Nonperturbative Landau gauge and infrared critical exponents in QCD

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We discuss Faddeev-Popov quantization at the nonperturbative level and show that Gribov's prescription of cutting off the functional integral at the Gribov horizon does not change the Schwinger-Dyson equations, but rather resolves an ambiguity in the solution of these equations. We note that Gribov's prescription is not exact, and we therefore turn to the method of stochastic quantization in its time-independent formulation, and recall the proof that it is correct at the nonperturbative level. The nonperturbative Landau gauge is derived as a limiting case, and it is found that it yields the Faddeev-Popov method in the Landau gauge with a cutoff at the Gribov horizon, plus a novel term that corrects for overcounting of Gribov copies inside the Gribov horizon. Nonperturbative but truncated coupled Schwinger-Dyson equations for the gluon and ghost propagators $D(k)$ and $G(k)$ in the Landau gauge are solved asymptotically in the infrared region. The infrared critical exponents or anomalous dimensions, defined by $D(k)\sim 1/(k^2)^{1+a_D}$ and $G(k)\sim 1/(k^2)^{1+a_G}$, are obtained in space-time dimensions $d=2$, 3, 4. Two possible solutions are obtained with the values, in $d=4$ dimensions, $a_G=1$, $a_D=-2$, or $a_G=(93-\sqrt{1201})/98\approx 0.595353$, $a_D=-2a_G$.

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I. INTRODUCTION

The problem of confinement in QCD presents a challenge to the theorist. One would like to understand how and why QCD describes a world of color-neutral hadrons with a mass gap, even though it appears perturbatively to be a theory of unconfined and massless gluons and quarks. A basic insight into the origin of the mass gap in gluodynamics was provided by Feynman [1], Gribov [2], and Cutkosky [3]. These authors proposed that the mass gap is produced by the drastic reduction of the physical configuration space in non-Abelian gauge theory that results from the physical identification, $A_1 \sim A_2$, of distinct but gauge-equivalent field configurations, $A_2={}^gA_1$. A simple analogy is the change in the spectrum of a free particle moving on a line, when points on the line are identified modulo 2π. The real line is reduced to the circle so the continuous spectrum becomes discrete. In analytic calculations, the appropriate identification of gauge-equivalent configurations requires nonperturbative gauge fixing, and in this article we approach the confinement problem by considering how the nonperturbative gauge fixing impacts the Schwinger-Dyson equations of gluodynamics and their solution. We discuss both the Faddeev-Popov formulation, for which the Gribov problem may have an approximate—but not an exact—solution, and stochastic gauge fixing which overcomes this difficulty [4]. We also briefly compare our results for the infrared critical exponents with numerical evaluations of QCD propagators. In this connection we note that stochastic gauge fixing of the type considered here has been implemented numerically with good statistics on impressively large lattices ($48^3\times 64$) [5,6,7,8]. This opens the exciting perspective of the close comparison of analytic and numerical calculations in this gauge. Stochastic quantization has also been adapted to Abelian projection [9], and convergence of the stochastic process has been studied theoretically [10].

We briefly review Faddeev-Popov quantization as a nonperturbative formulation. We note that the Faddeev-Popov weight $P_{\rm FP}(A)$ possesses *nodal surfaces* in A space where the Faddeev-Popov determinant vanishes, and that a cutoff of the functional integral on a nodal surface does not alter the Schwinger-Dyson equations, because it does not introduce boundary terms. As a result, Gribov's prescription to cut off the functional integral at the (first) Gribov horizon [2], a nodal a surface that completely surrounds the origin [11], does not change the Schwinger-Dyson equations of Faddeev-Popov theory at all, but rather it resolves an ambiguity in the solution of these equations.

We recall that Gribov's prescription is not in fact exact because there are Gribov copies inside the Gribov horizon that get overcounted. We then turn to the method of stochastic quantization as described by a time-independent diffusion equation in A space (so there is no fictitious "fifth time") [4]. This method by-passes the Gribov problem of choosing a representative on each gauge orbit because gauge fixing is replaced by the introduction of a "drift force" that is the harmless generator of a gauge transformation. We next derive a formulation of the Landau gauge, which is valid nonperturbatively, as a limiting case of stochastic quantization. It yields the Faddeev-Popov theory with a cutoff at the Gribov horizon, plus a novel term that corrects for overcounting inside the Gribov horizon.

However attractive a formulation may be that is valid at the nonperturbative level, it would remain largely ornamental without actual nonperturbative calculations. Fortunately, progress in finding approximate but nonperturbative solutions for the propagators in QCD has been achieved recently within the framework of Faddeev-Popov theory both in Coulomb gauge using the Hamiltonian formalism [12], and in Landau gauge by solving a truncated set of Schwinger-Dyson equations [13,14], and [15]. The Schwinger-Dyson approach is reviewed in [16]. In the latter part of the present

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article we solve the Schwinger-Dyson equations in the nonperturbative Landau gauge to obtain the infrared critical exponents or anomalous dimensions of the gluon and ghost propagators $D(k)$ and $G(k)$ in $d=2$, 3, and 4 space-time dimensions. The novel term is ignored here, in order to compare with other recent calculations, but we explicitly select the solution to the Schwinger-Dyson equations that vanishes outside the Gribov horizon. Although a truncation is necessarily required to solve these equations, nevertheless the values obtained for the infrared asymptotic dimensions agree with exact results for probability distributions that vanish outside the Gribov horizon [17], namely the vanishing of $D(k)$ at $k=0$, and an enhanced infrared singularity of $G(k)$. These properties also characterize the nonperturbative solutions of the Schwinger-Dyson equations in QCD obtained in recent studies [13,14,15], and we verify that they have also adopted the solution that vanishes outside the Gribov horizon.

II. FADDEEV-POPOV QUANTIZATION AT THE NONPERTURBATIVE LEVEL

The standard Faddeev-Popov (FP) Euclidean weight in Landau gauge is given by

$$P_{\rm FP}(A)=Q_{\rm FP}(A^{\rm tr})\,\delta(\partial_\mu A_\mu),$$
$$Q_{\rm FP}(A^{\rm tr})=N\exp[-S_{\rm YM}(A^{\rm tr})]\times\det[-\partial_\mu D_\mu(A^{\rm tr})] \tag{2.1}$$

with partition function

$$Z(J)=\int dA\,P_{\rm FP}(A)\exp(J,A)=\int dA^{\rm tr}Q_{\rm FP}(A^{\rm tr})\exp(J,A^{\rm tr}), \tag{2.2}$$

where $(J,A)\equiv\int d^4x J^a_\mu A^a_\mu$. It depends only on the transverse components of J and we write $Z=Z(J^{\rm tr})$. The complete set of Schwinger-Dyson (SD) equations reads

$$\frac{\delta\Sigma}{\delta A^{\rm tr}}(\delta/\delta J^{\rm tr})Z=J^{\rm tr}Z, \tag{2.3}$$

where $\Sigma(A^{\rm tr})\equiv S_{\rm YM}(A^{\rm tr})-{\rm Tr}\ln[-\partial_\mu D_\mu(A^{\rm tr})]$.

The Faddeev-Popov operator $-\partial_\mu D^{ac}_\mu(A^{\rm tr})=-\partial^2\delta^{ac}-f^{abc}A^{{\rm tr},b}_\mu\partial_\mu$ is Hermitian because $A^{\rm tr}$ is transverse, $\partial_\mu A^{\rm tr}_\mu=0$. However, it is not positive for every $A^{\rm tr}$. Because the Faddeev-Popov determinant is the product of nontrivial eigenvalues, $\det[-\partial_\mu D_\mu(A^{\rm tr})]=\Pi_n\lambda_n(A^{\rm tr})$, it vanishes together with $Q_{\rm FP}(A^{\rm tr})$ whenever any eigenvalue vanishes, and the equation $\lambda_n(A^{\rm tr})=0$ defines a *nodal surface* of $Q_{\rm FP}(A^{\rm tr})$ in $A^{\rm tr}$ space. The nodal surface where the lowest nontrivial eigenvalue vanishes, defined by $\lambda_0(A^{\rm tr})=0$, defines what is known as the (first) Gribov horizon. It forms the boundary of the region Ω, known as the Gribov region, with the defining property that all nontrivial eigenvalues of the Faddeev-Popov operator $-\partial_\mu D_\mu(A^{\rm tr})$ are positive. Clearly $Q_{\rm FP}(A^{\rm tr})$ is positive inside Ω and vanishes on $\partial\Omega$. It is known that the Gribov region has the following three properties: (i) it is a convex region of A space; (ii) it is bounded in every direction, and (iii) it includes the origin [11].

