{\phi}$. Then $H_{\text{free}}(\overline{\varphi})$ is also extended to $H_{\text{free}}(\overline{\varphi})+\overline{\varphi}^\dagger U^\dagger WU\overline{\varphi}$.

$$b_{i-\vec{k}-\sigma}(t)=\left[\frac{\sqrt{2\epsilon_{i\vec{k}}}}{H^{ii}_{\vec{k}\sigma}}v^{i\dagger}_{-\vec{k}-\sigma}\varphi_{i\vec{k}}(t)\right]^{\dagger}. \tag{2.11}$$

Since the Fourier component

$$\varphi_{i\vec{k}}(t)=\Sigma_{\sigma}(1/\sqrt{2\epsilon_{i\vec{k}}})[u^{i}_{\vec{k}\sigma}a_{i\vec{k}\sigma}(t)+v^{i}_{-\vec{k}\sigma}b^{\dagger}_{i-\vec{k}\sigma}(t)]$$

is obviously a linear combination of $\varphi_i(\vec{x},t)$, one can express ladder operators as linear combinations of the initial fields. Using the linearity of Eq. (2.7), we get

$$\begin{aligned}\tilde{a}_{\mu\vec{k}\sigma}(t)&=\frac{\sqrt{2\epsilon_{\mu\vec{k}}}}{H^{\mu\mu}_{\vec{k}\sigma}}u^{\mu\dagger}_{\vec{k}\sigma}[\Lambda(U,t)^{\dagger}\overline{\varphi}_{\vec{k}}(t)\Lambda(U,t)]_{\mu}\\&=\sum_j \frac{\sqrt{2\epsilon_{\mu\vec{k}}}}{H^{\mu\mu}_{\vec{k}\sigma}}u^{\mu\dagger}_{\vec{k}\sigma}U_{\mu j}\varphi_{j\vec{k}}(t),\\ \tilde{b}_{\mu-\vec{k}-\sigma}(t)&=\sum_j \frac{\sqrt{2\epsilon_{\mu\vec{k}}}}{H^{\mu\mu}_{\vec{k}\sigma}}U^{*}_{\mu j}\varphi^{\dagger}_{j\vec{k}}(t)v^{\mu}_{-\vec{k}-\sigma}.\end{aligned} \tag{2.12}$$

For the bosons, however, the ladder operators are not separated as in the fermion case, e.g.,

$$u^{i\dagger}_{\vec{k}\sigma}\varphi_{i\vec{k}}(t)=\frac{1}{\sqrt{2\epsilon_{i\vec{k}}}}[a_{i\vec{k}\sigma}(t)+h^{ii}_{\vec{k}\sigma}b^{\dagger}_{i-\vec{k}-\sigma}(t)] \tag{2.13}$$

and in general $h^{ii}_{\vec{k}\sigma}\neq 0$. Equation (2.13) implies that particles and antiparticles in the boson case cannot be distinguished unless time dynamics is considered. To deal with this problem, we define ladder operators for bosons by

$$\begin{aligned}a_{i\vec{k}\sigma}&=u^{i\dagger}_{\vec{k}\sigma}\left(\sqrt{\frac{\epsilon_{i\vec{k}}}{2}}\varphi_{i\vec{k}}(t)+\frac{1}{\sqrt{2\epsilon_{i\vec{k}}}}\dot{\varphi}_{i\vec{k}}(t)\right),\\ b^{\dagger}_{i-\vec{k}-\sigma}&=v^{i\dagger}_{-\vec{k}-\sigma}\left(\sqrt{\frac{\epsilon_{i\vec{k}}}{2}}\varphi_{i\vec{k}}(t)-\frac{1}{\sqrt{2\epsilon_{i\vec{k}}}}\dot{\varphi}_{i\vec{k}}(t)\right).\end{aligned} \tag{2.14}$$

With Eqs. (2.11) and (2.14), we then derive for fermions[2]

$$\begin{aligned}\tilde{a}_{\mu}&=\frac{\sqrt{2\epsilon_{\mu}}}{H^{\mu\mu}}\sum_{j,\sigma'}(u^{\mu\dagger}_{\vec{k}\sigma}u^{j}_{\vec{k}\sigma'}a_j+u^{\mu\dagger}_{\vec{k}\sigma}v^{j}_{-\vec{k}-\sigma'}b^{\dagger}_{-j})\frac{U_{\mu j}}{\sqrt{2\epsilon_j}}\\&=\sum_j\left(\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{H^{\mu j}}{H^{\mu\mu}}U_{\mu j}a_j+\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{h^{\mu j}}{H^{\mu\mu}}U_{\mu j}b^{\dagger}_{-j}\right),\end{aligned} \tag{2.15}$$

$$\begin{aligned}\tilde{b}_{-\mu}&=\frac{\sqrt{2\epsilon_{\mu}}}{H^{\mu\mu}}\sum_{j,\sigma'}[(v^{\mu\dagger}_{-\vec{k}-\sigma}u^{j}_{\vec{k}\sigma'})^{*}a^{\dagger}_j\\&\quad+(v^{\mu\dagger}_{-\vec{k}-\sigma}v^{j}_{-\vec{k}-\sigma'})^{*}b_{-j}]\frac{U^{*}_{\mu j}}{\sqrt{2\epsilon_j}}\\&=\sum_j\left(\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{(H^{\mu j})^{*}}{H^{\mu\mu}}U^{*}_{\mu j}b_{-j}\right.\\&\quad\left.-\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{(h^{\mu j})^{*}}{H^{\mu\mu}}U^{*}_{\mu j}a^{\dagger}_j\right),\end{aligned} \tag{2.16}$$

and for bosons,

$$\begin{aligned}\tilde{a}_{\mu}&=\frac{\sqrt{2\epsilon_{\mu}}}{2}\sum_{j,\sigma'}\left(u^{\mu\dagger}_{\vec{k}\sigma}u^{j}_{\vec{k}\sigma'}\frac{\epsilon_\mu+\epsilon_j}{\epsilon_\mu}a_j\right.\\&\quad\left.+u^{\mu\dagger}_{\vec{k}\sigma}v^{j}_{-\vec{k}-\sigma'}\frac{\epsilon_\mu-\epsilon_j}{\epsilon_\mu}b^{\dagger}_{-j}\right)\frac{U_{\mu j}}{\sqrt{2\epsilon_j}}\\&=\sum_j\left(\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}+\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2}H^{\mu j}U_{\mu j}a_j\right.\\&\quad\left.+\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}-\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2}h^{\mu j}U_{\mu j}b^{\dagger}_{-j}\right),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\tilde{b}_{-\mu}&=\frac{\sqrt{2\epsilon_{\mu}}}{2}\sum_{j,\sigma'}\left(v^{\mu\dagger}_{-\vec{k}-\sigma}u^{j}_{\vec{k}\sigma'}\frac{\epsilon_\mu-\epsilon_j}{\epsilon_\mu}a^{\dagger}_j\right.\\&\quad\left.+v^{\mu\dagger}_{-\vec{k}-\sigma}v^{j}_{-\vec{k}-\sigma'}\frac{\epsilon_\mu+\epsilon_j}{\epsilon_\mu}b_{-j}\right)^{*}\frac{U^{*}_{\mu j}}{\sqrt{2\epsilon_j}}\\&=\sum_j\left(\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}+\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2}(H^{\mu j})^{*}U^{*}_{\mu j}b_{-j}\right.\\&\quad\left.+\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}-\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2}(h^{\mu j})^{*}U^{*}_{\mu j}a^{\dagger}_j\right).\end{aligned} \tag{2.18}$$

Denoting the spin of the mixed fields as S, we can unify the expressions for both fermion and boson in an identical form as

$$\begin{aligned}\tilde{a}_{\mu}&=\sum_j(\alpha_{\mu j}a_j+\beta_{\mu j}b^{\dagger}_{-j}),\\ \tilde{b}_{-\mu}&=\sum_j[\alpha^{*}_{\mu j}b_{-j}+(-1)^{2S}\beta^{*}_{\mu j}a^{\dagger}_j],\end{aligned} \tag{2.19}$$

by defining

$$\alpha_{\mu j}=\gamma^{+}_{\mu j}U_{\mu j},\quad \beta_{\mu j}=\gamma^{-}_{\mu j}U_{\mu j}, \tag{2.20}$$

[2]Here, we abbreviate the notations $a_{j\vec{k}\sigma}$ and $b^{\dagger}_{j-\vec{k}-\sigma}$ as a_j and $b^{\dagger}_{-j}$, respectively. A similar abbreviation is used for $\tilde{a}_{\mu}$ and $\tilde{b}_{-\mu}$.

where

$$\gamma^{+}_{\mu j}=\begin{cases}\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{H^{\mu j}}{H^{\mu\mu}} & \text{fermions,}\\ H^{\mu j}\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}+\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2} & \text{bosons,}\end{cases}$$

$$\gamma^{-}_{\mu j}=\begin{cases}\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{h^{\mu j}}{H^{\mu\mu}} & \text{fermions,}\\ h^{\mu j}\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}-\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2} & \text{bosons.}\end{cases} \tag{2.21}$$

We also note from unitarity that

$$\begin{cases}|\alpha_{\mu j}|^2+|\beta_{\mu j}|^2=|U_{\mu j}|^2, & \text{fermions,}\\ |\alpha_{\mu j}|^2-|\beta_{\mu j}|^2=|U_{\mu j}|^2, & \text{bosons,}\end{cases} \tag{2.22}$$

so that one can treat $\alpha_{\mu j}$ and $\beta_{\mu j}$ as cosine and sine for fermions (cosh and sinh for bosons), respectively,

$$\alpha_{\mu j}=U_{\mu j}\begin{cases}\cos(\theta_{\mu j}) & \text{fermions,}\\ \cosh(\theta_{\mu j}) & \text{bosons,}\end{cases}$$

$$\beta_{\mu j}=U_{\mu j}\begin{cases}\sin(\theta_{\mu j}) & \text{fermions,}\\ \sinh(\theta_{\mu j}) & \text{bosons.}\end{cases} \tag{2.23}$$

From the fact that Eqs. (2.15)–(2.18) serve as the mixing group representation, one can conclude that

$$\theta_{\mu j}-\theta_{\mu j'}=\theta_{j'j} \tag{2.24}$$

regardless of m_μ. Using the formulas, presented in Appendix A, this can be explicitly verified for $S=0,\ \frac{1}{2}$, and 1 by calculating, for example, $\partial\gamma^{-}_{\mu j}/\partial m_\mu$. In every case, $\partial\gamma^{-}_{\mu j}/\partial m_\mu$ can be reduced to $\partial\gamma^{-}_{\mu j}/\partial m_\mu=\gamma^{+}_{\mu j}f(m_\mu)$, e.g., for fermions,

$$\frac{\partial\theta_{\mu j'}}{\partial m_\mu}-\frac{\partial\theta_{\mu j}}{\partial m_\mu}=\frac{\frac{\partial\sin(\theta_{\mu j'})}{\partial m_\mu}}{\cos(\theta_{\mu j'})}-\frac{\frac{\partial\sin(\theta_{\mu j})}{\partial m_\mu}}{\cos(\theta_{\mu j})}$$
$$=f(m_\mu)-f(m_\mu)=0$$

so that $\theta_{\mu j}=\theta_\mu-\theta_j$, where $\cos(\theta_\mu)=(1/2\sqrt{\epsilon_\mu})(\sqrt{\epsilon_\mu+m_\mu}+\sqrt{\epsilon_\mu-m_\mu})$ and $\sin(\theta_\mu)=(1/2\sqrt{\epsilon_\mu})(\sqrt{\epsilon_\mu+m_\mu}-\sqrt{\epsilon_\mu-m_\mu})$.

The introduced ladder operators are consistent with the representation of the mixing transformation in the Fock space:

$$\begin{aligned}|\alpha_\mu+1,t\rangle_f&=\tilde{a}^\dagger_\mu(t)|\alpha_\mu,t\rangle_f\\&=\Lambda(U,t)^\dagger a^\dagger_i(t)\Lambda(U,t)\Lambda(U,t)^\dagger|\alpha_i,t\rangle_m\\&=\Lambda(U,t)^\dagger|\alpha_i+1,t\rangle_m,\end{aligned} \tag{2.25}$$

and the flavor vacuum state satisfies

$$\tilde{a}_\mu(t)|0,t\rangle_f=\Lambda(U,t)^\dagger a_i(t)\Lambda(U,t)\Lambda(U,t)^\dagger|0\rangle_m=0. \tag{2.26}$$

While Eq. (2.12) may be viewed as the result of expanding flavor fields $\phi_\mu(x)$ in the basis parametrized by free-field mass m_i, it was noticed that one may as well expand flavor fields in the basis with the flavor mass parameters m_μ which correspond to choosing $u^\mu_{\vec{k}\sigma},v^\mu_{-\vec{k}\sigma}$ as free-field amplitudes with the flavor mass (m_μ) in Eqs. (2.11) and (2.14) [24].

In other words, for any $\Lambda(U,t)$, $\Lambda'(U,t)=I(t)^{-1}\Lambda(U,t)I(t)$, which can be obtained by means of a similarity transformation mixing $\tilde{a}_{\mu\vec{k}\sigma}(t)$ and $\tilde{b}^\dagger_{\mu-\vec{k}-\sigma}(t)$ but leaving their combination in $\phi(\vec{k})$ unchanged [i.e., $\phi_\mu(\vec{k},t)=I(t)^{-1}\phi_\mu(\vec{k},t)I(t)$], is also a representation of the mixing group. The ladder operators, defined by Eqs. (2.15)–(2.18), therefore depend on the choice of $I(t)$ or, equivalently, the "bare" mass m_μ assigned to the flavor fields, which is called as a mass parametrization.

Although there are different opinions about whether or not the measurable results of the theory depend on the mass parameters [23,24,26,27] we note that the mass parametrization problem indeed is not specific to the quantum mixing, but can be revealed in the free-field case as well as in the perturbation theory. As discussed in [27], when dealing with the free-field problem defined by the free Hamiltonian

$$:H_0:=\sum_{\vec{k}\sigma}(\epsilon_{\vec{k}}a^\dagger_{\vec{k}\sigma}a_{\vec{k}\sigma}+\epsilon_{\vec{k}}b^\dagger_{\vec{k}\sigma}b_{\vec{k}\sigma}), \tag{2.27}$$

one may still consider the change of the mass parametrization $m\to m_\mu$ defined in [26] by

$$\begin{pmatrix}\tilde{a}\\ \tilde{b}^\dagger\end{pmatrix}=I^{-1}(t)\begin{pmatrix}a\\ b^\dagger\end{pmatrix}I(t)$$
$$=\begin{pmatrix}e^{i(\tilde{\epsilon}_{\vec{k}}-\epsilon_{\vec{k}})t}\rho^*_{\vec{k}} & e^{i(\tilde{\epsilon}_{\vec{k}}+\epsilon_{\vec{k}})t}\lambda_{\vec{k}}\\ e^{-i(\tilde{\epsilon}_{\vec{k}}+\epsilon_{\vec{k}})t}\lambda^*_{\vec{k}} & e^{-i(\tilde{\epsilon}_{\vec{k}}-\epsilon_{\vec{k}})t}\rho_{\vec{k}}\end{pmatrix}\begin{pmatrix}a(0)\\ b^\dagger(0)\end{pmatrix}, \tag{2.28}$$

where $\tilde{\epsilon}_k=\sqrt{k^2+m_\mu^2}$ and $\epsilon_k=\sqrt{k^2+m^2}$. Indeed, as we observe in [27], the number operator for the free fields in this transformation is not conserved, e.g., for fermions,

$$\langle\tilde{N}\rangle=|\{\tilde{a},\tilde{a}^\dagger(t)\}|^2=||\rho_k|^2e^{-i\epsilon_k t}+|\lambda_k|^2e^{i\epsilon_k t}|^2, \tag{2.29}$$

which may lead to the obviously wrong conclusion that the number of particles in the free-field case is not an observable quantity.

