Semiclassical quantization of effective string theory and Regge trajectories

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(Received 7 February 2002; published 21 May 2002)

We begin with an effective string theory for long distance QCD, and evaluate the semiclassical expansion of this theory about a classical rotating string solution, taking into account the dynamics of the boundary of the string. We show that, after renormalization, the zero point energy of the string fluctuations remains finite when the masses of the quarks on the ends of the string approach zero. The theory is then conformally invariant in any spacetime dimension D. For $D=26$ the energy spectrum of the rotating string formally coincides with that of the open string in classical bosonic string theory. However, its physical origin is different. It is a semiclassical spectrum of an effective string theory valid only for large values of the angular momentum. For $D=4$, the first semiclassical correction adds the constant 1/12 to the classical Regge formula.

DOI: 10.1103/PhysRevD.65.094042 PACS number(s): 12.38.Aw, 11.25.Pm, 11.27.+d

I. INTRODUCTION

String models provide a simple picture of quark confinement, and have been used to understand the hadronic spectrum since well before QCD was established as the theory of strong interactions. A straight rotating string gives rise to linear Regge trajectories relating the angular momenta of mesons composed of light quarks to the squares of their masses. A fixed straight string gives a linear potential between heavy quarks, and the zero point energy of the long wavelength fluctuations of this string gives rise to a universal correction to the linear potential [1]. Excited states of a fluctuating string with fixed ends give potentials of hybrid mesons [2]. In this paper we calculate the effect of string fluctuations on the Regge trajectories of mesons.

In a previous paper [3], we derived an effective string theory of vortices, beginning with a field theory containing classical vortex solutions (dual superconducting vortices) [4–11]. The field theory itself was an effective field theory of long distance QCD, describing phenomena at distances greater than the radius of the flux tube whose center is the location of the vortex. The resulting effective string theory was obtained as a development of earlier work by many authors [1,12–22]. We then used this effective string theory to calculate the zero point energy of the string fluctuations around a straight, rotating string with quarks on its ends. The classical equations of motion determined the distance between the quarks in terms of their angular velocity ω, and the fluctuations of the ends of the string were not taken into account. The calculated zero point energy gave a correction to the classical formula for the leading Regge trajectory. For static quarks separated by a fixed distance R, the expression for the zero point energy reduced to $-\pi/12R$, the result of Lüscher for the contribution of string fluctuations to the static quark-antiquark potential. However, for rotating quarks the zero point energy diverged logarithmically as the quark mass m approached zero, and we were not able to calculate Regge trajectories for zero mass quarks.

In this paper, we show how to take the mass zero limit. We also treat the quark motion quantum mechanically, so that the boundaries of the string become dynamical variables which couple to the interior degrees of freedom of the string. We evaluate the contribution of string fluctuations to Regge trajectories in the limit of massless quarks, and in the limit where one quark is massless and the other is heavy. Finally, we generalize our expressions for Regge trajectories to D spacetime dimensions, and compare with the spectrum of the classical bosonic string.

II. OUTLINE

In Sec. III we review the results obtained in [3], giving the expression for the functional integral representation of the effective string theory. We give the expression for the contribution of string fluctuations to the Wilson loop of the effective string theory, calculated in the classical background of a world sheet with rotating quarks on its ends. This expression exhibits a logarithmic divergence as the quark mass goes to zero. In Sec. IV, we show how to remove the logarithmic divergence by renormalization, and take the zero mass limit.

In Sec. V, we take into account the quantum fluctuations of the positions of the quark and antiquark at the ends of the string. We obtain an effective Lagrangian for the rotating string from which the meson energy levels can be determined. This effective Lagrangian $L_{\mathrm{eff}}(\omega)$ is determined by a functional integral of the effective string theory evaluated in the steepest descent approximation about a classical rotating string solution. The action determining this path integral is the Nambu-Goto action, added to the action of the point particles on the ends of the string.

In Secs. VI–VIII, we expand the action to quadratic order in small fluctuations about the classical rotating string solution. These fluctuations separate into two classes, interior degrees of freedom determining the positions of the interior points of the string and boundary degrees of freedom determining the fluctuations of the positions of the quark-antiquark pair at the ends of the string. The fluctuations of the ends of the string excite the interior points, which in turn react back on the ends, producing an effective quark-antiquark interaction. The remaining interior string fluctuations are decoupled from the fluctuations of the positions of

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the quarks. In Sec. IX, we calculate the zero point energy of these interior fluctuations, generalizing the results of [3] to the case where the quark masses are not equal.

In Sec. X, we find the propagator for the boundary fluctuations from the effective boundary action. We show that for zero mass quarks, the poles in the propagator are at integral multiples of ω. These frequencies are the same as the frequencies of the harmonic oscillators determining the interior Lagrangian. That is, for massless quarks, we find that taking the fluctuations of the boundary into account does not change the spectrum of the excited states of the rotating string. We also evaluate the position of the poles in the propagator when one quark is heavy and the other is massless. We find that the spectrum in this case is shifted when boundary effects are taken into account.

In Sec. XI, we evaluate the results of Secs. IX and X in certain physical limits. Our results are valid when the masses of the quarks are either zero or very large and when the string length is large compared to its thickness. In Sec. XII, we calculate the Regge trajectories of mesons containing zero mass quarks. The ground state of the rotating string gives the leading Regge trajectory, and the excited states of the rotating string give rise to daughter Regge trajectories determining the spectrum of hybrid mesons. We also calculate the Regge trajectories for mesons composed of one heavy quark and one light quark.

In Sec. XIII, we extend the calculations of this paper to D spacetime dimensions, and compare with the spectrum of classical bosonic string theory.

III. PREVIOUS WORK

A. Effective string theory

In Ref. [3], we began with a quantum field theory having classical vortex solutions. The dual Abelian Higgs model is an example of such a field theory. The surface $\tilde{x}^\mu(\xi^1,\xi^2)$ of zeros of the complex dual Higgs field is the location of the vortex sheet, and electric flux is confined to tubes of radius a, where $a^{-1}=M$, the mass of the vector particle in the theory.

The path integral, which defines the Wilson loop $W[\Gamma]$ of the field theory, goes over all field configurations containing a vortex sheet bounded by the loop Γ formed by the world lines of the trajectories of the quark and antiquark on the ends of the vortex. The action $S_{\text{eff}}[\tilde{x}^\mu]$ of the effective string theory is obtained by first integrating only over field configurations containing a vortex on a particular surface $\tilde{x}^\mu$. The remaining integral over the surfaces $\tilde{x}^\mu$ then gives $W[\Gamma]$ the form of an effective string theory of vortices.

The action $S_{\text{eff}}[\tilde{x}^\mu]$ is invariant under reparametrizations $\xi^a \to \xi'^a(\xi)$, $a=1,2$, of the world sheet $\tilde{x}^\mu(\xi)$ of the vortex. We choose a particular parametrization of $\tilde{x}^\mu$ in terms of the amplitudes $f^a(\xi)$, $a=1,2$, of the two transverse fluctuations of the vortex:

$$\tilde{x}^\mu = x^\mu(f^1(\xi),f^2(\xi),\xi^1,\xi^2). \tag{3.1}$$

This gives $W[\Gamma]$ the form

$$W[\Gamma]=\int \mathcal{D}f^1\mathcal{D}f^2\Delta_{FP}e^{iS_{\text{eff}}[\tilde{x}^\mu]}, \tag{3.2}$$

where

$$\Delta_{FP}=\text{Det}\left[\frac{\epsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}}\frac{\partial x^\mu}{\partial f^1}\frac{\partial x^\nu}{\partial f^2}\frac{\partial \tilde{x}^\alpha}{\partial \xi^1}\frac{\partial \tilde{x}^\beta}{\partial \xi^2}\right] \tag{3.3}$$

is the Faddeev-Popov determinant produced by gauge fixing the reparametrization symmetry, and where $\sqrt{-g}$ is the square root of the determinant of the induced metric g_{ab}:

$$g_{ab}=\frac{\partial \tilde{x}^\mu}{\partial \xi^a}\frac{\partial \tilde{x}_\mu}{\partial \xi^b}. \tag{3.4}$$

The path integral (3.2) goes over string fluctuations with wavelengths greater than the radius $1/M$ of the flux tube. The measure of the path integral (3.2) is universal and parametrization invariant. The factor Δ_{FP} came from rewriting the original field theory path integral as a ratio of path integrals of two string theories [3,12].

The action $S_{\text{eff}}[\tilde{x}^\mu]$ can be expanded in powers of the extrinsic curvature tensor $\mathcal{K}^A_{ab}$ of the world sheet $\tilde{x}^\mu$:

$$S_{\text{eff}}[\tilde{x}^\mu]=-\sigma\int d^2\xi\sqrt{-g}-\beta\int d^2\xi\sqrt{-g}(\mathcal{K}^A_{ab})^2+\cdots. \tag{3.5}$$

The extrinsic curvature tensor is

$$\mathcal{K}^A_{ab}=n^A_\mu(\xi)\frac{\partial^2\tilde{x}^\mu}{\partial\xi^a\partial\xi^b}, \tag{3.6}$$

where $n^A_\mu(\xi)$, $A=1,2$, are vectors normal to the world sheet at the point $\tilde{x}^\mu(\xi)$. The string tension σ and the rigidity β are determined by the parameters of the underlying effective field theory.

The extrinsic curvature $\mathcal{K}^A_{ab}$ is of the order of magnitude of the angular velocity ω, and the expansion parameter in the semiclassical approximation is $\omega^2/\sigma \sim 1/J$, where J is the angular momentum of the rotating string. Therefore, in the region of large J where the effective theory is applicable, the action (3.5) can be replaced by the Nambu-Goto action S_{NG}:

$$S_{\text{eff}}[\tilde{x}^\mu]=S_{\text{NG}}=-\sigma\int d^2\xi\sqrt{-g}. \tag{3.7}$$

B. Semiclassical calculation in the background of a rotating string

Using Eqs. (3.2) and (3.7), we calculated $W[\Gamma]$ in the leading semiclassical approximation in the background of a world sheet generated by a straight string attached to quarks rotating with uniform angular velocity ω (see Fig. 1). The quarks have masses m_1 and m_2, move with velocities $v_1=\omega R_1$ and $v_2=\omega R_2$, and are separated by a fixed distance $R=R_1+R_2$. The parameters $\xi=(t,r)$ are the time t and the

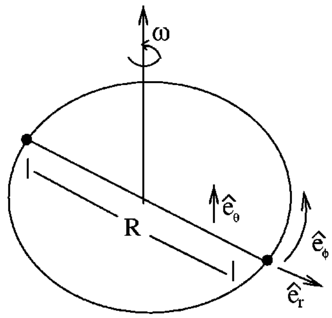

FIG. 1. The string coordinate system.

coordinate r, which runs along the string from $-R_1$ to R_2, so that the transverse velocity of the straight string is zero when $r=0$.

The amplitudes $f(\xi)$ of the transverse fluctuations are the spherical coordinates $\theta(r,t)$ and $\phi(r,t)$ of a point on the string. These angles are defined in an unconventional manner so that $\theta(r,t)=\phi(r,t)=0$ is a straight string rotating in the xy plane. The ends of the string are fixed to their classical trajectories:

$$\theta(-R_1,t)=\theta(R_2,t)=\phi(-R_1,t)=\phi(R_2,t)=0. \quad (3.8)$$

The fluctuating world sheet $\tilde{x}^\mu$ then has the parametrization $\tilde{x}^\mu(r,t)$ given by

$$\begin{aligned}\tilde{x}^\mu(r,t)&=x^\mu(\theta(r,t),\phi(r,t),r,t)\\&=t\hat{\mathbf{e}}_0^\mu+r\{\cos\theta(r,t)\cos[\phi(r,t)+\omega t]\hat{\mathbf{e}}_1^\mu\\&\quad+\cos\theta(r,t)\sin[\phi(r,t)+\omega t]\hat{\mathbf{e}}_2^\mu\\&\quad-\sin\theta(r,t)\hat{\mathbf{e}}_3^\mu\},\end{aligned} \quad (3.9)$$

where $\hat{\mathbf{e}}_\alpha^\mu$, $\alpha=0,\ldots,3$, are unit vectors along the four fixed spacetime axes, $\hat{\mathbf{e}}_\alpha^\mu=\delta_\alpha^\mu$.

The classical rotating straight string $\overline{x^\mu}(r,t)$ has the parametrization (3.9) with $\theta=\phi=0$:

$$\begin{aligned}\overline{x^\mu}(r,t)&=x^\mu(\theta(r,t)=0,\phi(r,t)=0,r,t)\\&=t\hat{\mathbf{e}}_0^\mu+r[\cos\omega t\hat{\mathbf{e}}_1^\mu+\sin\omega t\hat{\mathbf{e}}_2^\mu].\end{aligned} \quad (3.10)$$

The corresponding metric $\overline{g_{ab}}=g_{ab}[\overline{x^\mu}]$ and the classical action $S_{\rm NG}[\overline{x^\mu}]$ are independent of the time t, so that $W[\Gamma]$ has the form

$$W[\Gamma]=e^{iTL^{\rm string}(R_1,R_2,\omega)}, \quad (3.11)$$

where T is the elapsed time. For massless quarks, the ends of the string move with the velocity of light, and singularities appear in $L^{\rm string}$. To regulate these singularities, we retain the quark mass as a cutoff and take the massless limit at the end when evaluating physical quantities for massless quarks.

The Lagrangian $L^{\rm string}$ is the sum of a classical part $L_{\rm cl}^{\rm string}$ and a fluctuating part $L_{\rm fluc}^{\rm string}$,

$$L^{\rm string}=L_{\rm cl}^{\rm string}+L_{\rm fluc}^{\rm string}, \quad (3.12)$$

where

$$L_{\rm cl}^{\rm string}=-\frac{\sigma}{T}\int d^2\xi\sqrt{-\bar{g}}=-\sigma\int_{-R_1}^{R_2}dr\sqrt{1-r^2\omega^2}. \quad (3.13)$$

The effective Lagrangian for the quark-antiquark pair is obtained by adding quark mass terms to $L^{\rm string}$:

$$L_{\rm eff}(R_1,R_2,\omega)=-\sum_{i=1}^{2}m_i\sqrt{1-(\omega R_i)^2}+L^{\rm string}(R_1,R_2,\omega). \quad (3.14)$$

The effective Lagrangian is the sum of a classical part and a fluctuating part:

$$L_{\rm eff}(R_1,R_2,\omega)=L_{\rm cl}(R_1,R_2,\omega)+L_{\rm fluc}^{\rm string}(R_1,R_2,\omega), \quad (3.15)$$

where

$$\begin{aligned}L_{\rm cl}&=-\sum_{i=1}^{2}m_i\sqrt{1-(\omega R_i)^2}-\sigma\int_{-R_1}^{R_2}dr\sqrt{1-r^2\omega^2}\\&=-\sum_{i=1}^{2}\left[\sigma\frac{R_i}{2}\left(\frac{\arcsin(v_i)}{v_i}+\gamma_i^{-1}\right)+m_i\gamma_i^{-1}\right],\end{aligned} \quad (3.16)$$

with

$$\gamma_i=\frac{1}{\sqrt{1-v_i^2}},\quad v_i=\omega R_i. \quad (3.17)$$

The expression for $L_{\rm fluc}^{\rm string}$ is obtained from Eq. (3.11) and the semiclassical calculation of $W[\Gamma]$. It contains terms which are quadratically, linearly, and logarithmically divergent in the cutoff M. The quadratically divergent term is a renormalization of the string tension, the linearly divergent term is a renormalization of the quark mass, and the logarithmically divergent term is proportional to the integral of the scalar curvature over the whole world sheet [1]. After absorbing the quadratically and linearly divergent terms into renormalizations, we obtained an expression for $L_{\rm fluc}^{\rm string}$ [3]. The following is a generalization of that expression to the case of unequal quark masses (see Sec. IX):

$$\begin{aligned}L_{\rm fluc}^{\rm string}(R_1,R_2,\omega)&=\frac{\pi}{12R_p}-\sum_{i=1}^{2}\frac{\omega v_i\gamma_i}{\pi}\left[\ln\left(\frac{MR_i}{\gamma_i^2-1}\right)+1\right]\\&\quad+\frac{1}{2}\omega+\omega f(v_1,v_2),\end{aligned} \quad (3.18)$$

where R_p is the proper length of the string,

$$R_p = \frac{1}{\omega}(\arcsin v_1 + \arcsin v_2), \tag{3.19}$$

and

$$f(v_1,v_2) = -\frac{1}{\pi}\int_0^\infty ds \ln\left[\frac{s^2+(v_1\gamma_1+v_2\gamma_2)s\coth(sR_p\omega)+v_1\gamma_1 v_2\gamma_2}{(s+v_1\gamma_1)(s+v_2\gamma_2)}\right]. \tag{3.20}$$

The function $f(v_1,v_2)$ vanishes when v_1 and v_2 approach unity, so that the last term in Eq. (3.18) is small for relativistic quarks.

In the limit $\omega\rightarrow 0$, $R_p\rightarrow R_1+R_2=R$, and $L^{\text{string}}_{\text{fluc}}$ reduces to the result of Lüscher for the correction to the static quark-antiquark potential due to string fluctuations:

$$V_{\text{Lüscher}} = -L^{\text{string}}_{\text{fluc}}(R_1,R_2,\omega=0) = -\frac{\pi}{12R}. \tag{3.21}$$

For $\omega\neq 0$, $L^{\text{string}}_{\text{fluc}}$ contains a logarithmically divergent part. We simplify this term using the classical equation of motion,

$$\left.\frac{\partial L_{\text{cl}}}{\partial R_i}\right|_{R_i=\bar{R}_i} = 0, \tag{3.22}$$

to express $\bar{R}_i$ in terms of ω. Equation (3.22) gives the relation

$$\sigma\bar{R}_i = m_i(\bar{\gamma}_i^2 - 1), \tag{3.23}$$

where $\bar{\gamma}_i$ is equal to γ_i evaluated at $R_i=\bar{R}_i$. The solution of Eq. (3.23) for $\bar{R}_i$ as a function of ω is

$$\bar{R}_i = \frac{1}{\omega}\left[\sqrt{\left(\frac{m_i\omega}{2\sigma}\right)^2+1} - \frac{m_i\omega}{2\sigma}\right]. \tag{3.24}$$

Using the relation (3.23) in Eq. (3.18) gives

$$L^{\text{string}}_{\text{fluc}} = \frac{\pi}{12R_p} - \sum_{i=1}^{2}\frac{\omega v_i\bar{\gamma}_i}{\pi}\left[\ln\left(\frac{Mm_i}{\sigma}\right)+1\right] + \frac{1}{2}\omega + \omega f(v_1,v_2). \tag{3.25}$$

The logarithmically divergent quantity in the square brackets is independent of the dynamical parameter ω. This is important, because the quantity $\omega v_i\bar{\gamma}_i$ diverges when $m_i\rightarrow 0$.

In the next section, we will show that the term containing the logarithmic divergence can be absorbed by renormalization of a contribution to the string action called the geodesic curvature. When this divergence is removed, the theory will be finite in the $m_i\rightarrow 0$ limit. This renormalization was not done in [3], and is important because it will produce a finite limit of the theory for massless quarks.

IV. RENORMALIZATION OF THE GEODESIC CURVATURE

We now define the geodesic curvature [23], and renormalize Eq. (3.25). In the same way that the action for the string (3.5) can be expanded in powers of the extrinsic curvature, the action for the boundary can be expanded in powers of the geodesic curvature. Using the notation x_i^μ for the positions of the ends of the string,

$$x_i^\mu(t) \equiv \tilde{x}^\mu((-1)^i R_i(t),t), \tag{4.1}$$

the boundary part S_b of the action is

$$S_b = -\sum_{i=1}^{2} m_i\int dt\sqrt{-\dot{x}_i^{\mu 2}} - \sum_{i=1}^{2}\kappa_i\int dt\frac{\dot{x}_i^\mu}{\sqrt{-\dot{x}_i^{\mu 2}}} \times[(-1)^i t_{\mu\nu}]|_{r=(-1)^i R_i(t)}\frac{d}{dt}\frac{\dot{x}_i^\nu}{\sqrt{-\dot{x}_i^{\mu 2}}}, \tag{4.2}$$

where $t_{\mu\nu}$ is the antisymmetric string world sheet orientation tensor:

$$t^{\mu\nu} \equiv \frac{\epsilon^{ab}}{\sqrt{-g}}\frac{\partial\tilde{x}^\mu}{\partial\xi^a}\frac{\partial\tilde{x}^\nu}{\partial\xi^b}. \tag{4.3}$$

The first term in Eq. (4.2) is the quark mass term. The second is the contribution of the geodesic curvature of the boundary (the extrinsic curvature of the boundary in the plane of the string world sheet). The factor of $(-1)^i$ multiplying $t_{\mu\nu}$ is present so that $\dot{x}_i^\mu(-1)^i t_{\mu\nu}$ is always an outward pointing radial vector.

For a straight string rotating with angular velocity ω, the geodesic curvature is equal to $\omega v_i\gamma_i^2$. Inserting this in Eq. (4.2) and dropping the integral over time gives the boundary Lagrangian

$$L^{\text{boundary}} = -\sum_{i=1}^{2}\gamma_i^{-1}[m_i+\kappa_i\omega v_i\gamma_i^2+\cdots], \tag{4.4}$$

where κ_i is the coefficient of the first order term in this expansion. The logarithmic divergence in Eq. 3.25 can then be regarded as a renormalization of κ_i. In the limit where the quark is massless, we must take the $m_i\rightarrow 0$ limit before we take the cutoff M to infinity, since we have an effective theory. In the $m_i\rightarrow 0$ limit, the requirement that the action

(4.4) is finite then forces the renormalized value of κ_i to be zero (note that this does not prevent κ_i from being zero for nonzero m_i). The logarithmic divergence in M may therefore be absorbed into a renormalization of the geodesic curvature in the case where either $m_i=0$ or $v_i \ll 1$.