The existence of nodal surfaces of $Q_{\rm FP}(A^{\rm tr})$ implies that the solution of the Schwinger-Dyson (SD) equation for $Z(J^{\rm tr})$ is not unique. For if the integral (2.2) that defines $Z(J^{\rm tr})$ is cut off on any nodal surface, the same SD equation follows, without any boundary contribution. Moreover, since the SD equation for $Z(J^{\rm tr})$ is linear, any linear combination of two solutions is a solution. These ambiguities are reflected in corresponding ambiguities in the solution of the SD equation for $W(J^{\rm tr})\equiv\ln Z(J^{\rm tr})$, and for the effective action $\Gamma(A^{\rm tr})$, obtained from $W(J^{\rm tr})$ by the Legendre transformation.

[We illustrate these points with a baby model. We replace the field $A^{\rm tr}_\mu(x)$ by a real variable a, and the Faddeev-Popov weight $Q_{\rm FP}(A^{\rm tr})$ by $p(a)\equiv\exp(-\frac{1}{2}a^2)(1-g^2a^2)$, which vanishes on the "Gribov horizon" $a=\pm g^{-1}$. The partition function $Z_1(j)\equiv\int_{-\infty}^{+\infty}da\,p(a)\exp(ja)$ satisfies the SD equation

$$\frac{\partial\Sigma}{\partial a}\left(\frac{\partial}{\partial j}\right)Z_1(j)=\left[\frac{\partial}{\partial j}-g\left(1+g\frac{\partial}{\partial j}\right)^{-1}+g\left(1-g\frac{\partial}{\partial j}\right)^{-1}\right]Z_1(j)=jZ_1(j), \tag{2.4}$$

corresponding to the action $\Sigma=\frac{1}{2}a^2-\ln(1-g^2a^2)$. Suppose we restrict the integral to the Gribov region, $|a|\leqslant g^{-1}$, so the partition function is given instead by $Z_2(j)=\int_{-1/g}^{1/g}da\,p(a)\exp(ja)$. The change occurs only for $a^2>1/g^2$, so $Z_1(j)$ and $Z_2(j)$ have the same perturbative expansion. It is clear that $Z_1(j)$ and $Z_2(j)$ satisfy the *same* SD equation (2.4) without a boundary contribution, because $p(a)$ vanishes on the boundary $a=\pm1/g$. Moreover, because the SD equation for $Z(j)$ is linear, any linear combination, $Z(j)=\alpha Z_1(j)+\beta Z_2(j)$ also satisfies the same SD equation. Of course only one of them corresponds to a weight that is positive everywhere. This example easily generalizes to any number of dimensions.]

Gribov proposed to cut off the integral on the boundary $\partial\Omega$ of what we now call the Gribov region Ω, so the partition function is given by

$$Z_\Omega(J^{\rm tr})\equiv\int_\Omega dA^{\rm tr}Q_{\rm FP}(A^{\rm tr})\exp(J^{\rm tr},A^{\rm tr}), \tag{2.5}$$

for he conjectured that the region where the Faddeev-Popov operator is positive contains only one gauge copy on each gauge orbit. Since $\partial\Omega$ is a nodal surface, Z_Ω satisfies the *same* Schwinger-Dyson equation (2.3), $(\delta\Sigma/\delta A^{\rm tr})(\delta/\delta J^{\rm tr})Z_\Omega=J^{\rm tr}Z_\Omega$. Instead, Gribov's proposal selects a particular one out of a class of nonperturbative solutions of these equations.[1]

It is now known, however, that Gribov's conjecture is not exact. Indeed, there are Gribov copies inside the Gribov horizon.[2] We shall show, however, that an exact nonperturbative formulation yields Gribov's proposal plus a well-defined correction term that corrects for overcounting inside the Gribov horizon.

There is an alternative proposal to make the Faddeev-Popov method valid nonperturbatively. It is conjectured that if one sums over all Gribov copies using the signed Faddeev-Popov determinant $\det[-\partial_\mu D_\mu(A^{\rm tr})]$, then additional Gribov copies cancel in pairs, the reason being that the signed determinant counts the signed intersection number which is a topological invariant. This is presumably the outcome of Becchi-Rouet-Stora-Tyutin (BRST) quantization which has formal properties that suggest it may be valid nonperturbatively [18]. Moreover, this conjecture is supported by simple models [19] and [20]. However, it is not known at present how to turn this prescription into a nonperturbative calculational scheme in QCD, for example, by selecting a particular solution of the SD equations and moreover, if the measure is not everywhere positive there is the danger of delicate cancellations that may cause an approximate solution to be unreliable. On the other hand, the Gribov proposal is easily implemented, for example by requiring that the solution of the SD equation possess positivity properties.

III. TIME-INDEPENDENT STOCHASTIC QUANTIZATION

The difficulties with Faddeev-Popov gauge fixing pointed out by Gribov are by-passed by stochastic or bulk quantization. This method is a formalization of a Monte Carlo simulation [21], and in its most powerful formulation it makes use of a fictitious "fifth time" that corresponds to computer time or number of sweeps of the lattice in a Monte Carlo simulation. Despite the Gribov ambiguity, there is no problem of overcounting with gauge fixing in Monte Carlo simulations, which is achieved by a gauge transformation of choice after any sweep, nor is there one in stochastic or bulk quantization which relies on an infinitesimal gauge transformation. In the five-dimensional formulation, the field $A_\mu^a = A_\mu^a(x,t)$ depends on the four Euclidean coordinates x_μ and on the fifth time t. One may easily write down the SD equations for this formulation which involves a local five-dimensional action, and BRST invariances and Slavnov-Taylor identities are available to control divergences [22,23], and [24]. However the five-dimensional propagator $D = D(k^2,\omega)$ depends on two invariants k^2 and ω, which makes the solution of the SD equations in five dimensions more complicated, and we shall not attempt it here.

Instead we turn to an older four-dimensional formulation of stochastic quantization [4]. It is based on an analogy between the (formal) Euclidean weight $P_{\rm YM}(A) = \exp[-S_{\rm YM}(A)]$ and the Boltzmann distribution $P(x) = \exp[-V(x)]$. The latter is the solution of the time-independent diffusion equation

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i} + \frac{\partial V}{\partial x_i}\right) P = 0,$$

where the drift force is $K_i = -\partial V/\partial x_i$. We shall shortly consider more general drift forces K_i that are not necessarily conservative. The field-theoretic analog of this equation is

$$H_{\rm YM} P(A) \equiv -\int d^4x \, \frac{\delta}{\delta A_\mu(x)} \left(\frac{\delta}{\delta A_\mu(x)} + \frac{\delta S_{\rm YM}}{\delta A_\mu(x)} \right) \times P(A) = 0, \tag{3.1}$$

where the drift force is $K_{{\rm YM},\mu}(x) \equiv -\partial S_{\rm YM}/\delta A_\mu(x)$, which is solved by $P(A) = \exp(-S_{\rm YM})$.

For a gauge theory, this solution is not normalizable. However, for a gauge theory one may modify the drift force $K_{\rm YM} \to K_{\rm YM} + K_{\rm gt}$ by adding to it a "force" K_{gt} tangent to the gauge orbit, without changing the expectation value of gauge-invariant observables $O(A)$. Such a force has the form of an infinitesimal gauge transformation $K_{{\rm gt},\mu}^a \equiv (D_\mu v)^a$, where $v^a(x;A)$ is an element of the Lie algebra, and $(D_\mu v)^a = \partial_\mu v^a + f^{abc} A_\mu^b v^c$ is its gauge-covariant derivative. This force is not conservative, which means that it cannot be expressed as a gradient, $K_{{\rm gt},\mu} \neq -\partial \Sigma/\delta A_\mu$, so this method is not available in a local action formalism in four dimensions. The total drift force is given by

$$K_\mu \equiv K_{{\rm YM},\mu} + K_{{\rm gt},\mu} = -\frac{\delta S_{\rm YM}}{\delta A_\mu} + D_\mu v, \tag{3.2}$$

and $P(A)$ is the solution of the modified time-independent diffusion equation

[1] For partition function Z_Ω, the expectation value of $A^{\rm tr}$ in the presence of the source $J^{\rm tr}$ is $A_{\rm cl} \equiv \langle A^{\rm tr} \rangle_{J^{\rm tr}} = \delta W_\Omega / \delta J^{\rm tr}$, where $W_\Omega = \ln Z_\Omega$. Because the probability distribution in the presence of sources $Q(A^{\rm tr}) \exp(J^{\rm tr}, A^{\rm tr})$ is positive in the Gribov region Ω, and because Ω is convex, it follows that $A_{\rm cl}$ lies in Ω. Consequently the effective action $\Gamma_\Omega(A_{\rm cl})$, obtained by Legendre transform from W_Ω, is defined only on the Gribov region.