This can also be understood mathematically once we note that the above transformation is equivalent to the splitting of the initial Lagrangian into

$$L_0=L_0'+L_I'$$
$$=\int d^3p[\{(\hat{p}\psi)^\dagger(\hat{p}\psi)-m_\mu^2\psi^\dagger\psi\}$$
$$+(m_\mu^2-m^2)\psi^\dagger\psi], \tag{2.30}$$

where an additional self-interaction term, responsible for the oscillation of $\langle\tilde{N}\rangle$, appears. Physically, the transformation, given by Eq. (2.28), can be viewed as a redefinition of the physical one-particle states. The tilde quantities correspond then to some new quasiparticle objects so that the tilde number operator describes a different type of particles and thus it does not have to be invariant under such transformation. Nevertheless, the charge quantum number is still conserved in the transformation, given by Eq. (2.28). The situation here may be analogous to the representation of physical observables under the change of coordinate systems. Although the Casimir operator (e.g., $\vec{S}^2$ in the spin observables) must be independent of the coordinate system, other physical operators (e.g., S_x, S_y, and S_z) do depend on the coordinate system. To compare the eigenvalue of S_z between theory and experiment, one should first fix the coordinate system. Similarly, we think specific mass parameters should be selected from the physical reasoning to compare theoretical results (e.g., the occupation number expectation) with experiments.

From the above example, it is clear that the same mass parametrization problem is also present in the regular perturbation theory once one attempts to redefine the physical one-particle states as shown in Eq. (2.28). Indeed, in the free theory and the perturbation theory this issue is resolved by the presence of the mass scale of well-defined asymptotic physical states, which therefore fix the mass parameters. In the mixing problem, however, at least two feasible mass scales may be suggested either by the mass scale of the energy eigenstates or by the flavor mass scale which corresponds to no self-interaction term in the Hamiltonian, given by Eq. (2.2), and thus further discussion of this issue in the mixing problem is clearly necessary. We think the mass eigenvalues that can be measured from the experiments may be the natural choice for the mass scale in the given physical system.

In any case, all the above unified formulation for any number of fields with integer or half-integer spin holds for the arbitrary mass parameter m_μ when $\epsilon_i=\sqrt{k^2+m_i^2}$ and $\epsilon_\mu=\sqrt{k^2+m_\mu^2}$ in Eqs. (2.19)–(2.21) are understood as the energies of the free field φ_i and the flavor field ϕ_μ, respectively.

In the rest of this section, let us consider the explicit form of the flavor vacuum state. We obtain its structure by solving directly the infinite set of equations

$$\tilde{a}_\nu|0\rangle_f=0, \quad \tilde{b}_\nu|0\rangle_f=0. \tag{2.31}$$

We can express the flavor vacuum state as a linear combination of the mass eigenstates, i.e., in the most general form,

$$|0\rangle_f=\sum_{(n),(l)}\frac{1}{n_1!n_2!\cdots n_k!}B_{(n)(l)}(a_1^\dagger)^{n_1}\cdots(a_k^\dagger)^{n_k}(b_{-1}^\dagger)^{l_1}\cdots(b_{-k}^\dagger)^{l_k}|0\rangle_m, \tag{2.32}$$

with $(n)=(n_1n_2n_3\cdots)$ and $k=N$ for the mixing of N fields. After applying Eq. (2.31) to Eq. (2.32), we get an infinite set of equations given by

$$\sum_j(\alpha_{\mu j}B_{(n_j+1)(l)}+\beta_{\mu j}B_{(n)(l_j-1)})=0$$
$$\text{for all sets of } (n),(l), \tag{2.33}$$

where $(n_j\pm1)=(n_1n_2\cdots n_j\pm1\cdots)$. The solution of this problem is presented in Appendix B. For the flavor vacuum state, we find

$$|0\rangle_f=\frac{1}{\mathcal{Z}}\exp\left(\sum_{i,j=1}^N Z_{ij}a_i^\dagger b_{-j}^\dagger\right)|0\rangle_m, \tag{2.34}$$

where Z_{ij} is an (i,j) element of the matrix $\hat{Z}=-\hat{\alpha}^{-1}\hat{\beta}$. The normalization constant $\mathcal{Z}$ is fixed by ${}_f\langle0|0\rangle_f=1$; $\mathcal{Z}=\det^{1/2}(1+\hat{Z}\hat{Z}^\dagger)$ for fermions and $\mathcal{Z}=\det^{-1/2}(1-\hat{Z}\hat{Z}^\dagger)$ for bosons. The flavor Fock space is then built by applying the flavor-field creation operators $(\tilde{a}_\mu^\dagger,\tilde{b}_\nu^\dagger)$ to the vacuum state $|0\rangle_f$.

We see that the flavor vacuum state has a rich coherent structure. This situation is different from the perturbative quantum field theory, where the adiabatic enabling of interaction is present and $|0\rangle_{\text{interacting}}\sim|0\rangle_{\text{free}}$. The nonperturbative vacuum solution renders nontrivial effects in the flavor dynamics, as we will show in Sec. III. In particular, the normalization constant $\mathcal{Z}$ is always greater than 1 so that in the infinite volume limit, when the density of states is going to infinity, we have

$$\mathcal{Z}_{\text{tot}}=\exp\left(\frac{V}{(2\pi)^3}\int d\vec{k}\,\ln(\mathcal{Z}_{\vec{k}})\right)\to\infty. \tag{2.35}$$

Thus, any possible state for the flavor vacuum shall have an infinite norm in the free-field Fock space and therefore the flavor vacuum state cannot be found in original Fock space. The unitary inequivalence of the flavor Fock space and the original Fock space is therefore established, i.e., ${}_f\langle0|0\rangle_m=(1/\mathcal{Z}_{\text{tot}})\to0$. The effect is essentially due to an infinite number of momentum degrees of freedom, which is analogous to the existence of a phase transition in the infinite volume limit.

III. TIME DYNAMICS OF THE MIXED QUANTUM FIELDS

Now we have a closer look at the dynamics of quantum fields respresented by the ladder operators shown in Eq. (2.19). First of all, we note that only $a_{i\vec{k}\sigma}$ and $b_{i-\vec{k}-\sigma}$ operators and their conjugates are mixed together. We denote the set of quantum fields formed by all linear combinations of these operators and their products (algebra on $a_{i\vec{k}\sigma}$, $b_{i-\vec{k}-\sigma}$, and H.c.) as a cluster $\Omega_{\vec{k}\sigma}$ with a particular momentum $\vec{k}$ and a particular helicity σ. It follows that $\Omega_{\vec{k}\sigma}$'s are invariant under mixing transformation $\Lambda(U,t)$ and we thus can treat each cluster independently of each other.

The time dynamics of the flavor fields is determined by the nonequal time commutation/anticommutation relationships for boson/fermion fields that can be derived from Eq. (2.19) using the standard commutation/anticommutation relationships for the original ladder operators,

$$\begin{aligned}
F_{\mu\nu}(t)&=[\tilde{a}_\mu(t),\tilde{a}^\dagger_\nu]_\pm=\sum_{k,k'}(\alpha_{\mu k}\alpha^*_{\nu k'}[a_k e^{-i\epsilon_k t},a^\dagger_{k'}]_\pm+\beta_{\mu k}\beta^*_{\nu k'}[b^\dagger_{-k}e^{i\epsilon_k t},b_{-k'}]_\pm)\\
&=\sum_k[\alpha_{\mu k}\alpha^*_{\nu k}e^{-i\epsilon_k t}-(-1)^{2S}\beta_{\mu k}\beta^*_{\nu k}e^{i\epsilon_k t}],\\
&[\tilde{b}_{-\mu}(t),\tilde{b}^\dagger_{-\nu}]_\pm=F_{\nu\mu}(t),\\
G_{\mu\nu}(t)&=[\tilde{b}_{-\mu}(t),\tilde{a}_\nu]_\pm=\sum_{k,k'}\{\alpha^*_{\mu k}\beta_{\nu k'}[b_{-k}e^{-i\epsilon_k t},b^\dagger_{-k'}]_\pm+(-1)^{2S}\beta^*_{\mu k}\alpha_{\nu k'}[a^\dagger_k e^{i\epsilon_k t},a_{k'}]_\pm\}\\
&=\sum_k(\alpha^*_{\mu k}\beta_{\nu k}e^{-i\epsilon_k t}-\beta^*_{\mu k}\alpha_{\nu k}e^{i\epsilon_k t}).
\end{aligned}\tag{3.1}$$

The two matrices $\hat{F}$ and $\hat{G}$ represent the only nontrivial commutators/anticommutators in the sense that all others are either zero or can be written in terms of the elements of these matrices. It is useful to note that, for $t=0$, Eq. (3.1) shall be reduced to $F_{\mu\nu}(0)=\delta_{\mu\nu}$ and $G_{\mu\nu}(0)=0$. We also note that

$$\begin{aligned}
F_{\mu\nu}(t)^*&=F_{\nu\mu}(-t),\\
G_{\mu\nu}(t)^*&=-G_{\nu\mu}(t).
\end{aligned}\tag{3.2}$$

Equation (3.1) allows us to compute many mixing quantities directly. The time dynamics of the flavor-field ladder operators can be derived by writing them as $\tilde{a}_\mu(t)=\Sigma_\nu[f_{\mu\nu}\tilde{a}_\nu(0)+g_{\mu\nu}\tilde{b}^\dagger_{-\nu}(0)+\cdots]$. Then, one can get straightforwardly $f^*_{\mu\nu}=[\tilde{a}_\nu(0),\tilde{a}^\dagger_\mu(t)]_\pm=F_{\nu\mu}(-t)$ and $g_{\mu\nu}=[\tilde{b}_{-\nu}(0),\tilde{a}_\mu(t)]_\pm=G_{\nu\mu}(-t)$ while all other coefficients are zeros:

$$\begin{aligned}
\tilde{a}_\mu(t)&=\sum_\nu[F_{\mu\nu}(t)\tilde{a}_\nu+G_{\nu\mu}(-t)\tilde{b}^\dagger_{-\nu}],\\
\tilde{b}_{-\mu}(t)&=\sum_\nu[F_{\nu\mu}(t)\tilde{b}_{-\nu}+(-1)^{2S}G_{\mu\nu}(t)\tilde{a}^\dagger_\nu].
\end{aligned}\tag{3.3}$$

We now consider the condensate densities of the definite-mass particles in the flavor vacuum $[Z'_i={}_f\langle 0|a^\dagger_i(t)a_i(t)|0\rangle_f]$, the number of definite-flavor particles in the flavor vacuum $[Z_\nu={}_f\langle 0|\tilde{a}^\dagger_\nu(t)\tilde{a}_\nu(t)|0\rangle_f]$, and the particle number average for a single definite-flavor particle initial state, which is related in the Heisenberg picture to

$$\begin{aligned}
N_{\rho\nu\sigma}&={}_\mu\langle\rho|\tilde{a}^\dagger_\nu(t)\tilde{a}_\nu(t)|\sigma\rangle_\mu,\\
\overline{N}_{\rho\nu\sigma}&={}_\mu\langle\rho|\tilde{b}^\dagger_{-\nu}(t)\tilde{b}_{-\nu}(t)|\sigma\rangle_\mu.
\end{aligned}$$

The free-field particle condensates in the flavor vacuum state are computed from the explicit form of the ladder operators given by Eq. (2.19) as

$$Z'_i=\sum_j|\beta_{ij}|^2.\tag{3.4}$$

In the following, the particle-antiparticle symmetry should be accounted for, so that a corresponding antiparticle quantity can be found from the particle expression after a necessary substitution (particles→antiparticles and vice versa). Thus, the antiparticle condensate is given by the same quantity in Eq. (3.4). The definite-flavor particle condensates in the free-field vacuum are also given by Eq. (3.4).

Using Eq. (3.3), we get the flavor-field condensates in the flavor vacuum (Z_ν) as

$$Z_\nu(t)=\sum_\mu|G_{\nu\mu}(-t)|^2.\tag{3.5}$$

It is remarkable that this number is not zero but oscillates, demonstrating the oscillations of definite-flavor particles in the flavor vacuum. This effect reveals the unitary inequivalence of the flavor Fock spaces for different times due to the time dynamics of the flavor vacuum.

The evolution of the particle ($N_{\rho\nu\sigma}$) and antiparticle ($\overline{N}_{\rho\nu\sigma}$) number with flavor ν can be found using the standard technique of normal ordering, i.e., moving annihilation operators to the right side and creation operators to the left side of the expression. With this technique, we obtain

$$\begin{aligned}N_{\rho\nu\sigma}(t)&=[\tilde{a}_{\rho},\tilde{a}^{\dagger}_{\nu}(t)]_{\pm}[\tilde{a}_{\nu}(t),\tilde{a}^{\dagger}_{\sigma}]_{\pm}\\&\quad+\delta_{\rho\sigma}\langle 0|\tilde{a}^{\dagger}_{\nu}(t)\tilde{a}_{\nu}(t)|0\rangle\\&=F^{*}_{\nu\rho}(t)F_{\nu\sigma}(t)+\delta_{\rho\sigma}Z_{\nu}(t),\\\overline{N}_{\rho\nu\sigma}(t)&=(-1)^{2S}[\tilde{a}_{\rho},\tilde{b}_{-\nu}(t)]_{\pm}[\tilde{b}^{\dagger}_{-\nu}(t),\tilde{a}^{\dagger}_{\sigma}]_{\pm}\\&\quad+\delta_{\rho\sigma}\langle 0|\tilde{b}^{\dagger}_{\nu}(t)\tilde{b}_{\nu}(t)|0\rangle\\&=(-1)^{2S}G_{\nu\rho}(t)G_{\nu\sigma}(t)^{*}+\delta_{\rho\sigma}Z_{\nu}(t).\end{aligned}\tag{3.6}$$

The flavor charge $Q_{\rho\nu\sigma}=N_{\rho\nu\sigma}-\overline{N}_{\rho\nu\sigma}$ [22,23,26] is then given by

$$\begin{aligned}Q_{\rho\nu\sigma}&=N_{\rho\nu\sigma}-\overline{N}_{\rho\nu\sigma}\\&=F^{*}_{\nu\rho}(t)F_{\nu\sigma}(t)-(-1)^{2S}G_{\nu\rho}(t)G_{\nu\sigma}(t)^{*}.\end{aligned}\tag{3.7}$$

For a specific case of the number evolution in the beam with a fixed 3-momentum, we find

$$\begin{aligned}N_{\rho\nu\rho}&=\langle 0|\tilde{a}_{\rho}\tilde{a}^{\dagger}_{\nu}(t)\tilde{a}_{\nu}(t)\tilde{a}^{\dagger}_{\rho}|0\rangle=|F_{\nu\rho}(t)|^{2}+Z_{\nu}(t),\\\overline{N}_{\rho\nu\rho}&=\langle 0|\tilde{a}_{\rho}\tilde{b}^{\dagger}_{-\nu}(t)\tilde{b}_{-\nu}(t)\tilde{a}^{\dagger}_{\rho}|0\rangle\\&=(-1)^{2S}|G_{\nu\rho}(t)|^{2}+Z_{\nu}(t),\\Q_{\rho\nu\rho}&=|F_{\nu\rho}(t)|^{2}-(-1)^{2S}|G_{\nu\rho}(t)|^{2}.\end{aligned}\tag{3.8}$$

We note that $N_{\rho\nu\rho}$'s as well as $Q_{\rho\nu\rho}$'s are in general dependent on the choice of mass parameter m_{μ}.