Removing the terms in Eq. 3.25 proportional to $\omega v_i \gamma_i$ gives an expression for $L_{\text{fluc}}^{\text{string}}$ which is applicable in the massless quark limit:

$$L_{\text{fluc}}^{\text{string}}=\frac{\pi}{12 R_p}+\frac{1}{2}\omega+\omega f(v_1,v_2). \tag{4.5}$$

In the case of two light mesons with $m_1=m_2=0$ ($\overline{\gamma_1},\overline{\gamma_2}\to\infty$), Eqs. 3.19 and 3.20 give $R_p=\pi/\omega$ and $f(v_1,v_2)=0$, so that Eq. 4.5 becomes

$$L_{\text{fluc}}^{\text{string}}(\omega)|_{m_1=m_2=0}=\frac{\omega}{12}+\frac{\omega}{2}=\frac{7}{12}\omega. \tag{4.6}$$

In the case of one heavy and one light quark, $m_1\to\infty$ ($v_1\to 0$) and $m_2=0$ ($\overline{\gamma_2}\to\infty$), $R_p=\pi/2\omega$ and

$$f(v_1=0,v_2\to 1)=-\frac{1}{\pi}\int_0^\infty \ln\coth\left(\frac{\pi}{2}s\right)=-\frac{\omega}{4}, \tag{4.7}$$

so

$$L_{\text{fluc}}^{\text{string}}(\omega)|_{\substack{m_1\to\infty\\ m_2=0}}=\frac{\omega}{6}+\frac{\omega}{2}-\frac{\omega}{4}=\frac{5}{12}\omega. \tag{4.8}$$

V. FLUCTUATIONS IN THE MOTION OF THE QUARKS AT THE ENDS OF THE STRING

In the previous discussion, the quark-antiquark pair moved in a fixed classical trajectory in the xy plane [see Fig. 1 and Eqs. (3.8) and (3.24)]. We now take into account the fluctuations of the positions $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ of the quarks at the ends of the rotating string, so that these coordinates are no longer fixed by Eqs. (3.8) and (3.24). The radial coordinates $R_1(t)$ and $R_2(t)$, along with the angular coordinates $\theta(-R_1(t),t)$, $\phi(-R_1(t),t)$, $\theta(R_2(t),t)$ and $\phi(R_2(t),t)$, parametrize the end points $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ of the string in a reference frame rotating with angular velocity ω in the xy plane:

$$\begin{aligned}\vec{x}_1(t)=&-R_1(t)\{\cos\theta(-R_1(t),t)\cos[\phi(-R_1(t),t)+\omega t]\hat{\mathbf{e}}_1\\&+\cos\theta(-R_1(t),t)\sin[\phi(-R_1(t),t)+\omega t]\hat{\mathbf{e}}_2\\&-\sin\theta(-R_1(t),t)\hat{\mathbf{e}}_3\},\\ \vec{x}_2(t)=&R_2(t)\{\cos\theta(R_2(t),t)\cos[\phi(R_2(t),t)+\omega t]\hat{\mathbf{e}}_1\\&+\cos\theta(R_2(t),t)\sin[\phi(R_2(t),t)+\omega t]\hat{\mathbf{e}}_2\\&-\sin\theta(R_2(t),t)\hat{\mathbf{e}}_3\}.\end{aligned} \tag{5.1}$$

The values of the coordinates r and t at the ends of the string are determined by the equations $r=-R_1(t)$ and $r=R_2(t)$, so the string has the representation (3.9) with

$$-R_1(t)\leqslant r\leqslant R_2(t). \tag{5.2}$$

We extend the functional integral (3.2) to include a path integral over $\vec{x}_1(t)$ and $\vec{\mathbf{x}}_2(t)$, and add the action of the quarks to the string action (3.7). This extension replaces $W[\Gamma]$ by the "partition function" Z,

$$\begin{aligned}Z=&\frac{1}{Z_b}\int \mathcal{D}f^1(\xi)\mathcal{D}f^2(\xi)\mathcal{D}\vec{x}_1(t)\mathcal{D}\vec{\mathbf{x}}_2(t)\Delta_{FP}\\&\times\exp\left(-i\sigma\int d^2\xi\sqrt{-g}-i\sum_{i=1}^{2} m_i\int_{-T/2}^{T/2}dt\sqrt{1-\dot{\vec{x}}_i^2(t)}\right),\end{aligned} \tag{5.3}$$

where Z_b is the partition function of two free (scalar) quarks:

$$\begin{aligned}Z_b=&\int \mathcal{D}\vec{\mathbf{x}}_1(t)\mathcal{D}\vec{\mathbf{x}}_2(t)\\&\times\exp\left(-i\sum_{i=1}^{2} m_i\int_{-T/2}^{T/2}dt[\sqrt{1-\dot{\vec{x}}_i^2(t)}]\right).\end{aligned} \tag{5.4}$$

Dividing by Z_b removes the vacuum energy of the quarks.

The partition function Z sums over all string states. In choosing the parametrization (3.9) for $\tilde{x}^\mu(\xi)$, we have replaced Z with a partition function which contains a sum over those string states with a particular value of the average angular velocity ω. We denote this partition function by $Z(\omega)$.

Under the parametrization (3.9), the integration measure $\mathcal{D}f^1\mathcal{D}f^2\Delta_{FP}$ for the interior of the string becomes

$$\mathcal{D}f^1(\xi)\mathcal{D}f^2(\xi)\Delta_{FP}=\mathcal{D}[\sin\theta(r,t)]\mathcal{D}\phi(r,t)\text{Det}\left[\frac{r^2}{\sqrt{-g}}\right]. \tag{5.5}$$

The integration measure for the end points is

$$\begin{aligned}\mathcal{D}\vec{\mathbf{x}}_1\mathcal{D}\vec{\mathbf{x}}_2=&\mathcal{D}(\sin\theta)|_{r=-R_1(t)}\mathcal{D}\phi|_{r=-R_1(t)}\\&\times\mathcal{D}R_1\mathcal{D}(\sin\theta)|_{r=R_2(t)}\\&\times\mathcal{D}\phi|_{r=R_2(t)}\mathcal{D}R_2\text{Det}[R_1^2]\text{Det}[R_2^2].\end{aligned} \tag{5.6}$$

The path integral (5.3), with the parametrizations (3.9) and (5.1), is then

$$\begin{aligned}Z(\omega)=&\frac{1}{Z_b}\int \mathcal{D}(\sin\theta)\mathcal{D}\phi\mathcal{D}R_1\mathcal{D}R_2\text{Det}\left[\frac{r^2}{\sqrt{-g}}\right]\\&\times\text{Det}[R_1^2]\text{Det}[R_2^2]\exp\left(i\int_{-T/2}^{T/2}dtL[\tilde{x}^\mu]\right),\end{aligned} \tag{5.7}$$

where the Lagrangian $L[\tilde{x}^\mu]$ is

$$L[\tilde{x}^\mu] = -\sigma\int_{-R_1(t)}^{R_2(t)} dr\sqrt{-g} - \sum_{i=1}^{2} m_i\sqrt{-\dot{x}_i^{\mu 2}}, \quad (5.8)$$

where $\dot{x}_i^\mu$ is the time derivative of x_i^μ, defined in Eq. 4.1.

The functional integral (5.7) evaluated in the steepest descent approximation about the classical solution $\theta(r,t) = \phi(r,t) = 0$, $R_i(t) = \overline{R_i}$ determines the effective Lagrangian for the rotating quark-antiquark pair:

$$Z(\omega) = e^{iTL_{\rm eff}(\omega)}. \quad (5.9)$$

Equation 5.9 is the extension of Eq. 3.11 to include fluctuations of the boundary. The effective Lagrangian is the sum of a classical part and a fluctuating part:

$$L_{\rm eff}(\omega) = L_{\rm cl}(\omega) + L_{\rm fluc}(\omega), \quad (5.10)$$

where $L_{\rm cl}(\omega)$ is given by Eq. 3.16. The fluctuation part $L_{\rm fluc}$ contains contributions both from fluctuations of the interior of the string and from fluctuations of the boundary of the string.

In Appendix A, using the methods of Dashen, Hasslacher, and Neveu [24], we obtain from the full partition function Z a quantization condition on the angular momentum, and show how to find the energies of the physical meson states. We summarize these results here. The angular momentum J is

$$J = \frac{dL_{\rm eff}(\omega)}{d\omega}, \quad (5.11)$$

where $L_{\rm eff}(\omega)$ is determined by Eq. 5.9 in terms of the particular partition function (5.7). The angular momentum is fixed by the WKB quantization condition

$$J = l + \frac{1}{2}, \quad l = 0,1,2, \ldots . \quad (5.12)$$

The energy $E(\omega)$ is given by the corresponding Hamiltonian:

$$E(\omega) = \omega\frac{dL_{\rm eff}(\omega)}{d\omega} - L_{\rm eff}(\omega). \quad (5.13)$$

The energy is equal to the classical energy $E_{\rm cl}(\omega)$, plus a correction due to fluctuations. To first order in the perturbation $L_{\rm fluc}(\omega)$, the correction to the energy is minus the correction to the Lagrangian [25],

$$E(\overline{\omega}) = E_{\rm cl}(\overline{\omega}) - L_{\rm fluc}(\overline{\omega}), \quad (5.14)$$

where

$$E_{\rm cl}(\overline{\omega}) = \overline{\omega}\frac{dL_{\rm cl}(\overline{\omega})}{d\overline{\omega}} - L_{\rm cl}(\overline{\omega}), \quad (5.15)$$

and where $\overline{\omega}$ is given as a function of J by the classical relation

$$J = \frac{dL_{\rm cl}(\overline{\omega})}{d\overline{\omega}}. \quad (5.16)$$

The zero point energy of the fluctuations is then

$$E_{\rm fluc}(J) = -L_{\rm fluc}(\overline{\omega}(J)). \quad (5.17)$$

Equations 5.16 and 5.17 give the leading semiclassical correction to the energies of mesons on the leading Regge trajectory.

VI. QUADRATIC EXPANSION OF THE ACTION

To evaluate the effective Lagrangian $L_{\rm eff}(\omega)$ from Eq. 5.9, we must first expand the Lagrangian $L[\tilde{x}^\mu]$ to quadratic order in the small fluctuations about the classical solution. We call $r_1(t)$ and $r_2(t)$ the fluctuations of $R_1(t)$ and $R_2(t)$ about the classical values $\overline{R_1}$ and $\overline{R_2}$, Eq. 3.24:

$$R_1(t) = \overline{R_1} + r_1(t), \quad R_2(t) = \overline{R_2} + r_2(t). \quad (6.1)$$

To quadratic order in small fluctuations, the angular coordinates of the ends of the string $\theta((-1)^i R_i(t),t)$ and $\phi((-1)^i R_i(t),t)$ can be evaluated at the classical values of the $R_i(t)$.

The degrees of freedom in the Lagrangian can be viewed as a combination of string degrees of freedom $[\phi(r,t)$ and $\theta(r,t)$ for $-\overline{R_1} < r < \overline{R_2}]$ and quark degrees of freedom

$$\theta(-\overline{R_1},t),\ \theta(\overline{R_2},t),\ \phi(-\overline{R_1},t),\ \phi(\overline{R_2},t),\ r_1(t),\ r_2(t). \quad (6.2)$$

The quark degrees of freedom depend only upon t (and not r), and we refer to them as "boundary" degrees of freedom. The string degrees of freedom depend on both t and r, and we refer to them as the "interior" degrees of freedom. Equations 5.9 and (5.10) for $L_{\rm eff}(\omega)$ reduce to Eqs. 3.11 and (3.12) when the boundary degrees of freedom are set equal to zero, and become boundary conditions on the interior degrees of freedom. Inclusion of these boundary degrees of freedom gives an additional contribution to $L_{\rm eff}(\omega)$.

We now expand the Lagrangian (5.8) and the corresponding action $\int_{-T/2}^{T/2} dt L[\tilde{x}^\mu]$ to quadratic order in the small fluctuations $\theta(r,t)$, $\phi(r,t)$ and $r_i(t)$ about the classical solution $\theta = \phi = 0$, $r_i = 0$. We first evaluate the string tension term. Using the parametrization (3.9) of the world sheet, we obtain the tangent vectors to $\tilde{x}^\mu$:

$$\begin{aligned}\dot{x}^\mu = \hat{\mathbf{e}}_0^\mu - r\dot{\theta}[\sin\theta\cos(\phi+\omega t)\hat{\mathbf{e}}_1^\mu + \sin\theta\sin(\phi+\omega t)\hat{\mathbf{e}}_2^\mu \\ + \cos\theta\hat{\mathbf{e}}_3^\mu] + r(\dot{\phi}+\omega)\cos\theta[-\sin(\phi+\omega t)\hat{\mathbf{e}}_1^\mu \\ + \cos(\phi+\omega t)\hat{\mathbf{e}}_2^\mu],\end{aligned}$$

$$
\begin{aligned}
x'^{\mu} &\equiv \frac{\partial x^{\mu}}{\partial r} \\
&= [\cos\theta\cos(\phi+\omega t)\hat{\mathbf{e}}_1^{\mu} + \cos\theta\sin(\phi+\omega t)\hat{\mathbf{e}}_2^{\mu} \\
&\quad - \sin\theta\hat{\mathbf{e}}_3^{\mu}] - r\theta'[\sin\theta\cos(\phi+\omega t)\hat{\mathbf{e}}_1^{\mu} + \sin\theta\sin(\phi \\
&\quad + \omega t)\hat{\mathbf{e}}_2^{\mu} + \cos\theta\hat{\mathbf{e}}_3^{\mu}] + r\phi'\cos\theta[-\sin(\phi+\omega t)\hat{\mathbf{e}}_1^{\mu} \\
&\quad + \cos(\phi+\omega t)\hat{\mathbf{e}}_2^{\mu}]. \qquad (6.3)
\end{aligned}
$$

The components of the metric are

$$
\begin{aligned}
g_{tt} &= \dot{x}^{\mu}\dot{x}_{\mu} = -1 + r^2\dot{\theta}^2 + r^2(\dot{\phi}+\omega)^2\cos^2\theta, \\
g_{rt} &= \dot{x}^{\mu}x'_{\mu} = r^2\dot{\theta}\theta' + r^2(\dot{\phi}+\omega)\phi'\cos^2\theta, \\
g_{rr} &= x'^{\mu}x'_{\mu} = 1 + r^2\theta'^2 + r^2\phi'^2\cos^2\theta. \qquad (6.4)
\end{aligned}
$$

To quadratic order in θ and ϕ, the square root of the determinant of the metric is

$$
\begin{aligned}
\sqrt{-g} &= \gamma^{-1} - r^2\omega\gamma\dot{\phi} - \frac{1}{2}r^2\gamma(\dot{\theta}^2 - \omega^2\theta^2 - \gamma^{-2}\theta'^2) \\
&\quad - \frac{1}{2}r^2\gamma^3(\dot{\phi}^2 - \gamma^{-2}\phi'^2), \qquad (6.5)
\end{aligned}
$$

where $\gamma^{-1} = \sqrt{1 - r^2\omega^2}$.

To evaluate the string tension term to quadratic order, we must also expand the limits of integration $-R_1(t)$ and $R_2(t)$ about $-\overline{R}_1$ and $\overline{R}_2$ respectively. Using Eqs. (6.1) and (6.3), we obtain

$$
\begin{aligned}
\int_{-R_1(t)}^{R_2(t)} dr\sqrt{-g} &= \int_{-\overline{R}_1}^{\overline{R}_2} dr\Big[\gamma^{-1} - r^2\omega\gamma\dot{\phi} - \frac{1}{2}r^2\gamma(\dot{\theta}^2 - \omega^2\theta^2 \\
&\quad - \gamma^{-2}\theta'^2) - \frac{1}{2}r^2\gamma^3(\dot{\phi}^2 - \gamma^{-2}\phi'^2)\Big] \\
&\quad + \sum_i \Bigg[r_i(\gamma^{-1} - r^2\omega\gamma\dot{\phi})\Big|_{r=(-1)^i\overline{R}_i} \\
&\quad + \frac{1}{2}(-1)^i r_i^2 \frac{d\gamma^{-1}}{dr}\Big|_{r=(-1)^i\overline{R}_i}\Bigg]. \qquad (6.6)
\end{aligned}
$$

The second term in Eq. (6.6), which is linear in ϕ, is a perfect time derivative, and contributes a term in the action given by

$$
\begin{aligned}
&-\int_{-T/2}^{T/2} dt \int_{-\overline{R}_1}^{\overline{R}_2} dr\, r^2\gamma\omega\dot{\phi} \\
&\quad = -\int_{-\overline{R}_1}^{\overline{R}_2} dr\, r^2\gamma\omega\left[\phi\left(\frac{T}{2},r\right) - \phi\left(-\frac{T}{2},r\right)\right] = 0. \qquad (6.7)
\end{aligned}
$$

The quantity in square brackets is zero, since the angular velocity ω is defined to be the angle traversed by the string in time T, divided by the time T. The constraint (6.7) is the condition that the fluctuation $\dot{\phi}$ not contribute to first order to the angular momentum of the string, and hence contribute only to vibrational modes.

Next we expand the quark mass term in Eq. (5.8):

$$
\begin{aligned}
\sqrt{-(\dot{\tilde{x}}^{\mu} + \dot{r}\tilde{x}'^{\mu})^2}\Big|_{r=(-1)^iR_i(t)} &= \overline{\gamma}_i^{-1} - R_i(t)^2\omega\gamma\Big|_{r=(-1)^iR_i(t)} \frac{d\phi((-1)^iR_i(t),t)}{dt} \\
&\quad - \frac{1}{2}\overline{R}_i^2\overline{\gamma}_i(\dot{\theta}^2 - \omega^2\theta^2)\Big|_{r=(-1)^i\overline{R}_i} - \frac{1}{2}\overline{R}_i^2\overline{\gamma}_i^3\dot{\phi}^2\Big|_{r=(-1)^i\overline{R}_i} - \frac{1}{2}\overline{\gamma}_i\dot{R}_i(t)^2 \\
&= \overline{\gamma}_i^{-1} + (-1)^i r_i \frac{d(\gamma^{-1})}{dr}\Big|_{r=(-1)^i\overline{R}_i} + \frac{1}{2}r_i^2\frac{d^2(\gamma^{-1})}{dr^2}\Big|_{r=(-1)^i\overline{R}_i} \\
&\quad - \overline{R}_i^2\overline{\gamma}_i\omega\frac{d\phi(R_i(t),t)}{dt} - (-1)^i r_i\dot{\phi}\omega\frac{d(r^2\gamma)}{dr}\Big|_{r=(-1)^i\overline{R}_i} - \frac{1}{2}\overline{R}_i^2\overline{\gamma}_i(\dot{\theta}^2 - \omega^2\theta^2)\Big|_{r=(-1)^i\overline{R}_i} \\
&\quad - \frac{1}{2}\overline{R}_i^2\overline{\gamma}_i^3\dot{\phi}^2\Big|_{r=(-1)^i\overline{R}_i} - \frac{1}{2}\overline{\gamma}_i\dot{r}_i(t)^2. \qquad (6.8)
\end{aligned}
$$

Inserting Eqs. (6.6) and (6.8) into Eq. (5.8) and applying the constraint (6.7) gives

$$
\begin{aligned}
L[\tilde{x}^{\mu}] &= L_{\mathrm{cl}}(\omega) + \frac{1}{2}\sigma\int_{-\overline{R}_1}^{\overline{R}_2} dr[r^2\gamma(\dot{\theta}^2 - \omega^2\theta^2 - \gamma^{-2}\theta'^2) + r^2\gamma^3(\dot{\phi}^2 - \gamma^{-2}\phi'^2)] \\
&\quad + \sum_{i=1}^{2}\Bigg[m_i\overline{R}_i^2(\dot{\theta}^2 - \omega^2\theta^2)\Big|_{r=(-1)^i\overline{R}_i} + m_i\overline{R}_i^2\dot{\phi}^2\Big|_{r=(-1)^i\overline{R}_i} + (-\sigma\overline{\gamma}_i^{-1} + m_i\overline{\gamma}_i\overline{R}_i\omega^2)r_i + \frac{1}{2}m_i\overline{\gamma}_i\dot{r}_i^2 + \frac{1}{2}(\sigma\overline{\gamma}_i\overline{R}_i\omega^2 \\
&\quad + m_i\overline{\gamma}_i^3\omega^2)r_i^2 + [\sigma\overline{\gamma}_i\overline{R}_i^2\omega + m_i\overline{\gamma}_i(\overline{\gamma}_i^2 + 1)\overline{R}_i\omega]r_i\dot{\phi}\Big|_{r=(-1)^i\overline{R}_i}\Bigg]. \qquad (6.9)
\end{aligned}
$$

The term in Eq. (6.9) which is linear in r_i vanishes, because $\overline{R_i}$ satisfies the classical equation of motion (3.24). Replacing σ using Eq. (3.24) in the terms in Eq. (6.9) which contain r_i gives

$$L[\tilde{x}^\mu]=L_{\rm cl}(\omega)+\frac{1}{2}\sigma\int_{-\overline{R}_1}^{\overline{R}_2}dr[r^2\gamma(\dot{\theta}^2-\omega^2\theta^2-\gamma^{-2}\theta'^2)+r^2\gamma^3(\dot{\phi}^2-\gamma^{-2}\phi'^2)]+\sum_i m_i\overline{\gamma_i}\bigg[\frac{1}{2}\overline{R_i^2}(\dot{\theta}^2-\omega^2\theta^2)\bigg|_{r=(-1)^i\overline{R_i}}$$
$$+\frac{1}{2}\overline{R_i^2}\,\overline{\gamma_i^2}\dot{\phi}^2\bigg|_{r=(-1)^i\overline{R_i}}+\frac{1}{2}\dot{r}_i^2+\frac{1}{2}(2\overline{\gamma_i^2}-1)\omega^2r_i^2+2\overline{\gamma_i^2}\,\overline{R_i}\omega r_i\dot{\phi}\bigg|_{r=(-1)^i\overline{R_i}}\bigg]. \tag{6.10}$$

Equation (6.10) is the complete quadratic expansion of the Lagrangian, and it includes both interior and boundary degrees of freedom. The interior-interior interactions are all contained in the integral over r, and the boundary-boundary interactions in the last term are proportional to the quark masses. Interior-boundary interactions occur in the terms in the integral containing θ' and ϕ', which couple interior and boundary parts of θ and ϕ through the spatial derivative.

VII. COUPLING OF QUARKS TO EXTERNAL SOURCES

The effective Lagrangian (5.9) determines the energy (5.17) of the ground state of a rotating quark-antiquark pair having angular momentum J. We will also calculate the energies of the excited states of the mesons (hybrid mesons lying on daughter Regge trajectories) by examining the poles in the Green's function which describes the coupling of the end points of the string. To obtain this Green's function, we add to $L[\tilde{x}^\mu]$ a term $L_{\rm source}$ coupling the positions $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ of the quarks to external forces $\vec{\rho}^{\,i}(t)$, $i=1,2$:

$$L_{\rm source}=\sum_{i=1}^{2}\vec{\rho}^{\,i}(t)\cdot\vec{\mathbf{x}}_i(t), \tag{7.1}$$

where

$$\vec{\rho}^{\,i}=\rho^i_\phi\sin\delta_i\hat{\mathbf{e}}_1+\rho^i_\phi\cos\delta_i\hat{\mathbf{e}}_2+\rho^i_\theta\hat{\mathbf{e}}_3. \tag{7.2}$$

The sources ρ^i_θ couple to the fluctuations θ transverse to the plane of rotation, while ρ^i_ϕ and δ_i are the polar coordinates of the forces coupling to the fluctuations ϕ, r_1, and r_2 lying in this plane. The Lagrangian (6.10) couples the interior and boundary degrees of freedom, so that sources acting on the boundary will also couple to the interior degrees of freedom, and are capable of generating the excited states of the string.

Inserting the expression (5.1) for the quark coordinate $\vec{\mathbf{x}}_i$ into $L_{\rm source}$, and keeping the leading term in small fluctuations gives

$$L_{\rm source}=L^\theta_{\rm source}+L^\phi_{\rm source}, \tag{7.3}$$

where

$$L^\theta_{\rm source}=\sum_{i=1}^{2}(-1)^i\rho^i_\theta(t)\overline{R_i}\theta((-1)^i\overline{R_i},t), \tag{7.4}$$

and

$$L^\phi_{\rm source}=\sum_{i=1}^{2}(-1)^i\rho^i_\phi(t)\{r_i(t)\sin[\omega t+\delta_i(t)]+\overline{R_i}\phi((-1)^i\overline{R_i},t)\cos[\omega t+\delta_i(t)]\}. \tag{7.5}$$

The Lagrangian $L^\theta_{\rm source}$ gives the coupling of the source to the transverse fluctuations $\theta((-1)^i\overline{R_i},t)$, while $L^\phi_{\rm source}$ gives the coupling of the sources to the in plane degrees of freedom $r_i(t)$ and $\phi((-1)^i\overline{R_i},t)$ (see Fig. 1). The phases δ_i give the direction of the external force in the plane of rotation in the space-fixed system, and the angles $\omega t+\delta_i$ give the angle between this force and the instantaneous position of the rotating string.