[2] As shown in Sec. IV, they are given by ${}^gA$, where $g = g_{\rm min}(x)$ is any local minimum of the functional $F_A(g) = \int d^4x |{}^gA|^2$, where ${}^gA = g^{-1}Ag + g^{-1}\partial g$ is the transform of A by the local gauge transformation $g(x)$. In a lattice discretization, the link variable corresponding to the field $A(x)$ is generically a random field, so the minimization problem is of spin-glass type which is known to have many solutions. On the other hand, for a smooth configuration, such as the vacuum, $A = 0$, there are few solutions. Thus the number of copies is different for different orbits. Moreover, since the Faddeev-Popov weight is positive inside the Gribov horizon, there can be no cancellations to save the day. Note also that, in a lattice discretization, the variables that characterize a configuration take values in a compact space, so a minimizing configuration always exists, which shows that Ω contains at least one Gribov copy for each orbit.

$$HP=(H_{\rm YM}+H_{\rm gt})P$$
$$=-\int d^4x\,\frac{\delta}{\delta A_\mu}\left(\frac{\delta}{\delta A_\mu}+\frac{\delta S_{\rm YM}}{\delta A_\mu}-D_\mu v\right)P$$
$$=0. \tag{3.3}$$

It is easy to show [4] that the expectation value $\langle O\rangle=\int dA\,O(A)P(A)$ of gauge-invariant observables $O(A)$ is independent of v, using the fact that $H^\dagger_{\rm gt}$ is the generator of an infinitesimal local gauge transformation

$$H^\dagger_{\rm gt}=G(v)=\int d^4x\,v(x)G(x),$$
$$G(x)\equiv D_\mu\frac{\delta}{\delta A_\mu}, \tag{3.4}$$

and that $O(A)$ and $H_{\rm YM}$ are gauge invariant, $G(x)O=0$, and $[G(x),H_{\rm YM}]=0$.

The additional drift force $K_{{\rm gt},\mu}=D_\mu v$ must be chosen so that it is globally a restoring force along gauge orbits, thus preventing the escape of probability to infinity along the gauge orbit where $S_{\rm YM}$ is flat. This may be achieved by choosing $K_{\rm gt}$ to be in the direction of steepest descent, restricted to gauge orbit directions, of some conveniently chosen minimizing functional $F(A)$. A convenient choice is the Hilbert square norm, $F(A)=\|A\|^2=\int d^4x|A_\mu|^2$. For a generic infinitesimal variation restricted to gauge orbit directions $\delta A_\mu=\epsilon D_\mu\omega$, we have

$$\delta F(A)=2(A_\mu,\delta A_\mu)=2(A_\mu,\epsilon D_\mu\omega)=2\epsilon(A_\mu,\partial_\mu\omega)$$
$$=-2\epsilon(\partial_\mu A_\mu,\omega). \tag{3.5}$$

The direction of steepest descent of $\|A\|^2$, restricted to gauge orbit directions, is seen to be $\delta A_\mu=\epsilon D_\mu\omega$ for $\omega=\partial_\lambda A_\lambda$. Thus if we choose $v=a^{-1}\partial_\lambda A_\lambda$, where a is a positive gauge parameter, the drift force $K_{\tau,\mu}\equiv a^{-1}D_\mu\partial_\lambda A_\lambda$ points globally in the direction of steepest descent, restricted to gauge orbit directions, of the minimizing functional $\|A\|^2$. In the following we shall use the time-independent diffusion equation

$$HP\equiv-\int d^4x\,\frac{\delta}{\delta A_\mu}\left(\frac{\delta}{\delta A_\mu}+\frac{\delta S_{\rm YM}}{\delta A_\mu}-a^{-1}D_\mu\partial_\lambda A_\lambda\right)P$$
$$=0. \tag{3.6}$$

The five-dimensional formulation is based on the corresponding time-dependent diffusion equation

$$\partial P/\partial t=-HP. \tag{3.7}$$

IV. NONPERTURBATIVE LANDAU GAUGE

Because the gauge-fixing force points in the direction of steepest descent of the minimizing functional $F(A)=\|A\|^2$, restricted to gauge orbit directions, it follows that for large values of the gauge parameter a^{-1} the probability gets concentrated near the local minima of this functional restricted to gauge orbit variations.[3] At a local minimum the first variation vanishes for all ω, $\delta F(A)=-2\epsilon(\omega,\partial_\mu A_\mu)=0$, as we have just seen, so at a minimum, the Landau gauge condition $\partial_\mu A_\mu=0$, is satisfied. *In addition*, the second variation in gauge orbit directions is non-negative, $\delta^2F(A)=-2\epsilon(\omega,\partial_\mu\delta A_\mu)=-2\epsilon^2(\omega,\partial_\mu D_\mu\omega)\geqslant 0$, for all ω, which is the statement that the Faddeev-Popov operator $M(A)\equiv-\partial_\mu D_\mu(A)$ is positive. These are the defining properties of the Gribov region, and we conclude that in the limit in which the gauge parameter approaches zero, $a\to 0$, the probability $P(A)$ gets concentrated on transverse configurations $A=A^{\rm tr}$ that lie *inside* the Gribov horizon. We have noted above that there are Gribov copies inside the Gribov horizon. However, the present method does not require that the probability gets concentrated on any particular one of them such as, for example, the absolute minimum of the minimizing function, and for finite gauge parameter a, the gauge-fixing is ''soft'' in the sense that no particular gauge condition is imposed. For gauge-invariant observables, it does not matter how the probability is distributed along a gauge orbit, but only that it be correctly distributed between gauge orbits. This is assured because a harmless gauge transformation was introduced instead of gauge fixing.

We have noted that A becomes purely transverse in the limit $a\to 0$. We shall solve Eq. (3.6) in this limit by the Born-Oppenheimer method in order to obtain the nonperturbative Landau gauge. For small a, the longitudinal component of A is small and, as we shall see, it evolves rapidly compared to the transverse component. However, because of the factor a^{-1} in Eq. (3.6), the mean value of the longitudinal part of the gluon propagator strongly influences the transverse propagator in the limit $a\to 0$.

We decompose A into its transverse and longitudinal parts according to $A^b_\mu=A^{{\rm tr},b}_\mu+a^{1/2}\partial_\mu(\partial^2)^{-1}L^b$, so $\partial_\mu A^b_\mu=a^{1/2}L^b$, and $\delta/\delta A^b_\mu=(\delta/\delta A^{{\rm tr},b}_\mu)-a^{-1/2}\partial_\mu(\delta/\delta L^b)$. In terms of these variables, Eq. (3.6) reads

$$\int d^4x\left[\frac{\delta}{\delta A^{\rm tr}_\mu}\left(\frac{\delta}{\delta A^{\rm tr}_\mu}+\frac{\delta S_{\rm YM}}{\delta A_\mu}-a^{-1/2}(A_\mu\times L)\right)+a^{-1}\frac{\delta}{\delta L}\left((-\partial^2)\frac{\delta}{\delta L}+a^{1/2}\partial_\mu\frac{\delta S_{\rm YM}}{\delta A_\mu}-\partial_\mu D_\mu(A)L\right)\right]P=0, \tag{4.1}$$

where we have used the notation $(K\times L)^b=[K,L]^b=f^{bcd}K^cL^d$ for elements K and L of the Lie algebra. The leading terms in H are of order a^{-1}, $a^{-1/2}$, and a^0, and we expand $P=P_0+a^{1/2}P_1+\cdots$. The leading term, of order a^{-1}, reads

[3] These conditions define a local minimum at $g(x)=1$ of the functional on the gauge orbit through A defined by $F_A(g)=\int d^4x|{}^gA|^2$.

$$\int d^4x \frac{\delta}{\delta L}\left((-\partial^2)\frac{\delta}{\delta L} - \partial_\mu D_\mu(A^{\rm tr})L\right)P_0 = 0. \quad (4.2)$$