We may explicitly see this in the example of the charge operator. According to Eq. (3.8), we get

$$\begin{aligned}Q_{\mu\nu\mu}&=\sum_{k,k'}[\alpha_{\mu k}\alpha^{*}_{\nu k}e^{i\epsilon_{k}t}-(-1)^{2S}\beta_{\mu k}\beta^{*}_{\nu k}e^{-i\epsilon_{k}t}][\alpha^{*}_{\mu k'}\alpha_{\nu k'}e^{-i\epsilon_{k'}t}-(-1)^{2S}\beta^{*}_{\mu k'}\beta_{\nu k'}e^{i\epsilon_{k'}t}]-(-1)^{2S}\sum_{k,k'}(\alpha^{*}_{\nu k}\beta_{\mu k}e^{-i\epsilon_{k}t}\\&\quad-\beta^{*}_{\nu k}\alpha_{\mu k}e^{i\epsilon_{k}t})(\alpha_{\nu k'}\beta^{*}_{\mu k'}e^{i\epsilon_{k'}t}-\beta_{\nu k'}\alpha^{*}_{\mu k'}e^{-i\epsilon_{k'}t})\\&=\sum_{k,k'}e^{-i(\epsilon_{k'}-\epsilon_{k})t}[\alpha^{*}_{\mu k'}\alpha_{\nu k'}\alpha_{\mu k}\alpha^{*}_{\nu k}-(-1)^{2S}\beta_{\nu k'}\alpha^{*}_{\mu k'}\beta^{*}_{\nu k}\alpha_{\mu k}]+e^{i(\epsilon_{k'}-\epsilon_{k})t}[\beta_{\mu k}\beta^{*}_{\nu k}\beta^{*}_{\mu k'}\beta_{\nu k'}-(-1)^{2S}\\&\quad\times\alpha_{\nu k'}\beta^{*}_{\mu k'}\alpha^{*}_{\nu k}\beta_{\mu k}]-(-1)^{2S}e^{-i(\epsilon_{k'}+\epsilon_{k})t}(\beta_{\mu k}\beta^{*}_{\nu k}\alpha^{*}_{\mu k'}\alpha_{\nu k'}-\alpha^{*}_{\nu k}\beta_{\mu k}\beta_{\nu k'}\alpha^{*}_{\mu k'})-(-1)^{2S}e^{i(\epsilon_{k'}+\epsilon_{k})t}\\&\quad\times[\alpha_{\mu k}\alpha^{*}_{\nu k}\beta^{*}_{\mu k'}\beta_{\nu k'}-\beta^{*}_{\nu k}\alpha_{\mu k}\alpha_{\nu k'}\beta^{*}_{\mu k'}]\\&=\sum_{k,k'}e^{-i(\epsilon_{k'}-\epsilon_{k})t}\alpha^{*}_{\mu k'}\alpha_{\mu k}[\alpha_{\nu k'}\alpha^{*}_{\nu k}-(-1)^{2S}\beta_{\nu k'}\beta^{*}_{\nu k}]-(-1)^{2S}e^{i(\epsilon_{k'}-\epsilon_{k})t}\beta_{\mu k}\beta^{*}_{\mu k'}[\alpha_{\nu k'}\alpha^{*}_{\nu k}-(-1)^{2S}\beta^{*}_{\nu k}\beta_{\nu k'}]\\&\quad-(-1)^{2S}e^{-i(\epsilon_{k'}+\epsilon_{k})t}\beta_{\mu k}\alpha^{*}_{\mu k'}(\beta^{*}_{\nu k}\alpha_{\nu k'}-\alpha^{*}_{\nu k}\beta_{\nu k'})-(-1)^{2S}e^{i(\epsilon_{k'}+\epsilon_{k})t}\alpha_{\mu k}\beta^{*}_{\mu k'}(\alpha^{*}_{\nu k}\beta_{\nu k'}-\beta^{*}_{\nu k}\alpha_{\nu k'})\\&=\sum_{k,k'}[\alpha_{\nu k'}\alpha^{*}_{\nu k}-(-1)^{2S}\beta_{\nu k'}\beta^{*}_{\nu k}][e^{-i(\epsilon_{k'}-\epsilon_{k})t}\alpha^{*}_{\mu k'}\alpha_{\mu k}-(-1)^{2S}e^{i(\epsilon_{k'}-\epsilon_{k})t}\beta_{\mu k}\beta^{*}_{\mu k'}]-(-1)^{2S}(\beta^{*}_{\nu k}\alpha_{\nu k'}\\&\quad-\alpha^{*}_{\nu k}\beta_{\nu k'})(e^{-i(\epsilon_{k'}+\epsilon_{k})t}\beta_{\mu k}\alpha^{*}_{\mu k'}-e^{i(\epsilon_{k'}+\epsilon_{k})t}\alpha_{\mu k}\beta^{*}_{\mu k'}).\end{aligned}\tag{3.9}$$

Taking into account Eq. (2.23), we can write, e.g., for fermions ($S=\frac{1}{2}$)

$$\alpha_{\nu k'}\alpha^{*}_{\nu k}+\beta_{\nu k'}\beta^{*}_{\nu k}=U_{\nu k'}U^{*}_{\nu k}[\cos(\theta_{\nu k'})\cos(\theta_{\nu k})+\sin(\theta_{\nu k'})\sin(\theta_{\nu k})]=U_{\nu k'}U^{*}_{\nu k}\cos(\theta_{\nu k'}-\theta_{\nu k})=U_{\nu k'}U^{*}_{\nu k}\cos(\theta_{kk'}),$$

$$\beta^{*}_{\nu k}\alpha_{\nu k'}-\alpha^{*}_{\nu k}\beta_{\nu k'}=U_{\nu k'}U^{*}_{\nu k}[\cos(\theta_{\nu k'})\sin(\theta_{\nu k})-\cos(\theta_{\nu k'})\sin(\theta_{\nu k})]=U_{\nu k'}U^{*}_{\nu k}\sin(\theta_{\nu k}-\theta_{\nu k'})=U_{\nu k'}U^{*}_{\nu k}\sin(\theta_{k'k}).$$

Thus, we find

$$\begin{aligned}Q_{\mu\nu\mu}&=\sum_{k,k'}U_{\nu k'}U^{*}_{\nu k}U_{\mu k}U^{*}_{\mu k'}[\cos^{2}(\theta_{kk'})\cos(\omega_{k'k}t)+i\cos(\theta_{k'k})\cos(\theta_{\mu k}+\theta_{\mu k'})\sin(\omega_{kk'}t)\\&\quad+\sin^{2}(\theta_{k'k})\cos(\Omega_{k'k}t)-i\sin(\theta_{k'k})\sin(\theta_{\mu k}+\theta_{\mu k'})\sin(\Omega_{kk'}t)],\end{aligned}\tag{3.10}$$

where $\Omega_{ij}=\epsilon_{i}+\epsilon_{j}$ and $\omega_{ij}=\epsilon_{i}-\epsilon_{j}$. This can be rewritten as

$$Q_{\mu\nu\mu}=\sum_{k,k'}\mathrm{Re}(U_{\nu k'}U^*_{\nu k}U_{\mu k}U^*_{\mu k'})[\cos^2(\theta_{kk'})\cos(\omega_{k'k}t)-(-1)^{2S}\sin^2(\theta_{k'k})\cos(\Omega_{k'k}t)]+\sum_{k,k'}\mathrm{Im}(U_{\nu k'}U^*_{\nu k}U_{\mu k}U^*_{\mu k'})\times[\cos(\theta_{kk'})\cos(\theta_{\mu k}+\theta_{\mu k'})\sin(\omega_{k'k}t)-(-1)^{2S}\sin(\theta_{k'k})\sin(\theta_{\mu k}+\theta_{\mu k'})\sin(\Omega_{k'k}t)]. \quad (3.11)$$

This formula is also valid for bosons with the substitution of cos→cosh, sin→sinh.

We see now that $Q_{\mu\nu\mu}$ does not depend on the mass parameters only for real mixing matrices $U_{\mu k}$ [22,24]. Otherwise, there is a nontrivial mass dependence from the imaginary part of U. Interestingly, even in the latter case, there is no dependence on the mass of the flavor field ν (m_ν) but only on the mass of the initial flavor state μ.

We also note that Eq. (3.8) may be viewed as a superposition of the two terms: $\rho\to\nu$ propagation and background vacuum contribution Z_ν. Thus, one may introduce the particle-particle and particle-antiparticle propagation amplitudes, respectively, defined by

$$\mathcal{P}_{\rho\to\nu}(k,t)=[\tilde{a}_\nu(t),\tilde{a}^\dagger_\rho(0)]_\pm=F_{\nu\rho}(t),$$
$$\mathcal{P}_{\rho\to-\bar{\nu}}(k,t)=[\tilde{b}_{-\nu}(t),\tilde{a}_\rho(0)]_\pm=G_{\nu\rho}(t). \quad (3.12)$$

Indeed, such propagation amplitudes appear from the flavor-field Green function ${}_f\langle 0(t=0)|\phi_\nu(k,t)\phi^\dagger_\rho(k,0)|0(t=0)\rangle_f$ for $t>0$. Propagation functions, defined in this way, are clearly the Green functions of the mixed-field problem and obey the causality features relevant to such Green functions.[3]

IV. TWO-FIELD UNITARY MIXING

A. Vector meson mixing $(S=1)$

We now consider the unitary mixing of two fields with spin 1 (vector mesons). U(2) parametrization consists of four parameters: three phases that can be absorbed in the phase redefinition of fields and one essential real angle that is left, so that

$$U=\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}. \quad (4.1)$$

Using Appendix A, we then define $\gamma^\pm_{\mu i}=\frac{1}{2}[\sqrt{(\epsilon_\mu/\epsilon_i)}\pm\sqrt{(\epsilon_i/\epsilon_\mu)}]$ for $\sigma=\pm1$ and

$$\gamma^+_{\mu i}=\frac{1}{2}\,\frac{\epsilon_\mu\epsilon_i-k^2}{m_\mu m_i}\left(\sqrt{\frac{\epsilon_\mu}{\epsilon_i}}+\sqrt{\frac{\epsilon_i}{\epsilon_\mu}}\right),$$
$$\gamma^-_{\mu i}=\frac{1}{2}\,\frac{k^2+\epsilon_\mu\epsilon_i}{m_\mu m_i}\left(\sqrt{\frac{\epsilon_\mu}{\epsilon_i}}-\sqrt{\frac{\epsilon_i}{\epsilon_\mu}}\right) \quad (4.2)$$

for $\sigma=0$. For the free-field mass m_i basis, $\gamma^+_{12}=\gamma^+_{21}=\gamma_+$, $\gamma^-_{12}=-\gamma^-_{21}=\gamma_-$. We use this basis in Sec. IV.

The ladder mixing matrices α and β are given by

$$\alpha=\begin{pmatrix}\cos(\theta) & \gamma_+\sin(\theta)\\ -\gamma_+\sin(\theta) & \cos(\theta)\end{pmatrix},$$
$$\beta=\begin{pmatrix}0 & \gamma_-\sin(\theta)\\ -\gamma_-\sin(\theta) & 0\end{pmatrix}. \quad (4.3)$$

For the flavor charge oscillation, we then obtain the result that is not dependent on the mass parametrization:

$$Q_{111}=1+\sin^2(2\theta)\left[\gamma_-^2\sin^2\left(\frac{\Omega_{12}t}{2}\right)-\gamma_+^2\sin^2\left(\frac{\omega_{12}t}{2}\right)\right],$$
$$Q_{121}=\sin^2(2\theta)\left[\gamma_+^2\sin^2\left(\frac{\omega_{12}t}{2}\right)-\gamma_-^2\sin^2\left(\frac{\Omega_{12}t}{2}\right)\right]. \quad (4.4)$$

We see that this result, with an exception of greater complexity of $\gamma_\pm$, is identical to the case of spin 0 [26,27]. According to the above theory, in fact, this should be the case for the two-field mixing with any integer spin. For $S=1$ we see that an essential difference from the scalar/pseudoscalar meson mixing, such as the complication of momentum dependence of $\gamma_\pm$, occurs only for the mixing of longitudinally polarized particles. The mixing of transverse components is essentially the same as in the case of spin-zero particles.

The details of non-equal-time commutators are given by

$$F=\begin{Bmatrix}e^{-i\epsilon_1 t}\cos^2(\theta)+e^{-i\epsilon_2 t}\gamma_+^2\sin^2(\theta)-e^{i\epsilon_2 t}\gamma_-^2\sin^2(\theta); & \gamma_+\sin(\theta)\cos(\theta)(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t})\\ \gamma_+\sin(\theta)\cos(\theta)(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t}); & e^{-i\epsilon_2 t}\cos^2(\theta)+e^{-i\epsilon_1 t}\gamma_+^2\sin^2(\theta)-e^{i\epsilon_1 t}\gamma_-^2\sin^2(\theta)\end{Bmatrix}, \quad (4.5)$$

$$G=\begin{pmatrix}\gamma_+\gamma_-\sin^2(\theta)(e^{-i\epsilon_2 t}-e^{i\epsilon_2 t}) & \gamma_-\sin(\theta)\cos(\theta)(e^{-i\epsilon_1 t}-e^{i\epsilon_2 t})\\ \gamma_-\sin(\theta)\cos(\theta)(e^{-i\epsilon_2 t}-e^{i\epsilon_1 t}) & \gamma_+\gamma_-\sin^2(\theta)(e^{i\epsilon_1 t}-e^{-i\epsilon_1 t})\end{pmatrix}. \quad (4.6)$$

[3]See Refs. [21,24] for the discussion of the Green functions in the quantum theory of the mixed fields.

The condensates of free-field particles are

$$Z_1' = Z_2' = \gamma_-^2 \sin^2(\theta) \tag{4.7}$$

and the condensates of the flavor particles in the vacuum are

$$\begin{aligned} Z_1 &= 4\gamma_-^2 \sin^2(\theta)\left[\cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right) + \gamma_+^2 \sin^2(\theta)\sin^2\left(\frac{\Omega_{22}t}{2}\right)\right], \\ Z_2 &= 4\gamma_-^2 \sin^2(\theta)\left[\cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right) + \gamma_+^2 \sin^2(\theta)\sin^2\left(\frac{\Omega_{11}t}{2}\right)\right]. \end{aligned} \tag{4.8}$$

The flavor vacuum structure is defined by the matrix $\hat{Z}$:

$$\hat{Z} = \frac{-1}{[\cos^2(\theta) + \gamma_+^2 \sin^2(\theta)]}\begin{pmatrix} -\gamma_+\gamma_- \sin^2(\theta) & \gamma_- \cos(\theta)\sin(\theta) \\ \gamma_- \cos(\theta)\sin(\theta) & \gamma_+\gamma_- \sin^2(\theta) \end{pmatrix} \tag{4.9}$$

with the normalization constant being

$$\mathcal{Z} = \left(1 - \frac{\gamma_-^2 \sin^2(\theta)}{\cos^2(\theta) + \gamma_+^2 \sin^2(\theta)}\right)^{-1} = 1 + \gamma_-^2 \sin^2(\theta).$$

The time evolution of the flavor particle number (if No. 1 were emitted) is given by

$$\begin{aligned} N_{111} &= 1 + \sin^2(\theta)\left\{8\gamma_-^2 \cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right) - 4\gamma_+^2 \cos^2(\theta)\sin^2\left(\frac{\omega_{12}t}{2}\right) + 8\gamma_+^2\gamma_-^2 \sin^2(\theta)\sin^2\left(\frac{\Omega_{22}t}{2}\right)\right\}, \\ \overline{N}_{111} &= 4\gamma_-^2 \sin^2(\theta)\left[2\gamma_+^2 \sin^2(\theta)\sin^2\left(\frac{\Omega_{22}t}{2}\right) + \cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right)\right], \end{aligned} \tag{4.10}$$

$$\begin{aligned} N_{121} &= \sin^2(\theta)\left\{4\gamma_+^2 \cos^2(\theta)\sin^2\left(\frac{\omega_{12}t}{2}\right) + 4\gamma_-^2 \cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right) + 4\gamma_+^2\gamma_-^2 \sin^2(\theta)\sin^2\left(\frac{\Omega_{11}t}{2}\right)\right\}, \\ \overline{N}_{121} &= 4\gamma_-^2 \sin^2(\theta)\left[2\cos^2(\theta)\cos^2\left(\frac{\Omega_{12}t}{2}\right) + \gamma_+^2 \sin^2(\theta)\sin^2\left(\frac{\Omega_{11}t}{2}\right)\right]. \end{aligned} \tag{4.11}$$

Also we note that the scalar and pseudoscalar case follows immediately from the above presentation when $\gamma_{\mu i}^{\pm} = \frac{1}{2}[\sqrt{(\epsilon_\mu/\epsilon_i)} \pm \sqrt{(\epsilon_i/\epsilon_\mu)}]$. In this respect, the spin-zero mixing is equivalent to the mixing of transverse components of vector fields, described by Eqs. (4.4), (4.10), and (4.11). These results are consistent with the previously known results [26,27].