The Lagrangian (6.10) does not couple the traverse degrees of freedom $\theta(r,t)$ to the in-plane degrees of freedom $\phi(r,t)$ and $r_i(t)$, so we can treat them independently. We can write

$$L+L_{\rm source}=L_{\rm cl}+L_\theta+L_\phi, \tag{7.6}$$

where

$$L_\theta=\frac{1}{2}\sigma\int_{-\overline{R}_1}^{\overline{R}_2}dr\,\gamma r^2(\dot{\theta}^2-\omega^2\theta^2-\gamma^{-2}\theta'^2)+\hat{L}_\theta,$$
$$L_\phi=\frac{1}{2}\sigma\int_{-\overline{R}_1}^{\overline{R}_2}dr\,\gamma^3r^2(\dot{\phi}^2-\gamma^{-2}\phi'^2)+\hat{L}_\phi, \tag{7.7}$$

and

$$\hat{L}_\phi=\sum_i m_i\overline{\gamma_i}\bigg[\frac{1}{2}\dot{r}_i^2+\frac{1}{2}(2\overline{\gamma_i^2}-1)\omega^2r_i^2+2\overline{R_i}\omega\overline{\gamma_i^2}r_i\dot{\phi}+\frac{1}{2}\overline{R_i^2}\,\overline{\gamma_i^2}\dot{\phi}^2\bigg]\bigg|_{r=(-1)^i\overline{R_i}}$$
$$+\sum_i(-1)^i\rho^i_\phi[r_i\sin(\omega t+\delta_i)+\overline{R_i}\phi\big|_{r=(-1)^i\overline{R_i}}\times\cos(\omega t+\delta_i)],$$

$$\hat{L}_\theta=\sum_{i=1}^{2}\left[\frac{1}{2}m_i\overline{R}_i^2\overline{\gamma}_i(\dot{\theta}^2-\omega^2\theta^2)\right.$$
$$\left.\left.+(-1)^i\rho_\theta^i\overline{R}_i\theta\right]\right|_{r=(-1)^i\overline{R}_i}. \tag{7.8}$$

The quantities $\hat{L}_\theta$ and $\hat{L}_\phi$ contain the quark mass terms and the source terms, and depend only on the boundary values (6.2). The remaining terms in Eq. (7.7) are the contributions of the string Lagrangian to L_θ and L_ϕ, and they depend upon both the interior and boundary degrees of freedom. In the next section we will decouple the interior and boundary degrees of freedom, and will obtain an expression for the action as a sum of an interior contribution and a boundary contribution. We will do this separately in each "sector" (θ and ϕ) using a common procedure.

VIII. DECOUPLING THE INTERIOR FROM THE BOUNDARY

We write the two equations (7.7) for L_θ and L_ϕ as specializations of an equation for L_ψ ($\psi=\theta,\phi$):

$$L_\psi=\frac{1}{2}\sigma\int_{-\overline{R}_1}^{\overline{R}_2}dr\Sigma(r)[\dot{\psi}^2(r,t)-C\psi^2(r,t)-\gamma^{-2}\psi'^2(r,t)]$$
$$+\hat{L}_\psi. \tag{8.1}$$

The constant C is ω^2 in the θ sector, and zero in the ϕ sector. The function $\Sigma(r)$ is

$$\Sigma(r)|_{L_\theta}=\gamma r^2,\quad \Sigma(r)|_{L_\phi}=\gamma^3r^2. \tag{8.2}$$

The action S_ψ for each sector can be expressed in terms of the Fourier transform of ψ with respect to time,

$$\tilde{\psi}(r,\nu)\equiv\int dte^{-i\nu t}\psi(r,t). \tag{8.3}$$

$$S_\psi=\int dtL_\psi=-\frac{\sigma}{2}\int\frac{d\nu}{2\pi}\int_{-\overline{R}_1}^{\overline{R}_2}dr\tilde{\psi}^*\left[-\frac{\partial}{\partial r}\Sigma(r)\gamma^{-2}\frac{\partial}{\partial r}\right.$$
$$\left.-(\nu^2-C)\Sigma(r)\right]\tilde{\psi}-\frac{\sigma}{2}\sum_{i=1}^{2}(-1)^i\Sigma[(-1)^i\overline{R}_i]$$
$$\times\overline{\gamma}_i^{-2}\tilde{\psi}_{b,i}^*\left(\frac{\partial\tilde{\psi}}{\partial r}\right)\Bigg|_{r=(-1)^i\overline{R}_i}+\int dt\hat{L}_\psi, \tag{8.4}$$

where $\tilde{\psi}_{b,i}$ are the values of $\tilde{\psi}$ evaluated at the ends $(-1)^i\overline{R}_i$ of the string:

$$\tilde{\psi}_{b,i}(\nu)\equiv\tilde{\psi}((-1)^i\overline{R}_i,\nu). \tag{8.5}$$

We next define the "boundary part" $\tilde{\psi}_B(r,\nu)$ of $\tilde{\psi}(r,\nu)$ to be the solution to the differential equation

$$\left[-\frac{\partial}{\partial r}\Sigma(r)\gamma^{-2}\frac{\partial}{\partial r}-(\nu^2-C)\Sigma(r)\right]\tilde{\psi}_B(r,\nu)=0, \tag{8.6}$$

satisfying the boundary conditions

$$\tilde{\psi}_B((-1)^i\overline{R}_i,\nu)=\tilde{\psi}_{b,i}(\nu). \tag{8.7}$$

We define the "interior part" $\tilde{\psi}_I(r,\nu)$ of $\tilde{\psi}$ as

$$\tilde{\psi}_I(r,\nu)=\tilde{\psi}(r,\nu)-\tilde{\psi}_B(r,\nu). \tag{8.8}$$

As a result of the boundary condition (8.7), $\tilde{\psi}_B$ is entirely determined by the $\tilde{\psi}_{b,i}$. Equation (8.6) guarantees that $\tilde{\psi}_B$ does not couple to $\tilde{\psi}_I$. Notice that, by definition,

$$\tilde{\psi}_I((-1)^i\overline{R}_i,\nu)=0, \tag{8.9}$$

so that the fields $\tilde{\psi}_I$ do not involve the boundary fluctuations.

Replacing $\tilde{\psi}$ in Eq. (8.4) with $\tilde{\psi}_B$ and $\tilde{\psi}_I$ and integrating by parts yields

$$S_\psi=S_{I,\psi}+S_{B,\psi}, \tag{8.10}$$

with

$$S_{I,\psi}=-\frac{\sigma}{2}\int\frac{d\nu}{2\pi}\int_{-\overline{R}_1}^{\overline{R}_2}dr\tilde{\psi}_I^*\left[-\frac{\partial}{\partial r}\Sigma(r)\gamma^{-2}\frac{\partial}{\partial r}\right.$$
$$\left.-(\nu^2-C)\Sigma(r)\right]\tilde{\psi}_I \tag{8.11}$$

and

$$S_{B,\psi}=-\frac{\sigma}{2}\sum_{i=1}^{2}(-1)^i\Sigma[(-1)^i\overline{R}_i]\overline{\gamma}_i^{-2}$$
$$\times\int\frac{d\nu}{2\pi}\tilde{\psi}_{b,i}^*\left(\frac{\partial\tilde{\psi}_B}{\partial r}\right)\Bigg|_{r=(-1)^i\overline{R}_i}+\int dt\hat{L}_\psi. \tag{8.12}$$

The interior action depends only on the interior degrees of freedom [the values of $\tilde{\psi}(r,\nu)$ for $-\overline{R}_1<r<\overline{R}_2$], while the boundary action only depends of $\tilde{\psi}_{b,i}(\nu)$.

To express $\tilde{\psi}_B$ in terms of the $\tilde{\psi}_{b,i}$, we use the Green's function $G(r,r',\nu)$ satisfying the equation

$$\left[-\frac{\partial}{\partial r}\Sigma(r)\gamma^{-2}\frac{\partial}{\partial r}-(\nu^2-C)\Sigma(r)\right]G(r,r',\nu)=\delta(r-r'), \tag{8.13}$$

for $-\overline{R}_1<r<\overline{R}_2$, and the boundary conditions

$$G((-1)^i\overline{R}_i,r',\nu)=0. \tag{8.14}$$

The solution of Eq. (8.6) with boundary conditions (8.7) is

$$\tilde{\psi}_B(r,\nu)=\sum_{i=1}^{2}(-1)^{i+1}\Sigma[(-1)^i\bar{R}_i]\bar{\gamma}_i^{-2}\tilde{\psi}_{b,i}(\nu)\times\frac{\partial}{\partial r'}G(r,r',\nu)\bigg|_{r'=(-1)^i\bar{R}_i}. \tag{8.15}$$

Inserting the expression (8.15) for $\tilde{\psi}_B$ into the definition (8.12) of $S_{B,\psi}$, we find

$$S_{B,\psi}=\int dt\hat{L}_\psi-\frac{\sigma}{2}\sum_{i,j=1}^{2}\int\frac{d\nu}{2\pi}\tilde{\psi}^*_{b,i}G^{ij}_\psi(\nu)\tilde{\psi}_{b,j}, \tag{8.16}$$

where

$$G^{ij}_\psi(\nu)\equiv(-1)^{i+j}\Sigma[(-1)^i\bar{R}_i]\Sigma[(-1)^j\bar{R}_j]\bar{\gamma}_i^{-2}\bar{\gamma}_j^{-2}\times\frac{\partial^2}{\partial r\partial r'}G(r,r',\nu)\bigg|_{\substack{r'=(-1)^i\bar{R}_i\\ r=(-1)^j\bar{R}_j}}. \tag{8.17}$$

Equation (8.16) gives the boundary action in terms of $\tilde{\psi}_{b,i}(\nu)$ and the functions $G^{ij}_\psi(\nu)$ which are evaluated in Appendix B [Eqs. (B12) and (B17)]. The term involving G^{ij}_ψ in Eq. (8.16) represents the "back reaction" of the interior degrees of freedom to the boundary variables.

Inserting Eq. (7.8) into Eq. (8.16) gives the boundary actions $S_{B,\theta}$ and $S_{B,\phi}$:

$$S_{B,\theta}=\int\frac{d\nu}{2\pi}\Bigg\{\sum_{i=1}^{2}\frac{1}{2}m_i\bar{R}_i^2\bar{\gamma}_i(\nu^2-\omega^2)|\tilde{\theta}_{b,i}(\nu)|^2-\frac{\sigma}{2}\sum_{i,j=1}^{2}\tilde{\theta}^*_{b,i}(\nu)G^{ij}_\theta(\nu)\tilde{\theta}_{b,j}(\nu)+\sum_{i=1}^{2}(-1)^i\tilde{\rho}^i_\theta{}^*(\nu)\bar{R}_i\tilde{\theta}_{b,i}(\nu)\Bigg\} \tag{8.18}$$

and

$$S_{B,\phi}=\int\frac{d\nu}{2\pi}\Bigg\{\sum_{i=1}^{2}m_i\bar{\gamma}_i\bigg[\frac{1}{2}[\nu^2+(2\bar{\gamma}_i^2-1)\omega^2]|\tilde{r}_i(\nu)|^2-2\bar{R}_i\omega\bar{\gamma}_i^2\nu\,\mathrm{Im}[\tilde{r}_i^*(\nu)\tilde{\phi}_{b,i}(\nu)]+\frac{1}{2}\bar{R}_i^2\bar{\gamma}_i^2\nu^2|\tilde{\phi}_{b,i}(\nu)|^2\bigg]+\sum_{i=1}^{2}\frac{(-1)^i}{2}[\tilde{\rho}^i_\phi{}^*(\nu+\omega)[\bar{R}_i\tilde{\phi}_{b,i}(\nu)-i\tilde{r}_i(\nu)]+\tilde{\rho}^i_\phi{}^*(\nu-\omega)[\bar{R}_i\tilde{\phi}_{b,i}(\nu)+i\tilde{r}_i(\nu)]]-\frac{\sigma}{2}\sum_{i,j=1}^{2}\tilde{\phi}^*_{b,i}(\nu)G^{ij}_\phi(\nu)\tilde{\phi}_{b,j}(\nu)\Bigg\}. \tag{8.19}$$

We have introduced the Fourier transforms of r_i and the sources ρ^i_θ and ρ^i_ϕ:

$$\tilde{r}_i(\nu)\equiv\int dte^{-i\nu t}r_i(t),\quad\tilde{\rho}^i_\theta(\nu)\equiv\int dte^{-i\nu t}\rho^i_\theta(t),\quad\tilde{\rho}^i_\phi(\nu)\equiv\int dte^{-i\nu t-i\delta_i(t)}\rho^i_\phi(t). \tag{8.20}$$

The function $\tilde{\rho}^i_\phi(\nu)$ incorporates two degrees of freedom, $\rho^i_\phi(t)$ and $\delta_i(t)$, so it is an arbitrary complex function. The other two Fourier transforms satisfy the reality conditions

$$\tilde{r}_i(-\nu)=\tilde{r}_i^*(\nu),\quad\tilde{\rho}^i_\theta(-\nu)=\tilde{\rho}^i_\theta{}^*(\nu). \tag{8.21}$$

Using the definition (8.2) of $\Sigma(r)$ and C in Eq. (8.11) gives the interior actions

$$S_{I,\theta}=-\frac{\sigma}{2}\int\frac{d\nu}{2\pi}\int_{-\bar{R}_1}^{\bar{R}_2}dr\,\tilde{\theta}_I^*\bigg[-\frac{\partial}{\partial r}\gamma^{-1}r^2\frac{\partial}{\partial r}-(\nu^2-\omega^2)\gamma r^2\bigg]\tilde{\theta}_I \tag{8.22}$$

and

$$S_{I,\phi}=-\frac{\sigma}{2}\int\frac{d\nu}{2\pi}\int_{-\bar{R}_1}^{\bar{R}_2}dr\,\tilde{\phi}_I^*\bigg[-\frac{\partial}{\partial r}\gamma r^2\frac{\partial}{\partial r}-\nu^2\gamma^3r^2\bigg]\tilde{\phi}_I. \tag{8.23}$$

The total action is the sum of the independent contributions,

$$\int dt(L+L_{\mathrm{source}})=\int dtL_{\mathrm{cl}}+S_{I,\theta}+S_{I,\phi}+S_{B,\theta}+S_{B,\phi}, \tag{8.24}$$

and the partition function $Z(\omega)$ is a corresponding product:

$$Z(\omega)=e^{iTL_{\mathrm{cl}}(\omega)}Z_I(\omega)Z_B(\omega). \tag{8.25}$$

The interior partition function is

$$Z_I(\omega)=\int\mathcal{D}\theta_Ie^{iS_{I,\theta}}\int\mathcal{D}\phi_Ie^{iS_{I,\phi}}\mathrm{Det}\bigg[\frac{r^2}{\sqrt{-g}}\bigg], \tag{8.26}$$

where $\mathrm{Det}[r^2/\sqrt{-g}]$ is replaced by its classical value. The boundary partition function is

$$Z_B(\omega)=\frac{1}{Z_b}\int\prod_{i=1}^{2}[\mathcal{D}\theta_{b,i}\mathcal{D}\phi_{b,i}\mathcal{D}r_i\mathrm{Det}[\bar{R}_i^2]]e^{iS_{B,\theta}+iS_{B,\phi}}. \tag{8.27}$$

We will evaluate these two parts separately in the next two sections.

IX. EVALUATION OF THE INTERIOR PARTITION FUNCTION $Z_I(\omega)$

In this section, we evaluate $Z_I(\omega)$ and derive Eq. (3.25) for $L^{\mathrm{string}}_{\mathrm{fluc}}$, generalizing the results of [3] to the case where the quark masses are unequal. The interior action depends exclusively on the functions $\tilde{\theta}_I(r,\nu)$ and $\tilde{\phi}_I(r,\nu)$, which vanish at $r=(-1)^i\overline{R_i}$. We simplify $S_{I,\theta}$ and $S_{I,\phi}$ by changing coordinates from r to

$$x=\frac{1}{\omega}\arcsin \omega r. \tag{9.1}$$

We also change our integration variables $\tilde{\theta}_I$ and $\tilde{\phi}_I$ to differently normalized functions

$$\tilde{\theta}_I(r,\nu)=\frac{1}{r}k(x,t),$$

$$\tilde{\phi}_I(r,\nu)=\frac{1}{\gamma r}f(x,t). \tag{9.2}$$

The components of the action become

$$S_{I,\theta}=-\frac{\sigma}{2}\int\frac{d\nu}{2\pi}\int_{-X_1}^{X_2}dx k^*\left[-\frac{\partial^2}{\partial x^2}-\nu^2\right]k,$$

$$S_{I,\phi}=-\frac{\sigma}{2}\int\frac{d\nu}{2\pi}\int_{-X_1}^{X_2}dx f^*\left[-\frac{\partial^2}{\partial x^2}+2\omega^2\sec^2\omega x-\nu^2\right]f, \tag{9.3}$$

where the limits of integration are

$$X_i=\frac{1}{\omega}\arcsin \omega\overline{R_i}. \tag{9.4}$$

In terms of the new variables (9.2), the interior partition function (8.26) is

$$Z_I(\omega)=\int \mathcal{D}k\mathcal{D}f e^{iS_{I,\theta}+iS_{I,\phi}}. \tag{9.5}$$

Doing the integrals over f and k gives

$$Z_I(\omega)=\mathrm{Det}^{-1/2}[-\nabla^2]\mathrm{Det}^{-1/2}[-\nabla^2+2\omega^2\sec^2\omega x], \tag{9.6}$$

where $-\nabla^2$ is the Laplacian in the x,t coordinate system. This coordinate system is conformally flat, i.e. $g_{xx}=-g_{tt}$, $g_{xt}=0$, so we can use the result *et al.* [1], that for a static string in the large time limit,

$$\mathrm{Det}^{-1}[-\nabla^2]=e^{(-\pi/12)(T/R)}\quad:\ \text{static quark background}, \tag{9.7}$$

where R is the length of the string and T is the time elapsed. In Eq. (9.3), the string length X_1+X_2 obtained from Eq. (9.4) is R_p, given by Eq. (3.19), which is the "proper length" of a relativistic rotating string. Making the replacement $R=R_p$ in Eq. (9.7) gives

$$\mathrm{Det}^{-1}[-\nabla^2]=e^{-(\pi/12)\,(T/R_p)}\quad:\ \text{rotating quark background}. \tag{9.8}$$

We therefore see that

$$Z_I(\omega)=e^{(-\pi/12)(T/R_p)}\mathrm{Det}^{-1/2}\left[\frac{-\nabla^2+2\omega^2\sec^2\omega x}{-\nabla^2}\right]\equiv e^{-TL_{\mathrm{fluc}}^{\mathrm{string}}(\omega)}, \tag{9.9}$$

so that

$$L_{\mathrm{fluc}}^{\mathrm{string}}(\omega)=\frac{\pi}{12R_p}-\frac{i}{T}\mathrm{Tr}\log\left[\frac{-\nabla^2+2\omega^2\sec^2\omega x}{-\nabla^2}\right] \tag{9.10}$$

is the contribution of the interior degrees of freedom to L_{fluc}.

We can express the trace in Eq. (9.10) in terms of the eigenvalues μ_n and λ_n determined by the spatial boundary problems

$$\left(-\frac{\partial^2}{\partial x^2}+2\omega^2\sec^2\omega x\right)f_n(x)=\mu_n f_n(x),$$

$$-\frac{\partial^2}{\partial x^2}k_n(x)=\lambda_n k_n(x), \tag{9.11}$$

where $-X_1<x<X_2$, and $f_n(x)$ and $k_n(x)$ vanish at the boundaries. The differential equations (9.11) are identical to Eqs. (B14) and (B8), and the eigenvalues μ_n and λ_n, as well as the corresponding eigenfunctions, are given by Eqs. (C17), (C18), (C31), and (C32). The eigenvalues are obtained from the equations

$$\tan(\sqrt{\mu_n}R_p)=\sqrt{\mu_n}\omega\frac{v_1\overline{\gamma_1}+v_2\overline{\gamma_2}}{\mu_n-\omega^2 v_1\overline{\gamma_1}v_2\overline{\gamma_2}},$$

$$\sqrt{\lambda_n}=\frac{\pi n}{R_p}. \tag{9.12}$$

The solution $\mu_n=\omega^2$ to the first of these two equations does not produce a valid eigenvalue, as the corresponding eigenfunction vanishes everywhere.

Taking a Fourier transform in time and performing a Wick rotation on Eq. (9.10) gives

$$L_{\mathrm{fluc}}^{\mathrm{string}}(\omega)=\frac{\pi}{12R_p}+\int\frac{d\nu}{2\pi}\sum_{n=1}^{\Lambda R_p}\pi\ln\left[\frac{\nu^2+\mu_n}{\nu^2+\lambda_n}\right]=\frac{\pi}{12R_p}+\sum_{n=1}^{\Lambda R_p}\pi\left[\sqrt{\mu_n}-\frac{\pi n}{R_p}\right]. \tag{9.13}$$

The sum over n is logarithmically divergent, so we have imposed a cutoff, restricting ourselves to spatial eigenvalues less than Λ. Since our evaluation of the determinants took place in the x coordinate system, this is a cutoff in the x coordinate space. It is related to the cutoff M in the r coordinate space by the equation

$$\Lambda=M\frac{\partial r}{\partial x}=M\gamma^{-1}. \tag{9.14}$$

Using the methods of [3] to evaluate the sum (9.13) gives the result (3.25) for $L_{\rm fluc}^{\rm string}(\omega)$.