This is solved by P_0 that is Gaussian in L,

$$P_0(A^{\rm tr},L) = Q(A^{\rm tr})(\det X)^{-1/2} \times \exp[-1/2(L,XL)], \quad (4.3)$$

where $X^{bc}(x,y;A^{\rm tr})$ is a symmetric kernel. Equation (4.2) is satisfied provided X satisfies $[L,X(-\partial^2)XL]-(L,XML)=0$ identically for all L, and $\mathrm{tr}[(-\partial^2)X-M]=0$. Here $M=M(A^{\rm tr})\equiv -\partial_\mu D_\mu(A^{\rm tr})$ is the Faddeev-Popov operator that is symmetric for A transverse. The first equation yields $X(-2\partial^2)X = XM+MX$, or $MY+YM=-2\partial^2$ for $Y\equiv X^{-1}$. Moreover, when this equation is satisfied it implies that the second equation is also satisfied. The equation for Y is linear. To solve it we take matrix elements in the basis provided by the eigenfunctions of the Faddeev-Popov operator $M(A^{\rm tr})u_n = \lambda_n u_n$, where $\lambda_n=\lambda_n(A^{\rm tr})$, and obtain

$$(u_m,X^{-1}u_n) = (u_n,Yu_m) = (\lambda_m+\lambda_m)^{-1}(u_n,(-2\partial^2)u_m). \quad (4.4)$$

We see that the Gaussian solution $P_0(A^{\rm tr},L)$ is normalizable in L only when all the eigenvalues $\lambda_n(A^{\rm tr})$ are positive, namely, for $A^{\rm tr}$ inside the Gribov region. However, we have seen above that in the limit $a\to 0$ the solution $P(A)$ is supported inside the Gribov region. Thus the coefficient function $Q(A^{\rm tr})$ carries a factor $\theta(\lambda_0(A^{\rm tr}))$, which restricts the support of P_0 to this region. Finally we note that for $A^{\rm tr}$ in the Gribov region, Y may be written

$$X^{-1} = Y = \int_0^\infty dt \exp(-Mt)(-2\partial^2)\exp(-Mt). \quad (4.5)$$

This representation shows explicitly that X is a positive operator for $A^{\rm tr}$ inside the Gribov region.

To determine $Q(A^{\rm tr})$ we substitute Eq. (4.3) into Eq. (4.1), and integrate over L. This kills the term in $\delta/\delta L$. It also kills the term of order $a^{-1/2}$ in $a^{-1/2}A_\mu \times L = a^{-1/2}[A_\mu^{\rm tr} + a^{1/2}\partial_\mu(\partial^2)^{-1}L]\times L$ because this term is odd in L. This gives in the limit $a\to 0$, the finite equation for $Q(A^{\rm tr})$,

$$\int d^4x \frac{\delta}{\delta A_\mu^{\rm tr}}\left(\frac{\delta}{\delta A_\mu^{\rm tr}} + \frac{\delta S_{\rm YM}}{\delta A_\mu}(A^{\rm tr}) - K_{{\rm gt\ eff},\mu}(A^{\rm tr})\right)Q(A^{\rm tr}) = 0. \quad (4.6)$$

Here $K_{\rm gt\ eff}$ is the average over L of the gauge-transformation force, with weight $(\det X)^{-1/2}\exp[-1/2(L,XL)]$, namely,

$$\begin{aligned} K^b_{{\rm gt\ eff},\mu}(x;A^{\rm tr}) &\equiv \langle f^{bcd}[\partial_\mu(\partial^2)^{-1}L^c](x)L^d(x)\rangle \\ &= f^{bcd}\partial_\mu(\partial^2)^{-1}Y^{cd}(x,y;A^{\rm tr})|_{y=x} \\ &= \int_0^\infty dt f^{bcd}\partial_\mu(\partial^2)^{-1}[\exp(-Mt)(-2\partial^2) \\ &\quad \times \exp(-Mt)]^{cd}(x,y)|_{y=x}, \end{aligned} \quad (4.7)$$

and we have used $Y^{cd}(x,y;A^{\rm tr}) = \langle L^c(x)L^d(y)\rangle$. We now take the limit $a\to 0$, namely $P(A)\to \lim_{a\to 0}P(A) = \lim_{a\to 0}P_0(A)$. With $L=a^{-1/2}\partial_\mu A_\mu$ this gives

$$P(A) = Q(A^{\rm tr})\,\delta(\partial_\mu A_\mu), \quad (4.8)$$

where $Q(A^{\rm tr})$ is the solution of Eq. (4.6). This defines the nonperturbative Landau gauge.

To exhibit the relation between the nonperturbative Landau gauge and the Faddeev-Popov theory, we decompose $K_{\rm gt\ eff}$, given in Eq. (4.7), according to

$$\begin{aligned} K_{\rm gt\ eff} &= K_1 + K_2, \\ K^b_{1,\mu}(x) &= -f^{bcd}\partial_\mu(M^{-1})^{cd}(x,y)|_{y=x}, \\ K^b_{2,\mu}(x) &= \int_0^\infty dt\, f^{bcd}\partial_\mu(\partial^2)^{-1}\{[2\partial^2,\exp(-Mt)] \\ &\quad \times \exp(-Mt)\}^{cd}(x,y)|_{y=x}. \end{aligned} \quad (4.9)$$

The first term may be written

$$K^b_{1,\mu}(x) = \frac{\delta(\mathrm{tr}\ln M)}{\delta A_\mu^{{\rm tr},b}(x)} = -\frac{\delta\Sigma}{\delta A_\mu^{{\rm tr},b}(x)}, \quad (4.10)$$

so K_1 is a conservative drift force, derived from an action $\Sigma \equiv \mathrm{tr}\ln M = -(\ln\det M)$ that precisely reproduces the Faddeev-Popov determinant. So if K_2 were neglected we regain the Faddeev-Popov theory, with the added stipulation to choose the solution that vanishes outside the Gribov horizon.

The second term may be simplified using the identity

$$[\partial^2,\exp(-Mt)] = -\int_0^t ds \exp(-Ms)[\partial^2,M] \times \exp[-M(t-s)], \quad (4.11)$$

which gives

$$\begin{aligned} K^b_{2,\mu}(x) &= -\int_0^\infty ds\, f^{bcd}\partial_\mu(\partial^2)^{-1} \\ &\quad \times\{\exp(-Ms)[\partial^2,M] \\ &\quad \times \exp(-Ms)M^{-1}\}^{cd}(x,y)|_{y=x}, \end{aligned} \quad (4.12)$$

where $M=M(A^{\rm tr})$. The "drift force" K_2 is a novel term. Its presence is required to correct the overcounting, discussed in Sec. II, that occurs when the Faddeev-Popov theory is cut off at the Gribov horizon.[4]

[4]In our derivation we used the Born-Oppenheimer method that is nonperturbative in g in order to obtain the $a\to 0$ limit at finite g. So the presence of the new term K_2 is not in contradiction with the fact that the Faddeev-Popov theory provides a formal perturbative expansion that has all the correct properties, including perturbative unitarity.

V. SCHWINGER-DYSON EQUATIONS

The partition function is defined by

$$Z(J)=\int dA\, Q(A^{\rm tr})\,\delta(\partial_\mu A_\mu)\exp(J,A)$$
$$=\int dA^{\rm tr}Q(A^{\rm tr})\exp(J,A^{\rm tr}). \tag{5.1}$$

It depends only on the transverse component $J_\mu^{\rm tr}$ of J_μ (on-shell gauge condition), and we write $Z=Z(J^{\rm tr})$. Generally, in the Faddeev-Popov approach, one relaxes the transversality condition, by writing $\delta(\partial_\mu A_\mu)=\int db\, \exp(i\int d^4 xb\partial_\mu A_\mu)$, and then one uses Slavnov-Taylor identities to determine longitudinal parts of vertices. However, these identities have not yet been derived in the present four-dimensional stochastic approach, and we shall solve the SD equations using the on-shell formalism for the gauge condition. The on-shell correlation functions, such as propagators, are the same as the off-shell ones, but the vertices (one-particle irreducible functions) are strictly transverse. Renormalization theory is not well articulated at present in the on-shell formalism, but we shall not encounter ultraviolet divergences in the SD equations in the infrared limit. Moreover, we shall see that in this limit the SD equations are invariant under the renormalization group.[5]

The partition function $Z(J^{\rm tr})$, which is the generating functional of (transverse) correlation functions, is the Fourier transform of the probability distribution $Q(A^{\rm tr})$. Consequently the SD equation for $Z(J^{\rm tr})$ is simply the diffusion equation (4.6), expressed in terms of the Fourier-transformed variables,

$$\int d^4x J_\mu^{\rm tr}\left[J_\mu^{\rm tr}+K_{{\rm tot\ eff},\mu}\left(\frac{\delta}{\delta J^{\rm tr}}\right)\right]Z(J^{\rm tr})=0. \tag{5.2}$$

Here we have introduced the total effective (tot eff) drift force

$$K_{{\rm tot\ eff},\mu}(A^{\rm tr})\equiv-\frac{\delta S_{\rm YM}}{\delta A_\mu}(A^{\rm tr})+K_{{\rm gt\ eff},\mu}(A^{\rm tr}). \tag{5.3}$$