B. Fermion mixings ($S=\frac{1}{2}$)

We also present here the calculations for $S=\frac{1}{2}$ case. For the consistent notation with the previous works [16,28],[4] we define

$$U = \frac{\sqrt{(\epsilon_1 + m_1)(\epsilon_2 + m_2)} + \sqrt{(\epsilon_1 - m_1)(\epsilon_2 - m_2)}}{2\sqrt{\epsilon_1\epsilon_2}}, \tag{4.12}$$

$$V = \sigma\frac{\sqrt{(\epsilon_1 - m_1)(\epsilon_2 + m_2)} - \sqrt{(\epsilon_1 + m_1)(\epsilon_2 - m_2)}}{2\sqrt{\epsilon_1\epsilon_2}}.$$

The charge fluctuations are then given by

$$Q_{111} = 1 - \sin^2(2\theta)\left[U^2 \sin^2\left(\frac{\omega_{12}t}{2}\right) + V^2 \sin^2\left(\frac{\Omega_{12}t}{2}\right)\right], \tag{4.13}$$

[4]In our notation, $U=\gamma_+$, $V=\gamma_-$.

$$Q_{121}=\sin^2(2\theta)\left[U^2\sin^2\left(\frac{\omega_{12}t}{2}\right)+V^2\sin^2\left(\frac{\Omega_{12}t}{2}\right)\right],$$

and the ladder mixing matrices are

$$\alpha=\begin{pmatrix}\cos(\theta) & U\sin(\theta)\\ -U\sin(\theta) & \cos(\theta)\end{pmatrix},$$
$$\beta=\begin{pmatrix}0 & V\sin(\theta)\\ V\sin(\theta) & 0\end{pmatrix}, \tag{4.14}$$

which are the same as the previously known results [16,28].

We can also give more details on the fermion mixing dynamics. The non-equal-time anticommutators are given by

$$F=\begin{Bmatrix}e^{-i\epsilon_1 t}\cos^2(\theta)+e^{-i\epsilon_2 t}U^2\sin^2(\theta)+e^{i\epsilon_2 t}V^2\sin^2(\theta); & U\sin(\theta)\cos(\theta)(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t})\\ U\sin(\theta)\cos(\theta)(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t}); & e^{-i\epsilon_2 t}\cos^2(\theta)+e^{-i\epsilon_1 t}U^2\sin^2(\theta)+e^{i\epsilon_1 t}V^2\sin^2(\theta)\end{Bmatrix}, \tag{4.15}$$

$$G=\begin{pmatrix}UV\sin^2(\theta)(e^{-i\epsilon_2 t}-e^{i\epsilon_2 t}) & V\sin(\theta)\cos(\theta)(e^{-i\epsilon_1 t}-e^{i\epsilon_2 t})\\ V\sin(\theta)\cos(\theta)(e^{-i\epsilon_2 t}-e^{i\epsilon_1 t}) & UV\sin^2(\theta)(e^{i\epsilon_1 t}-e^{-i\epsilon_1 t})\end{pmatrix}. \tag{4.16}$$

The condensates of the free-field particles are

$$Z_1'=Z_2'=V^2\sin^2(\theta) \tag{4.17}$$

and the condensates of the flavor particles are

$$Z_1=4V^2\sin^2(\theta)\left[\cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right)+U^2\sin^2(\theta)\sin^2\left(\frac{\Omega_{22}t}{2}\right)\right],$$
$$Z_2=4V^2\sin^2(\theta)\left[\cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right)+U^2\sin^2(\theta)\sin^2\left(\frac{\Omega_{11}t}{2}\right)\right]. \tag{4.18}$$

The vacuum structure is defined by the matrix $\hat{Z}$:

$$\hat{Z}=\frac{-1}{\cos^2(\theta)+U^2\sin^2(\theta)}\begin{pmatrix}-UV\sin^2(\theta) & V\cos(\theta)\sin(\theta)\\ V\cos(\theta)\sin(\theta) & UV\sin^2(\theta)\end{pmatrix}$$

with the normalization constant being

$$\mathcal{Z}=\frac{1}{\cos^2(\theta)+U^2\sin^2(\theta)}=\frac{1}{1-V^2\sin^2(\theta)}.$$

The time evolution of the flavor particle number (if No. 1 were emitted) is then given by

$$N_{111}=1-4U^2\sin^2(\theta)\cos^2(\theta)\sin^2\left(\frac{\omega_{12}t}{2}\right),$$
$$\overline{N}_{111}=4V^2\sin^2(\theta)\cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right), \tag{4.19}$$

$$N_{121}=4\sin^2(\theta)\left\{U^2\cos^2(\theta)\sin^2\left(\frac{\omega_{12}t}{2}\right)+V^2\cos^2(\theta)\sin^2\left(\frac{\Omega_{12}t}{2}\right)+U^2V^2\sin^2(\theta)\sin^2\left(\frac{\Omega_{11}t}{2}\right)\right\},$$
$$\overline{N}_{121}=4U^2V^2\sin^4(\theta)\sin^2\left(\frac{\Omega_{11}t}{2}\right). \tag{4.20}$$

V. CONCLUSION

The quantum field mixing effects may be understood by considering the interplay between the two Fock spaces of the free fields and the interacting fields. As demonstrated in the two-field mixing treatment, this interplay is highly nontrivial and gives rise to a deviation from the simple quantum-mechanical approach due to the high-frequency oscillations and the antiparticle component in the system.

We have now extended the previous results and presented a solution without approximations for the quantum field theory of mixings in the arbitrary number of fields with boson or fermion statistics. As one might have expected from the previous two-field treatment [15,21,26,27], all results fall into the same scheme and can be easily unified. We investigated the field time dynamics by calculating unequal-time commutators and discussed the propagation functions. We found an explicit solution for the interacting field Fock space and the corresponding vacuum structure that turned out to be a generalized coherent state. We then showed the unitary inequivalence between the mixed-field Fock space and the free-field Fock space in the infinite volume limit. After we built a formal calculational framework, we applied it to solve mixing dynamics of two vector mesons $(S=1)$ and fermions $(S=\frac{1}{2})$. We found that the scalar/pseudoscalar $(S=0)$ boson mixing is the same as the mixing of transverse components of the vector fields, while for the longitudinal component of the vector field we found a richer momentum dependence than in the spin-zero case.

However, from the application of our approach to three-fermion/boson mixing cases, which we summarize in Appendix C, we saw a very complicated structure of more general results. Oscillation formulas typically involve all possible low-frequency and high-frequency combination terms. The amplitudes of the oscillation terms are essentially momentum-dependent. We have also discussed the existence of the coherent antiparticle beam generated from the starting definite-flavor particle beam and presented its dynamics.

Our general approach does not require us to use any specific continuous parametrization of the mixing group but directly takes the values of matrix elements. This allows an analysis to be carried out in a unified closed form, as shown in Secs. II and III. In general, it may be preferable to solve the mixing problems without going through the intermediate parametrization step for the mixing matrix. Even if one wants to use a specific parametrization scheme for the mixing matrix, it is rather straightforward to formulate our general framework into a symbolic calculation system, like MAPLE or MATHEMATICA, and carry out extensive calculations involving mixing parameters in a short period of time. Examples of such calculations are shown in Appendix C.

The physical application of the above formalism can be seen in investigating the neutrino mixing, mixing of gauge vector bosons governed by the Weinberg angle in the electroweak theory, as well as vector mesons such as ρ and ω. It also seems possible to apply these results to consider nonperturbative quark-mixing effects in the standard model and provide partial summation of the regular perturbation theory in mixing degrees of freedom. For this purpose, considering the covariant form of the above theory might be of great interest. Consideration along this line is in progress.

ACKNOWLEDGMENTS

We thank Professor G. Vitiello and Dr. M. Blasone for helpful discussions. This work was supported by a grant from the U.S. Department of Energy (DE-FG02-96ER 40947). The North Carolina Supercomputing Center and the National Energy Research Scientific Computer Center are also acknowledged for the grant of computing time.

APPENDIX A: ESSENTIAL CASES OF MIXING FIELD PARAMETERS

The most essential cases in modern particle physics are scalar or pseudoscalar (spin 0), vector (spin 1) boson fields, and spin-$\frac{1}{2}$ fermion fields. For these cases, mixing theory parameters are explicitly derived from quantum field theory [20,29]. We then have for scalar or pseudoscalar fields (spin 0)

$$u_{\vec{k},0}=v_{\vec{k},0}=1, \tag{A1}$$

and for vector fields (spin 1)

$$u_{\vec{k},0}=v_{\vec{k},0}=\left(\frac{k}{m},i\frac{\epsilon(k)}{m}\vec{n}\right),$$

$$u_{\vec{k},\pm 1}=v_{\vec{k},\pm 1}=(0,i\vec{n}_{\pm}), \tag{A2}$$

where $\vec{n}=\vec{k}/k=\vec{e}_z$ and $\vec{n}_{\pm}=\mp(1/\sqrt{2})(\vec{e}_x \pm i\vec{e}_y)$ form a spherical basis. For bispinor fields (spin $\frac{1}{2}$), we use the standard representation of the γ matrices given by

$$\gamma^0=\begin{pmatrix}\hat{I} & 0\\ 0 & -\hat{I}\end{pmatrix}, \quad \vec{\gamma}=\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}, \tag{A3}$$

and the corresponding representations of spinors,

$$\begin{aligned} u_{\vec{k},\sigma}&=[\sqrt{\epsilon(k)+m}\,\omega_\sigma,\sqrt{\epsilon(k)-m}(\vec{n}\cdot\vec{\sigma})\omega_\sigma],\\ v_{-\vec{k},\sigma}&=[-\sqrt{\epsilon(k)-m}(\vec{n}\cdot\vec{\sigma})\omega_{-\sigma},\sqrt{\epsilon(k)+m}\,\omega_{-\sigma}], \end{aligned} \tag{A4}$$

where ω_σ is spinor satisfying $(\vec{n}\cdot\vec{\sigma})\omega_\sigma=\sigma\cdot\omega_\sigma$ and σ takes values ± 1.

The H and h matrix parameters are then for the scalar case

$$H^{\mu j}=h^{\mu j}=1. \tag{A5}$$

For spin 1,

$$\begin{cases} H^{\mu j}_{\vec{k},0} = \dfrac{\epsilon_\mu(k)\epsilon_j(k)-k^2}{m_\mu m_j}, \\ h^{\mu j}_{\vec{k},0} = \dfrac{\epsilon_\mu(k)\epsilon_j(k)+k^2}{m_\mu m_j}, \end{cases} \quad \sigma=0 \tag{A6}$$

$$H^{\mu j}_{\vec{k},\pm} = h^{\mu j}_{\vec{k},\pm} = 1, \sigma = \pm 1,$$

and for spin $\frac{1}{2}$,

$$H^{\mu j}_{\vec{k},\sigma} = \sqrt{[\epsilon_\mu(k)+m_\mu][\epsilon_j(k)+m_j]} + \sqrt{[\epsilon_\mu(k)-m_\mu][\epsilon_j(k)-m_j]}, \tag{A7}$$

$$h^{\mu j}_{\vec{k},\sigma} = \sigma\{\sqrt{[\epsilon_\mu(k)-m_\mu][\epsilon_j(k)+m_j]} - \sqrt{[\epsilon_\mu(k)+m_\mu][\epsilon_j(k)-m_j]}\}.$$

APPENDIX B: THE FLAVOR VACUUM STATE

In this appendix, we explicitly solve the flavor vacuum structure. We first consider the boson case.

We write the sought flavor vacuum state as the most general linear combination from the original-field Fock space,

$$|0\rangle_f = \sum_{(n),(l)} \frac{1}{n_1! n_2! \cdots n_k!} B_{(n)(l)} (a_1^\dagger)^{n_1} \cdots (a_k^\dagger)^{n_k} (b_{-1}^\dagger)^{l_1} \cdots (b_{-k}^\dagger)^{l_k} |0\rangle_m, \tag{B1}$$

where $\kappa = N$ for the mixing of N fields. From the particle/antiparticle symmetry, the part of Eq. (2.31) involving antiparticle annihilation operators results in a dependent set of equations and thus can be omitted. Expanding Eq. (2.31), we find

$$\sum_j (\alpha_{ij} B_{(n_j+1)(l)} + \beta_{ij} B_{(n)(l_j-1)}) = 0 \quad \text{for all } (n),(l) \tag{B2}$$

where (n_j+1) notation stands for $(n_1, n_2, \ldots, n_j+1, \ldots, n_k)$ and k is the number of flavor fields. To solve this infinite set of equations, we introduce symbolic operators which decrease the subscript index of B coefficients, i.e., $d_{-j} B_{(n)(l)} = B_{(n)(l_j-1)}$. Then solving each set of equations in Eq. (B2) with respect to $B_{(n_j+1)(l)}$, we find

$$B_{(n_i+1)(l)} = \left(\sum_j Z_{ij} d_{-j} \right) B_{(n)(l)}$$

and consequently

$$B_{(n)(l)} = \prod_i \left(\sum_j Z_{ij} d_{-j} \right)^{n_i} B_{(0)(l)} \tag{B3}$$

with matrix $\hat{Z} = -\hat{\alpha}^{-1} \cdot \hat{\beta}$. Considering the momentum conservation and the original Eq. (B2), it can be shown that only $B_{(0)(l=0)}$ must be nonzero among all (l). Thus, applying symbolic operators d_{-j} and leaving only terms $B_{(0)(0)}$ in the expansion, we get

$$B_{(n)(l)} = \sum_{\substack{(j_p^i) \\ \left\{ \begin{smallmatrix} \Sigma_p j_p^i = n_i \\ \Sigma_i j_p^i = l_p \end{smallmatrix} \right\}}} \prod_i \frac{n_i!}{j_1^i! \cdots j_k^i!} Z_{i1}^{j_1^i} \cdots Z_{ik}^{j_k^i} B_{(0)(0)}. \tag{B4}$$

It is possible to rewrite this complicated expression in the more compact form

$$|0\rangle_f = \frac{1}{\mathcal{Z}} \sum_{(k)} \prod_i \frac{1}{k_i!} \left(\sum_j Z_{ij} a_i^\dagger b_{-j}^\dagger \right)^{k_i} |0\rangle_m \tag{B5}$$

that can be shown directly by expanding the above expression. It also can be argued that to obtain $B_{(n)(l)}$ from Eq. (B5) one needs to leave only those terms in the expansion that give the correct power of particle and antiparticle creation operators, i.e., total powers of all $a_i^\dagger$'s are n_i's and $b_i^\dagger$'s are l_i. But this is the same as extracting $B_{(n)(l)}$ from Eq. (B3). The constant $\mathcal{Z}$ is introduced instead of $B_{(0)(0)}$ and serves as a normalization factor determined by ${}_f\langle 0|0\rangle_f = 1$.