X. EVALUATING THE BOUNDARY PARTITION FUNCTION $Z_B(\omega)$

We begin our evaluation of $Z_B(\omega)$, Eq. (8.27), by writing down the explicit form of the partition function. Inserting the expressions (8.18) and (8.19) gives Z_B the form

$$Z_B(\omega)=\frac{1}{Z_b}\int\prod_{i=1}^{2}[\mathcal{D}\tilde{\theta}_{b,i}\mathcal{D}\tilde{\phi}_{b,i}\mathcal{D}\tilde{r}_i\mathrm{Det}[\overline{R}_i^2]]\exp\Bigg\{i\int\frac{d\nu}{2\pi}-\frac{\sigma}{2}\sum_{i,j=1}^{2}[\tilde{\theta}^*_{b,i}(\nu)G_\theta^{ij}(\nu)\tilde{\theta}_{b,j}(\nu)+\tilde{\phi}^*_{b,i}(\nu)G_\phi^{ij}(\nu)\tilde{\phi}_{b,j}(\nu)]$$
$$+\sum_{i=1}^{2}m_i\overline{\gamma_i}\Bigg(\frac{1}{2}[\nu^2+(2\overline{\gamma_i^2}-1)\omega^2]|\tilde{r}_i(\nu)|^2-2\overline{R}_i\omega\overline{\gamma_i^2}\nu\mathrm{Im}[\tilde{r}_i^*(\nu)\tilde{\phi}_{b,i}(\nu)]+\frac{1}{2}\overline{R}_i^2(\nu^2-\omega^2)|\tilde{\theta}_{b,i}(\nu)|^2$$
$$+\frac{1}{2}\overline{R}_i^2\overline{\gamma_i^2}\nu^2|\tilde{\phi}_{b,i}(\nu)|^2\Bigg)+\sum_{i=1}^{2}(-1)^i\Bigg[\tilde{\rho}_\theta^i*(\nu)\overline{R}_i\tilde{\theta}_{b,i}(\nu)+\frac{1}{2}\tilde{\rho}_\phi^i*(\nu+\omega)[\overline{R}_i\tilde{\phi}_{b,i}(\nu)-i\tilde{r}_i(\nu)]+\frac{1}{2}\tilde{\rho}_\phi^i*(\nu$$
$$-\omega)[\overline{R}_i\tilde{\phi}_{b,i}(\nu)+i\tilde{r}_i(\nu)]\Bigg]\Bigg\}. \tag{10.1}$$

Doing the integral over the $\tilde{r}_i$ gives

$$\int\prod_{i=1}^{2}\mathcal{D}\tilde{r}_i\exp\Bigg\{i\int\frac{d\nu}{2\pi}\sum_{i=1}^{2}m_i\overline{\gamma_i}\Bigg[\frac{1}{2}[\nu^2+(2\overline{\gamma_i^2}-1)\omega^2]|\tilde{r}_i(\nu)|^2-2\overline{R}_i\omega\overline{\gamma_i^2}\nu\mathrm{Im}[\tilde{r}_i^*(\nu)\tilde{\phi}_{b,i}(\nu)]+\frac{(-1)^i}{m_i\overline{\gamma_i}}[-i\tilde{\rho}_\theta^i*(\nu+\omega)$$
$$+i\tilde{\rho}_\theta^i*(\nu-\omega)]\tilde{r}_i(\nu)\Bigg]\Bigg\}$$
$$=\mathrm{Det}^{-1/2}\Bigg[\prod_{i=1}^{2}m_i\overline{\gamma_i}[\nu^2+(2\overline{\gamma_i^2}-1)\omega^2]\Bigg]\exp\Bigg\{i\int\frac{d\nu}{2\pi}\sum_{i=1}^{2}\Bigg[-2m_i\overline{R}_i^2\overline{\gamma_i^5}\frac{\nu^2\omega^2}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}|\tilde{\phi}_{b,i}(\nu)|^2$$
$$+\sum_{i=1}^{2}\frac{(-1)^i}{2}\overline{R}_i\mathrm{Re}\Bigg(\tilde{\rho}_\phi^i*(\nu+\omega)\frac{-2\overline{\gamma_i^2}\omega\nu}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}+\tilde{\rho}_\phi^i*(\nu-\omega)\frac{2\overline{\gamma_i^2}\omega\nu}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}\Bigg)$$
$$\times\tilde{\phi}_{b,i}(\nu)-\frac{1}{8}\sum_{i=1}^{2}\frac{1}{m_i\overline{\gamma_i}\overline{R}_i^2}[\nu^2+(2\overline{\gamma_i^2}-1)\omega^2]^{-1}|\tilde{\rho}_\phi^i(\nu+\omega)-\tilde{\rho}_\phi^i(\nu-\omega)|^2\Bigg]\Bigg\}. \tag{10.2}$$

Inserting Eq. (10.2) into Eq. (10.1), and using the fact that Z_b^{-1} is equal to $\mathrm{Det}^3[\nu^2]$, up to an overall constant, gives the following expression for $Z_B(\omega)$:

$$Z_B(\omega)=(\mathrm{const})\int\prod_{i=1}^{2}[\mathcal{D}\tilde{\theta}_{b,i}\mathcal{D}\tilde{\phi}_{b,i}]\mathrm{Det}^{-1/2}\Bigg[\prod_{i=1}^{2}[\nu^2+(2\overline{\gamma_i^2}-1)\omega^2]\Bigg]\mathrm{Det}^3[\nu^2]$$
$$\times\exp\Bigg\{i\int\frac{d\nu}{2\pi}\Bigg[-\frac{1}{2}\sum_{i,j=1}^{2}[\tilde{\theta}^*_{b,i}(\nu)\Gamma_\theta^{ij-1}(\nu)\tilde{\theta}_{b,j}(\nu)+\tilde{\phi}^*_{b,i}(\nu)\Gamma_\phi^{ij-1}(\nu)\tilde{\phi}_{b,j}(\nu)]$$
$$+\sum_{i=1}^{2}(-1)^i\overline{R}_i\mathrm{Re}\Bigg[\frac{1}{2}\Bigg(\tilde{\rho}_\phi^i*(\nu+\omega)\frac{\nu^2-\omega^2-2\overline{\gamma_i^2}\omega(\nu-\omega)}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}+\tilde{\rho}_\phi^i*(\nu-\omega)\frac{\nu^2-\omega^2+2\overline{\gamma_i^2}\omega(\nu+\omega)}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}\Bigg)$$
$$\times\tilde{\phi}_{b,i}(\nu)+[\tilde{\rho}_\theta^i*(\nu)\tilde{\theta}_{b,i}(\nu)]\Bigg]-\frac{1}{8}\sum_{i=1}^{2}\frac{1}{m_i\overline{\gamma_i}\overline{R}_i^2}[\nu^2+(2\overline{\gamma_i^2}-1)\omega^2]^{-1}|\tilde{\rho}_\phi^i(\nu+\omega)-\tilde{\rho}_\phi^i(\nu-\omega)|^2\Bigg]\Bigg\}, \tag{10.3}$$

where

$$\Gamma_\theta^{ij-1}(\nu)=\delta_{ij}m_i\overline{\gamma_i}\overline{R}_i^2(\nu^2-\overline{\omega^2})-\sigma G_\theta^{ij}(\nu) \tag{10.4}$$

and

$$\Gamma_{\phi}^{ij-1}(\nu)=\delta_{ij}m_i\overline{\gamma}_i^3\overline{R}_i^2\nu^2\frac{\nu^2-(2\overline{\gamma}_i^2+1)\omega^2}{\nu^2+(2\overline{\gamma}_i^2-1)\omega^2}-\sigma G_{\phi}^{ij}(\nu). \tag{10.5}$$

The quantities Γ^{ij-1} are the coefficients of the quadratic terms in $\tilde{\theta}_{b,i}$ and $\tilde{\phi}_{b,i}$. They determine the contribution of the boundary fluctuations to the string energy, and are closely related to the physical propagator for driven oscillations of the string modes. Inserting the explicit forms (B12) and (B17) of G_{θ}^{ij} and G_{ϕ}^{ij}, and using Eq. (3.23) to replace m_i, we find

$$\Gamma_{\theta}^{ij-1}(\nu)=\frac{\nu\sigma v_i v_j}{\omega^2}\left[\delta_{ij}\left(\frac{\nu}{\omega v_i\overline{\gamma}_i}-\cot(\nu R_p)\right)+(1-\delta_{ij})\csc(\nu R_p)\right] \tag{10.6}$$

and

$$\Gamma_{\phi}^{ij-1}(\nu)=\delta_{ij}\frac{2\sigma\nu^2 v_i\overline{\gamma}_i(\nu^2-\omega^2)}{\omega^3[\nu^2+(2\overline{\gamma}_i^2-1)\omega^2]}-\frac{\sigma\nu}{\omega^3}(\nu^2-\omega^2)\frac{\delta_{ij}[\nu v_i\overline{\gamma}_i\sin(\nu R_p)-\omega v_1\overline{\gamma}_1 v_2\overline{\gamma}_2\cos(\nu R_p)]+(1-\delta_{ij})\omega v_i\overline{\gamma}_1 v_2\overline{\gamma}_2}{(\nu^2-\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2)\sin(\nu R_p)-\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)\cos(\nu R_p)}. \tag{10.7}$$

Doing the $\tilde{\theta}_{b,i}$ and $\tilde{\phi}_{b,i}$ integrations in Eq. (10.3) gives

$$Z_B(\omega)=e^{iS_{\text{boundary}}+iS_{\text{sources}}}, \tag{10.8}$$

where

$$e^{iS_{\text{boundary}}}=\text{Det}^{-1/2}\left[\prod_{i=1}^{2}[\nu^2+(2\overline{\gamma}_i^2-1)\omega^2]\right]\frac{\text{Det}^{-1/2}[\Gamma_{\theta}^{ij-1}]}{\text{Det}^{-1/2}[\delta_{ij}m_i\overline{R}_i^2\overline{\gamma}_i]}\frac{\text{Det}^{-1/2}[\Gamma_{\phi}^{ij-1}]}{\text{Det}^{-1/2}[\delta_{ij}m_i\overline{R}_i^2\overline{\gamma}_i^3]}\text{Det}^3[\nu^2] \tag{10.9}$$

defines the normalized boundary action. In the limit of large elapsed time T, S_{boundary} is proportional to T, so we define the effective boundary Lagrangian

$$L_{\text{boundary}}(\omega)=\lim_{T\to\infty}\frac{-i}{T}S_{\text{boundary}}(\omega). \tag{10.10}$$

We evaluate $L_{\text{boundary}}(\omega)$ in Appendix C.

The source terms in Eq. (10.8) are

$$\begin{aligned}S_{\text{sources}}=\sum_{i,j=1}^{2}\int\frac{d\nu}{2\pi}(-1)^{i+j}\overline{R}_i\overline{R}_j\Bigg[&\frac{1}{2}\tilde{\rho}_{\theta}^{i}*(\nu)\Gamma_{\theta}^{ij}(\nu)\tilde{\rho}_{\theta}^{j}(\nu)+\frac{1}{8}\Gamma_{\phi}^{ij}(\nu)\left(\tilde{\rho}_{\phi}^{i}*(\nu+\omega)\frac{\nu^2-\omega^2-2\overline{\gamma}_j^2\omega(\nu-\omega)}{\nu^2+(2\overline{\gamma}_j^2-1)\omega^2}\right.\\&\left.+\tilde{\rho}_{\phi}^{i}*(\nu-\omega)\frac{\nu^2-\omega^2+2\overline{\gamma}_j^2\omega(\nu+\omega)}{\nu^2+(2\overline{\gamma}_j^2-1)\omega^2}\right)\left(\tilde{\rho}_{\phi}^{j}(\nu+\omega)\frac{\nu^2-\omega^2-2\overline{\gamma}_i^2\omega(\nu-\omega)}{\nu^2+(2\overline{\gamma}_i^2-1)\omega^2}+\tilde{\rho}_{\phi}^{j}(\nu-\omega)\frac{\nu^2-\omega^2+2\overline{\gamma}_i^2\omega(\nu+\omega)}{\nu^2+(2\overline{\gamma}_i^2-1)\omega^2}\right)\\&+\frac{\delta_{ij}}{m_i\overline{\gamma}_i\overline{R}_i^2}[\nu^2+(2\overline{\gamma}_i^2-1)\omega^2]^{-1}\frac{1}{8}|\tilde{\rho}_{\phi}^{i}(\nu+\omega)-\tilde{\rho}_{\phi}^{i}(\nu-\omega)|^2\Bigg].\end{aligned} \tag{10.11}$$

The first two terms in Eq. (10.11) were produced by the $\tilde{\theta}_{b,i}$ and $\tilde{\phi}_{b,i}$ integrals. The third was produced by the $\tilde{r}_i$ integral, and does not couple the two ends of the string to each other.

XI. EXPLICIT EVALUATIONS OF L_{fluc} AND S_{sources}

Combining Eqs. (8.25), (9.9), and (10.8) gives

$$Z(\omega)=e^{iTL_{\text{cl}}(\omega)+L_{\text{fluc}}(\omega)+iS_{\text{sources}}}, \tag{11.1}$$

where $L_{\text{cl}}(\omega)$ is defined by Eq. (3.16), S_{sources} is defined by Eq. (10.11), and L_{fluc} is

$$L_{\text{fluc}}(\omega)=L_{\text{fluc}}^{\text{string}}(\omega)+L_{\text{boundary}}(\omega). \tag{11.2}$$

The terms on the right-hand side of Eq. (11.2) are defined by Eqs. (4.5) and (10.10).

We now evaluate the partition function for appropriate physical limits of the quark masses. As we previously discussed in Sec. IV, because of the renormalization of the geodesic curvature, our results are only valid when the quark masses are either very large or exactly zero. The case where both masses are large is relevant only to the evaluation of the potential. We therefore have two physical limits: (1) the light-light case, where $m_1 = m_2 = 0$ $(\overline{\gamma_1}, \overline{\gamma_2} \to \infty)$, and (2) the heavy-light case, where $m_1 \to \infty$ $(v_1 \to 0)$ and $m_2 = 0$ $(\overline{\gamma_2} \to \infty)$. We now evaluate $L_{\rm fluc}(\omega)$ and $S_{\rm sources}$ in these limits. This will give us the zero point energy and the excitation energies of the fluctuations.

We begin with $L_{\rm fluc}$, and its two parts $L_{\rm fluc}^{\rm string}$ and $L_{\rm boundary}$. In the light-light limit $L_{\rm fluc}^{\rm string}$ is $(7/12)\omega$, and in the heavy-light limit it is $(5/12)\omega$ [see Eqs. (4.6) and (4.8)]. We evaluate $L_{\rm boundary}$ in Appendix C, and find, in the light-light limit,

$$L_{\rm boundary} = 0 \tag{11.3}$$

and, in the heavy-light limit,

$$L_{\rm boundary} = -\frac{1}{4}\omega. \tag{11.4}$$

Thus, in the light-light limit,

$$L_{\rm fluc}(\omega) = \frac{7}{12}\omega \tag{11.5}$$

and, in the heavy-light limit,

$$L_{\rm fluc}(\omega) = \frac{1}{6}\omega. \tag{11.6}$$

We next evaluate $S_{\rm sources}$ in these limits. Consider first the excitations in the θ sector. We set ρ_ϕ^i equal to zero in Eq. (10.11) to obtain the the θ sector propagator:

$$K_\theta^{ij}(\nu) \equiv (-1)^{i+j}\overline{R_i}\,\overline{R_j}\,\Gamma_\theta^{ij}(\nu). \tag{11.7}$$

The propagator $K_\theta^{ij}(\nu)$ has a simple pole wherever ν is equal to the energy of one of the excited modes. There is also a double pole at $\nu=0$ due to the invariance of the Lagrangian under a translation of the string in the direction perpendicular to the plane of rotation. This translation mode does not correspond to an excited state of the string.

A general excited state will include multiple excitations of each mode, so its energy will be a sum of multiples of the energy of each mode. Only single excitations will appear in the propagator $K_\theta^{ij}(\nu)$, because of the harmonic oscillator selection rules. We obtain an explicit form for $K_\theta^{ij}(\nu)$ by inverting $\Gamma_\theta^{ij}(\nu)^{-1}$ (in the sense of inverting a two by two matrix). In the light-light case, Eq. (10.6) becomes

$$\Gamma_\theta^{ij\,-1}(\nu) = \frac{\nu\sigma}{\omega^2}\left[-\delta_{ij}\cot\left(\pi\frac{\nu}{\omega}\right) + (1-\delta_{ij})\csc\left(\pi\frac{\nu}{\omega}\right)\right], \tag{11.8}$$

so the θ propagator is

$$K_\theta^{ij}(\nu) = \frac{1}{\nu\sigma}\left[\delta_{ij}\cot\left(\pi\frac{\nu}{\omega}\right) - (1-\delta_{ij})\csc\left(\pi\frac{\nu}{\omega}\right)\right]. \tag{11.9}$$

This has a double pole at $\nu=0$ due to the translation mode in the direction perpendicular to the plane of rotation, which does not correspond to an excited string state.

The single poles at $\nu = k\omega$ for $\nu \neq 0$ are due to the excited states of the string. These are the same poles as appear in $G_\theta^{ij}(\nu)$, Eq. (B12), and consequently in $K_\theta^{ij\,-1}(\nu)$, in the limit of massless quarks. The locations of these poles are the same in $K_\theta^{ij}(\nu)$ and $K_\theta^{ij\,-1}(\nu)$ because $\det K_\theta^{ij}(\nu) = -1/\nu^2\sigma^2$ in the massless quark limit.

In the heavy-light case, the components of $\Gamma_\theta^{ij\,-1}$ are

$$\Gamma_\theta^{11-1}(\nu) = \frac{\nu^2\sigma}{\omega^3}v_1 + O(v_1^2),$$

$$\Gamma_\theta^{22-1}(\nu) = -\frac{\nu\sigma}{\omega^2}\cot\left(\frac{\pi\nu}{2\omega}\right) + O(v_1),$$

$$\Gamma_\theta^{12-1}(\nu) = \frac{\nu\sigma}{\omega^2}v_1\csc\left(\frac{\pi\nu}{2\omega}\right) + O(v_1^2). \tag{11.10}$$

Inverting the 2×2 matrix (11.10) gives, up to an overall normalization, the components of the θ propagator:

$$K_\theta^{11}(\nu) = \frac{\omega}{\nu^2\sigma}v_1 + O(v_1^2),$$

$$K_\theta^{22}(\nu) = -\frac{1}{\nu\sigma}\tan\left(\frac{\pi\nu}{2\omega}\right) + O(v_1),$$

$$K_\theta^{12}(\nu) = \frac{\omega}{\nu^2\sigma}\sec\left(\frac{\pi\nu}{2\omega}\right)v_1 + O(v_1^2). \tag{11.11}$$

We see that, in the limit $m_1 \to \infty$ $(v_1 \to 0)$, only the light-light component $K_\theta^{22}(\nu)$ is nonvanishing. It has poles at

$$\nu = (2k+1)\omega, \quad k = 0,1,2,\ldots, \tag{11.12}$$

corresponding to the normal modes of a string with one end fixed. The remaining components of the propagator are proportional to the heavy quark velocity. The heavy-heavy component $K_\theta^{11}(\nu)$ contains only a double pole at $\nu=0$, corresponding to the translation mode, and the heavy-light component $K_\theta^{12}(\nu)$ has poles corresponding both to excited vibrational states and to the translation mode.

We next obtain the ϕ sector propagator by examining Eq. (10.11). With some rearranging of terms, Eq. (10.11) is

$$S_{\rm sources}=\sum_{i,j=1}^{2}\int\frac{d\nu}{2\pi}\Bigg\{\frac{1}{2}\tilde{\rho}_\theta^i{}^*(\nu)K_\theta^{ij}(\nu)\tilde{\rho}_\theta^j(\nu)+\frac{1}{2}\tilde{\rho}_\phi^i{}^*(\nu)K_\phi^{ij}(\nu)\tilde{\rho}_\phi^j(\nu)+\frac{1}{4}{\rm Re}\Bigg[\tilde{\rho}_\phi^i{}^*(\nu+\omega)\Bigg((-1)^{i+j}\overline{R_i}\overline{R_j}$$
$$\times\frac{\nu^2-\omega^2-2\overline{\gamma_i^2}\omega(\nu-\omega)}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}\Gamma_\phi^{ij}(\nu)\frac{\nu^2-\omega^2+2\overline{\gamma_j^2}\omega(\nu+\omega)}{\nu^2+(2\overline{\gamma_j^2}-1)\omega^2}-\frac{\delta_{ij}}{m_i\overline{\gamma_i}}\frac{1}{\nu^2+(2\overline{\gamma_i^2}-1)\omega^2}\Bigg)\tilde{\rho}_\phi^j(\nu-\omega)\Bigg]\Bigg\},\quad(11.13)$$

where $K_\phi^{ij}(\nu)$ is

$$K_\phi^{ij}(\nu)\equiv\frac{\delta_{ij}}{4m_i\overline{\gamma_i}}\left(\frac{1}{\nu^2-2\nu\omega+2\overline{\gamma_i^2}\omega^2}+\frac{1}{\nu^2+2\nu\omega+2\overline{\gamma_i^2}\omega^2}\right)+\frac{(-1)^{i+j}}{4}\overline{R_i}\overline{R_j}$$
$$\times\Bigg[\frac{\nu^2-2\nu\omega-2\overline{\gamma_i^2}\omega(\nu-2\omega)}{\nu^2-2\nu\omega+2\overline{\gamma_i^2}\omega^2}\,\frac{\nu^2-2\nu\omega-2\overline{\gamma_j^2}\omega(\nu-2\omega)}{\nu^2-2\nu\omega+2\overline{\gamma_j^2}\omega^2}\Gamma_\phi^{ij}(\nu-\omega)$$
$$+\frac{\nu^2+2\nu\omega+2\overline{\gamma_i^2}\omega(\nu+2\omega)}{\nu^2+2\nu\omega+2\overline{\gamma_i^2}\omega^2}\,\frac{\nu^2+2\nu\omega+2\overline{\gamma_j^2}\omega(\nu+2\omega)}{\nu^2+2\nu\omega+2\overline{\gamma_j^2}\omega^2}\Gamma_\phi^{ij}(\nu+\omega)\Bigg].\quad(11.14)$$

In some of the terms in Eq. (11.13) we have translated the integration variable ν by $\pm\omega$ to make the argument of $\tilde{\rho}_\phi^i$ be ν instead of $\nu\pm\omega$. There are three terms in Eq. (11.13):

(1) The $K_\theta^{ij}(\nu)$ term. This is the θ sector propagator which we examined earlier.

(2) The $K_\phi^{ij}(\nu)$ term, where the arguments of the two $\tilde{\rho}_\phi^i$ factors are equal. This is the ϕ sector propagator.

(3) The $\tilde{\rho}_\phi^i{}^*(\nu+\omega)\tilde{\rho}_\phi^j(\nu-\omega)$ term, where the arguments of the two $\tilde{\rho}_\phi^i$ factors differ by 2ω. Since the natural frequency of the classical rotating string is ω, driving it at a frequency ν produces sidebands at $\nu\pm2\omega$ (to leading semiclassical order). The third term in Eq. (11.13) is a manifestation of this effect.

The poles in $K_\phi^{ij}(\nu)$ give the energies of the excited modes of the string. We now consider how the poles in $K_\phi^{ij}(\nu)$ relate to the poles in $\Gamma_\phi^{ij}(\nu)$. Since Γ_ϕ^{ij} appears in Eq. (11.14) with the argument $\nu\pm\omega$, the double pole in K_ϕ^{ij} at $\nu=0$ due to the translation modes of the string will appear in Γ_ϕ^{ij} as double poles at $\nu=\pm\omega$. Likewise, the single poles in K_ϕ^{ij} giving the energies of singly excited states of the string will be shifted by ω when they appear in Γ_ϕ^{ij}.

In the light-light limit, $\Gamma_\phi^{ij\,-1}$, Eq. (10.7), is

$$\Gamma_\phi^{ij\,-1}(\nu)=\frac{\sigma\nu(\nu^2-\omega^2)}{\omega^4}\left[-\delta_{ij}\cot\left(\pi\frac{\nu}{\omega}\right)+(1-\delta_{ij})\csc\left(\pi\frac{\nu}{\omega}\right)\right],\quad(11.15)$$

so Γ_ϕ^{ij} is

$$\Gamma_\phi^{ij}(\nu)=\frac{\omega^4}{\sigma\nu(\nu^2-\omega^2)}\left[\delta_{ij}\cot\left(\pi\frac{\nu}{\omega}\right)+(1-\delta_{ij})\csc\left(\pi\frac{\nu}{\omega}\right)\right].\quad(11.16)$$

The factors $[\nu(\nu^2-\omega^2)]^{-1}$, combined with the trigonometric functions in Eq. (11.16), produce double poles in $\Gamma_\phi^{ij}(\nu)$ at $\nu=0,\pm\omega$, and single poles at $\nu=k\omega$ for $k=\pm2,\pm3,\ldots$.