Only the transverse component of $K_{{\rm tot\ eff},\mu}$ appears in the following. The free energy $W(J^{\rm tr})\equiv\ln Z(J^{\rm tr})$, which is the generating functional of connected correlation functions, satisfies the SD equation

$$\int d^4x\, J_\mu^{\rm tr}\left[J_\mu^{\rm tr}+K_{{\rm tot\ eff},\mu}\left(\frac{\delta W(J^{\rm tr})}{\delta J^{\rm tr}}+\frac{\delta}{\delta J^{\rm tr}}\right)\right]=0. \tag{5.4}$$

The effective action is obtained by Legendre transformation, $\Gamma(A^{\rm tr})=(J^{\rm tr},A^{\rm tr})-W(J^{\rm tr})$ by inverting $A^{\rm tr}=\delta W/\delta J^{\rm tr}$. It satisfies the SD equation

$$\int d^4x\,\frac{\delta T(A^{\rm tr})}{\delta A_\mu^{\rm tr}}\left[\frac{\delta\Gamma(A^{\rm tr})}{\delta A_\mu^{\rm tr}}+K_{{\rm tot\ eff},\mu}\left(A^{\rm tr}+\mathcal{D}(A^{\rm tr})\,\frac{\delta}{\delta A^{\rm tr}}\right)\right]=0, \tag{5.5}$$

where the argument of $K_{\rm tot\ eff}$ is written in matrix notation, and is given explicitly by $A_\mu^{\rm tr}(x)+\int d^4y\mathcal{D}_{\mu\nu}(x,y;A^{\rm tr})[\delta/\delta A_\nu^{\rm tr}(y)]$. Here $\mathcal{D}_{\mu\nu}(x,y;A^{\rm tr})=[\delta A_\nu^{\rm tr}(y)/\delta J_\nu^{\rm tr}(x)]$ is the gluon propagator in the presence of sources.

To obtain the SD equation for the propagator, we expand in powers of $A^{\rm tr}$,

$$\frac{\delta\Gamma(A^{\rm tr})}{\delta A_\mu^{\rm tr}}=(D^{-1}A^{\rm tr})_\mu+\cdots,$$

$$K^{\rm tr}_{{\rm tot\ eff},\mu}\left(A^{\rm tr}+\mathcal{D}(A^{\rm tr})\,\frac{\delta}{\delta A^{\rm tr}}\right)=-(RA^{\rm tr})_\mu+\cdots, \tag{5.6}$$

where we have again used matrix notation. Here $D_{\mu\nu}=D_{\mu\nu}(x-y)$ is the gluon propagator in the absence of sources, and

$$R_{\mu\nu}(x-y)\equiv-\left[\frac{\delta}{\delta A_\nu^{\rm tr}(y)}K^{\rm tr}_{{\rm tot\ eff},\mu}\right.$$
$$\left.\times\left(A^{\rm tr}+\mathcal{D}(A^{\rm tr})\,\frac{\delta}{\delta A^{\rm tr}}\right)(x)\right]\Bigg|_{A^{\rm tr}=0}. \tag{5.7}$$

Both D and R are identically transverse, and in momentum space, by virtue of Lorentz invariance, are of the form $D_{\mu\nu}(k)=D(k^2)[\delta_{\mu\nu}-k_\mu k_\nu/k^2]$ and $R_{\mu\nu}(k)=R(k^2)[\delta_{\mu\nu}-k_\mu k_\nu/k^2]$. Upon equating terms quadratic in $A^{\rm tr}$ in Eq. (5.5) we obtain

$$\int d^4x[D^{-1}A_\mu^{\rm tr}](x)[(D^{-1}-R)A_\mu^{\rm tr}](x)=0, \tag{5.8}$$

which we write in matrix notation as

$$(D^{-1}A^{\rm tr},[D^{-1}-R]A^{\rm tr})=0. \tag{5.9}$$

This holds identically in $A^{\rm tr}$. From the expressions for $D_{\mu\nu}(k)$ and $R_{\mu\nu}(k)$, we see that both are symmetric operators that commute, $DR=RD$. As a result, the operator appearing in the last equation is symmetric and must vanish,

$$D^{-1}(D^{-1}-R)=0. \tag{5.10}$$

This gives the SD equation for the gluon propagator

$$D^{-1}=R, \tag{5.11}$$

where R is given in Eq. (5.7).

[5]It should also be noted that it is not known at present how to maintain the Slavnov-Taylor identities exactly at the nonperturbative level in the off-shell formalism, although methods for dealing with this have been proposed [25].

VI. SOLUTION OF SD EQUATION IN THE INFRARED

Recall the decomposition $K_{\rm gt\,eff}=K_1+K_2$, where $K_1=\delta(\mathrm{tr}\ln M)/\delta A^{\rm tr}(x)$ is the drift force that, in the absence of K_2, describes the Faddeev-Popov theory in the Landau gauge. Since it is not without interest to solve the Faddeev-Popov theory nonperturbatively in the Landau gauge, and in order to compare our results with other authors, we shall here ignore K_2, the novel term that corrects the overcounting that occurs when the Faddeev-Popov theory is cut off at the Gribov horizon.

The remaining drift force, K_1, describes Faddeev-Popov theory in the Landau gauge. We have seen in Sec. II that there is an ambiguity in the solution of the SD equations of the Faddeev-Popov theory, with no clear prescription to resolve it at the nonperturbative level. Fortunately the present derivation provides the additional information that is needed to resolve this ambiguity: we must choose the solution of the SD equations that vanishes outside the Gribov horizon because, as we have seen, $Q(A^{\rm tr})$ vanishes outside the Gribov horizon in the limit $a\to 0$. With this choice it is likely that qualitative features of the exact theory (with K_2) will be preserved.

With neglect of K_2 we may write directly the familiar SD equations of the Faddeev-Popov theory in Landau gauge, in an arbitrary number d of Euclidean dimensions

$$D^{-1}_{\mu\nu}(k)=(\delta_{\mu\nu}k^2-k_\mu k_\nu)+Ng^2(2\pi)^{-d}\int d^dp\, G(p+k)\times(p+k)_\mu G(p)\Gamma_\nu(p,k)+(\text{gluon loops}), \quad (6.1)$$

$$G^{-1}(p)=p^2-Ng^2(2\pi)^{-d}\int d^dk\, G(k+p)\times(k+p)_\mu D_{\mu\nu}(k)\Gamma_\nu(p,k), \quad (6.2)$$

where $G(p)$ is the ghost propagator, $\Gamma_\nu(p,k)$ is the full ghost-ghost-gluon vertex. In the Landau gauge, a factorization of the external ghost momentum occurs, so the ghost-ghost-gluon vertex is of the form $\Gamma_\nu(p,k)=\Gamma_{\nu,\lambda}(p,k)p_\lambda$. As a result there is no independent renormalization of $\Gamma_\mu(p,k)$, and the renormalization constants in Landau gauge are related by $Z_g^2Z_3\tilde{Z}_3^2=1$, where $g_0=Z_gg_r$, $D_0=Z_3D_r$, and $G_0=\tilde{Z}_3G_r$. So far, we have written the SD equations for unrenormalized quantities, with the index 0 suppressed.

We must select the solution to these equations that corresponds to a probability distribution $Q(A^{\rm tr})$ that vanishes outside the Gribov horizon. To do so, it is sufficient to impose any property that holds for this distribution, provided only that it determines a unique solution of the SD equations. Besides positivity, which will be discussed in the concluding section, there are two exact properties that hold for a probability distribution $P(A^{\rm tr})$ that vanishes outside the Gribov horizon: (i) the horizon condition and (ii) the vanishing of the gluon propagator at $k=0$ [17][6]

$$\lim_{k\to 0} D(k)=0. \quad (6.3)$$

The horizon condition (i) is equivalent to the statement that $G(p)$ diverges more rapidly than $1/p^2$, or

$$\lim_{p\to 0}[p^2G(p)]^{-1}=0. \quad (6.4)$$

Indeed if we divide the SD equation (6.2) by p^2, and impose this condition, we obtain

$$\delta_{\mu\lambda}=Ng^2(2\pi)^{-d}\int d^dk\, G(k)D_{\mu\nu}(k)\Gamma_{\nu\lambda}(0,k). \quad (6.5)$$

This is the nonperturbative statement that the ghost self-energy, which is of the form $\Sigma(p)=p_\mu\Sigma_{\mu\lambda}(p)p_\lambda$ because of the factorization of the external ghost momentum, exactly cancels the tree level term at $p=0$,

$$\delta_{\mu\lambda}=\Sigma_{\mu\lambda}(0). \quad (6.6)$$

Equations (6.5) and (6.6) are the form of the horizon condition given in [26–28].[7] We will see that it is sufficient to apply either condition (i) or (ii), and the other condition then follows automatically. The horizon condition allows us to write the SD equation for the ghost propagator, (6.2), in the form

$$G^{-1}(p)=Ng^2(2\pi)^{-d}\int d^dk\, p_\mu D_{\mu\nu}(k)\times[\Gamma_{\nu,\lambda}(0,k)G(k)-\Gamma_{\nu,\lambda}(p,k)G(k+p)]p_\lambda, \quad (6.7)$$

where we have used $k_\mu D_{\mu\nu}(k)=0$. This equation was solved numerically in three-dimensions in [28], using an assumed form for $D(k)$.