Equation (B5) can be further simplified as

$$\begin{aligned} |0\rangle_f &= \frac{1}{\mathcal{Z}} \sum_{(k)} \prod_i \frac{1}{k_i!} \left(\sum_j Z_{ij} a_i^\dagger b_{-j}^\dagger \right)^{k_i} |0\rangle_m \\ &= \frac{1}{\mathcal{Z}} \prod_i \sum_{k_i=0}^{\infty} \frac{1}{k_i!} \left(\sum_j Z_{ij} a_i^\dagger b_{-j}^\dagger \right)^{k_i} |0\rangle_m \\ &= \frac{1}{\mathcal{Z}} \exp\left(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger \right) |0\rangle_m. \end{aligned} \tag{B6}$$

Let us now proceed to the fermion case. We employ the same idea as the symbolic shifting operators. If $\hat{C}_{(n)(l)}$ stands for a creation operator for the fermion state $|(n),(l)\rangle$, we want then

$$\begin{aligned} a_i B_{(n_i+1)(l)} \hat{C}_{(n_i+1)(l)} |0\rangle_m &= \pm B_{(n_i+1)(l)} \hat{C}_{(n)(l)} |0\rangle_m \\ &= d_{+i} B_{(n)(l)} \hat{C}_{(n)(l)} |0\rangle_m, \\ b_i^\dagger B_{(n)(l_i-1)} \hat{C}_{(n)(l_i-1)} |0\rangle_m &= \pm B_{(n)(l_i-1)} \hat{C}_{(n)(l)} |0\rangle_m \\ &= d_{-i} B_{(n)(l)} \hat{C}_{(n)(l)} |0\rangle_m, \end{aligned} \tag{B7}$$

with correct sign. Equation (B2) then can be written in the form

$$\sum_j (\alpha_{ij}d_{+j}+\beta_{ij}d_{-j})B_{(n)(l)}=0, \tag{B8}$$

which binds together the shifting operators that increase and decrease the index. This set can be solved as

$$d_{+i}[B_{(n)(l)}]=\sum_j Z_{ij}d_{-j}[B_{(n)(l)}] \tag{B9}$$

with the same matrix $\hat{Z}$ presented in the boson case. From the definition of shifting operators it can be inferred that they obey the anticommutation property (i.e., $d_{\pm i}d_{\pm j}=-d_{\pm j}d_{\pm i}$) and thus it can be shown further that for $i_1>i_2>\cdots>i_n$,

$$\begin{aligned} d_{+i_n}d_{+i_{n-1}}\cdots d_{+i_1}B_{(0)(l)}&=B_{(i)(l)},\\ d_{-i_1}d_{-i_2}\cdots d_{-i_l}B_{(n)(l)}&=B_{(n)(l-i)}, \end{aligned} \tag{B10}$$

so that the solution can be written again as

$$B_{(n)(l)}=\prod_i \left(\sum_j Z_{ij}d_{-j}\right)^{n_i}B_{(0)(l)}, \tag{B11}$$

where only $B_{(0)(0)}$ survives. Here, n_i can be only 0 or 1 and the anticommutation rules for the ordering are applied. It is remarkable that Eq. (B5) can still be used for the fermion vacuum. This can be verified by a direct expansion with the anticommutation nature of ladder operators. Thus, for either the boson or fermion case, the flavor vacuum state can be written as

$$|0\rangle_f=\frac{1}{\mathcal{Z}}\exp\left(\sum_{i,j=1}^{N} Z_{ij}a_i^\dagger b_{-j}^\dagger\right)|0\rangle_m. \tag{B12}$$

We now proceed to find the normalization constant $\mathcal{Z}$. To do this, we consider

$$\begin{aligned} \||0\rangle_f|^2&=\left|\exp\left(\sum_{i,j=1}^{N} Z_{ij}a_i^\dagger b_{-j}^\dagger\right)|0\rangle_m\right|^2\\ &=\sum_L \frac{1}{L!^2}\left|\left(\sum_{i,j=1}^{N} Z_{ij}a_i^\dagger b_{-j}^\dagger\right)^L|0\rangle_m\right|^2, \end{aligned} \tag{B13}$$

where we use the fact that the states of $(\Sigma_{i,j=1}^{N}Z_{ij}a_i^\dagger b_{-j}^\dagger)^L|0\rangle_m$ are orthogonal for different L's. We then employ the fact that matrix $\hat{Z}$ can be transformed to a diagonal form with two unitary transformations, i.e.,

$$Z'=\begin{pmatrix} x_1 & 0 & \dots\\ 0 & \ddots & 0\\ \dots & 0 & x_N \end{pmatrix}=UZV^\dagger. \tag{B14}$$

We can now introduce additional unitary transformations of $a'^\dagger=U^\dagger a^\dagger$, $b'^\dagger=V^\dagger b^\dagger$ to make $\Sigma_{i,j=1}^{N}Z_{ij}a_i^\dagger b_{-j}^\dagger=\Sigma_{i=1}^{N}Z'_{ii}a_i'^\dagger b_{-i}'^\dagger$, where a'_i, b'_{-j} satisfy the standard commutation/anticommutation relationship. Then, using the binomial formula to expand $(\Sigma_{i=1}^{N}Z'_{ii}a_i'^\dagger b_{-i}'^\dagger)^L$, we find

$$\begin{aligned} &\sum_L \frac{1}{L!^2}\left|\left(\sum_{i=1}^{N} Z'_{ii}a_i'^\dagger b_{-i}'^\dagger\right)^L|0\rangle_m\right|^2\\ &=\sum_L \frac{1}{L!^2}\sum_{n_1+\cdots+n_N=L} L!^2\prod_{j=1}^{N}\frac{1}{n_j!^2}|(Z'_{jj}a_j'^\dagger b_{-j}'^\dagger)^{n_j}|0\rangle_m|^2\\ &=\sum_L \sum_{n_1+\cdots+n_N=L}\prod_{j=1}^{N}\frac{n_j!^2}{n_j!^2}|Z'^{n_j}_{jj}|^2\\ &=\sum_{n_1,\dots,n_N}\lambda_1^{n_1}\cdots\lambda_N^{n_N}, \end{aligned} \tag{B15}$$

where λ_i's are eigenvalues of $ZZ^\dagger$. The summation limits in Eq. (B15) are different for fermions and bosons. For bosons, n_i runs from 0 to ∞, while for fermions they can only be 0 or 1. In either case, the sum can be evaluated to give

$$\begin{aligned} &\left|\exp\left(\sum_{i,j=1}^{N} Z_{ij}a_i^\dagger b_{-j}^\dagger\right)|0\rangle_m\right|^2\\ &=\begin{cases} \prod_i (1+\lambda_i) & \text{fermions}\\ \prod_i \frac{1}{1-\lambda_i} & \text{bosons} \end{cases}\\ &=\begin{cases} \det(\hat{1}+ZZ^\dagger) & \text{fermions}\\ \det^{-1}(\hat{1}-ZZ^\dagger) & \text{bosons.} \end{cases} \end{aligned} \tag{B16}$$

APPENDIX C: UNITARY MIXING OF THREE FIELDS IN WOLFENSTEIN PARAMETRIZATION

We now present an application of the above general formalism to the specific case of mixing of three quantum fields. Calculations were carried out with the help of the MATHEMATICA 3 symbolic calculational system.

We note that all time-dependent quantities in this section are presented in the form of matrices, each entry of which corresponds to a certain $\Omega_{ij}=\omega_i+\omega_j$ or $\omega_{ij}=\omega_i-\omega_j$ frequency. This means that each quantity is presented in the form

$$P=2\,\mathrm{Re}\left(\sum_{ij}[P_{ij}^{\Omega}e^{-i\Omega_{ij}t}+P_{ij}^{\omega}e^{-i\omega_{ij}t}]\right), \tag{C1}$$

where P^Ω and P^ω matrices are written as follows:

$$\begin{aligned} P^\Omega&=\{\{P_{11}^\Omega,P_{12}^\Omega,P_{13}^\Omega\},\{P_{21}^\Omega,P_{22}^\Omega,P_{23}^\Omega\},\{P_{31}^\Omega,P_{32}^\Omega,P_{33}^\Omega\}\},\\ P^\omega&=\{\{P_{11}^\omega,P_{12}^\omega,P_{13}^\omega\},\{P_{21}^\omega,P_{22}^\omega,P_{23}^\omega\},\{P_{31}^\omega,P_{32}^\omega,P_{33}^\omega\}\}. \end{aligned} \tag{C2}$$

Since the diagonal elements of P^{ω} correspond to the same constant term $\omega_{ii}=0$, we can collect the diagonal elements of P^{ω} as $\mathrm{Sp}(P^{\omega})=P^{\omega}_{11}+P^{\omega}_{22}+P^{\omega}_{33}$ and express only the off-diagonal elements as

$$\tilde{P}^{\omega}=\{\{0,P^{\omega}_{12},P^{\omega}_{13}\},\{P^{\omega}_{21},0,P^{\omega}_{23}\},\{P^{\omega}_{31},P^{\omega}_{32},0\}\}. \quad \text{(C3)}$$

1. The case of three fermion fields

We now show our results for the unitary mixing of three fields with spin $\frac{1}{2}$ (bispinors). Although an explicit parametrization is not needed in our formalism, we may use the Wolfenstein parametrization as an explicit form of a mixing matrix,

$$U=\begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \quad \text{(C4)}$$

All results are then computed to a few lowest orders in λ.

For the bispinors, we redefine our H and h matrices as $H^{ij}\to H^{ij}/(2\sqrt{\epsilon_i\epsilon_j})$, $h^{ij}\to h^{ij}/(2\sqrt{\epsilon_i\epsilon_j})$ so that

$$H=\begin{pmatrix} 1 & u_{12} & u_{13} \\ u_{12} & 1 & u_{23} \\ u_{13} & u_{23} & 1 \end{pmatrix},$$

$$h=\begin{pmatrix} 0 & v_{12} & v_{13} \\ -v_{12} & 0 & v_{23} \\ -v_{13} & -v_{23} & 0 \end{pmatrix}. \quad \text{(C5)}$$

Also, u_{ij}, v_{ij} are defined in the same way as in the two-field mixing

$$u_{ij}=\frac{\sqrt{(\epsilon_i+m_i)(\epsilon_j+m_j)}+\sqrt{(\epsilon_i-m_i)(\epsilon_j-m_j)}}{2\sqrt{\epsilon_i\epsilon_j}},$$

$$v_{ij}=\sigma\frac{\sqrt{(\epsilon_i-m_i)(\epsilon_j+m_j)}-\sqrt{(\epsilon_i+m_i)(\epsilon_j-m_j)}}{2\sqrt{\epsilon_i\epsilon_j}}. \quad \text{(C6)}$$

Then, the structure of the ladder operators is described by α and β matrices,

$$\alpha=\begin{pmatrix} 1-\lambda^2/2 & u_{12}\lambda & u_{13}A\lambda^3(\rho-i\eta) \\ -u_{12}\lambda & 1-\lambda^2/2 & u_{23}A\lambda^2 \\ u_{13}A\lambda^3(1-\rho-i\eta) & -u_{23}A\lambda^2 & 1 \end{pmatrix},$$

$$\beta=\begin{pmatrix} 0 & v_{12}\lambda & v_{13}A\lambda^3(\rho-i\eta) \\ v_{12}\lambda & 0 & v_{23}A\lambda^2 \\ -v_{13}A\lambda^3(1-\rho-i\eta) & v_{23}A\lambda^2 & 0 \end{pmatrix}. \quad \text{(C7)}$$

To make the results more compact, we define $c=A(\rho-i\eta)$, $e=-A(1-\rho-i\eta)$, and $a=A$ so that

$$\alpha=\begin{pmatrix} 1-\lambda^2/2 & u_{12}\lambda & u_{13}c\lambda^3 \\ -u_{12}\lambda & 1-\lambda^2/2 & u_{23}a\lambda^2 \\ -u_{13}e\lambda^3 & -u_{23}a\lambda^2 & 1 \end{pmatrix},$$

$$\beta=\begin{pmatrix} 0 & v_{12}\lambda & v_{13}c\lambda^3 \\ v_{12}\lambda & 0 & v_{23}a\lambda^2 \\ v_{13}e\lambda^3 & v_{23}a\lambda^2 & 0 \end{pmatrix}. \quad \text{(C8)}$$

For the case when the No. 2 flavor particle was initially present, the flavor charge oscillation formulas are as follows. The flavor charge fluctuation, $Q_{212}(t)$, is given by

$$Q^{\Omega}_{212}=\Bigg\{\Bigg\{0,-\frac{\lambda^2 v_{12}^2}{2}(1-\lambda^2),-\frac{\lambda^6 a v_{13}c^*}{2}(u_{23}v_{12}+u_{12}v_{23})\Bigg\},$$
$$\Bigg\{-\frac{\lambda^2 v_{12}^2}{2}(1-\lambda^2),0,\frac{\lambda^6 a v_{23}}{2}(u_{12}v_{13}c^*-u_{13}v_{12}c)\Bigg\},$$
$$\Bigg\{-\frac{\lambda^6 a v_{13}c^*}{2}(u_{23}v_{12}+u_{12}v_{23}),\frac{\lambda^6 a v_{23}}{2}(u_{12}v_{13}c^*-u_{13}v_{12}c),0\Bigg\}\Bigg\}, \quad \text{(C9)}$$

$$\tilde{Q}^{\omega}_{212}=\left\{\left\{0,-\frac{\lambda^2 u_{12}^2}{2}(1-\lambda^2),\frac{\lambda^6 a u_{13}c}{2}(-u_{12}u_{23}+v_{12}v_{23})\right\},\right.$$
$$\left\{-\frac{\lambda^2 u_{12}^2}{2}(1-\lambda^2),0,\frac{\lambda^6 a u_{23}}{2}(u_{12}u_{13}c+v_{12}v_{13}c^*)\right\},$$
$$\left.\left\{\frac{\lambda^6 a u_{13}c^*}{2}(-u_{12}u_{23}+v_{12}v_{23}),\frac{\lambda^6 a u_{23}}{2}(u_{12}u_{13}c^*+v_{12}v_{13}c),0\right\}\right\}, \tag{C10}$$
$$\mathrm{Sp}(Q^{\omega}_{212})=\lambda^2(u_{12}^2+v_{12}^2)(1-\lambda^2).$$