The ϕ sector propagator, written in terms of Γ_ϕ^{ij}, is

$$K_\phi^{ij}(\nu)=\frac{(-1)^{i+j}}{4}\left(\frac{(\nu-2\omega)^2}{\omega^4}\Gamma_\phi^{ij}(\nu-\omega)+\frac{(\nu+2\omega)^2}{\omega^4}\times\Gamma_\phi^{ij}(\nu+\omega)\right).\quad(11.17)$$

The double poles in $\Gamma_\phi^{ij}(\nu)$ at $\nu=\pm\omega$ do not produce poles in the propagator $K_\phi^{ij}(\nu)$ due to the prefactors in Eq. (11.17). Instead, they produce a double pole at $\nu=0$. This double pole is the effect of the two translation modes in the directions parallel to the plane of string rotation, which are present in the propagator for all values of the quark mass.

The double pole in $\Gamma_\phi^{ij}(\nu)$ at $\nu=0$ produces double poles in $K_\phi^{ij}(\nu)$ at $\nu=\pm\omega$. These poles arise for the following reason: the condition (6.7) that the average of the fluctuations of the angular momentum of the string vanish means that the zero frequency component of the motion of the string is constrained. This constraint removes from the motion the zero frequency component of a global rotation. This global rotation is contained in the $\tilde{\phi}_{b,i}(\nu)$ boundary degrees of freedom. However, we have coupled all components of $\tilde{\phi}_{b,i}(\nu)$ to the sources $\tilde{\rho}_\phi^i(\nu)$. Elimination of the coupling to the zero frequency component of the global rotational mode will remove the double poles of $K_\phi^{ij}(\nu)$ at $\nu=\pm\omega$.

The single poles of $\Gamma_\phi^{ij}(\nu)$ at $\nu=\pm k\omega$ for $k\geqslant2$ produce single poles in the propagator $K_\phi^{ij}(\nu)$ at

$$\nu=\nu_k=k\omega,\quad\text{for } k=1,2,\ldots .\quad(11.18)$$

These are the true vibrational frequencies of the motion of the string in the plane of rotation, and give the energies of the excited states of the mesons corresponding to the excitation of a single quanta of frequency ν_k. The spectrum of excited states in the ϕ sector is then the same as in the θ sector, for a meson composed of zero mass quarks.

In the heavy-light case, Eq. (10.7) is

$$\Gamma_\phi^{11-1}(\nu)=\frac{\sigma}{\omega^3}\frac{(\nu^2-\omega^2)^2}{\nu^2+\omega^2}v_1+O(v_1^2),$$

$$\Gamma_\phi^{22-1}(\nu)=\frac{\sigma\nu(\nu^2-\omega^2)}{\omega^4}\tan\left(\frac{\pi\nu}{2\omega}\right)+O(v_1),$$

$$\Gamma_\phi^{12-1}(\nu)=-\frac{\sigma}{\omega^3}(\nu^2-\omega^2)^2 v_1\sec\left(\frac{\pi\nu}{2\omega}\right)+O(v_1^2), \tag{11.19}$$

so the components of Γ_ϕ^{ij} are

$$\Gamma_\phi^{11}(\nu)=\frac{\omega^3}{\sigma}\frac{\nu^2+\omega^2}{(\nu^2-\omega^2)^2}v_1^{-1}+O(1),$$

$$\Gamma_\phi^{22}(\nu)=\frac{\omega^4}{\sigma\nu(\nu^2-\omega^2)}\cot\left(\frac{\pi\nu}{2\omega}\right)+O(v_1),$$

$$\Gamma_\phi^{12}(\nu)=-\frac{\omega^4}{\sigma\nu}\frac{\nu^2+\omega^2}{\nu^2-\omega^2}\csc\left(\frac{\pi\nu}{2\omega}\right)+O(v_1). \tag{11.20}$$

The components of the ϕ sector propagator are

$$K_\phi^{11}(\nu)=\frac{\omega v_1}{4\sigma}\left(\frac{1}{\nu^2-2\nu\omega+2\omega^2}+\frac{1}{\nu^2+2\nu\omega+2\omega^2}\right)+\frac{(\nu-2\omega)^4v_1^2}{4\omega^2(\nu^2-2\nu\omega+2\omega^2)^2}\Gamma_\phi^{11}(\nu-\omega)+\frac{(\nu+2\omega)^4v_1^2}{4\omega^2(\nu^2+2\nu\omega+2\omega^2)^2}\Gamma_\phi^{11}(\nu+\omega)+O(v_1^2),$$

$$K_\phi^{22}(\nu)=\frac{(\nu-2\omega)^2}{4\omega^4}\Gamma_\phi^{22}(\nu-\omega)+\frac{(\nu+2\omega)^2}{4\omega^4}\Gamma_\phi^{22}(\nu+\omega)+O(v_1),$$

$$K_\phi^{12}(\nu)=\frac{(\nu-2\omega)^3v_1}{4\omega^3(\nu^2-2\nu\omega+2\omega^2)}\Gamma_\phi^{12}(\nu-\omega)-\frac{(\nu+2\omega)^3v_1}{4\omega^3(\nu^2+2\nu\omega+2\omega^2)}\Gamma_\phi^{12}(\nu+\omega)+O(v_1^2). \tag{11.21}$$

Just as in the θ sector, the heavy end propagator $K_\phi^{11}(\nu)$ only couples to the translation mode as $m_1\to\infty$. In this limit, the only poles in Γ_ϕ^{11} are double poles at $\nu=\pm\omega$, so the only pole in K_ϕ^{11} is a double pole at $\nu=0$. At the light quark end, Γ_ϕ^{22} does not have poles at $\nu=\pm\omega$, as the cotangent vanishes there. Just as in the θ sector, in the heavy-light limit the light end propagator K_ϕ^{22} does not couple to the translation modes of the string. The simple poles in Γ_ϕ^{22} at even multiples of ω produce poles in K_ϕ^{22} at $\nu=\pm(2n+1)\omega$.

The spectrum of singly excited string modes in the ϕ sector is then the same as in the θ sector, and the degeneracy of the light-light excitations is repeated in the heavy-light excitations. The remaining pole in Γ_ϕ^{22}, a double pole at $\nu=0$, is present for reasons already discussed in considering the light-light case. The heavy-light component of the propagator $K_\phi^{12}(\nu)$ couples to both light end and heavy end modes, just as it did in the θ sector.

XII. MESON SPECTRUM

In this section, we use the results (11.5) and (11.6), as well as the energies of the string excited states, to derive the meson Regge trajectories to leading semiclassical order. We begin by calculating classical Regge trajectories from the classical Lagrangian (3.16). The angular momentum of the meson and its energy $E_{\rm cl}(\omega)$ are given by Eqs. (5.16) and (5.15):

$$J=\frac{\partial L_{\rm cl}}{\partial\omega}=\sum_i\left[\sigma\frac{R_i^2}{2v_i}\left(\frac{\arcsin v_i}{v_i}-\gamma_i^{-1}\right)+m_iR_iv_i\gamma_i\right],$$

$$E=\omega\frac{\partial L_{cl}}{\partial\omega}-L_{cl}=\sum_i\left(\sigma R_i\frac{\arcsin v_i}{v_i}+m_i\gamma_i\right). \tag{12.1}$$

From the classical equation of motion we derived Eq. (3.23), which shows that R_i is proportional to γ_i^2 for large γ_i. Evaluating Eqs. (3.16) and (12.1) in the limit of massless quarks, where the quark velocity v_i goes to one, yields the classical results

$$L_{\rm cl}(\omega)=-\frac{\pi\sigma}{2\omega},\quad J=\frac{\pi\sigma}{2\omega^2},\quad E_{\rm cl}=\frac{\pi\sigma}{\omega},\quad J=\frac{E_{\rm cl}^2}{2\pi\sigma}. \tag{12.2}$$

In the heavy-light case v_1 goes to zero and v_2 goes to one, so

$$J=\frac{(E_{\rm cl}-m_1)^2}{\pi\sigma}. \tag{12.3}$$

We now include the correction (5.17) to the energy due to fluctuations. In the light-light case Eqs. (5.17), (11.5), and (12.2) give

$$E(J)=E_{cl}(\omega)-L_{\rm fluc}(\omega)=\frac{\pi\sigma}{\omega}-\frac{7}{12}\omega, \tag{12.4}$$

for the case of two light quarks. The value of ω is given as a function of J through the classical relation $\omega=\sqrt{\pi\sigma/2J}$. Squaring both sides of Eq. (12.4) and dropping the term quadratic in $L_{\rm fluc}$ yields

$$J=\frac{E^2}{2\pi\sigma}+\frac{7}{12}+O\left(\frac{\sigma}{E^2}\right). \tag{12.5}$$

Using the WKB quantization condition $J=l+1/2$ in Eq. (12.5) gives the leading Regge trajectory, relating the angular momentum quantum number l to the meson energy E:

$$l=\frac{E^2}{2\pi\sigma}+\frac{1}{12}+O\left(\frac{\sigma}{E^2}\right). \tag{12.6}$$

In the heavy-light case, Eqs. (11.6) and (12.3) give the Regge trajectory

$$l=\frac{(E-m_1)^2}{\pi\sigma}+\frac{1}{6}-\frac{1}{2}=\frac{(E-m_1)^2}{\pi\sigma}-\frac{1}{3}. \tag{12.7}$$

The energies of the excited states of the light mesons are obtained by adding the excitation energies $n\omega$ to Eq. (12.4):

$$E_n(\omega)=\frac{\pi\sigma}{\omega}-\frac{7}{12}\omega+n\omega. \tag{12.8}$$

Since there are many combinations of string normal modes which give the same n (e.g., a doubly excited $k=1$ mode and a singly excited $k=2$ mode each give $n=2$), the spectrum is highly degenerate. There are two $n=1$ trajectories, each corresponding to a single excitation of one of the $k=1$ normal modes. Higher values of n have higher degeneracies.

From Eq. (12.8) for the nth excited hybrid energy level we derive the ‘‘daughter’’ Regge trajectory:

$$l=\frac{E^2}{2\pi\sigma}+\frac{1}{12}-n+O\left(\frac{\sigma}{E^2}\right). \tag{12.9}$$

In the heavy-light case, the normal modes with frequencies of $(2k+1)\omega$ can combine to form states with excitation energies $n\omega$ for any n, though the degeneracies are different. The ‘‘daughter’’ Regge trajectories of Eq. (12.7) are then

$$l=\frac{(E-m_1)^2}{\pi\sigma}-\frac{1}{3}-n+O\left(\frac{\sigma}{(E-m_1)^2}\right). \tag{12.10}$$

Equations (12.9) and (12.10) give the leading semiclassical correction to the classical Regge formulas (12.2) and (12.3). To compute the $O(\sigma/E^2)$ corrections to this result, it would be necessary to compute the contribution of two loop vacuum diagrams in the two dimensional field theory to find the energy of the lowest lying trajectory, and to compute one loop corrections to the propagators to find the energies of the excited states.

XIII. GENERALIZATION TO $D\neq 4$ DIMENSIONS

It is interesting to compare the results (12.6) with the corresponding result from classical bosonic string theory. To do this, we need to generalize Eq. (12.6) to D dimensions. This dependence comes from the dependence of $L_{\rm eff}(\omega)$ on D, which in turn comes from $L_{\rm fluc}^{\rm string}(\omega)$ and $L_{\rm boundary}(\omega)$, since the classical string energy is independent of D.

Our calculation of $L_{\rm boundary}(\omega)$ separated the boundary fluctuations perpendicular to the plane of rotation of the string and the fluctuations in that plane. Neither of these contributed to $L_{\rm boundary}(\omega)$ (see Appendix C). Going from four dimensions to D dimensions only adds $D-4$ additional directions for perpendicular fluctuations. These fluctuations will each give the same contribution to $L_{\rm boundary}(\omega)$ that the fluctuations perpendicular to the plane of rotation did in the four dimensional case, namely zero. The function $L_{\rm boundary}(\omega)$ is therefore zero, independent of D.

The function $L_{\rm fluc}^{\rm string}(\omega)$ was derived from $Z_I(\omega)$, which was expressed in Eq. (9.6) as the product of two determinants, one due to string modes perpendicular to the plane of rotation and one due to string modes in the plane of rotation. Just as in the case of $L_{\rm boundary}(\omega)$, adding more dimensions adds additional string modes perpendicular to the plane of rotation, so the generalization of Eq. (9.6) to D dimensions is

$$\begin{aligned}Z_I(\omega)&={\rm Det}^{-(D-3)/2}[-\nabla^2]{\rm Det}^{-1/2}[-\nabla^2+2\omega^2\sec^2\omega x]\\&={\rm Det}^{-(D-2)/2}[-\nabla^2]{\rm Det}^{-1/2}\left[\frac{-\nabla^2+2\omega^2\sec^2\omega x}{-\nabla^2}\right].\end{aligned} \tag{13.1}$$

The first of the determinants produces a term $(D-2)\pi/24R_p$ in $L_{\rm fluc}^{\rm string}(\omega)$, equal to the Lüscher term in D dimensions, with the length R of the string replaced with its proper length R_p [see Eq. (3.19)]. We have already evaluated the second determinant, which, after renormalization, gave the second and third terms in Eq. (4.5). For massless quarks, $R_p=\pi/\omega$, and the contribution of the second determinant is $\omega/2$, so that the generalization of Eq. (12.6) to D dimensions is

$$E_n(\omega)=\frac{\pi\sigma}{\omega}-\frac{D-2}{24}\omega-\frac{\omega}{2}+n\omega. \tag{13.2}$$

This can be rewritten to get E^2 as a function of l:

$$E^2=2\pi\sigma\left[l-\frac{D-2}{24}+n+O\left(\frac{1}{l}\right)\right]. \tag{13.3}$$

In 26 dimensions, the equation for the energy is

$$E^2=2\pi\sigma\left[l-1+n+O\left(\frac{1}{l}\right)\right]. \tag{13.4}$$

The spectrum (13.4) coincides with the spectrum of open strings in classical bosonic string theory. However, in our approach Eq. (13.4) is valid only in the leading semiclassical approximation, so it cannot be used for $l=0$, where it would yield the scalar tachyon of the open bosonic string.

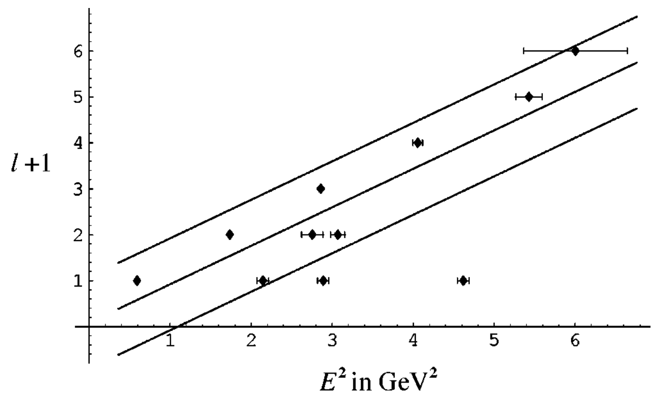

FIG. 2. Regge trajectories (12.9) with $n=0,1,2$, and meson masses in the ρ-a_2 sector.

XIV. COMPARISON WITH MESON MASSES

In Fig. 2, we plot the leading trajectory and first two daughters [Eq. (12.6) for $n=0,1,2$] using the string tension $\sigma=(0.436)^2$ GeV^{-2}, corresponding to a value α' $\alpha'=0.89$ GeV^{-2} for the slope of the ρ trajectory. The plotted points are meson masses on the ρ-a_2 trajectory, and on possible daughters of this trajectory. We have added one to the orbital angular momentum to account for the spin of the quarks ($J=l+s=l+1$). The plotted points lie to the right of the leading trajectory, so the predicted masses are too low. This may be due to the fact that we are using scalar instead of fermionic quarks. In any case, the semiclassical correction is small, and the leading trajectory lies close to the classical one.

We therefore compare the differences in the squared masses between the lowest lying meson for each l and higher energy states with the predictions of the semiclassical formula (12.6):

$$\frac{m_{\text{excited}}^2-m_{\text{lowest}}^2}{2\pi\sigma}=n, \quad n=1,2,\ldots . \tag{14.1}$$

This energy difference is entirely due to the excited states of the string, and therefore may not be so sensitive to the kind of quarks used in the model. The values of the mass difference (14.1) are shown in Table I for the excited states of the ρ ($l=0$) and a_2 ($l=1$). For $n=1$, the semiclassical string theory predicts 2 degenerate states. This double degeneracy will be broken by higher order corrections, and we expect the predicted $n=1$ mass to lie halfway between the physical masses of the two $n=1$ particles. This works very well for the a_2 meson, where averaging the masses of the two excited states gives $\Delta m^2/2\pi\sigma=0.98$, compared to the predicted value of unity. For the ρ mesons, with $l=0$, the semiclassical theory is not applicable, and the excited states in Table I are not readily identified with the predictions of Eq. (14.1).

XV. SUMMARY AND CONCLUSIONS

(1) Beginning with an effective string theory of vortices which describes long distance QCD, we have calculated, in the semiclassical approximation, the effect of string fluctuations on Regge trajectories, both for mesons containing light (zero mass) quarks, and for mesons containing one heavy and one light quark. The semiclassical correction to the leading Regge trajectory for light quarks adds a constant $(D-2)/24$ to the classical Regge formula. The small size of this semiclassical correction for $D=4$ could explain why Regge trajectories are linear at values of l of order 1.

(2) These results depended on two extensions of our previous work:

(a) The renormalization of the geodesic curvature in the semiclassical expansion about a rotating string solution, needed to take the zero quark mass limit.

(b) The decoupling of the boundary and interior degrees of freedom of the string to obtain the back reaction of the interior degrees of freedom on the boundary.

(3) The spectrum of the energies of the excited states formally coincides with the spectrum of the open string of bosonic string theory in its critical dimension $D=26$. Here, we obtained this spectrum for any D from the semiclassical expansion of an effective string theory. The functional determinant Δ_{FP} determining the measure for the path integral (3.2) made the theory conformally invariant in the limit of zero mass quarks. Perhaps this quantization of effective string theory might prove useful towards efforts in quantizing fundamental string theories in non-critical dimensions.

(4) We treated the light quarks as massless scalar particles. This is appropriate at best for determining the energies of excited states of the string, where the dependence on the boundary of the string is small. The effect of chiral symmetry breaking, generating a constituent quark mass, must play a dominant role in determining the masses of mesons which are ground states of quark-antiquark systems. However, this constituent mass should approximately cancel in the mass differences between mesons on the leading and first daughter trajectories. The effective string theory should then describe the excitation energies of the mesons.

ACKNOWLEDGMENTS

We would like to thank D. Gromes and A. Kaidalov for very helpful conversations.

TABLE I. Squared mass differences for the excited states of the ρ trajectory.

l	Lowest state	Excited state	m_{lowest}	m_{excited}	Δm^2	$\Delta m^2/2\pi\sigma$
1	ρ	$\rho(1450)$	0.769 GeV	1.465 GeV	1.555 GeV^2	1.30
1	ρ	$\rho(1700)$	0.769 GeV	1.700 GeV	2.299 GeV^2	1.92
1	ρ	$\rho(2150)$	0.769 GeV	2.149 GeV	4.027 GeV^2	3.37
2	$a_2(1320)$	$a_2(1660)$	1.318 GeV	1.660 GeV	1.019 GeV^2	0.85
2	$a_2(1320)$	$a_2(1750)$	1.318 GeV	1.752 GeV	1.332 GeV^2	1.12

APPENDIX A: QUANTIZING THE ANGULAR MOMENTUM

In this appendix, we quantize the angular momentum of the string semiclassically, using the semiclassical methods of Dashen, Hasslacher, and Neveu [24] (DHN) for obtaining the energies of periodic orbits. DHN find the energies of these states by looking at the trace of the propagator

$$G(E)=i\,\mathrm{tr}\int_0^\infty dT e^{i(E-H)T}, \tag{A1}$$

where H is the Hamiltonian. The operator on the right hand side is defined in terms of a partition function with periodic boundary conditions. In our case, it is

$$e^{-iHT}\equiv Z^{\text{periodic}}=\frac{1}{Z_b}\int \mathcal{D}f^1(\xi)\mathcal{D}f^2(\xi)\mathcal{D}\vec{\mathbf{x}}_1(t)\mathcal{D}\vec{\mathbf{x}}_2(t)\Delta_{FP} \times\exp\Bigg(-i\sigma\int d^2\xi\sqrt{-g} -i\sum_{i=1}^{2} m_i\int_{-T/2}^{T/2}dt\sqrt{1-\dot{\vec{\mathbf{x}}}_i^2(t)}\Bigg), \tag{A2}$$

where the variables f^i and $\vec{\mathbf{x}}_i$ are required to satisfy the boundary conditions

$$f^i|_{T/2}=f^i|_{-T/2}, \quad \vec{\mathbf{x}}_i|_{T/2}=\vec{\mathbf{x}}_i|_{-T/2}. \tag{A3}$$

1. Euler angles

The first step in quantizing the angular momentum is to introduce collective coordinates for the rotational degrees of freedom of the string. We parametrize the rigid body rotations of a straight string using the Euler angles α, β, and γ, defined by the rotation matrix M:

$$M=\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & \cos\alpha\cos\beta\sin\gamma+\sin\alpha\cos\gamma & \cos\alpha\sin\beta\\ -\sin\alpha\cos\beta\cos\gamma-\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & -\sin\alpha\sin\beta\\ -\sin\beta\cos\gamma & -\sin\beta\sin\gamma & \cos\beta \end{pmatrix}. \tag{A4}$$

The angles α,β, and γ are functions of the time t. The rate of change of the matrix M acting on a fixed vector $\hat{n}$ defines the angular velocity $\vec{\omega}$:

$$\frac{d}{dt}(M\hat{n})=\dot{M}\hat{n}\equiv\vec{\omega}\times(M\hat{n}). \tag{A5}$$

Since $\hat{n}$ is an arbitrary vector, we find

$$\begin{aligned}\vec{\omega}[\gamma,\beta,\alpha]&\equiv-\frac{1}{2}\hat{\mathbf{e}}_i\epsilon^{ijk}(\dot{M}M^{-1})^{jk}\\ &=-\dot{\alpha}\hat{\mathbf{e}}_3+\dot{\beta}(\cos\alpha\hat{\mathbf{e}}_2+\sin\alpha\hat{\mathbf{e}}_1)\\ &\quad-\dot{\gamma}(\cos\beta\hat{\mathbf{e}}_3-\sin\beta\sin\alpha\hat{\mathbf{e}}_2\\ &\quad+\sin\beta\cos\alpha\hat{\mathbf{e}}_1).\end{aligned} \tag{A6}$$

In the limit of small fluctuations about a straight rotating string, the string only has two rotational degrees of freedom. Classically, the angular velocity about the axis of the string must be zero. Any contribution to this component of the angular velocity must be of quadratic order in small fluctuations about the classical solution, since it takes one fluctuation to give the string a moment of inertia about its own axis, and another to give it rotation about that axis. Therefore, to quadratic order,

$$\vec{\omega}\cdot\vec{x}_0=0, \tag{A7}$$

where $\vec{x}_0$ is the classical position of the string.