We wish to determine the asymptotic (as) form of the propators at low momentum, $G^{\rm as}(p^2)$, and $D^{\rm as}_{\mu\nu}(k)=D^{\rm as}(k^2)P^{\rm tr}_{\mu\nu}(k)$, where $P^{\rm tr}_{\mu\nu}(k)=\delta_{\mu\nu}-k_\mu k_\nu/k^2$ is the transverse projector. For this purpose we let the external momenta in the SD equations be asymptotically small compared to QCD mass scales. In this case the loop integration will be dominated by asymptotically small loop momenta, so the propagators inside the integrals may also be replaced by their asymptotic values. This is true provided that the resulting integrals converge, as will be verified. We shall also truncate the SD equations by neglecting transverse vertex corrections, as usual, in order to obtain a closed system of equations, $\Gamma^{\rm tr}_\nu(p,k)\to P^{\rm tr}_{\nu\mu}(k)p_\mu$. Such truncations may, possibly, be

[6]The vanishing of the gluon propagator at $k=0$ results from the proximity of the Gribov horizon in infrared directions.

[7]In a space of high dimension N the probability distribution within a smooth surface such as a sphere $r<R$ gets concentrated near the surface $r=R$ because of the entropy or phase-space factor $r^{N-1}dr$. The horizon condition is the statement that the probability distribution within the Gribov horizon is concentrated on the Gribov horizon because the dimension N of A space diverges with the volume V in, say, a lattice discretization.

justified *a posteriori* by calculating corrections to see if they are small. Because $D_{\mu\nu}(k)$ is transverse, the SD equation for the ghost propagator simplifies to

$$(G^{\rm as})^{-1}(p^2)=Ng^2(2\pi)^{-d}\int d^dk(k^2)^{-1}\times[p^2k^2-(p\cdot k)^2]D^{\rm as}(k^2)\times\{G^{\rm as}(k^2)-G^{\rm as}[(k+p)^2]\}. \quad (6.8)$$

This equation is invariant under renormalization because of the identity $Z_g^2Z_3\tilde{Z}_3^2=1$. This allows us to take all quantities in Eq. (6.8) to be renormalized ones, with suppression of the index r.

Because the asymptotic infrared limit is a critical limit, the asymptotic propagators obey simple power laws,

$$D^{\rm as}(k^2)=c_D\mu^{2\alpha_D}(p^2)^{-(1+\alpha_D)},$$
$$gG^{\rm as}(p^2)=c_G\mu^{2\alpha_G+(4-d)/2}(p^2)^{-(1+\alpha_G)}, \quad (6.9)$$

according to standard renormalization-group arguments. Here α_D and α_G are infrared critical exponents or anomalous dimensions that we shall determine, while μ is a mass scale, and c_D and c_G are dimensionless parameters. The horizon condition (6.4) implies $\alpha_G>0$, whereas Eq. (6.3) implies $\alpha_D<-1$. Upon changing the integration variable according to $k_\mu=|p|k'_\mu$, and equating like powers of p we obtain

$$\alpha_D+2\alpha_G=-(4-d)/2. \quad (6.10)$$

The integral is an ultraviolet convergent for $d-2(1+\alpha_D)-2(1+\alpha_G)-2<0$, where two powers of k are gained because of the difference $\{G^{\rm as}(k)^2-G^{\rm as}[(k+p)^2]\}$. With $\alpha_D=-2\alpha_G-(4-d)/2$, this gives $\alpha_G<1$ as the condition for ultraviolet convergence, so $0<\alpha_G<1$.

We now turn to the SD equation for the gluon propagator (6.1). In the exact Faddeev-Popov theory with off-shell gauge condition, the right-hand side of Eq. (6.1) is exactly transverse in k on both free Lorentz indices μ and ν by virtue of the Slavnov-Taylor identities. This allows us to apply transverse projectors $P^{\rm tr}_{\mu'\mu}(k)=\delta_{\mu\nu}-k_\mu k_\nu/k^2$ and $P^{\rm tr}_{\nu'\nu}(k)$ to these indices. In our derivation, with the on-shell gauge condition, the projectors are automatically applied. As a result, since the gluon propagators are transverse, only the transverse parts of the vertices contribute on the right-hand side. We therefore make the truncation approximation of replacing these transverse vertices by their tree-level expressions. We now estimate the various terms on the right-hand side of the SD equation (6.1) for $D(k)$. We just concluded from the horizon condition and the SD equation for $G(p)$ that $\alpha_G>0$ and $\alpha_D<0$. As a result, on the right-hand side of Eq. (6.1), the ghost loop that we have written explicitly is more singular in the infrared than the gluon loops. Moreover, in the infrared, $D^{-1}(k)\sim(k^2)^{(1+\alpha_D)}$ is more singular at $k=0$ than the tree-level term $\sim k^2$ because $\alpha_D<0$. We now let the external momentum k have an asymptotically small value, so the loop integration is dominated by asymptotically small values of the integration variable p (provided the resulting integral converges). We take the asymptotic infrared limit of Eq. (6.1) with external projectors and obtain

$$[D^{\rm as}(k^2)]^{-1}P^{\rm tr}_{\mu\nu}(k)=Ng^2(2\pi)^{-d}P^{\rm tr}_{\mu\lambda}(k)\int d^dp\, p_\lambda G^{\rm as}[(k+p)^2]\times G^{\rm as}(p^2)p_\kappa P^{\rm tr}_{\kappa\nu}(k). \quad (6.11)$$

We take the trace on Lorentz indices and obtain

$$[D^{\rm as}(k^2)]^{-1}=Ng^2(2\pi)^{-d}[(d-1)k^2]^{-1}\times\int d^dp[p^2k^2-(p\cdot k)^2]\times G^{\rm as}[(k+p)^2]G^{\rm as}(p^2). \quad (6.12)$$

Like the ghost equation (6.8), this equation is invariant under renormalization because of the identity $Z_g^2Z_3\tilde{Z}_3^2=1$, and we may again take all quantities to be renormalized with suppression of the index r. We substitute the power laws (6.10) into this equation. By using the power-counting argument that was used for the ghost propagator, we again obtain the relation of the infrared critical exponents $\alpha_D+2\alpha_G=-(4-d)/2$. This integral converges in the ultraviolet for $d-2<4(1+\alpha_G)$, or $\alpha_G>(d-2)/4$.

The gluon and ghost SD equations now read

$$(c_Dc_G^2)^{-1}=I_D(\alpha_G)=I_G(\alpha_G), \quad (6.13)$$

where

$$I_D(\alpha_G)\equiv N(2\pi)^{-d}(d-1)^{-1}(k^2)^{-(2+\alpha_D)}\times\int d^dp[p^2k^2-(p\cdot k)^2]\times[(k+p)^2]^{-(1+\alpha_G)}[(p)^2]^{-(1+\alpha_G)}, \quad (6.14)$$

$$I_G(\alpha_G)\equiv N(2\pi)^{-d}(p^2)^{-(1+\alpha_G)}\times\int d^dk[p^2k^2-(p\cdot k)^2](k^2)^{-(2+\alpha_D)}\times\{(k^2)^{-(1+\alpha_G)}-[(k+p)^2]^{-(1+\alpha_G)}\}, \quad (6.15)$$

and it is understood that $\alpha_D\equiv-2\alpha_G-(4-d)/2$. The critical exponent α_G is determined by the equality (6.13). The integrals $I_D(\alpha_G)$ and $I_G(\alpha_G)$ are evaluated in the Appendix, without angular approximation, in arbitrary Euclidean dimension d.