Similarly, $Q_{222}(t)$ and $Q_{232}(t)$ are given by

$$Q^{\Omega}_{222}=\left\{\left\{0,\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{a^2(u_{23}v_{12}+u_{12}v_{23})^2}{2}\lambda^6\right\},\right.$$
$$\left\{\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),0,\frac{a^2 v_{23}^2\lambda^4}{2}(1-\lambda^2)\right\},$$
$$\left.\left\{\frac{a^2(u_{23}v_{12}+u_{12}v_{23})^2}{2}\lambda^6,\frac{a^2 v_{23}^2\lambda^4}{2}(1-\lambda^2),0\right\}\right\}, \tag{C11}$$

$$\tilde{Q}^{\omega}_{222}=\left\{\left\{0,\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{a^2(u_{23}u_{12}-v_{12}v_{23})^2}{2}\lambda^6\right\},\right.$$
$$\left\{\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),0,\frac{a^2 u_{23}^2\lambda^4}{2}(1-\lambda^2)\right\},$$
$$\left.\left\{\frac{a^2(u_{23}u_{12}-v_{12}v_{23})^2}{2}\lambda^6,\frac{a^2 u_{23}^2\lambda^4}{2}(1-\lambda^2),0\right\}\right\}, \tag{C12}$$

$$\mathrm{Sp}(Q^{\omega}_{222})=\frac{1}{2}-\lambda^2+\frac{1}{4}(3+2u_{12}^4+4u_{12}^2v_{12}^2+2v_{12}^4)\lambda^4,$$

and

$$Q^{\Omega}_{232}=\left\{\left\{0,\frac{a v_{12}\lambda^6}{2}(e u_{13}v_{23}-u_{23}v_{13}e^*),\frac{\lambda^6 a v_{13}e^*}{2}(u_{23}v_{12}+u_{12}v_{23})\right\},\right.$$
$$\left\{\frac{a v_{12}\lambda^6}{2}(e u_{13}v_{23}-u_{23}v_{13}e^*),0,-\frac{a^2 v_{23}^2\lambda^4}{2}+\frac{a^2 v_{23}^2\lambda^6}{4}\right\},$$
$$\left.\left\{\frac{\lambda^6 a v_{13}e^*}{2}(u_{23}v_{12}+u_{12}v_{23}),-\frac{a^2 v_{23}^2\lambda^4}{2}+\frac{a^2 v_{23}^2\lambda^6}{4},0\right\}\right\}, \tag{C13}$$

$$\tilde{Q}^{\omega}_{232}=\left\{\left\{0,-\frac{a u_{12}\lambda^6}{2}(e v_{13}v_{23}+u_{13}u_{23}e^*),\frac{\lambda^6 a u_{13}e^*}{2}(u_{12}u_{23}-v_{12}v_{23})\right\},\right.$$
$$\left\{-\frac{a u_{12}\lambda^6}{2}(e^* v_{13}v_{23}+u_{13}u_{23}e),0,-\frac{a^2 u_{23}^2\lambda^4}{2}+\frac{a^2 u_{23}^2\lambda^6}{4}\right\},$$
$$\left.\left\{\frac{\lambda^6 a u_{13}e}{2}(u_{12}u_{23}-v_{12}v_{23}),-\frac{a^2 u_{23}^2\lambda^4}{2}+\frac{a^2 u_{23}^2\lambda^6}{4},0\right\}\right\}, \tag{C14}$$
$$\mathrm{Sp}(Q^{\omega}_{232})=a^2(u_{23}^2+v_{23}^2)\lambda^4(1-\lambda^2/2),$$

respectively.

In more details, the dynamics is given by the following quantities. The non-equal-time anticommutators are given by

$$
\begin{aligned}
&F_{11}(t)=e^{-i\epsilon_1 t}+\lambda^2(-e^{-i\epsilon_1 t}+u_{12}^2 e^{-i\epsilon_2 t}+v_{12}^2 e^{i\epsilon_2 t}),\\
&F_{12}(t)=F_{21}(t)=\lambda u_{12}(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t})+\lambda^3\frac{u_{12}}{2}(e^{-i\epsilon_1 t}-e^{-i\epsilon_2 t}),\\
&F_{13}(t)=F_{31}(-t)^*=\lambda^3[u_{13}(ce^{-i\epsilon_3 t}-e^* e^{-i\epsilon_1 t})-au_{12}u_{23}e^{-i\epsilon_2 t}+av_{12}v_{23}e^{i\epsilon_2 t}]\\
&F_{22}(t)=e^{-i\epsilon_2 t}+\lambda^2(-e^{-i\epsilon_2 t}+u_{12}^2 e^{-i\epsilon_1 t}+v_{12}^2 e^{i\epsilon_1 t}),\\
&F_{23}(t)=F_{32}(t)=\lambda^2 a u_{23}(e^{-i\epsilon_3 t}-e^{-i\epsilon_2 t}),\\
&F_{33}(t)=e^{-i\epsilon_3 t};
\end{aligned}
\tag{C15}
$$

$$
\begin{aligned}
&G_{11}(t)=\lambda^2 u_{12}v_{12}(e^{-i\epsilon_2 t}-e^{i\epsilon_2 t}),\\
&G_{12}(t)=-[G_{21}(t)]^*=\lambda v_{12}(e^{-i\epsilon_1 t}-e^{i\epsilon_2 t})+\lambda^3\frac{v_{12}}{2}(-e^{-i\epsilon_1 t}+e^{i\epsilon_2 t}),\\
&G_{13}(t)=-[G_{31}(t)]^*=\lambda^3[v_{13}(ee^{-i\epsilon_1 t}-c^* e^{i\epsilon_3 t})+au_{23}v_{12}e^{i\epsilon_2 t}+au_{12}v_{23}e^{-i\epsilon_2 t}],\\
&G_{22}(t)=\lambda^2 u_{12}v_{12}(-e^{-i\epsilon_1 t}+e^{i\epsilon_1 t}),\\
&G_{23}(t)=-(G_{32}(t))^*=\lambda^2 a v_{23}(e^{-i\epsilon_2 t}-e^{i\epsilon_3 t}),\\
&G_{33}(t)=\lambda^4 a^2 u_{23}v_{23}(e^{i\epsilon_2 t}-e^{-i\epsilon_2 t}).
\end{aligned}
\tag{C16}
$$

The vacuum structure is defined by the $\hat{Z}$ matrix:

$$
\begin{aligned}
&Z_{11}=u_{12}v_{12}\lambda^2+u_{12}(1-u_{12}^2)v_{12}\lambda^4,\\
&Z_{12}=-v_{12}\lambda-(\tfrac{1}{2}-u_{12}^2)v_{12}\lambda^3=Z_{21},\\
&Z_{13}=-(cv_{13}-au_{12}v_{23})\lambda^3,\quad Z_{31}=-(au_{23}v_{12}+ev_{13})\lambda^3,\\
&Z_{22}=-Z_{11}+a^2u_{23}v_{23}\lambda^4,\\
&Z_{23}=-av_{23}\lambda^2-[cu_{12}v_{13}+a(\tfrac{1}{2}-u_{12}^2)v_{23}]\lambda^4,\\
&Z_{32}=-av_{23}\lambda^2-(eu_{13}+au_{12}u_{23})v_{12}\lambda^4,\\
&Z_{33}=-a^2u_{23}v_{23}\lambda^4.
\end{aligned}
\tag{C17}
$$

The normalization constant is obtained as $\mathcal{Z}\approx 1+v_{12}^2\lambda^2+(v_{12}^2+a^2v_{23}^2-v_{12}^2u_{12}^2)\lambda^4+\cdots$.

If the particle of sort No. 2 is originally present, then for the particle of sort No. 1 the mixing quantities are as follows. The free-field particle condensate is

$$
Z_1'=v_{12}^2\lambda^2 \tag{C18}
$$

and the flavor particle condensate is

$$
\begin{aligned}
Z_1^{\Omega}=\Bigg\{&\left\{0,-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),-\frac{\lambda^6}{2}v_{13}c^*(au_{23}v_{12}+v_{13}e^*)\right\},\\
&\left\{-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),-\lambda^4u_{12}^2v_{12}^2,-\frac{\lambda^6 av_{23}}{2}(cu_{13}v_{12}+u_{12}v_{13}c^*)\right\},
\end{aligned}
$$

$$\left\{-\frac{\lambda^6}{2}v_{13}c^*(au_{23}v_{12}+v_{13}e^*),-\frac{\lambda^6}{2}av_{23}(cu_{13}v_{12}+u_{12}v_{13}c^*),-a^2cu_{13}u_{23}v_{13}v_{23}c^*\lambda^{10}\right\}\bigg\}, \quad (C19)$$

$$\tilde{Z}_1^{\omega}=\left\{\left\{0,\frac{aeu_{12}v_{13}v_{23}}{2}\lambda^6,\frac{acu_{13}v_{12}v_{23}}{2}\lambda^6\right\},\left\{\frac{ae^*u_{12}v_{13}v_{23}}{2}\lambda^6,0,\frac{\lambda^8 v_{12}v_{13}c^*}{2}\left(2cu_{12}u_{13}-\frac{1}{2}au_{23}\right)\right\},\right.$$

$$\left.\left\{\frac{ac^*u_{13}v_{12}v_{23}}{2}\lambda^6,\frac{\lambda^8 v_{12}v_{13}c}{2}(2c^*u_{12}u_{13}-\tfrac{1}{2}au_{23}),0\right\}\right\}; \quad (C20)$$

$$\mathrm{Sp}(Z_1^{\omega})=\lambda^2 v_{12}^2-(1-u_{12}^2)v_{12}^2\lambda^4.$$

The flavor particle number fluctuations are given by $N_{212}(t)=|F_{12}(t)|^2+Z_1(t)$:

$$N_{212}^{\Omega}=\left\{\left\{-\lambda^4u_{12}^2v_{12}^2,-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),\frac{\lambda^6a}{2}[v_{13}v_{23}u_{12}(e^*-c^*)-v_{12}v_{13}u_{23}c^*+2av_{12}v_{23}u_{12}u_{23}]\right\},\right.$$

$$\left\{-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),0,-\frac{a^2v_{23}^2}{2}\lambda^4\right\},$$

$$\left.\left\{\frac{\lambda^6a}{2}[v_{13}v_{23}u_{12}(e^*-c^*)-v_{12}v_{13}u_{23}c^*+2av_{12}v_{23}u_{12}u_{23}],-\frac{a^2v_{23}^2}{2}\lambda^4,-a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \quad (C21)$$

$$\tilde{N}_{212}^{\omega}=\left\{\left\{0,-\lambda^2u_{12}^2(1-\lambda^2),\frac{\lambda^6a}{2}[v_{12}v_{23}(-u_{13}e^*+u_{13}c-2au_{12}u_{23})-cu_{12}u_{13}u_{23}]\right\},\right.$$

$$\left\{-\lambda^2u_{12}^2(1-\lambda^2),0,\frac{\lambda^6au_{23}}{2}(cu_{12}u_{13}+c^*v_{12}v_{13})\right\},$$

$$\left.\left\{\frac{\lambda^6a}{2}[v_{12}v_{23}(-u_{13}e+u_{13}c^*-2au_{12}u_{23})-c^*u_{12}u_{13}u_{23}],\frac{\lambda^6au_{23}}{2}(c^*u_{12}u_{13}+cv_{12}v_{13}),0\right\}\right\}; \quad (C22)$$

$$\mathrm{Sp}(N_{212}^{\omega})=\lambda^2(u_{12}^2+v_{12}^2)+\lambda^4[a^2v_{23}^2-v_{12}^2-u_{12}^2(1-v_{12}^2)],$$

and the flavor antiparticle number fluctuations, $\overline{N}_{212}(t)$, are given by

$$\overline{N}_{212}^{\Omega}=\left\{\left\{-u_{12}^2v_{12}^2\lambda^4,\frac{\lambda^6aeu_{13}v_{12}v_{23}}{2},\frac{\lambda^6au_{12}v_{23}}{2}(2au_{23}v_{12}+v_{13}e^*)\right\},\right.$$

$$\left\{\frac{\lambda^6aeu_{13}v_{12}v_{23}}{2},0,-\frac{a^2v_{23}^2\lambda^4}{2}\right\},$$

$$\left.\left\{\frac{\lambda^6au_{12}v_{23}}{2}(2au_{23}v_{12}+v_{13}e^*),-\frac{a^2v_{23}^2\lambda^4}{2},-a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \quad (C23)$$

$$\tilde{\overline{N}}_{212}^{\omega}=\left\{\left\{0,-\frac{aeu_{12}v_{13}v_{23}\lambda^6}{2},-\frac{av_{12}v_{23}\lambda^6}{2}(2au_{12}u_{23}+u_{13}e^*)\right\},\right.$$

$$\left.\left\{-\frac{ae^*u_{12}v_{13}v_{23}\lambda^6}{2},0,0\right\},\left\{-\frac{av_{12}v_{23}\lambda^6}{2}(2au_{12}u_{23}+u_{13}e),0,0\right\}\right\}; \quad (C24)$$

$$\mathrm{Sp}(\overline{N}_{212}^{\omega})=\lambda^4(u_{12}^2v_{12}^2+a^2v_{23}^2).$$

In the same initial condition, we obtain the following for the particle of sort No. 2. The free-field particle condensate is

$$Z_2'=v_{12}^2\lambda^2+a^2v_{23}^2\lambda^4 \tag{C25}$$

and the flavor particle condensate is

$$\begin{aligned}Z_2^{\Omega}=\Bigg\{&\Bigg\{-u_{12}^2v_{12}^2\lambda^4,-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{\lambda^6 a}{2}(2au_{12}u_{23}v_{12}v_{23}-c^*u_{23}v_{12}v_{13}+e^*u_{12}v_{13}v_{23})\Bigg\},\\&\Bigg\{-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),0,-\frac{a^2v_{23}^2}{2}\lambda^4\Bigg\},\\&\Bigg\{\frac{\lambda^6 a}{2}(2au_{12}u_{23}v_{12}v_{23}-c^*u_{23}v_{12}v_{13}+e^*u_{12}v_{13}v_{23}),-\frac{a^2v_{23}^2}{2}\lambda^4,-a^4u_{23}^2v_{23}^2\lambda^8\Bigg\}\Bigg\};\end{aligned} \tag{C26}$$

$$\begin{aligned}\tilde{Z}_2^{\omega}=\Bigg\{&\Bigg\{0,-\frac{aeu_{12}v_{13}v_{23}\lambda^6}{2},\frac{av_{12}v_{23}\lambda^6}{2}(cu_{13}-2au_{12}u_{23}-e^*u_{13})\Bigg\},\\&\Bigg\{-\frac{ae^*u_{12}v_{13}v_{23}\lambda^6}{2},0,\frac{au_{23}v_{12}v_{13}c^*\lambda^6}{2}\Bigg\},\\&\Bigg\{\frac{av_{12}v_{23}\lambda^6}{2}(c^*u_{13}-2au_{12}u_{23}-eu_{13}),\frac{au_{23}v_{12}v_{13}c\lambda^6}{2},0\Bigg\}\Bigg\};\end{aligned} \tag{C27}$$

$$\mathrm{Sp}(Z_2^{\omega})=v_{12}^2\lambda^2+[a^2v_{23}^2+v_{12}^2(u_{12}^2-1)]\lambda^4.$$