The two physical angular degrees of freedom are the Euler angles $\alpha(t)$ and $\beta(t)$ determining the orientation of the vector $\hat{x}_0$, chosen to be the $\hat{\mathbf{e}}_3'$ axis in the body fixed frame:

$$\hat{x}_0=\cos\alpha\sin\beta\hat{\mathbf{e}}_1-\sin\alpha\sin\beta\hat{\mathbf{e}}_2+\cos\beta\hat{\mathbf{e}}_3. \tag{A8}$$

The condition $\hat{x}_0\cdot\vec{\omega}=0$, that there be no rotations about the string axis, is

$$\dot{\alpha}\cos\beta+\dot{\gamma}=0. \tag{A9}$$

This means that γ is superfluous. Substituting for γ using Eq. (A9) gives

$$\begin{aligned}\vec{\omega}[\beta,\alpha]=&-\dot{\alpha}\sin\beta(\sin\beta\hat{\mathbf{e}}_3+\sin\alpha\cos\beta\hat{\mathbf{e}}_2-\cos\alpha\cos\beta\hat{\mathbf{e}}_1)\\ &+\dot{\beta}(\cos\alpha\hat{\mathbf{e}}_2+\sin\alpha\hat{\mathbf{e}}_1).\end{aligned} \tag{A10}$$

To introduce the Euler angles into the functional integral (A2), we must define α and β as functionals of the string position $\tilde{x}^\mu$. Let the function $\vec{\Omega}[\tilde{x}^\mu](t)$ be the angular velocity of the string $\tilde{x}^\mu$ at time t. The form of this function is not needed for our calculation. We fix $\vec{\omega}[\alpha,\beta]$ at all times by inserting a factor of 1 into the partition function:

$$1=\int \mathcal{D}\beta\mathcal{D}\alpha\,\delta^{(2)}[\vec{\omega}-\vec{\Omega}[\tilde{x}^{\mu}]]\mathrm{Det}\left[\epsilon^{ijk}\hat{x}_0^i\frac{\partial\omega^j}{\partial\alpha}\frac{\partial\omega^k}{\partial\beta}\right]. \tag{A11}$$

Because the argument of the δ function only has two nonzero components, (A11) only contains two δ functions. Inserting the definition (A10) of $\vec{\omega}$ in the determinant gives

$$1=\int \mathcal{D}(\cos\beta)\mathcal{D}\alpha\,\delta^{(2)}[\vec{\omega}-\vec{\Omega}[\tilde{x}^{\mu}]]\mathrm{Det}\left[-\frac{d^2}{dt^2}\right]. \tag{A12}$$

Inserting the factor (A12) into the partition function allows us to write the center of mass partition function in terms of rotational degrees of freedom α and β:

$$Z^{\text{periodic}}=\frac{1}{Z_b}\int \mathcal{D}(\cos\beta)\mathcal{D}\alpha\mathcal{D}f_1\mathcal{D}f_2\mathcal{D}\vec{x}_1\mathcal{D}\vec{x}_2\Delta_{FP}$$
$$\times\delta^{(2)}[\vec{\omega}-\vec{\Omega}[\tilde{x}^{\mu}]]\mathrm{Det}\left[-\frac{d^2}{dt^2}\right]$$
$$\times\exp\left(i\int dtL[x^{\mu}]\right). \tag{A13}$$

α is the angle between the y axis and the normal to the plane of rotation. β is the (angular) position of the end of the string in the plane of rotation.

2. Extracting the sum over classical solutions

We evaluate the partition function (A13) semiclassically. It will then contain a sum over all classical solutions of period T. We now explicitly extract this sum from the functional integral. The classical solutions in (A13) correspond to motion where the axis $\hat{x}_0(t)$ of the string rotates with uniform angular velocity. We can parametrize these solutions by the Euler angles

$$\alpha=\text{const},\quad \beta=\omega t. \tag{A14}$$

The constant ω satisfies the equation

$$\omega=\frac{2\pi n}{T} \tag{A15}$$

for some integer n. We can always make a global rotation to ensure that the classical solution for α and β has the form (A14), so we write

$$Z^{\text{periodic}}=\int \mathcal{D}(\cos\beta)\mathcal{D}\alpha\mathrm{Det}\left[-\frac{d^2}{dt^2}\right]e^{iS_{\text{rotate}}[\alpha,\beta]}, \tag{A16}$$

where

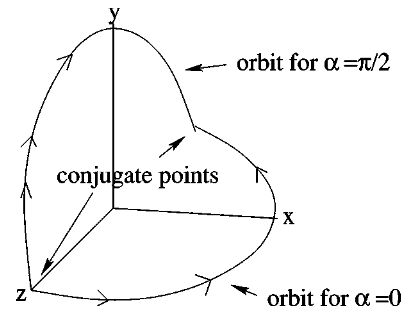

FIG. 3. Orbits of the end of the string for different values of α.

$$e^{iS_{\text{rotate}}[\alpha,\beta]}\equiv\frac{1}{Z_b}\int dR^{ij}\int \mathcal{D}f_1\mathcal{D}f_2\mathcal{D}\vec{x}_1\mathcal{D}\vec{x}_2\Delta_{FP}$$
$$\times\delta^{(2)}(R^{ij}\hat{x}_0^j-\hat{\mathbf{e}}_3^i)$$
$$\times\delta^{(2)}[\vec{\omega}^i-R^{ij}\vec{\Omega}^j[\tilde{x}^{\mu}]]\exp\left(i\int dtL[x^{\mu}]\right). \tag{A17}$$

To enable us to extract the sum over classical solutions, we will explicitly divide α and β into parts which change the classical solution and parts which perturb the fields away from the classical solution. We first divide the field $\alpha(t)$ into fluctuations which change the classical solution and fluctuations which move the field away from its classical value. Classically, $\alpha(t)$ can be any constant α_0. The choice of this constant determines which of the planes passing through $\beta=0$ will contain the string rotation. We note that, since $\vec{\omega}^2=\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2$, α always appears in the action in terms of the form $\dot{\alpha}\sin\beta$, which is classically zero. We define a new variable

$$a=\dot{\alpha}\sin\beta. \tag{A18}$$

In terms of the function $a(t)$, $\alpha(t)$ is

$$\alpha(t)=\alpha_0+\int_0^t dt'\frac{a(t')}{\sin\beta(t')}. \tag{A19}$$

Because a only depends on $\dot{\alpha}$, its classical value is independent of the choice of the constant α_0.

The factor of $\sin\beta$ in the definition of a produces a complication. Comparing the inverse propagators of α and a, we see that

$$\frac{\partial^2 S}{\partial\alpha(t)\partial\alpha(t')}=-\frac{d}{dt}\sin\beta(t)\frac{\partial^2 S}{\partial a(t)\partial a(t')}\sin\beta(t')\frac{d}{dt'}. \tag{A20}$$

When β is a multiple of π, the inverse propagator of α vanishes, but the same is not true of the inverse propagator of a. At these points, the end of the string is at either the point $\beta=0$ or its antipode $\beta=\pi$, independent of the value of α (see Fig. 3). These points are called conjugate points. At these points classical trajectories with different values of α_0

meet. As a result of the singularity in the propagator of α, the partition function picks up a phase of $\pi/2$ at each of these points when we do the α integral, analogous to the phase shift at a WKB turning point [26]. The a propagator does not have this singularity, so the partition function will not receive a phase shift from the a integral. In changing variables from α to a, we must add this phase shift to the partition function. The integration measure for α is

$$\mathcal{D}\alpha = d\alpha_0 \mathcal{D}a \mathrm{Det}^{-1/2}[-\partial_t^2]\mathrm{Det}^{-1}[\sin\beta]\exp\left(i/2\int_{-T/2}^{T/2} dt\dot{\beta}\right). \tag{A21}$$

The argument of the exponential is equal to $\pi/2$ multiplied by the number of times β passes through a multiple of π, which is the phase shift. The integral over α_0 is present because a is independent of the constant part of α. α_0 varies between 0 and 2π. In terms of these new variables, the partition function is

$$Z^{\mathrm{periodic}} = \int d\alpha_0 \mathcal{D}\beta \mathcal{D}a \mathrm{Det}^{1/2}\left[-\frac{d^2}{dt^2}\right]\exp\left(iS_{\mathrm{rotate}}[\alpha,\beta] + i/2\int_{-T/2}^{T/2} dt\dot{\beta}\right). \tag{A22}$$

We next extract the sum over the classical frequencies $\omega = 2\pi n/T$ from the integral over β. Let $b(t) = \dot{\beta}(t) - \omega$, and $\beta_0 = \beta(t=0)$. Then $\beta(t)$ is given by

$$\beta(t) = \beta_0 + \omega t + \int_0^t dt' b(t'). \tag{A23}$$

As a result of the boundary conditions on β, $b(t)$ is subject to the restriction

$$\int_{-T/2}^{T/2} dt b = \beta(T/2) - \beta(-T/2) - \omega T = 0. \tag{A24}$$

The function $b(t)$ is also independent of β_0.

The change of variables from β to b produces a change in the functional integration measure:

$$\mathcal{D}\beta = \sum_{\omega = 2\pi n/T} \mathcal{D}b\, d\beta_0 \mathrm{Det}^{-1/2}[-\partial_t^2] 2\sqrt{\frac{\pi}{T}}. \tag{A25}$$

The factor of $2\sqrt{\pi/T}$ appears in Eq. (A25) because the integral has periodic boundary conditions. The definition of the determinant of $-\partial_t^2$ is [27]

$$\mathrm{Det}^{-1/2}[-\partial_t^2] = \int ds_1 \cdots ds_j \exp\left\{-\frac{s_1^2}{t_1} - \frac{(s_2-s_1)^2}{t_2-t_1} - \cdots - \frac{(s_j - s_{j-1})^2}{t_j - t_{j-1}}\right\}. \tag{A26}$$

However, in the case of periodic boundary conditions, $s_j = s_0 = 0$. Because of this, we need a Lagrange multiplier to identify the values of s at these two points:

$$\mathrm{Det}^{-1/2}_{\mathrm{periodic}}[-\partial_t^2] = \int d\lambda \int ds_1 \cdots ds_j \exp\left\{-\frac{s_1^2}{t_1} - \frac{(s_2-s_1)^2}{t_2-t_1} - \cdots - \frac{(s_j - s_{j-1})^2}{t_j - t_{j-1}} + i\lambda s_j\right\}. \tag{A27}$$

As a result of the additional term in the action, we must translate each of the s_i to be able to do the integral. This translation is

$$s_i \to s_i + i\frac{\lambda}{2}t_i. \tag{A28}$$

The effect of this translation on each s^2 term is

$$-\frac{(s_i - s_{i-1})^2}{t_i - t_{i-i}} \to -\frac{(s_i - s_{i-1})^2}{t_i - t_{i-i}} - i\lambda(s_i - s_{i-1}) + \frac{\lambda^2}{4}(t_i - t_{i-i}), \tag{A29}$$

while the effect on the λs_j term is

$$i\lambda s_j \to i\lambda s_j - \frac{\lambda^2}{2}t_j. \tag{A30}$$

The total effect of this transformation is that

$$\begin{aligned}\mathrm{Det}^{-1/2}_{\mathrm{periodic}}[-\partial_t^2] &= \int d\lambda \int ds_1 \cdots ds_j \exp\left\{-\frac{s_1^2}{t_1} - \frac{(s_2-s_1)^2}{t_2-t_1} - \cdots - \frac{(s_j - s_{j-1})^2}{t_j - t_{j-1}} - \frac{\lambda^2}{4}t_j\right\} \\ &= \mathrm{Det}^{-1/2}[-\partial_t^2]\int d\lambda \exp\left\{-\frac{\lambda^2}{4}T\right\} \\ &= \mathrm{Det}^{-1/2}[-\partial_t^2] 2\sqrt{\frac{\pi}{T}}.\end{aligned} \tag{A31}$$

This gives the factor we have included in Eq. (A25). A more general version of this derivation, valid for all Gaussian functional integrals, is done in [10]. Making the change of variables (A23) and implementing the restriction (A24) gives the partition function the form

$$Z^{\mathrm{periodic}} = 2\sqrt{\frac{\pi}{T}}\sum_{\omega=2\pi n/T}\int d\alpha_0 d\beta_0 \mathcal{D}a\mathcal{D}b\,\delta\left(\int_{-T/2}^{T/2} dt b(t)\right) e^{iS_{\mathrm{rotate}}[\alpha,\beta] + iT\omega/2}. \tag{A32}$$

Equation (A32) gives Z^{periodic} as a sum over semiclassical integrals about classical solutions:

$$Z^{\mathrm{periodic}} = 2\sqrt{\frac{\pi}{T}}\sum_{\omega=2\pi n/T} e^{iT\omega/2} Z(\omega), \tag{A33}$$

where $Z(\omega)$ is

$$
\begin{aligned}
Z(\omega) \equiv \frac{1}{Z_b} \int & d\alpha_0 d\beta_0 \mathcal{D}a \mathcal{D}b dR^{ij} \mathcal{D}f_1 \mathcal{D}f_2 \mathcal{D}\vec{x}_1 \mathcal{D}\vec{x}_2 \\
& \times \Delta_{FP} \delta^{(2)}(R^{ij}\hat{x}_0^j - \hat{\mathbf{e}}_3^i) \\
& \times \delta\left(\int_{-T/2}^{T/2} dt b(t) \right) \delta^{(2)}[\vec{\omega}^i - R^{ij}\vec{\Omega}^j[\tilde{x}^\mu]] \\
& \times \exp\left(i \int dt L[x^\mu] \right).
\end{aligned}
\tag{A34}
$$

Doing the integrations over a, b, α_0, and β_0 in Eq. (A34) gives

$$
\begin{aligned}
Z(\omega) = \frac{1}{Z_b} \int & \mathcal{D}f_1 \mathcal{D}f_2 \mathcal{D}\vec{x}_1 \mathcal{D}\vec{x}_2 \Delta_{FP} \\
& \times \delta(\omega - |\langle \vec{\Omega}[\tilde{x}^\mu] \rangle|) \exp\left(i \int dt L[x^\mu] \right).
\end{aligned}
\tag{A35}
$$

The constraints that have been placed on α and β restrict $Z(\omega)$ to those string configurations with angular velocity ω. The expression (A35) is equivalent to the partition function $Z(\omega)$ defined in Eq. (5.7), up to the δ function:

$$
\delta(\omega - |\langle \vec{\Omega}[\tilde{x}^\mu] \rangle|). \tag{A36}
$$

This δ function implements the boundary condition (6.7) as a constraint on $\tilde{x}^\mu$. The constraint is due to the fact that, in obtaining the sum over classical solutions, we have removed the zero frequency component of one degree of freedom from the partition function.

3. Summing over classical solutions

We now insert the sum over classical solutions (A33) into Eq. (A1) and use Eq. (5.9). This expresses the propagator $G(E)$ in terms of $L_{\rm eff}(\omega)$:

$$
G(E) = i \int_0^\infty dT 2 \sqrt{\frac{\pi}{T}} \sum_{\omega = 2\pi n/T} e^{iT[E + \omega/2 + L_{\rm eff}(\omega)]}. \tag{A37}
$$

We evaluate of the T integral by the method of stationary phase. The poles in the propagator $G(E)$ appear at those values of E for which the sum over n diverges. Therefore, we must approximate the T integral in a way which is valid for large n. As n becomes large, the classical solution will consist of many orbits at some frequency ω determined by the energy. Therefore, T is large in the large n limit, and the phase in the exponential fluctuates wildly. We do the T integral by expanding about the stationary point of this phase. This point defines T as a function of E:

$$
\begin{aligned}
& \frac{d}{dT}\left[T\left(L_{\rm eff}(\omega) + E + \frac{\omega}{2} \right) \right] \\
& \quad = L_{\rm eff}(\omega) + E + \frac{\omega}{2} + T \frac{d\omega}{dT}\left(\frac{dL_{\rm eff}}{d\omega} + \frac{1}{2} \right) = 0.
\end{aligned}
\tag{A38}
$$

The definition $\omega = 2\pi n/T$ gives

$$
\frac{d\omega}{dT} = -\frac{\omega}{T}, \tag{A39}
$$

so the energy is

$$
E = \omega \frac{dL_{\rm eff}}{d\omega} - L_{\rm eff}(\omega). \tag{A40}
$$

Equation (A40) implicitly defines ω as a function of E, so ω is independent of n, while T is proportional to n. The integral also produces a factor of

$$
\begin{aligned}
& \sqrt{\pi}\left\{ \frac{1}{2} \frac{d^2}{dT^2}\left[T\left(L_{\rm eff}(\omega) + E + \frac{\omega}{2} \right) \right] \right\}^{-1/2} \\
& \quad = \sqrt{2\pi T}\left(\omega^2 \frac{dL_{\rm eff}}{d\omega^2} \right)^{-1/2},
\end{aligned}
\tag{A41}
$$

which cancels the factor of $T^{-1/2}$ in Eq. (A37). The propagator is

$$
G(E) = 2i\pi\sqrt{2} \sum_{n=1}^\infty \left(\omega^2 \frac{dL_{\rm eff}}{d\omega^2} \right)^{-1/2} e^{iT[L_{\rm eff}(\omega) + E + \omega/2]}. \tag{A42}
$$

To do the sum over n, we make the n dependence explicit by writing $T = 2\pi n/\omega$ everywhere:

$$
\begin{aligned}
G(E) = 16 i \pi^3 \sqrt{2} & \left(\omega^2 \frac{dL_{\rm eff}}{d\omega^2} \right)^{-1/2} \\
& \times \sum_{n=1}^\infty \exp\left\{ 2\pi i n\left(\frac{L_{\rm eff}(\omega)}{\omega} + \frac{E}{\omega} + \frac{1}{2} \right) \right\}.
\end{aligned}
\tag{A43}
$$

Doing the sum, and replacing E using Eq. (A40), gives

$$
\begin{aligned}
& \sum_{n=1}^\infty \exp\left\{ 2\pi i n\left(\frac{L_{\rm eff}(\omega)}{\omega} + \frac{E}{\omega} + \frac{1}{2} \right) \right\} \\
& \quad = \frac{1}{1 - \exp\left\{ 2\pi i\left(\frac{dL_{\rm eff}}{d\omega} + \frac{1}{2} \right) \right\}} - 1.
\end{aligned}
\tag{A44}
$$

Therefore, the propagator has poles whenever $dL_{\rm eff}/d\omega$ is an odd half integer:

$$\frac{dL_{\rm eff}}{d\omega}=l+\frac{1}{2}. \tag{A45}$$

This is the WKB quantization condition for angular momentum. It tells us the angular velocities of the angular momentum states. Equation (A40) then gives the energies of those states. The poles in the propagator $G(E)$ come from the divergence of the sum over orbits for large numbers of orbits (large T), so $L_{\rm eff}(\omega)$ is defined by taking the large T limit:

$$L_{\rm eff}(\omega)\equiv\lim_{T\to\infty}\frac{-i}{T}\ln Z(\omega). \tag{A46}$$

APPENDIX B: EVALUATION OF $G^{ij}(\nu)$

We will now evaluate Eq. (8.17),

$$G^{ij}_{\psi}(\nu)\equiv(-1)^{i+j}\Sigma[(-1)^i\overline{R_i}]\Sigma[(-1)^j\overline{R_j}]\overline{\gamma_i}^{-2}\overline{\gamma_j}^{-2}\times\frac{\partial^2}{\partial r\partial r'}G(r,r',\nu)\Big|_{\substack{r'=(-1)^i\overline{R_i}\\ r=(-1)^j\overline{R_j}}}. \tag{B1}$$

The Green's function $G(r,r',\nu)$ can be written in terms of functions $l_i(r,\nu)$,

$$G(r,r',\nu)=-\frac{l_1(r_>)l_2(r_<)}{\Sigma(r)\gamma^{-2}W[l_1,l_2]}, \tag{B2}$$

and their Wronskian,

$$W[l_1,l_2]=\left(\frac{\partial l_1(r)}{\partial r}l_2(r)-\frac{\partial l_2(r)}{\partial r}l_1(r)\right). \tag{B3}$$

The functions $l_i(r,\nu)$ satisfy the differential equation

$$\left(-\frac{\partial}{\partial r}\gamma^{-2}\Sigma(r)\frac{\partial}{\partial r}-\Sigma(r)(\nu^2-C)\right)l_i(r,\nu)=0, \tag{B4}$$

with the boundary conditions

$$l_i((-1)^j\overline{R_j},\nu)=\delta_{ij}. \tag{B5}$$

We will evaluate G^{ij}_{ψ} in both the θ sector, where $C=\omega^2$ and $\Sigma(r)=\gamma r^2$, and the ϕ sector, where $C=0$ and $\Sigma(r)=\gamma^3r^2$.

We begin with the θ sector. The first thing we do is change variables. We use the "proper length" coordinate x, previously defined Eq. (9.1) to be

$$x=\frac{1}{\omega}\arcsin(\omega r). \tag{B6}$$

We also change the normalization of the functions $l_i^{\theta}(r,\nu)$. We define the functions $q_i^{\theta}(x,\nu)$:

$$q_i^{\theta}(x,\nu)=\frac{1}{r}l_i^{\theta}(r,\nu). \tag{B7}$$

The q_i^{θ} satisfy the differential equation

$$\left(\nu^2+\frac{\partial^2}{\partial x^2}\right)q_i^{\theta}(x,\nu)=0 \tag{B8}$$

and the boundary conditions

$$q_i^{\theta}\left(\frac{(-1)^i}{\omega}\arcsin v_i,\nu\right)=\delta_{ij}(-1)^i\overline{R_i}. \tag{B9}$$

Therefore, the q_i^{θ} are

$$q_i^{\theta}(x,\nu)=\overline{R_i}\,\frac{\sin\left(\nu x+(-1)^i\frac{\nu}{\omega}\arcsin v_{\hat{i}}\right)}{\sin(\nu R_p)}, \tag{B10}$$

where $v_{\hat{i}}$ is the "other" velocity, i.e. $v_{\hat{2}}=v_1$ and $v_{\hat{1}}=v_2$, and R_p is given by Eq. (3.19).