VII. DETERMINATION OF INFRARED CRITICAL EXPONENTS

To determine the critical exponent α_G, we substitute the formulas for $I_D(\alpha_G)$ and $I_G(\alpha_G)$, given in the Appendix, into the equation $I_D(\alpha_G)=I_G(\alpha_G)$ and obtain, for $\alpha\equiv\alpha_G$,

$$f_d(\alpha) \equiv \frac{(d-1)\pi}{\sin(\pi\alpha)} \frac{\Gamma(1+2\alpha)}{\Gamma(-2\alpha+d/2)\Gamma(1+\alpha+d/2)} \times \frac{\Gamma(d-2\alpha)}{\Gamma(-\alpha+d/2)\Gamma(1+2\alpha-d/2)} = 1. \tag{7.1}$$

We take the dimension d of space-time in the interval $2 \leqslant d \leqslant 4$. The integrals $I_D(\alpha)$ and $I_D(\alpha)$ are both convergent in the ultraviolet only for α in the interval $0<(d-2)/4<\alpha<1$, so the equation which determines α holds only in this interval. However, whereas $I_D(\alpha)$ is manifestly positive throughout this interval, the expression for $I_G(\alpha)$ is negative for $\alpha > d/4$, because $1/\Gamma(-2\alpha+d/2)$ changes sign at $\alpha = d/4$, so we look for a solution only in the reduced interval

$$0 \leqslant (d-2)/4 \leqslant \alpha \leqslant d/4 \leqslant 1. \tag{7.2}$$

The identity, $\Gamma(-2\alpha+d/2)\Gamma(1+2\alpha-d/2) = \pi/\sin[\pi(-2\alpha+d/2)]$, gives

$$f_d(\alpha) \equiv \frac{(d-1)\sin[\pi(-2\alpha+d/2)]}{\sin(\pi\alpha)} \times \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha+d/2)} \frac{\Gamma(d-2\alpha)}{\Gamma(-\alpha+d/2)} = 1. \tag{7.3}$$

For the case of physical interest, $d=4$, the allowed interval is $1/2 \leqslant \alpha \leqslant 1$, and the function $f_4(\alpha)$ contains the factor $\sin[\pi(-2\alpha+2)]/\sin(\pi\alpha) = \sin[\pi(-2\alpha)]/\sin(\pi\alpha)$ which is of the indeterminate form 0/0 at $\alpha=1$. To control this (and a similar indeterminacy for $d=2$ at $\alpha=0$), we first consider d in the range $2<d<4$, and then take the limit $d\to 4$ (and $d\to 2$). For d in this range, one sees that the function $f_d(\alpha)$ is *positive* and finite, $f_d(\alpha)>0$, for α in the interior of the allowed interval $(d-2)/4<\alpha<d/4$, but vanishes at *both* end points $f_d[(d-2)/4] = f_d(d/4) = 0$ because of the factor $\sin[\pi(-2\alpha+d/2)]$. It follows that the equation $f_d(\alpha)=1$ has an *even* number of solutions (if any)[8] for $2<d<4$. We now set $d=4$ and obtain

$$f_4(\alpha) \equiv \frac{-3\sin(2\pi\alpha)}{\sin(\pi\alpha)} \frac{\Gamma(1+2\alpha)\Gamma(4-2\alpha)}{\Gamma(3+\alpha)\Gamma(2-\alpha)} = 1. \tag{7.4}$$

We use

$$\Gamma(1+2\alpha)\Gamma(4-2\alpha) = (3-2\alpha)(2-2\alpha)(1-2\alpha)2\alpha\pi/\sin(2\pi\alpha), \tag{7.5}$$

$$\Gamma(3+\alpha)\Gamma(2-\alpha) = (2+\alpha)(1+\alpha)\alpha(1-ga)\pi/\sin(\pi\alpha),$$

and obtain

$$f_4(\alpha) = 12\frac{(3-2\alpha)(2\alpha-1)}{(2+\alpha)(1+\alpha)} = 1, \tag{7.6}$$

where we have used $[\sin(2\pi\alpha)/\sin(\pi\alpha)][\sin(\pi\alpha)/\sin(2\pi\alpha)] = 1$, which is valid only for $1/2<\alpha<1$. This yields a quadratic equation with roots $\alpha = [93 \pm \sqrt{(1201)}]/98 \approx [93 \pm 34.66]/98$. Only *one* root $\alpha \approx 0.5953$ lies in the interval $1/2<\alpha<1$. On the other hand, we have just seen that for $2<d<4$, there is an *even* number of roots. The resolution is that for $d=4-\epsilon$ there are two roots, and the second root is given by $\alpha = 1-O(\epsilon)$, so in the limit $d\to 4$, there is a second root at $\alpha=1$.

We conclude that the infrared critical exponents $\alpha=\alpha_G$ and $\alpha_D = -2\alpha_D-(4-d)/2$ are given, in $d=4$ dimensions, by two possible sets of values

$$\alpha_G = 1, \quad \alpha_D = -2,$$
$$\alpha_G = [93-\sqrt{(1201)}]/98 \approx 0.5953, \tag{7.7}$$
$$\alpha_D = -[93-\sqrt{(1201)}]/49 \approx -1.1906.$$

In the same way one finds for $d=2$,

$$\alpha_G = 0, \quad \alpha_D = -1,$$
$$\alpha_G = 1/5, \quad \alpha_D = -7/5. \tag{7.8}$$

For $d=3$, one obtains the equation,

$$f3(\alpha) = \frac{32\alpha(1-\alpha)[1-\cot^2(\pi\alpha)]}{(3+2\alpha)(1+2\alpha)} = 1$$

with roots in the interval $1/4 \leqslant \alpha \leqslant 3/4$, given by

$$\alpha_G = 1/2, \quad \alpha_D = -3/2$$
$$\alpha_G \approx 0.3976, \quad \alpha_D \approx -1.2952. \tag{7.9}$$

We expect that in each case one of the roots is spurious, and arises because Eq. (7.3) does not express the full content of the theory.

We note that in each case one solution corresponds to $\alpha_G = (d-2)/2 = 0$, 1/2, and 1, for $d=2$, 3, and 4, which gives $G(k) \sim 1/(k^2)^{d/2}$. This may be too infrared singular to be acceptable. But for $d=2$, the other solution, with $\alpha_G = 1/5$, is even more infrared singular, which suggests that for $d=2$ the first solution may be preferred namely, $\alpha_D=-1$ and $\alpha_G=0$, which may make the case $d=2$ pathological in the Landau gauge. This case is exactly solvable in the axial gauge because the nonlinear term in the $d=2$ Yang-Mills field is absent in this gauge, and gives an area law at the classical level. There can, of course, be no physical gluons in $d=1+1$ dimensions even in the free theory which may thus

[8] From numerical plots it appears that for $2<d<4$ there are always two distinct real roots in the range $(d-2)/4<\alpha<d/4$, except possibly near $d\approx 2.662$ where there may be a double root near $\alpha \approx 0.33095$.

be considered confining. Clearly the case $d=2$ in the Landau gauge requires a more detailed investigation that we do not attempt here.

VIII. DISCUSSION AND CONCLUSION

We have seen that because the Faddeev-Popov weight $P_{\mathrm{FP}}(A)$ contains nodal surfaces, the SD equations corresponding to the Faddeev-Popov method are ambiguous, and in practice one does not know how to select an exact and globally correct solution. Gribov's proposal, to cut off the Faddeev-Popov integral at the first nodal surface, produces a positive probability distribution, but it is not exact because it overcounts some gauge orbits, although it may give a useful approximation.

By contrast the method of stochastic quantization bypasses the Gribov problem of selecting a single representative in each gauge orbit. Instead the diffusion equation in A space, Eq. (3.6), contains an additional "drift force" $a^{-1}D_\mu \partial \cdot A$ that is a harmless generator of a gauge transformation. The corresponding DS equation that defines the nonperturbative Landau gauge was obtained by solving the limit $a\to 0$ of this equation by the Born-Oppenheimer method. The limiting probability distribution $Q(A^{\mathrm{tr}})$ was shown to vanish outside the Gribov horizon. It is determined by a diffusion equation that contains the novel term K_2, Eq. (4.9), that corrects the Faddeev-Popov distribution cutoff at the Gribov horizon for overcounting inside the Gribov horizon.