The flavor particle number fluctuations, $N_{222}(t)=|F_{22}(t)|^2+Z_2(t)$, are given by

$$\begin{aligned}N_{222}^{\Omega}=\Bigg\{&\Bigg\{-eu_{12}u_{13}v_{12}v_{13}e^*\lambda^8,\frac{aeu_{13}v_{12}v_{23}}{2}\lambda^6,\frac{\lambda^6 a}{2}[a(u_{23}v_{12}+u_{12}v_{23})^2-u_{23}v_{12}v_{13}c^*+u_{12}v_{13}v_{23}e^*]\Bigg\},\\&\Bigg\{\frac{aeu_{13}v_{12}v_{23}}{2}\lambda^6,0,-\frac{\lambda^6 av_{23}}{2}\left(cu_{13}v_{12}+\frac{av_{23}}{2}\right)\Bigg\},\\&\Bigg\{\frac{\lambda^6 a}{2}[a(u_{23}v_{12}+u_{12}v_{23})^2-u_{23}v_{12}v_{13}c^*+u_{12}v_{13}v_{23}e^*],\\&-\frac{\lambda^6 av_{23}}{2}\left(cu_{13}v_{12}+\frac{av_{23}}{2}\right),-a^2cu_{13}u_{23}v_{13}v_{23}c^*\lambda^{10}\Bigg\}\Bigg\};\end{aligned} \tag{C28}$$

$$\begin{aligned}\tilde{N}_{222}^{\omega}=\Bigg\{&\Bigg\{0,\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{\lambda^6 a}{2}[a(u_{12}u_{23}-v_{12}v_{23})^2+u_{13}v_{12}v_{23}c-u_{13}v_{12}v_{23}e^*]\Bigg\},\\&\Bigg\{\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),0,\frac{a^2u_{23}^2\lambda^4}{2}\Bigg\},\\&\Bigg\{\frac{\lambda^6 a}{2}[a(u_{12}u_{23}-v_{12}v_{23})^2+u_{13}v_{12}v_{23}c-u_{13}v_{12}v_{23}e^*],\frac{a^2u_{23}^2\lambda^4}{2},0\Bigg\}\Bigg\}\end{aligned} \tag{C29}$$

$$\mathrm{Sp}(N_{222}^{\omega})=\frac{1}{2}+(v_{12}^2-1)\lambda^2+\left(\frac{3}{4}+\frac{u_{12}^4}{2}-v_{12}^2+u_{12}^2v_{12}^2+\frac{v_{12}^4}{2}+a^2v_{23}^2\right)\lambda^4,$$

and the flavor antiparticle number fluctuations, $\overline{N}_{222}(t)$, are given by

$$\overline{N}_{222}^{\Omega}=\Bigg\{\Bigg\{-eu_{12}u_{13}v_{12}v_{13}e^*\lambda^8,-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{\lambda^6 a}{2}v_{13}(-u_{23}v_{12}c^*+u_{12}v_{23}e^*)\Bigg\},$$

$$\left\{-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),0,-\frac{a^2v_{23}^2\lambda^4}{2}\right\},\left\{\frac{\lambda^6 a}{2}v_{13}(-u_{23}v_{12}c^*+u_{12}v_{23}e^*),\right.$$
$$\left.\left.-\frac{a^2v_{23}^2\lambda^4}{2},-a^2cc^*u_{13}u_{23}v_{13}v_{23}\lambda^{10}\right\}\right\}; \tag{C30}$$

$$\tilde{\tilde{N}}^{\omega}_{222}=\left\{\left\{0,-\frac{aeu_{12}v_{13}v_{23}}{2}\lambda^6,\frac{au_{13}v_{12}v_{23}\lambda^6}{2}(c-e^*)\right\},\right.$$
$$\left\{-\frac{ae^*u_{12}v_{13}v_{23}}{2}\lambda^6,0,\frac{au_{23}v_{12}v_{13}c^*\lambda^6}{2}\right\},$$
$$\left.\left\{\frac{au_{13}v_{12}v_{23}\lambda^6}{2}(c^*-e),\frac{au_{23}v_{12}v_{13}c\lambda^6}{2},0\right\}\right\};$$
$$\mathrm{Sp}(\overline{N}^{\omega}_{222})=v_{12}^2\lambda^2+(a^2v_{23}^2-v_{12}^2)\lambda^4. \tag{C31}$$

Again in the same initial condition, the mixing quantities for the particle of sort No. 3 are as follows. The free-field particle condensate is

$$Z_1'=a^2v_{23}^2\lambda^4 \tag{C32}$$

and the flavor particle condensate is

$$Z^{\Omega}_{232}=\left\{\left\{-eu_{12}u_{13}v_{12}v_{13}e^*\lambda^8,\frac{\lambda^6 av_{12}}{2}(u_{13}v_{23}e+u_{23}v_{13}e^*),\frac{\lambda^6 v_{13}e^*}{2}(au_{12}v_{23}-v_{13}c^*)\right\},\right.$$
$$\left\{\frac{\lambda^6 av_{12}}{2}(u_{13}v_{23}e+u_{23}v_{13}e^*),a^2u_{12}u_{23}v_{12}v_{23}\lambda^6,-\frac{\lambda^4a^2v_{23}^2}{2}\right\}$$
$$\left.\left\{\frac{\lambda^6 v_{13}e^*}{2}(au_{12}v_{23}-v_{13}c^*),-\frac{\lambda^4a^2v_{23}^2}{2},0\right\}\right\}; \tag{C33}$$

$$\tilde{Z}^{\omega}_{232}=\left\{\left\{0,a^2eu_{13}u_{23}v_{13}v_{23}e^*\lambda^{10},-\frac{au_{13}v_{12}v_{23}e^*}{2}\lambda^6\right\},\right.$$
$$\left\{a^2eu_{13}u_{23}v_{13}v_{23}e^*\lambda^{10},0,-\frac{au_{23}v_{12}v_{13}c^*}{2}\lambda^6\right\},$$
$$\left.\left\{-\frac{au_{13}v_{12}v_{23}e}{2}\lambda^6,-\frac{au_{23}v_{12}v_{13}c}{2}\lambda^6,0\right\}\right\};$$
$$\mathrm{Sp}(Z^{\omega}_{232})=a^2v_{23}^2\lambda^4. \tag{C34}$$

The flavor particle number fluctuations, $N_{232}(t)=|F_{32}(t)|^2+Z_3(t)$, are given by

$$N^{\Omega}_{232}=\left\{\left\{-\lambda^4u_{12}^2v_{12}^2,-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),\frac{\lambda^6 a}{2}[2av_{12}v_{23}u_{12}u_{23}-u_{23}v_{12}v_{13}c^*+v_{13}(u_{23}v_{12}+u_{12}v_{23})e^*]\right\},\right.$$
$$\left\{-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),0,-\frac{a^2v_{23}^2}{2}\lambda^4\right\},$$
$$\left.\left\{\frac{\lambda^6 a}{2}[2av_{12}v_{23}u_{12}u_{23}-u_{23}v_{12}v_{13}c^*+v_{13}(u_{23}v_{12}+u_{12}v_{23})e^*],-\frac{a^2v_{23}^2}{2}\lambda^4,-a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \tag{C35}$$

$$\tilde{N}^{\omega}_{232}=\left\{\left\{0,-\frac{au_{12}\lambda^6}{2}(v_{13}v_{23}e+u_{13}u_{23}e^*),\frac{\lambda^6 a}{2}\{v_{12}v_{23}[u_{13}(c-e^*)-2au_{12}u_{23}]+u_{12}u_{13}u_{23}e^*\}\right\},\right.$$

$$\left\{-\frac{au_{12}\lambda^6}{2}(u_{13}u_{23}e+v_{13}v_{23}e^*),0,-\frac{a^2u_{23}^2\lambda^4}{2}\right\},$$

$$\left\{\frac{\lambda^6 a}{2}\{v_{12}v_{23}[u_{13}(c^*-e)-2au_{12}u_{23}]+u_{12}u_{13}u_{23}e\},-\frac{a^2u_{23}^2\lambda^4}{2},0\right\}\Bigg\};$$

$$\mathrm{Sp}(N_{232}^{\omega})=\lambda^2v_{12}^2+\lambda^4[a^2(v_{23}^2+u_{23}^2)-v_{12}^2(1-u_{12}^2)]. \tag{C36}$$

Similarly, the flavor antiparticle number fluctuations, $\overline{N}_{232}(t)$, are given by

$$\overline{N}_{232}^{\Omega}=\left\{\left\{-u_{12}^2v_{12}^2\lambda^4,-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{\lambda^6 av_{12}u_{23}}{2}(2av_{23}u_{12}-v_{13}c^*)\right\},\right.$$

$$\left\{-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),0,-\frac{acu_{13}v_{12}v_{23}\lambda^6}{2}\right\},$$

$$\left.\left\{\frac{\lambda^6 av_{12}u_{23}}{2}(2av_{23}u_{12}-v_{13}c^*),-\frac{acu_{13}v_{12}v_{23}\lambda^6}{2},-a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \tag{C37}$$

$$\tilde{\overline{N}}_{232}^{\omega}=\left\{\left\{0,0,-\frac{av_{12}v_{23}\lambda^6}{2}(2au_{12}u_{23}-u_{13}c)\right\},\right.$$

$$\left\{0,0,\frac{au_{23}v_{12}v_{13}c^*\lambda^6}{2}\right\},$$

$$\left.\left\{-\frac{av_{12}v_{23}\lambda^6}{2}(2au_{12}u_{23}-u_{13}c^*),\frac{au_{23}v_{12}v_{13}c\lambda^6}{2},0\right\}\right\};$$

$$\mathrm{Sp}(\overline{N}_{232}^{\omega})=v_{12}^2\lambda^2+\lambda^4v_{12}^2(u_{12}^2-1). \tag{C38}$$

2. The three-boson-field case

We now consider the application to bosons. The boson case is not much different from the fermion case. With the use of the $\gamma_{ij}^{+},\gamma_{ij}^{-}$ matrices, one can write the ladder mixing matrices as

$$\alpha=\begin{pmatrix}1-\lambda^2/2 & \gamma_{12}^{+}\lambda & \gamma_{13}^{+}A\lambda^3(\rho-i\eta)\\ -\gamma_{12}^{+}\lambda & 1-\lambda^2/2 & \gamma_{23}^{+}A\lambda^2\\ \gamma_{13}^{+}A\lambda^3(1-\rho-i\eta) & -\gamma_{23}^{+}A\lambda^2 & 1\end{pmatrix},$$

$$\beta=\begin{pmatrix}0 & \gamma_{12}^{-}\lambda & \gamma_{13}^{-}A\lambda^3(\rho-i\eta)\\ \gamma_{12}^{-}\lambda & 0 & \gamma_{23}^{-}A\lambda^2\\ -\gamma_{13}^{-}A\lambda^3(1-\rho-i\eta) & \gamma_{23}^{-}A\lambda^2 & 0\end{pmatrix}. \tag{C39}$$

We see that α and β indeed have the same form as in the fermion case with the correspondence $\gamma_{ij}^{+}\to u_{ij}$ and $\gamma_{ij}^{-}\to v_{ij}$:

$$\alpha=\begin{pmatrix}1-\lambda^2/2 & u_{12}\lambda & u_{13}c\lambda^3\\ -u_{12}\lambda & 1-\lambda^2/2 & u_{23}a\lambda^2\\ -u_{13}e\lambda^3 & -u_{23}a\lambda^2 & 1\end{pmatrix},$$

$$\beta=\begin{pmatrix}0 & v_{12}\lambda & v_{13}c\lambda^3\\ v_{12}\lambda & 0 & v_{23}a\lambda^2\\ v_{13}e\lambda^3 & v_{23}a\lambda^2 & 0\end{pmatrix}. \tag{C40}$$

This shows that the only difference appears in the quantities that have explicit spin dependence, i.e., $F_{\mu\nu}$ and everything involving $F_{\mu\nu}$. As a rule, the quantities for the boson case can be obtained from the fermion formulas by simply changing the signs in the terms quadratic in v_{ij}. We summarize them below.

The oscillation formulas are as follows. For the particle of sort No. 1, $Q_{212}(t)$ is

$$Q^{\Omega}_{212}=\left\{\left\{0,\frac{\lambda^2 v_{12}^2}{2}(1-\lambda^2),\frac{\lambda^6 a v_{13}c^*}{2}(u_{23}v_{12}+u_{12}v_{23})\right\},\right.$$
$$\left\{\frac{\lambda^2 v_{12}^2}{2}(1-\lambda^2),0,-\frac{\lambda^6 a v_{23}}{2}(u_{12}v_{13}c^*-u_{13}v_{12}c)\right\},$$
$$\left.\left\{\frac{\lambda^6 a v_{13}c^*}{2}(u_{23}v_{12}+u_{12}v_{23}),-\frac{\lambda^6 a v_{23}}{2}(u_{12}v_{13}c^*-u_{13}v_{12}c),0\right\}\right\}; \quad \text{(C41)}$$

$$\tilde{Q}^{\omega}_{212}=\left\{\left\{0,-\frac{\lambda^2 u_{12}^2}{2}(1-\lambda^2),-\frac{\lambda^6 a u_{13}c}{2}(u_{12}u_{23}+v_{12}v_{23})\right\},\right.$$
$$\left\{-\frac{\lambda^2 u_{12}^2}{2}(1-\lambda^2),0,\frac{\lambda^6 a u_{23}}{2}(u_{12}u_{13}c-v_{12}v_{13}c^*)\right\},$$
$$\left.\left\{-\frac{\lambda^6 a u_{13}c^*}{2}(u_{12}u_{23}+v_{12}v_{23}),\frac{\lambda^6 a u_{23}}{2}(u_{12}u_{13}c^*-v_{12}v_{13}c),0\right\}\right\};$$
$$\text{Sp}(Q^{\omega}_{212})=\lambda^2(u_{12}^2-v_{12}^2)(1-\lambda^2). \quad \text{(C42)}$$

For the particle of sort No. 2, $Q_{222}(t)$ is

$$Q^{\Omega}_{222}=\left\{\left\{0,-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),-\frac{a^2(u_{23}v_{12}+u_{12}v_{23})^2}{2}\lambda^6\right\},\right.$$
$$\left\{-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),0,-\frac{a^2 v_{23}^2\lambda^4}{2}(1-\lambda^2)\right\},$$
$$\left.\left\{-\frac{a^2(u_{23}v_{12}+u_{12}v_{23})^2}{2}\lambda^6,-\frac{a^2 v_{23}^2\lambda^4}{2}(1-\lambda^2),0\right\}\right\}; \quad \text{(C43)}$$

$$\tilde{Q}^{\omega}_{222}=\left\{\left\{0,\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{a^2(u_{23}u_{12}+v_{12}v_{23})^2}{2}\lambda^6\right\},\right.$$
$$\left\{\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),0,\frac{a^2 u_{23}^2\lambda^4}{2}(1-\lambda^2)\right\},$$
$$\left.\left\{\frac{a^2(u_{23}u_{12}+v_{12}v_{23})^2}{2}\lambda^6,\frac{a^2 u_{23}^2\lambda^4}{2}(1-\lambda^2),0\right\}\right\};$$
$$\text{Sp}(Q^{\omega}_{222})=\frac{1}{2}-\lambda^2+\frac{1}{4}(3+2u_{12}^4-4u_{12}^2v_{12}^2+2v_{12}^4)\lambda^4. \quad \text{(C44)}$$

For the particle of sort No. 3, $Q_{232}(t)$ is

$$Q^{\Omega}_{232}=\left\{\left\{0,-\frac{av_{12}\lambda^6}{2}(eu_{13}v_{23}-u_{23}v_{13}e^*),-\frac{\lambda^6 av_{13}e^*}{2}(u_{23}v_{12}+u_{12}v_{23})\right\},\right.$$
$$\left\{-\frac{av_{12}\lambda^6}{2}(eu_{13}v_{23}-u_{23}v_{13}e^*),0,\frac{a^2v_{23}^2\lambda^4}{2}-\frac{a^2v_{23}^2\lambda^6}{4}\right\},$$
$$\left.\left\{-\frac{\lambda^6 av_{13}e^*}{2}(u_{23}v_{12}+u_{12}v_{23}),\frac{a^2v_{23}^2\lambda^4}{2}-\frac{a^2v_{23}^2\lambda^6}{4},0\right\}\right\}; \tag{C45}$$

$$\tilde{Q}^{\omega}_{232}=\left\{\left\{0,\frac{au_{12}\lambda^6}{2}(ev_{13}v_{23}-u_{13}u_{23}e^*),\frac{\lambda^6 au_{13}e^*}{2}(u_{12}u_{23}+v_{12}v_{23})\right\},\right.$$
$$\left\{\frac{au_{12}\lambda^6}{2}(e^*v_{13}v_{23}-u_{13}u_{23}e),0,-\frac{a^2u_{23}^2\lambda^4}{2}+\frac{a^2u_{23}^2\lambda^6}{4}\right\},$$
$$\left.\left\{\frac{\lambda^6 au_{13}e}{2}(u_{12}u_{23}+v_{12}v_{23}),-\frac{a^2u_{23}^2\lambda^4}{2}+\frac{a^2u_{23}^2\lambda^6}{4},0\right\}\right\};$$
$$\mathrm{Sp}(Q^{\omega}_{232})=a^2(u_{23}^2-v_{23}^2)\lambda^4(1-\lambda^2/2). \tag{C46}$$