The functions $l_i^{\theta}(r,\nu)$ are therefore

$$l_i^{\theta}(r,\nu)=\frac{\overline{R_i}}{r}\,\frac{\sin\left(\frac{\nu}{\omega}[\arcsin(\omega r)+(-1)^i\arcsin v_{\hat{i}}]\right)}{\sin(\nu R_p)}, \tag{B11}$$

and $G^{ij}_{\theta}(\nu)$ is

$$G^{ij}_{\theta}(\nu)=\delta_{ij}\left(-\frac{\overline{R_i}}{\overline{\gamma_i}}+\nu\overline{R_i^2}\cot(\nu R_p)\right)-(1-\delta_{ij})\nu\overline{R_i}\,\overline{R_j}\csc(\nu R_p). \tag{B12}$$

Next, consider the ϕ sector. We define the functions $q_i^{\phi}(x,\nu)$,:

$$l_i^{\phi}(r,\nu)=\frac{1}{\gamma r}q_i^{\phi}(x,\nu). \tag{B13}$$

The q_i^{ϕ} satisfy the differential equation

$$\left(\nu^2+\frac{\partial^2}{\partial x^2}-2\omega^2\sec^2(\omega x)\right)q_i^{\phi}(x,\nu)=0, \tag{B14}$$

with boundary conditions

$$q_i^{\phi}\left(\frac{(-1)^j}{\omega}\arcsin v_j,\nu\right)=\delta_{ij}(-1)^i\overline{R_i}\,\overline{\gamma_i}. \tag{B15}$$

The q_i^{ϕ} are related to the eigenfunctions (C31), and are given by

$$q_i^{\phi}(x,\nu)=\overline{R}_i\overline{\gamma}_i\left[[\nu^2-(-1)^i\omega^2 v_{\hat{i}}\overline{\gamma}_{\hat{i}}\tan(\omega x)](-1)^i\sin\left(\nu x+(-1)^i\frac{\nu}{\omega}\arcsin v_{\hat{i}}\right)-\nu\omega[(-1)^i\tan(\omega x)+v_{\hat{i}}\overline{\gamma}_{\hat{i}}]\right.$$

$$\left.\times\cos\left(\nu x+(-1)^i\frac{\nu}{\omega}\arcsin v_{\hat{i}}\right)\right][(\nu^2-\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2)\sin(\nu R_p)-\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)\cos(\nu R_p)]^{-1}. \quad \text{(B16)}$$

This gives the value of $G_{\phi}^{ij}(\nu)$:

$$G_{\phi}^{ij}(\nu)=-\delta_{ij}\overline{\gamma}_i\overline{R}_i\frac{\nu^2}{\omega^2}+\frac{\nu}{\omega^3}(\nu^2-\omega^2)\frac{\delta_{ij}[\nu v_i\overline{\gamma}_i\sin(\nu R_p)-\omega v_1\overline{\gamma}_1 v_2\overline{\gamma}_2\cos(\nu R_p)]+(1-\delta_{ij})\omega v_i\overline{\gamma}_1 v_2\overline{\gamma}_2}{(\nu^2-\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2)\sin(\nu R_p)-\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)\cos(\nu R_p)}. \quad \text{(B17)}$$

APPENDIX C: EVALUATION OF L_{boundary}

We now evaluate the contribution L_{boundary} of the boundary degrees of freedom to the effective Lagrangian:

$$L_{\text{boundary}}=\lim_{T\to\infty}\frac{-i}{T}\left\{-\frac{1}{2}\sum_{i=1}^{2}\text{Tr}\ln\left[\frac{\nu^2+(2\overline{\gamma}_i^2-1)\omega^2}{\nu^2}\right]-\frac{1}{2}\text{Tr}\ln\left[\frac{\Gamma_{\theta}^{ij-1}}{\delta_{ij}m_i\overline{R}_i^2\overline{\gamma}_i\nu^2}\right]-\frac{1}{2}\text{Tr}\ln\left[\frac{\Gamma_{\phi}^{ij-1}}{\delta_{ij}m_i\overline{R}_i^2\overline{\gamma}_i^3\nu^2}\right]\right\}. \quad \text{(C1)}$$

Converting the traces to integrals, inserting the explicit form (10.4) of Γ_{θ}^{ij-1} and (10.5) of Γ_{ϕ}^{ij-1}, and Wick rotating $\nu\to -i\nu$ gives L_{boundary} the form

$$L_{\text{boundary}}=-\frac{1}{2}\int\frac{d\nu}{2\pi}\left\{\text{tr}\ln\left[\delta_{ij}\frac{\nu^2-(2\overline{\gamma}_i^2-1)\omega^2}{\nu^2}\right]+\text{tr}\ln\left[\delta_{ij}\frac{\nu^2+\omega^2}{\nu^2}+\frac{\omega^3}{\nu^2}\sqrt{\frac{\overline{\gamma}_i\overline{\gamma}_j}{v_i v_j}}G_{\theta}^{ij}(-i\nu)\right]+\text{tr}\ln\left[\delta_{ij}\frac{\nu^2+(2\overline{\gamma}_i^2+1)\omega^2}{\nu^2-(2\overline{\gamma}_i^2-1)\omega^2}\right.\right.$$

$$\left.\left.+\frac{\omega^3}{\nu^2}\frac{1}{\sqrt{v_i\overline{\gamma}_i v_j\overline{\gamma}_j}}G_{\phi}^{ij}(-i\nu)\right]\right\}. \quad \text{(C2)}$$

The traces in Eq. (C2) are over the indices i,j.

The integral over the first trace in Eq. (C2) is zero. The integrals over the second and third terms are logarithmically divergent in the cutoff on the wavelengths of string modes. Three of these modes are translation modes, so their contribution to L_{boundary} should not be included in our calculation of meson masses. Normally these modes would contribute nothing to L_{boundary}, since they appear at $\nu=0$. However, two of these modes are in the ϕ sector, and as a result of the frequency shifting in that sector, they appear as poles in Γ_{ϕ}^{ij} at $\nu=\pm\omega$. These modes contribute to L_{boundary} as harmonic oscillators with frequency ω, so the contribution of the translation modes to L_{boundary} is

$$L_{\text{boundary}}^{\text{translation}}=0-\frac{\omega}{2}-\frac{\omega}{2}=-\omega. \quad \text{(C3)}$$

Subtracting this contribution from L_{boundary} gives

$$L_{\text{boundary}}=-\frac{1}{2}\int\frac{d\nu}{2\pi}\left\{\text{tr}\ln\left[\delta_{ij}\frac{\nu^2+\omega^2}{\nu^2}+\frac{\omega^3}{\nu^2}\sqrt{\frac{\overline{\gamma}_i\overline{\gamma}_j}{v_i v_j}}G_{\theta}^{ij}(-i\nu)\right]+\text{tr}\ln\left[\delta_{ij}\frac{\nu^2+(2\overline{\gamma}_i^2+1)\omega^2}{\nu^2-(2\overline{\gamma}_i^2-1)\omega^2}+\frac{\omega^3}{\nu^2}\frac{1}{\sqrt{v_i\overline{\gamma}_i v_j\overline{\gamma}_j}}G_{\phi}^{ij}(-i\nu)\right]\right\}$$

$$+\omega. \quad \text{(C4)}$$

We see that there is a logarithmic divergence in L_{boundary} by noting that, without a cutoff, $G_{\theta}^{ij}(-i\nu)$ and $G_{\phi}^{ij}(-i\nu)$ are

$$G_{\theta}^{ij}(-i\nu)=\delta_{ij}\left(-\frac{\overline{R}_i}{\overline{\gamma}_i}+\nu\overline{R}_i^2\coth(\nu R_p)\right)-(1-\delta_{ij})\nu\overline{R}_i\overline{R}_j\text{csch}(\nu R_p),$$

$$G_{\phi}^{ij}(-i\nu)=\delta_{ij}\overline{\gamma}_i\overline{R}_i\frac{\nu^2}{\omega^2}-\frac{\nu}{\omega^3}(\nu^2+\omega^2)\frac{\delta_{ij}[\nu v_i\overline{\gamma}_i\sinh(\nu R_p)+\omega v_1\overline{\gamma}_1 v_2\overline{\gamma}_2\cosh(\nu R_p)]-(1-\delta_{ij})\omega v_i\overline{\gamma}_1 v_2\overline{\gamma}_2}{(\nu^2+\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2)\sinh(\nu R_p)+\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)\cosh(\nu R_p)}. \quad \text{(C5)}$$

The functions (C5) are proportional to ν in the large ν limit. We pull this divergence outside of the logarithm by adding and subtracting the trace of G^{ij} from the integrals. This breaks Eq. (C4) into four parts,

$$L_{\text{boundary}}=L_\theta^{\text{log term}}+L_\phi^{\text{log term}}+L_\theta^{\text{cutoff}}+L_\phi^{\text{cutoff}}, \tag{C6}$$

where

$$L_\theta^{\text{log term}}=-\frac{1}{2}\int\frac{d\nu}{2\pi}\Bigg\{\text{tr}\ln\Bigg[\delta_{ij}\Bigg(1+\frac{\omega v_i\bar{\gamma}_i}{\nu}\coth(\nu R_p)\Bigg)-(1-\delta_{ij})\frac{\omega}{\nu}\sqrt{v_1\bar{\gamma}_1 v_2\bar{\gamma}_2}\,\text{csch}(\nu R_p)\Bigg]$$
$$-\sum_{i=1}^{2}\frac{\omega^3}{\nu^2}v_i\bar{\gamma}_i\Bigg(\nu\bar{R}_i^2\coth(\nu R_p)-\frac{\bar{R}_i^2}{R_p}\Bigg)\Bigg\},$$

$$L_\phi^{\text{log term}}=-\frac{1}{2}\int\frac{d\nu}{2\pi}\Bigg\{\text{tr}\ln\Bigg[2\delta_{ij}\frac{\nu^2+\omega^2}{\nu^2-(2\bar{\gamma}_i^2-1)\omega^2}-(\nu^2+\omega^2)$$
$$\times\frac{\delta_{ij}\Bigg(\sinh(\nu R_p)+\dfrac{\omega v_1\bar{\gamma}_1 v_2\bar{\gamma}_2}{\nu v_i\bar{\gamma}_i}\cosh(\nu R_p)\Bigg)-(1-\delta_{ij})\dfrac{\omega}{\nu}\sqrt{v_i\bar{\gamma}_1 v_2\bar{\gamma}_2}}{(\nu^2+\omega^2 v_1\bar{\gamma}_1 v_2\bar{\gamma}_2)\sinh(\nu R_p)+\nu\omega(v_1\bar{\gamma}_1+v_2\bar{\gamma}_2)\cosh(\nu R_p)}\Bigg]-\sum_{i=1}^{2}\frac{\omega^3}{\nu^2}\frac{1}{v_i\bar{\gamma}_i}\Bigg[\bar{\gamma}_i\bar{R}_i\frac{\nu^2}{\omega^2}$$
$$+\frac{v_1\bar{\gamma}_1 v_2\bar{\gamma}_2}{\omega(v_1\bar{\gamma}_1+v_2\bar{\gamma}_2)+\omega^2 R_p v_1\bar{\gamma}_1 v_2\bar{\gamma}_2}-\frac{\nu}{\omega^3}(\nu^2+\omega^2)$$
$$\times\frac{\nu v_i\bar{\gamma}_i\sinh(\nu R_p)+\omega v_1\bar{\gamma}_1 v_2\bar{\gamma}_2\cosh(\nu R_p)}{(\nu^2+\omega^2 v_1\bar{\gamma}_1 v_2\bar{\gamma}_2)\sinh(\nu R_p)+\nu\omega(v_1\bar{\gamma}_1+v_2\bar{\gamma}_2)\cosh(\nu R_p)}\Bigg]\Bigg\}+\omega, \tag{C7}$$

and where

$$L_\theta^{\text{cutoff}}=-\frac{1}{2}\int\frac{d\nu}{2\pi}\sum_{i=1}^{2}\frac{\omega^3}{\nu^2}\frac{\bar{\gamma}_i}{v_i}[G_\theta^{ii}(-i\nu)-G_\theta^{ii}(0)],$$

$$L_\phi^{\text{cutoff}}=-\frac{1}{2}\int\frac{d\nu}{2\pi}\sum_{i=1}^{2}\frac{\omega^3}{\nu^2}\frac{1}{v_i\bar{\gamma}_i}[G_\phi^{ii}(-i\nu)-G_\phi^{ii}(0)]. \tag{C8}$$

We have grouped the term ω with the ϕ sector, since it was introduced to cancel the contribution of the translation modes in the ϕ sector. The presence of $G_\theta^{ii}(0)$ and $G_\phi^{ii}(0)$ in the cutoff terms removes a divergence at $\nu=0$. The logarithmic divergence is now entirely in L_θ^{cutoff} and L_ϕ^{cutoff}. Since $L_\theta^{\text{log term}}$ and $L_\phi^{\text{log term}}$ are cutoff independent, we have inserted the explicit functions (C5) into their definitions (C8).

Naive insertion of the functions Eq. (C5) into L_θ^{cutoff} and L_ϕ^{cutoff} in (C4) would make these integrals divergent. We must therefore include the dependence of the Green's function $G(r,r',\nu)$ on the cutoff Λ in the definitions of G_θ^{ij} and G_ϕ^{ij}. We determine the cutoff dependence of $G(r,r',\nu)$ by writing it as a sum of functions $s_n(r)$,

$$G(r,r',\nu)=\sum_{n=1}^{n_{\max}}\frac{1}{\chi_n-\nu^2}\frac{s_n(r)s_n(r')}{\int_{-\bar{R}_1}^{\bar{R}_2}dr''\Sigma(r'')s_n^2(r'')}, \tag{C9}$$

which satisfy the eigenfunction equation

$$\Bigg(-\frac{\partial}{\partial r}\Sigma(r)\gamma^{-2}\frac{\partial}{\partial r}+\Sigma(r)(\chi_n+C)\Bigg)s_n(r)=0, \tag{C10}$$

with eigenvalue χ_n and boundary conditions

$$s_n(-\bar{R}_1)=s_n(\bar{R}_2)=0. \tag{C11}$$

The upper limit of the sum in Eq. (C9) is defined by the equation

$$\chi_{n_{\max}} \leqslant \Lambda^2 < \chi_{n_{\max}+1}. \tag{C12}$$

Inserting Eq. (C9) in the definition (8.17) for the G^{ij} gives

$$G_{\psi}^{ij}(\nu) \equiv (-1)^{i+j}\Sigma[(-1)^i\bar{R}_i]\Sigma[(-1)^j\bar{R}_j]\bar{\gamma}_i^{-2}\bar{\gamma}_j^{-2}\sum_{n=1}^{n_{\max}}\frac{1}{\chi_n-\nu^2}\frac{s_n'[(-1)^i\bar{R}_i]s_n'[(-1)^i\bar{R}_j]}{\int_{-\bar{R}_1}^{\bar{R}_2}dr''\Sigma(r'')s_n^2(r'')}. \tag{C13}$$

The cutoff dependent part of L_{boundary} has the form

$$L_{\psi}^{\text{cutoff}} = -\frac{1}{2}\int\frac{d\nu}{2\pi}\sum_{i=1}^{2}\frac{\omega^3}{\nu^2}\frac{v_i\bar{\gamma}_i^2}{\omega^2\Sigma[(-1)^i\bar{R}_i]}[G_{\psi}^{ii}(-i\nu)-G_{\psi}^{ii}(0)]. \tag{C14}$$

Inserting Eq. (C13) into Eq. (C14) gives

$$L_{\psi}^{\text{cutoff}} = -\frac{1}{2}\int\frac{d\nu}{2\pi}\sum_{i=1}^{2}\frac{\omega}{\nu^2}v_i\bar{\gamma}_i^{-2}\Sigma[(-1)^i\bar{R}_i]\sum_{n=1}^{n_{\max}}\frac{\{s_n'[(-1)^i\bar{R}_i]\}^2}{\int_{-\bar{R}_1}^{\bar{R}_2}dr''\Sigma(r'')s_n^2(r'')}\left(\frac{1}{\chi_n+\nu^2}-\frac{1}{\chi_n}\right)$$

$$= -\frac{\omega}{2}\sum_{i=1}^{2}v_i\bar{\gamma}_i^{-2}\Sigma[(-1)^i\bar{R}_i]\sum_{n=1}^{n_{\max}}\chi_n^{-3/2}\frac{\{s_n'[(-1)^i\bar{R}_i]\}^2}{\int_{-\bar{R}_1}^{\bar{R}_2}dr''\Sigma(r'')s_n^2(r'')}. \tag{C15}$$

For the θ sector, the eigenfunctions $s_n(r)$ are

$$s_n(r) = \frac{1}{r}k_n\left(\frac{\arcsin\omega r}{\omega}\right), \tag{C16}$$

where k_n is defined by

$$k_n(x) = \sin\left(\frac{\pi n}{R_p}(x+X_1)\right), \tag{C17}$$

and the eigenvalues χ_n are

$$\chi_n = \left(\frac{\pi n}{R_p}\right)^2. \tag{C18}$$

Replacing the $s_n(r)$ with Eq. (C16) gives

$$L_{\theta}^{\text{cutoff}} = -\frac{1}{2}\sum_{i=1}^{2}\bar{\gamma}_i v_i\omega\sum_{n=1}^{\Lambda R_p/\pi}\frac{1}{\pi n}. \tag{C19}$$

Replacing the sum over n with a contour integral with poles at $z = \pi n/R_p$ gives

$$L_{\theta}^{\text{cutoff}} = -\frac{1}{4\pi i}\sum_{i=1}^{2}\bar{\gamma}_i v_i\omega\int dz\frac{1}{z}\cot(R_p z). \tag{C20}$$

The contour runs along the line $\text{Re}\,z = \pi/(2R_p)$ and along a semicircle where $|z| = \Lambda$, the cutoff, and the real part of z is positive.

We divide Eq. (C20) into two integrals over the parts of the contour to get

$$L_{\theta}^{\text{cutoff}} = -\frac{1}{4\pi}\sum_{i=1}^{2}\bar{\gamma}_i v_i\omega\left[\int_{-\sqrt{\Lambda^2-\pi^2/4R_p^2}}^{\sqrt{\Lambda^2-\pi^2/4R_p^2}}dy\frac{1}{y-i\dfrac{\pi}{2R_p}}\tanh(R_p y)+\int_{-\arccos(\pi/2R_p\Lambda)}^{\arccos(\pi/2R_p\Lambda)}d\theta\cot(\Lambda R_p e^{i\theta})\right]. \tag{C21}$$

The cotangent in the θ integral is proportional to the sign of θ for large Λ, so the θ integral vanishes. The integral over y is real, because the imaginary part of the integrand changes sign when $y \to -y$:

$$L_\theta^{\text{cutoff}} = -\frac{1}{2\pi}\sum_i \overline{\gamma}_i v_i \omega \int_0^{\sqrt{\Lambda^2-\pi^2/4R_p^2}} dy \frac{y}{y^2+\frac{\pi^2}{4R_p^2}} \tanh(R_p y). \tag{C22}$$

We extract the cutoff dependence from the integral to get

$$L_\theta^{\text{cutoff}} = -\sum_i \frac{\omega v_i \overline{\gamma}_i}{2\pi}\left[\ln\left(\frac{2MR_p}{\pi\overline{\gamma}_i}\right) + \int_0^\infty du \frac{u}{u^2+\frac{\pi^2}{4}}(\tanh u - 1)\right]. \tag{C23}$$

We have replaced Λ, the cutoff for the coordinate x, by $M/\overline{\gamma}_i$, where M is the cutoff for the physical coordinate r, as we did in Eq. (9.14). We have also changed integration variables to $u = yR_p$.

The logarithmic dependence of Eq. (C23) on M is removed by renormalization of the coefficient κ of the geodesic curvature term defined in Eq. (4.2). Since the geodesic curvature diverges in the small quark mass limit, our choice of renormalization point determines the coefficient of the leading term in the effective Lagrangian in that limit. In the final result, the renormalized geodesic curvature term will cancel the divergence of the semiclassical corrections in the small quark mass limit. We therefore choose the renormalization point for which there is no small quark mass divergence in the semiclassical corrections. We must evaluate both $L_\theta^{\text{log term}}$ and L_θ^{cutoff}, since both contribute to the small mass limit divergence.

The term $L_\theta^{\text{log term}}$ in L_{boundary} is

$$L_\theta^{\text{log term}} = -\frac{1}{2}\int \frac{d\nu}{2\pi}\left\{\ln\left[1+\frac{\omega}{\nu}(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)\coth(\nu R_p)+\frac{\omega^2}{\nu^2}v_1\overline{\gamma}_1 v_2\overline{\gamma}_2\right] - \frac{\omega v_i \overline{\gamma}_i}{\nu}\sum_i\left[\coth(\nu R_p)-\frac{1}{\nu R_p}\right]\right\}. \tag{C24}$$

We can simplify this integral by rewriting the integrand:

$$L_\theta^{\text{log term}} = -\int_0^\infty \frac{d\nu}{2\pi}\ln\left[1+\frac{\omega\nu(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)}{\nu^2+\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)+\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2}[\coth(\nu R_p)-1]\right]$$
$$+\int_0^\infty \frac{d\nu}{2\pi}\left\{\ln\left[\left(1+\frac{\omega}{\nu}v_1\overline{\gamma}_1\right)\left(1+\frac{\omega}{\nu}v_2\overline{\gamma}_2\right)\right] - \sum_i \frac{\omega v_i \overline{\gamma}_i}{\nu}\left[\coth(\nu R_p)-\frac{1}{\nu R_p}\right]\right\}. \tag{C25}$$

Integrating by parts in the second integral gives

$$L_\theta^{\text{log term}} = -\int_0^\infty \frac{d\nu}{2\pi}\ln\left[1+\frac{\omega\nu(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)}{\nu^2+\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)+\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2}[\coth(\nu R_p)-1]\right] + \sum_i \frac{\omega v_i\overline{\gamma}_i}{2\pi}\left\{\left[\frac{\nu+\omega v_i\overline{\gamma}_i}{\omega v_i\overline{\gamma}_i}\ln(\nu+\omega v_i\overline{\gamma}_i)\right.\right.$$
$$\left.\left. -\frac{\nu\ln\nu}{\omega v_i\overline{\gamma}_i} - \ln(\nu R_p)\left(\coth(\nu R_p)-\frac{1}{\nu R_p}\right)\right]\Bigg|_0^\infty - R_p\int_0^\infty d\nu \ln(\nu R_p)\left[\operatorname{csch}^2(\nu R_p)-\frac{1}{\nu^2 R_p^2}\right]\right\}. \tag{C26}$$

Evaluating this expression and making the change of variables $t = \nu R_p$ gives

$$L_\theta^{\text{log term}} = -\int_0^\infty \frac{d\nu}{2\pi}\ln\left[1+\frac{\omega\nu(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)}{\nu^2+\nu\omega(v_1\overline{\gamma}_1+v_2\overline{\gamma}_2)+\omega^2 v_1\overline{\gamma}_1 v_2\overline{\gamma}_2}[\coth(\nu R_p)-1]\right]$$
$$+\sum_i \frac{\omega v_i\overline{\gamma}_i}{2\pi}\left\{-\ln(R_p\omega v_i\overline{\gamma}_i)+1-\int_0^\infty dt \ln t\left[\operatorname{csch}^2(t)-\frac{1}{t^2}\right]\right\}. \tag{C27}$$

The sum of L_θ^{cutoff} and $L_\theta^{\text{log term}}$ is

$$L_{\theta}^{\rm cutoff}+L_{\theta}^{\rm log\ term}=-\int_0^{\infty}\frac{d\nu}{2\pi}\ln\left[1+\frac{\omega\nu(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})}{\nu^2+\nu\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})+\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2}}[\coth(\nu R_p)-1]\right]$$
$$+\sum_i\frac{\omega v_i\overline{\gamma_i}}{2\pi}\left\{\ln\left(\frac{2Mm_i}{\pi\sigma}\right)+1+\int_0^{\infty}du\frac{u}{u^2+\frac{\pi^2}{4}}(\tanh u-1)-\int_0^{\infty}dt\ln t\left[\operatorname{csch}^2(t)-\frac{1}{t^2}\right]\right\}, \qquad (C28)$$

where we have used the classical equation of motion (3.23) to simplify the argument of the logarithm. The terms in the sum are renormalizations of the geodesic curvature, so, after renormalization,

$$L_{\theta}^{\rm cutoff}+L_{\theta}^{\rm log\ term}=-\int_0^{\infty}\frac{d\nu}{2\pi}\ln\left[1+\frac{\omega\nu(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})}{\nu^2+\nu\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})+\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2}}[\coth(\nu R_p)-1]\right], \qquad (C29)$$

For two massless quarks the integral (C29) is zero, and for one massless and one heavy quark it has the value $\omega/8+O(v_{\rm heavy})$.

We next evaluate $L_{\phi}^{\rm cutoff}$, using the formula (C15) as we did for $L_{\theta}^{\rm cutoff}$. For the ϕ sector, the eigenfunctions $s_n(r)$ are

$$s_n(r)=\frac{1}{\gamma r}f_n\left(\frac{\arcsin\omega r}{\omega}\right), \qquad (C30)$$

where f_n is defined by

$$f_n(x)=\sqrt{\chi_n}\cos(\sqrt{\chi_n}x+\delta_n)+\omega\tan(\omega x)\sin(\sqrt{\chi_n}x+\delta_n), \qquad (C31)$$

and the eigenvalues χ_n satisfy the equation

$$\tan(\sqrt{\chi_n}R_p)=\sqrt{\chi_n}\omega\frac{v_1\overline{\gamma_1}+v_2\overline{\gamma_2}}{\chi_n-\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2}}. \qquad (C32)$$

The solution $\chi_n=\omega^2$ to Eq. (C32) is not a valid eigenvalue, as it causes Eq. (C31) to vanish everywhere. The phases δ_n are

$$\delta_n=\frac{\sqrt{\chi_n}}{\omega}\arcsin v_1+\arctan\left(\frac{\sqrt{\chi_n}}{\omega v_1\overline{\gamma_1}}\right)$$
$$=-\frac{\sqrt{\chi_n}}{\omega}\arcsin v_2-\arctan\left(\frac{\sqrt{\chi_n}}{\omega v_2\overline{\gamma_2}}\right). \qquad (C33)$$

The definition (C32) of χ_n makes the two definitions for δ_n equivalent.

Replacing the $s_n(r)$ with Eq. (C30) gives

$$L_{\phi}^{\rm cutoff}=-\frac{1}{2}\sum_i\overline{\gamma_i}v_i\omega\sum_n\frac{1}{\sqrt{\nu_n}}\frac{(\nu_n-\omega^2)}{(\nu_n+\omega^2v_i^2\overline{\gamma_i^2})}\left[R_p+\frac{\omega v_1\overline{\gamma_1}}{\nu_n+\omega^2v_1^2\overline{\gamma_1^2}}+\frac{\omega v_2\overline{\gamma_2}}{\nu_n+\omega^2v_2^2\overline{\gamma_2^2}}\right]^{-1}$$
$$=\frac{1}{2}\sum_n(\nu_n-\omega^2)\frac{1}{\sqrt{\nu_n}}\left[1-\frac{R_p}{R_p+\frac{\omega v_1\overline{\gamma_1}}{\nu_n+\omega^2v_1^2\overline{\gamma_1^2}}+\frac{\omega v_2\overline{\gamma_2}}{\nu_n+\omega^2v_2^2\overline{\gamma_2^2}}}\right]. \qquad (C34)$$

The function

$$F(z)=\frac{d}{dz}\ln\left[\frac{(z^2-\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2})\sin(R_pz)-\omega z(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\cos(R_pz)}{z^2-\omega^2}\right] \qquad (C35)$$

has poles of the residue one at $z=\pm\sqrt{\nu_n}$. We rewrite the sum (C34) as a contour integral

$$L_\phi^{\rm cutoff}=-\frac{1}{4\pi i}\int dz(z^2-\omega^2)\frac{1}{z}\left[1-\frac{R_p}{R_p+\frac{\omega v_1\overline{\gamma_1}}{z^2+\omega^2v_1^2\overline{\gamma_1^2}}+\frac{\omega v_2\overline{\gamma_2}}{z^2+\omega^2v_2^2\overline{\gamma_2^2}}}\right]F(z). \tag{C36}$$

The contour is the same as for Eq. (C20). The poles in the term in square brackets in Eq. (C36) all lie on the imaginary axis, which lies outside of the integration contour. Evaluating $F(z)$ gives

$$L_\phi^{\rm cutoff}=-\frac{1}{4\pi i}\int dz(z^2-\omega^2)\frac{1}{z}\left[1-\frac{R_p}{R_p+\frac{\omega v_1\overline{\gamma_1}}{z^2+\omega^2v_1^2\overline{\gamma_1^2}}+\frac{\omega v_2\overline{\gamma_2}}{z^2+\omega^2v_2^2\overline{\gamma_2^2}}}\right]\Bigg\{\{[2+R_p\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})]z\sin(R_pz)+[z^2R_p$$
$$-\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})-\omega^2R_pv_1\overline{\gamma_1}v_2\overline{\gamma_2}]\cos(R_pz)\}[(z^2-\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2})\sin(R_pz)-\omega z(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\cos(R_pz)]^{-1}$$
$$-\frac{2z}{z^2-\omega^2}\Bigg\}. \tag{C37}$$

The term $2z/(z^2-\omega^2)$ does not contribute to the integral, since its poles are canceled by the factor of $z^2-\omega^2$ in the integrand.

We divide the integral over z into an integral over the line at $\mathrm{Re}\, z=\pi/(2R_p)$ and an integral over the semicircle at $|z|=\Lambda$. The integral over the semicircle vanishes, and the integral over the line is

$$L_\phi^{\rm cutoff}=-\frac{1}{4\pi}\int_{-\sqrt{\Lambda^2-\pi^2/4R_p^2}}^{\sqrt{\Lambda^2-\pi^2/4R_p^2}}dy\left[\left(y-i\frac{\pi}{2R_p}\right)^2+\omega^2\right]\frac{1}{y-i\frac{\pi}{2R_p}}$$
$$\times\left[R_p\left(R_p-\frac{\omega v_1\overline{\gamma_1}}{\left(y-i\frac{\pi}{2R_p}\right)^2-\omega^2v_1^2\overline{\gamma_1^2}}-\frac{\omega v_2\overline{\gamma_2}}{\left(y-i\frac{\pi}{2R_p}\right)^2-\omega^2v_2^2\overline{\gamma_2^2}}\right)^{-1}-1\right]$$
$$\times\left\{\left[\left(y-i\frac{\pi}{2R_p}\right)^2R_p+\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})+\omega^2R_pv_1\overline{\gamma_1}v_2\overline{\gamma_2}\right]\sinh(R_py)\right.$$
$$\left.+[2+R_p\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})]\left(y-i\frac{\pi}{2R_p}\right)\cosh(R_py)\right\}$$
$$\times\left\{\left[\left(y-i\frac{\pi}{2R_p}\right)^2+\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2}\right]\cosh(R_py)+\omega\left(y-i\frac{\pi}{2R_p}\right)(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\sinh(R_py)\right\}^{-1}. \tag{C38}$$

Extracting the divergent part gives

$$L_\phi^{\rm cutoff}=-\sum_i\frac{v_i^2\overline{\gamma_i}}{2\pi\overline{R_i}}\ln\left(\frac{2MR_p}{\pi\overline{\gamma_i}}\right)+\frac{1}{2\pi}\int_0^\infty dy\,\mathrm{Re}\left(\left[\left(y-i\frac{\pi}{2R_p}\right)^2+\omega^2\right]\frac{1}{y-i\frac{\pi}{2R_p}}\right.$$
$$\times\frac{\frac{\omega v_1\overline{\gamma_1}}{(y-i\pi/2R_p)^2-\omega^2v_1^2\overline{\gamma_1^2}}+\frac{\omega v_2\overline{\gamma_2}}{(y-i\pi/2R_p)^2-\omega^2v_2^2\overline{\gamma_2^2}}}{R_p-\frac{\omega v_1\overline{\gamma_1}}{(y-i\pi/2R_p)^2-\omega^2v_1^2\overline{\gamma_1^2}}-\frac{\omega v_2\overline{\gamma_2}}{(y-i\pi/2R_p)^2-\omega^2v_2^2\overline{\gamma_2^2}}}\left\{\left[\left(y-i\frac{\pi}{2R_p}\right)^2R_p+\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\right.\right.$$
$$\left.\left.+\omega^2R_pv_1\overline{\gamma_1}v_2\overline{\gamma_2}\right]\sinh(R_py)+(2+R_p\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2}))\left(y-i\frac{\pi}{2R_p}\right)\cosh(R_py)\right\}\left\{\left[\left(y-i\frac{\pi}{2R_p}\right)^2\right.\right.$$
$$\left.\left.\left.+\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2}\right]\cosh(R_py)+\omega\left(y-i\frac{\pi}{2R_p}\right)(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\sinh(R_py)\right\}^{-1}-\sum_i\frac{\omega v_i\overline{\gamma_i}}{y-i\frac{\pi}{2R_p}}\right), \tag{C39}$$

where we have replaced Λ with $M/\overline{\gamma_i}$ just as we did with $L_\theta^{\rm cutoff}$.

We are interested in the value of $L_\theta^{\rm cutoff}$ in two limits, one where $\overline{\gamma_1},\overline{\gamma_2}\to\infty$, and one where $v_1\ll 1$, $\overline{\gamma_2}\to\infty$. In the first limit, the integral (C39) is dominated by the region where y is large. If y is of order ω, the term in the integrand which contains the hyperbolic functions is of order $\overline{\gamma_i}^{-1}$, so we can make the approximation

$$\sinh(R_p y)\simeq\cosh(R_p y)\simeq\frac{1}{2}e^{R_p y} \tag{C40}$$

to simplify the integrand:

$$L_\phi^{\rm cutoff}=-\sum_i \frac{\omega v_i\overline{\gamma_i}}{2\pi}\ln\left(\frac{2MR_p}{\pi\overline{\gamma_i}}\right)+\frac{1}{2\pi}{\rm Re}\int_0^\infty \frac{dy}{y-i\dfrac{\pi}{2R_p}}\left\{\left[\left(y-i\frac{\pi}{2R_p}\right)^2+\omega^2\right]\right.$$
$$\left.\times\left(\sum_i \frac{\omega v_i\overline{\gamma_i}}{\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2 v_i^2\overline{\gamma_i^2}}\right)\frac{R_p+\sum_i \dfrac{1}{y-i\dfrac{\pi}{2R_p}+\omega v_i\overline{\gamma_i}}}{R_p-\sum_i \dfrac{\omega v_i\overline{\gamma_i}}{\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2 v_i^2\overline{\gamma_i^2}}}-\sum_i \omega v_i\overline{\gamma_i}\right\}+O(\overline{\gamma_i}^{-1}). \tag{C41}$$

The terms in the integrand can be rearranged to extract the most important part of the integrand for large y:

$$L_\phi^{\rm cutoff}=-\sum_i \frac{\omega v_i\overline{\gamma_i}}{2\pi}\left[\ln\left(\frac{2MR_p}{\pi\overline{\gamma_i}}\right)+{\rm Re}\int_0^\infty \frac{dy}{y-i\dfrac{\pi}{2R_p}}\left(\frac{\left(y-i\dfrac{\pi}{2R_p}\right)^2+\omega^2}{\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2v_i^2\overline{\gamma_i^2}}-1\right)\right]+\frac{1}{2\pi}{\rm Re}\int_0^\infty dy\left[\left(y-i\frac{\pi}{2R_p}\right)^2+\omega^2\right]$$
$$\times\left(\sum_i \frac{\omega v_i\overline{\gamma_i}}{\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2 v_i^2\overline{\gamma_i^2}}\right)\left(\sum_i \frac{1}{\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2 v_i^2\overline{\gamma_i^2}}\right)\left(R_p-\sum_i \frac{\omega v_i\overline{\gamma_i}}{\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2 v_i^2\overline{\gamma_i^2}}\right)^{-1}+O(\overline{\gamma_i}^{-1}). \tag{C42}$$

The second integral in Eq. (C42) is zero. We can rewrite it as an integral from $-\infty$ to ∞, since the real part of the integrand is symmetric when $y\to -y$. We can then convert it into an integral over a closed contour by adding a semicircle which passes through $y=-i\infty$. The integrand has no poles inside this contour, so the integral is zero. The first integral in Eq. (C42) can be done exactly, giving

$$L_\phi^{\rm cutoff}=-\sum_i \frac{\omega v_i\overline{\gamma_i}}{2\pi}\ln\left(\frac{M}{\omega v_i\overline{\gamma_i^2}}\right)+O(\overline{\gamma_1}^{-1},\overline{\gamma_2}^{-1}), \tag{C43}$$

in the limit $\overline{\gamma_1},\overline{\gamma_2}\to\infty$.

In the limit $v_1\ll 1$, $\overline{\gamma_2}\to\infty$, $L_\phi^{\rm cutoff}$ splits into two parts. One part, dominated by $y=\omega v_2\overline{\gamma_2}+O(1)$, is evaluated the same way as in the light-light case. The other part is dominated by small y and is handled differently. Using the result (C43) to evaluate the first part and taking the limit $\overline{\gamma_2}\to\infty$ in the second part gives

$$L_\phi^{\rm cutoff} = -\frac{\omega v_1\overline{\gamma_1}}{2\pi}\ln\left(\frac{2MR_p}{\pi\overline{\gamma_1}}\right) - \frac{\omega v_2\overline{\gamma_2}}{2\pi}\ln\left(\frac{M}{\omega v_2\overline{\gamma_2^2}}\right) + \frac{\omega v_1\overline{\gamma_1}}{2\pi}\int_0^\infty \frac{dy}{y-i\dfrac{\pi}{2R_p}}{\rm Re}\left\{\frac{\left(y-i\dfrac{\pi}{2R_p}\right)^2+\omega^2}{R_p\left[\left(y-i\dfrac{\pi}{2R_p}\right)^2-\omega^2v_1^2\overline{\gamma_1^2}\right]-\omega v_1\overline{\gamma_1}}\right.$$
$$\left.\times\frac{(\omega+\omega^2R_pv_1\overline{\gamma_1})\sinh(R_py)+R_p\omega\left(y-i\dfrac{\pi}{2R_p}\right)\cosh(R_py)}{\omega^2v_1\overline{\gamma_1}\cosh(R_py)+\omega\left(y-i\dfrac{\pi}{2R_p}\right)\sinh(R_py)}-1\right\}+O(\overline{\gamma_2}^{-1}). \tag{C44}$$

Since $y-i\pi/2R_p$ is always at least of order 1, the integrand in Eq. (C44) is of order 1, and the term containing the integral is of order v_1. The difference between the first two terms in Eq. (C44) and the terms in Eq. (C43) is also of order v_1. Therefore, in the heavy-light limit,

$$L_\phi^{\rm cutoff} = -\sum_i \frac{\omega v_i\overline{\gamma_i}}{2\pi}\ln\left(\frac{M}{\omega v_i\overline{\gamma_i^2}}\right)+O(v_1,\overline{\gamma_2}^{-1}). \tag{C45}$$

We next evaluate $L_\phi^{\rm log\ term}$. We can simplify the integral by separating it into two parts which are small for $\nu \gg \omega$ and one part which does not contain hyperbolic functions. We break the ν integral into three pieces:

$$L_\phi^{\rm log\ term} = -I_1-I_2-I_3, \tag{C46}$$

where

$$I_1=\sum_i\int_0^\infty\frac{d\nu}{2\pi}\left[\ln\left(\frac{(\nu^2+\omega^2)(\nu^2+2\nu\omega v_i\overline{\gamma_i}+(2\overline{\gamma_i^2}-1)\omega^2)}{(\nu^2-(2\overline{\gamma_i^2}-1)\omega^2)\nu(\nu+\omega v_i\overline{\gamma_i})}\right)-\frac{v_i\overline{\gamma_i}\omega}{\nu+v_i\overline{\gamma_i}\omega}\right]-\omega,$$

$$I_2=\int_0^\infty\frac{d\nu}{2\pi}\left\{{\rm tr}\ln\left[2\delta_{ij}\frac{\nu^2+\omega^2}{\nu^2-(2\overline{\gamma_i^2}-1)\omega^2}-(\nu^2+\omega^2)\right.\right.$$
$$\left.\times\frac{\delta_{ij}\left(\sinh(\nu R_p)+\dfrac{\omega v_1\overline{\gamma_1}v_2\overline{\gamma_2}}{\nu v_i\overline{\gamma_i}}\cosh(\nu R_p)\right)-(1-\delta_{ij})\dfrac{\omega}{\nu}\sqrt{v_i\overline{\gamma_1}v_2\overline{\gamma_2}}}{(\nu^2+\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2})\sinh(\nu R_p)+\nu\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\cosh(\nu R_p)}\right]$$
$$\left.-\sum_i\ln\left(\frac{(\nu^2+\omega^2)[\nu^2+2\nu\omega v_i\overline{\gamma_i}+(2\overline{\gamma_i^2}-1)\omega^2]}{[\nu^2-(2\overline{\gamma_i^2}-1)\omega^2]\nu(\nu+\omega v_i\overline{\gamma_i})}\right)\right\},$$

$$I_3=-\sum_i\int_0^\infty\frac{d\nu}{2\pi}\left\{\frac{\omega^3}{\nu^2}\frac{1}{v_i\overline{\gamma_i}}\left[\overline{\gamma_i}\overline{R_i}\frac{\nu^2}{\omega^2}+\frac{v_1\overline{\gamma_1}v_2\overline{\gamma_2}}{\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})+\omega^2R_pv_1\overline{\gamma_1}v_2\overline{\gamma_2}}-\frac{\nu}{\omega^3}(\nu^2+\omega^2)\right.\right.$$
$$\left.\left.\times\frac{\nu v_i\overline{\gamma_i}\sinh(\nu R_p)+\omega v_1\overline{\gamma_1}v_2\overline{\gamma_2}\cosh(\nu R_p)}{(\nu^2+\omega^2v_1\overline{\gamma_1}v_2\overline{\gamma_2})\sinh(\nu R_p)+\nu\omega(v_1\overline{\gamma_1}+v_2\overline{\gamma_2})\cosh(\nu R_p)}\right]-\frac{\omega v_i\overline{\gamma_i}}{\nu+v_i\overline{\gamma_i}\omega}\right\}. \tag{C47}$$

The second and third integrals are dominated by the region where ν is small. We can evaluate I_1 exactly, since the argument of the logarithm factorizes. We find

$$I_1=\sum_i\left\{\frac{\omega v_i\overline{\gamma_i}}{2\pi}\left[1-\ln\left(\frac{2\overline{\gamma_i^2}-1}{\overline{\gamma_i^2}-1}\right)+\frac{\pi}{v_i}+\frac{2}{v_i}\arctan(v_i)\right]\right\}. \tag{C48}$$

In both the light-light and heavy-light limits we take the limit $\overline{\gamma_2}\to\infty$. In this limit, the three integrals (C47) are

$$I_1=\frac{\omega v_1\overline{\gamma_1}}{2\pi}\left[1-\ln\left(\frac{2\overline{\gamma}_1^2-1}{\overline{\gamma}_1^2-1}\right)+\frac{\pi}{v_1}+\frac{2}{v_1}\arctan(v_1)\right]+\frac{\omega v_2\overline{\gamma_2}}{2\pi}\left(1-\ln 2+\frac{3}{2}\pi\right)+O(\overline{\gamma}_2^{-1}),$$

$$I_2=\frac{\omega}{2\pi}\int_0^\infty ds\,\ln\left[\frac{(s+v_1\overline{\gamma_1})[(s^2+2\overline{\gamma}_1^2-1)\sinh(s\omega R_p)+2sv_i\overline{\gamma_i}\cosh(s\omega R_p)]}{[s\cosh(s\omega R_p)+v_1\overline{\gamma_1}\sinh(s\omega R_p)](s^2+2sv_1\overline{\gamma_1}+2\overline{\gamma}_1^2-1)}\right]+O(\overline{\gamma}_2^{-1}),$$

$$I_3=-\frac{\omega}{2\pi}\int_0^\infty ds\left[\frac{s}{s+v_1\overline{\gamma_1}}-\frac{(s^2+1)\cosh(s\omega R_p)}{s^2\cosh(s\omega R_p)+sv_1\overline{\gamma_1}\sinh(s\omega R_p)}+\frac{1}{s^2}\frac{1}{1+\omega R_p v_1\overline{\gamma_1}}\right]+O(\overline{\gamma}_2^{-1}), \tag{C49}$$

where we have changed integration variables in I_2 and I_3 to $s=\nu/\omega$.

In the light-light limit, we take $\overline{\gamma_1}\to\infty$ and find

$$I_1=\sum_i\frac{\omega v_i\overline{\gamma_i}}{2\pi}\left(1-\ln 2+\frac{3}{2}\pi\right)+O(\overline{\gamma}_1^{-1},\overline{\gamma}_2^{-1}),$$

$$I_2=O(\overline{\gamma}_1^{-1},\overline{\gamma}_2^{-1}),$$

$$I_3=O(\overline{\gamma}_1^{-1},\overline{\gamma}_2^{-1}), \tag{C50}$$

so $L_\phi^{\text{log term}}$ is

$$L_\phi^{\text{log term}}=-\sum_i\frac{\omega v_i\overline{\gamma_i}}{2\pi}\left(1-\ln 2+\frac{3}{2}\pi\right)+O(\overline{\gamma}_1^{-1},\overline{\gamma}_2^{-1}) \tag{C51}$$

in this limit. The combination $L_\phi^{\text{log term}}+L_\phi^{\text{cutoff}}$ is then

$$L_\phi^{\text{log term}}+L_\phi^{\text{cutoff}}=-\sum_i\frac{\omega v_i\overline{\gamma_i}}{2\pi}\left[\ln\left(\frac{M}{\omega v_i\overline{\gamma}_i^2}\right)+1-\ln 2+\frac{3}{2}\pi\right]+O(\overline{\gamma}_1^{-1},\overline{\gamma}_2^{-1}). \tag{C52}$$

After renormalizing the geodesic curvature, we obtain

$$L_\phi^{\text{log term}}+L_\phi^{\text{cutoff}}=0, \tag{C53}$$

in the limit of massless quarks.

In the heavy-light limit, we take $v_1\to 0$ in Eq. (C49) and find

$$I_1=\frac{\omega v_2\overline{\gamma_2}}{2\pi}\left(1-\ln 2+\frac{3}{2}\pi\right)+\frac{1}{2}\omega+O(v_1\ln v_1,\overline{\gamma}_2^{-1}),$$

$$I_2=\frac{\omega}{2\pi}\int_0^\infty ds\,\ln\tanh\left(\frac{\pi}{2}s\right)+O(v_1,\overline{\gamma}_2^{-1})=-\frac{1}{8}\omega+O(v_1,\overline{\gamma}_2^{-1}),$$

$$I_3=O(v_1,\overline{\gamma}_2^{-1}). \tag{C54}$$

Thus, in this limit, after renormalizing the geodesic curvature,

$$L_\phi^{\text{log term}}+L_\phi^{\text{cutoff}}=-\frac{3}{8}\omega+O(v_1\ln v_1,\overline{\gamma}_2^{-1}). \tag{C55}$$

Combining Eq. (C29) with Eq. (C53) gives L^{boundary} in the light-light limit,

$$L_{\text{boundary}}=0, \tag{C56}$$

and combining Eq. (C29) with Eq. (C55) gives L^{boundary} in the heavy-light limit:

$$L_{\text{boundary}}=-\frac{1}{4}\omega. \tag{C57}$$

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