The non-equal-time commutators are given by

$$F_{11}(t)=e^{-i\epsilon_1 t}+\lambda^2(-e^{-i\epsilon_1 t}+u_{12}^2e^{-i\epsilon_2 t}-v_{12}^2e^{i\epsilon_2 t}),$$
$$F_{12}(t)=F_{21}(t)=\lambda u_{12}(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t})+\lambda^3\frac{u_{12}}{2}(e^{-i\epsilon_1 t}-e^{-i\epsilon_2 t}),$$
$$F_{13}(t)=[F_{31}(-t)]^*=\lambda^3(cu_{13}e^{-i\epsilon_3 t}-au_{12}u_{23}e^{-i\epsilon_2 t}-av_{12}v_{23}e^{i\epsilon_2 t}-e^*u_{13}e^{-i\epsilon_1 t}), \tag{C47}$$
$$F_{22}(t)=e^{-i\epsilon_2 t}+\lambda^2(-e^{-i\epsilon_2 t}+u_{12}^2e^{-i\epsilon_1 t}-v_{12}^2e^{i\epsilon_1 t}),$$
$$F_{23}(t)=F_{32}(t)=\lambda^2 au_{23}(e^{-i\epsilon_3 t}-e^{-i\epsilon_2 t}),$$
$$F_{33}(t)=e^{-i\epsilon_3 t};$$
$$G_{11}(t)=\lambda^2u_{12}v_{12}(e^{-i\epsilon_2 t}-e^{i\epsilon_2 t}),$$
$$G_{12}(t)=-[G_{21}(t)]^*=\lambda v_{12}(e^{-i\epsilon_1 t}-e^{i\epsilon_2 t})+\lambda^3\frac{v_{12}}{2}(-e^{-i\epsilon_1 t}+e^{i\epsilon_2 t}),$$
$$G_{13}(t)=-[G_{31}(t)]^*=\lambda^3(au_{23}v_{12}e^{i\epsilon_2 t}+ev_{13}\cdot e^{-i\epsilon_1 t}+au_{12}v_{23}e^{-i\epsilon_2 t}-v_{13}c^*e^{i\epsilon_3 t}), \tag{C48}$$
$$G_{22}(t)=\lambda^2u_{12}v_{12}(-e^{-i\epsilon_1 t}+e^{i\epsilon_1 t}),$$
$$G_{23}(t)=-[G_{32}(t)]^*=\lambda^2 av_{23}(e^{-i\epsilon_2 t}-e^{i\epsilon_3 t}),$$
$$G_{33}(t)=\lambda^4a^2u_{23}v_{23}(e^{i\epsilon_2 t}-e^{-i\epsilon_2 t}).$$

The vacuum structure is given by the fermion $\hat{Z}$ [Eq. (C17)] with the normalization constant

$$\mathcal{Z}\approx 1+v_{12}^2\lambda^2+(v_{12}^2+a^2v_{23}^2+v_{12}^4-v_{12}^2u_{12}^2)\lambda^4+\cdots.$$

If the particle of sort No. 2 was emitted initially, then the particle of sort No. 1 has the following free-field particle condensate:

$$Z_1'=v_{12}^2\lambda^2 \tag{C49}$$

and the flavor particle condensate identical to the fermion case, i.e., Eqs. (C19) and (C20).

The flavor particle number fluctuation $N_{212}(t)$ is given by

$$N^{\Omega}_{212}=\left\{\left\{-\lambda^4u^2_{12}v^2_{12},-\lambda^2\frac{v^2_{12}}{2}(1-\lambda^2),\frac{\lambda^6a}{2}[2au_{12}u_{23}v_{12}v_{23}+u_{12}v_{13}v_{23}(e^*+c^*)-u_{23}v_{12}v_{13}c^*]\right\},\right.$$
$$\left\{-\lambda^2\frac{v^2_{12}}{2}(1-\lambda^2),0,-\frac{a^2v^2_{23}\lambda^4}{2}\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}[2au_{12}u_{23}v_{12}v_{23}+u_{12}v_{13}v_{23}(e^*+c^*)-u_{23}v_{12}v_{13}c^*],-\frac{a^2v^2_{23}\lambda^4}{2},-a^4u^2_{23}v^2_{23}\lambda^8\right\}\right\}; \tag{C50}$$

$$\tilde{N}^{\omega}_{212}=\left\{\left\{0,-\lambda^2u^2_{12}(1-\lambda^2),-\frac{\lambda^6a}{2}[v_{12}v_{23}(u_{13}e^*-u_{13}c+2au_{12}u_{23})+cu_{12}u_{13}u_{23}]\right\},\right.$$
$$\left\{-\lambda^2u^2_{12}(1-\lambda^2),0,\frac{\lambda^6au_{23}}{2}(cu_{12}u_{13}+c^*v_{12}v_{13})\right\},$$
$$\left.\left\{-\frac{\lambda^6a}{2}[v_{12}v_{23}(u_{13}e-u_{13}c^*+2au_{12}u_{23})+c^*u_{12}u_{13}u_{23}],\frac{\lambda^6au_{23}}{2}(c^*u_{12}u_{13}+cv_{12}v_{13}),0\right\}\right\};$$
$$\mathrm{Sp}(N^{\omega}_{212})=\lambda^2(u^2_{12}+v^2_{12})+\lambda^4[a^2v^2_{23}-v^2_{12}-u^2_{12}(1-v^2_{12})]. \tag{C51}$$

The flavor antiparticle number fluctuation $\overline{N}_{212}(t)$ is given by

$$\overline{N}^{\Omega}_{212}=\left\{\left\{-u^2_{12}v^2_{12}\lambda^4,-v^2_{12}\lambda^2(1-\lambda^2),\frac{\lambda^6a}{2}[-2u_{23}v_{12}v_{13}c^*+u_{12}v_{23}(2au_{23}v_{12}+v_{13}e^*)]\right\},\right.$$
$$\left\{-v^2_{12}\lambda^2(1-\lambda^2),0,-\frac{a^2v^2_{23}\lambda^4}{2}\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}[-2u_{23}v_{12}v_{13}c^*+u_{12}v_{23}(2au_{23}v_{12}+v_{13}e^*)],-\frac{a^2v^2_{23}\lambda^4}{2},-a^4u^2_{23}v^2_{23}\lambda^8\right\}\right\}; \tag{C52}$$

$$\tilde{\overline{N}}^{\omega}_{212}=\left\{\left\{0,-\frac{aeu_{12}v_{13}v_{23}\lambda^6}{2},-\frac{av_{12}v_{23}\lambda^6}{2}(-2cu_{13+}2au_{12}u_{23}+u_{13}e^*)\right\},\right.$$
$$\left\{-\frac{ae^*u_{12}v_{13}v_{23}\lambda^6}{2},0,au_{23}v_{12}v_{13}c^*\lambda^6\right\},$$
$$\left.\left\{-\frac{av_{12}v_{23}\lambda^6}{2}(-2c^*u_{13+}2au_{12}u_{23}+u_{13}e),au_{23}v_{12}v_{13}c\lambda^6,0\right\}\right\}; \tag{C53}$$

$$\mathrm{Sp}(\overline{N}^{\omega}_{212})=2v^2_{12}\lambda^2+\lambda^4(u^2_{12}v^2_{12}+a^2v^2_{23}-2v^2_{12}).$$

For the same initial condition, the particle of sort No. 2 has the free-field condensate given by

$$Z'_2=v^2_{12}\lambda^2+a^2v^2_{23}\lambda^4 \tag{C54}$$

and the flavor particle condensate identical to Eqs. (C26) and (C27).

The flavor particle number fluctuation $N_{222}(t)$ is

$$N_{222}^{\Omega}=\left\{\left\{-2u_{12}^2v_{12}^2\lambda^4,-v_{12}^2\lambda^2(1-\lambda^2),-\frac{\lambda^6a}{2}[a(u_{23}v_{12}-u_{12}v_{23})^2+u_{23}v_{12}v_{13}c^*-u_{12}v_{13}v_{23}e^*]\right\},\right.$$
$$\{-v_{12}^2\lambda^2(1-\lambda^2),0,-a^2v_{23}^2\lambda^4\},$$
$$\left.\left\{-\frac{\lambda^6a}{2}[a(u_{23}v_{12}-u_{12}v_{23})^2+u_{23}v_{12}v_{13}c^*-u_{12}v_{13}v_{23}e^*],-a^2v_{23}^2\lambda^4,-2a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \tag{C55}$$

$$\tilde{N}_{222}^{\omega}=\left\{\left\{0,\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{\lambda^6a}{2}[cu_{13}v_{12}v_{23}+a(u_{12}u_{23}-v_{12}v_{23})^2-u_{13}v_{12}v_{23}e^*]\right\},\right.$$
$$\left\{\frac{u_{12}^2\lambda^2}{2}(1-\lambda^2),0,\frac{a^2u_{23}^2\lambda^4}{2}\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}[c^*u_{13}v_{12}v_{23}+a(u_{12}u_{23}-v_{12}v_{23})^2-u_{13}v_{12}v_{23}e],\frac{a^2u_{23}^2\lambda^4}{2},0\right\}\right\}; \tag{C56}$$

$$\mathrm{Sp}(N_{222}^{\omega})=\frac{1}{2}+(v_{12}^2-1)\lambda^2+\left(\frac{3}{4}+\frac{u_{12}^4}{2}-v_{12}^2+u_{12}^2v_{12}^2+\frac{v_{12}^4}{2}+a^2v_{23}^2\right)\lambda^4.$$

The flavor antiparticle number fluctuation $\overline{N}_{222}(t)$ is

$$\overline{N}_{222}^{\Omega}=\left\{\left\{-2u_{12}^2v_{12}^2\lambda^4,-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),\frac{\lambda^6a}{2}[-u_{23}v_{12}v_{13}c^*+u_{12}v_{23}(4au_{23}v_{12}+v_{13}e^*)]\right\},\right.$$
$$\left\{-\frac{v_{12}^2\lambda^2}{2}(1-\lambda^2),0,-\frac{a^2v_{23}^2\lambda^4}{2}\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}[-u_{23}v_{12}v_{13}c^*+u_{12}v_{23}(4au_{23}v_{12}+v_{13}e^*)],-\frac{a^2v_{23}^2\lambda^4}{2},-2a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \tag{C57}$$

$$\tilde{\overline{N}}_{222}^{\omega}=\left\{\left\{0,-\frac{aeu_{12}v_{13}v_{23}}{2}\lambda^6,\frac{av_{12}v_{23}\lambda^6}{2}[(c-e^*)u_{13}-4au_{12}u_{23}]\right\},\right.$$
$$\left\{-\frac{ae^*u_{12}v_{13}v_{23}}{2}\lambda^6,0,\frac{au_{23}v_{12}v_{13}c^*\lambda^6}{2}\right\},$$
$$\left.\left\{\frac{av_{12}v_{23}\lambda^6}{2}[(c^*-e)u_{13}-4au_{12}u_{23}],\frac{au_{23}v_{12}v_{13}c\lambda^6}{2},0\right\}\right\};$$
$$\mathrm{Sp}(\overline{N}_{222}^{\omega})=v_{12}^2\lambda^2+(a^2v_{23}^2-v_{12}^2+2u_{12}^2v_{12}^2)\lambda^4. \tag{C58}$$

Finally, for the same initial condition, the particle of sort No. 3 has the free-field particle condensate given by

$$Z_1'=a^2v_{23}^2\lambda^4 \tag{C59}$$

and the flavor particle condensate identical to Eqs. (C33) and (C34).

The flavor particle number fluctuation $N_{232}(t)$ is given by

$$N_{232}^{\Omega}=\left\{\left\{-\lambda^4u_{12}^2v_{12}^2,-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),\frac{\lambda^6a}{2}[-u_{23}v_{12}v_{13}(c^*+e^*)+2au_{12}u_{23}v_{12}v_{23}+e^*u_{12}v_{13}v_{23}]\right\},\right.$$
$$\left\{-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),0,-\frac{a^2v_{23}^2}{2}\lambda^4\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}[-u_{23}v_{12}v_{13}(c^*+e^*)+2au_{12}u_{23}v_{12}v_{23}+e^*u_{12}v_{13}v_{23}],-\frac{a^2v_{23}^2}{2}\lambda^4,-a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \tag{C60}$$

$$\tilde{N}_{232}^{\omega}=\left\{\left\{0,-\frac{au_{12}\lambda^6}{2}(v_{13}v_{23}e+u_{13}u_{23}e^*),\frac{\lambda^6a}{2}\{v_{12}v_{23}[u_{13}(c-e^*)-2au_{12}u_{23}]+u_{13}u_{12}u_{23}e^*\}\right\},\right.$$
$$\left\{-\frac{au_{12}\lambda^6}{2}(v_{13}v_{23}e^*+u_{13}u_{23}e),0,-\frac{a^2u_{23}^2\lambda^4}{2}\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}\{v_{12}v_{23}[u_{13}(c^*-e)-2au_{12}u_{23}]+u_{13}u_{12}u_{23}e\},-\frac{a^2u_{23}^2\lambda^4}{2},0\right\}\right\}; \tag{C61}$$

$$\mathrm{Sp}(N_{232}^{\omega})=\lambda^2v_{12}^2+\lambda^4[a^2(u_{23}^2+v_{23}^2)-v_{12}^2(1-u_{12}^2)].$$

The flavor antiparticle number fluctuation $\bar{N}_{232}(t)$ is given by

$$\bar{N}_{232}^{\Omega}=\left\{\left\{-\lambda^4u_{12}^2v_{12}^2,-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),\frac{\lambda^6a}{2}(-u_{23}v_{12}v_{13}c^*+2au_{12}u_{23}v_{12}v_{23}+2e^*u_{12}v_{13}v_{23})\right\},\right.$$
$$\left\{-\lambda^2\frac{v_{12}^2}{2}(1-\lambda^2),0,-a^2v_{23}^2\lambda^4\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}(-u_{23}v_{12}v_{13}c^*+2au_{12}u_{23}v_{12}v_{23}+2e^*u_{12}v_{13}v_{23}),-a^2v_{23}^2\lambda^4,-a^4u_{23}^2v_{23}^2\lambda^8\right\}\right\}; \tag{C62}$$

$$\tilde{\bar{N}}_{232}^{\omega}=\left\{\left\{0,-\lambda^6au_{12}v_{13}v_{23}e,\frac{\lambda^6a}{2}v_{12}v_{23}[u_{13}(c-2e^*)-2au_{12}u_{23}]\right\},\right.$$
$$\left\{-\lambda^6au_{12}v_{13}v_{23}e^*,0,\frac{\lambda^6au_{23}}{2}c^*v_{12}v_{13}\right\},$$
$$\left.\left\{\frac{\lambda^6a}{2}v_{12}v_{23}[u_{13}(c^*-2e)-2au_{12}u_{23}],\frac{\lambda^6au_{23}}{2}cv_{12}v_{13},0\right\}\right\};$$
$$\mathrm{Sp}(N_{232}^{\omega})=\lambda^2v_{12}^2+\lambda^4[2a^2v_{23}^2-v_{12}^2(1-u_{12}^2)]. \tag{C63}$$

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