[We may mention here an alternative approach. The Landau gauge is the singular limit $a\to 0$ of more regular gauges, and contains a nonlocal effective drift force $K_{\mathrm{gt\ eff}}$, Eq. (4.9). For this reason it may be preferable to calculate with gauge parameter a finite, so the drift force, $K_\mu = -\delta S_{\mathrm{YM}}/\delta A_\mu + a^{-1}D_\mu \mu \cdot A$, remains local, and there is no horizon outside of which the probability distribution vanishes exactly. In this case the SD equation (5.5) for the effective action Γ gets replaced by

$$\int d^4x \, \frac{\delta\Gamma(A)}{\delta A_\mu}\left[\frac{\delta\Gamma(A)}{\delta A_\mu} + K_\mu\left(A + \mathcal{D}(A)\,\frac{\delta}{\delta A}\right)\right] = 0. \tag{8.1}$$

The gluon propagator is given by $D^{-1}=R$, where $R = -(\delta/\delta A)K[A+\mathcal{D}(A)(\delta/\delta A)]|_{A=0}$, as in Eq. (5.7). One would hope to solve the SD equations for the full propagators in this approach, and not just their infrared asymptotic limit. An advantage of this approach is that the solution for a finite value of the gauge parameter a could be directly compared with the numerical lattice data of [7] and [8] that is taken with stochastic gauge fixing and gauge parameter $a = 0.1$. To control ultraviolet divergences, it will be necessary to develop Ward-type identities appropriate to this scheme. They were not needed in the present calculation because no ultraviolet divergences appeared in the infrared limit. Such identities in the BRST form are available in the five-dimensional scheme that is based on the time-dependent diffusion equation [22,23,24], and alternatively one may attempt to solve the SD equations of the five-dimensional scheme nonperturbatively.]

In the second part of the article, where we calculated the infrared critical exponents, we have, however, ignored the new term K_2 in order to compare with other authors, and because it is not without interest to calculate the infrared critical exponents nonperturbatively in the Faddeev-Popov theory with a cutoff at the Gribov horizon.

It is noteworthy that all our values for the critical exponents in $d=2$, 3, and 4 dimensions agree with the exact results for a probability distribution that is cutoff at the Gribov horizon namely, the vanishing [17] of the gluon propagator $D(k)\to 0$ as $k\to 0$, and the enhancement [26,27,28] of the ghost propagator $[k^2G(k)]^{-1}\to 0$ (except for the first solution in $d=2$ which is marginal, with $\alpha_G=0$ and $a_D = -1$). The vanishing of $D(k)$ at $k=0$ is counterintuitive, and has no other explanation than the proximity of the Gribov horizon in infrared directions. This suppresses the infrared components $A(k)$ of the gluon field, and thus of the gluon propagator $D(k)=\langle |A(k)|^2\rangle$. Since our calculation involves a truncation of the SD equations which is an uncontrolled approximation, the stability of our results should be tested by estimating corrections. As for the future, an immediate challenge is to include the effect of the new term K_2, Eq. (4.9), that was not evaluated in the present calculation. One must also introduce quarks.

We wish to compare our values of the infrared asymptotic dimensions with those reported by [13,14], and [15]. But first we must verify whether they also selected the solution of the SD equations of Faddeev-Popov theory that vanishes outside the Gribov horizon. Note that to obtain a particular solution it is sufficient to require any one of its properties, provided that this requirement selects a unique solution. Indeed a unique solution was obtained in [13] by requiring that both the gluon and ghost propagators $D(k)$ and $G(k)$ be positive. These properties by no means follow from the Faddeev-Popov weight (2.1) that oscillates in sign, whereas restriction to the Gribov region does imply the positivity of both $G(k)$ and $D(k)$. So in fact the restriction to the Gribov region is also implemented in this way in [13]. Likewise the assumptions made in [14] and [15] to obtain a solution of the SD equations are equivalent to the horizon condition, Eq. (6.4), that we imposed in Sec. VI.

It is reassuring that the values given in Eq. (7.7) for $d=4$ agree qualitatively with the values reported in [13], namely $\alpha_G=[61-\sqrt{(1897)}]/19\approx 0.92$ and $\alpha_D = -2\alpha_G \approx -1.84$, in the sense that the gluon propagator $D(k)\sim 1/(k^2)^{1+a_D}$ vanishes at $k=0$, and the ghost propagator $G(k)\sim 1/(k^2)^{1+a_G}$ is enhanced. This may be an indication that these qualitative features of the solution are not merely an artifact of the approximations made. For the two treatments of the SD equations are quite different. Indeed in [13], the gauge condition is treated off-shell, by imposing the Slavnov-Taylor identities to determine longitudinal parts of vertices, and by using the method of [25] to adjust the gluon propagator. On the other hand, we have treated the gauge condition on shell, so only transverse quantities occur. There is a similar qualitative agreement for $d=4$ with the values reported in [14], $a_G=[77-\sqrt{(2281)}]/38\approx 0.769479$, and $a_G = -2a_G$, where an angular approximation was made.

The approximations made in [15] appear to be similar to ours, although the method of solution is quite different. The value reported there for $d=4$, $a_G=1$ and $a_D=-2$, agrees with our first solution.[9]

We also wish to compare our results with numerical Monte Carlo studies of propagators in Landau gauge. Numerical gauge fixing to the Landau gauge is achieved by minimizing, with respect to gauge transformations, the lattice analog of $F_A(g)=\int d^dx|{}^gA|^2$, which indubitably produces configurations that lie inside the Gribov horizon. This gauge fixing, like stochastic gauge fixing, has a Euclidean weight that is everywhere positive, without overcounting. However, it is not in the class of Faddeev-Popov gauges for which the determinent alternates in sign, so a comparison with analytic calculations by the Faddeev-Popov method does not have a completely clear interpretation.

The infrared behavior of the lattice propagators is very sensitive to finite-volume effects, and control of the volume dependence at fixed $\beta=2N/g_0^2$ is required. In particular $D(k)$ does not and should not vanish at $k=0$ at any finite lattice volume, but only when extrapolated to infinite volume. We have not attempted here to fit the data of [7] and [8] without an estimate of the finite-volume correction and the effect of the finite gauge parameter, but this is a promising avenue for future comparison of numerical and analytic results. However, we do note that it was reported in [7], with stochastic gauge fixing at gauge parameter $a=0.1$ (with Landau gauge at $a=0$), that a fit to the Gribov formula, $D(k)=Zk^2[(k^2)^2+M^4]^{-1}$, (strong infrared suppression) can explain the gross feature of the data. Recent studies in the Landau gauge at finite lattice volume indicate a suppression of the gluon propagator in the infrared [30], and are not incompatible with an enhancement of the ghost propagator [31]. The infrared behavior of the lattice gluon propagator $D(k)$ has been studied in SU(2) gauge theory in the Landau gauge in $d=3$ Euclidean dimensions [32]. It was found that $D(k)$ has a maximum at $k\approx 350$ MeV (normalized to the physical value of the string tension) that is practically β independent, and that $D(k)$ decreases as k decreases below this value. This decrease is interpreted as resulting from the proximity of the Gribov horizon in infrared directions. A similar behavior is expected for the three-dimensionally transverse part of the gluon propagator in the Coulomb gauge, in four Euclidean dimensions. This has been observed, and an extrapolation to infinite lattice volume at fixed β was in fact found, notably, to be consistent with the *vanishing* of $D(k)$ at $k=0$ [33] and [34]. We emphasize that this behavior is not seen at finite lattice volume but only in the extrapolation to infinite lattice volume, at fixed β. For this reason it is important to extend the lattice calculations in Landau, Coulomb, and stochastic gauges to larger volumes, and to extrapolate to infinite lattice volume before attempting a fit to continuum formulas.

[9]After the completion of this article, L. von Smekal kindly informed me that the first value, $\alpha_G=[93-\sqrt{(1201)}]/98$ in $d=4$ dimensions, was also obtained by C. Lerche [29].

So what have we learned about propagators and the confinement problem in QCD? We may summarize results qualitatively by the statement that in the infrared region in nonperturbative Landau gauge there is strong suppression or vanishing of the would-be physical gluon propagator, and strong enhancement of unphysical propagators.[10] This is true both for the analytic solutions of the Schwinger-Dyson equations obtained in [13,16,15,29], and here, and for the numerical lattice data just discussed, with similar numerical data in Coulomb gauge. We expect that these qualitative features will stand the test of time. They provide a simple intuitive picture of confinement in which the suppressed massless physical gluon disappears from the physical spectrum while the enhanced unphysical components provide a long-range color-confining force. (This long-range force should also confine quarks, but that has not been addressed here.) As discussed previously [35] both features may be understood as the result of the restriction to the Gribov region, which results from the identification of gauge-equivalent configurations. The infrared suppression of the transverse gluon propagator results from the proximity of the Gribov horizon in infrared directions, while the enhancement of the unphysical components is an entropy effect that results from high population in the neighborhood of the Gribov horizon, where the inverse Faddeev-Popov operator is enhanced.

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APPENDIX: EVALUATION OF INTEGRALS

To evaluate the gluon self-energy $I_D(\alpha_G)$, Eq. (6.14), we write

$$1/[(p-k)^2]^{1+\alpha_G}=\Gamma^{-1}(1+\alpha_G)\int_0^\infty dx\, x^{\alpha_G} \times\exp[-x(p-k)^2], \tag{A1}$$

and similarly for $1/[(p)^2]^{1+\alpha_G}$. This gives

$$I_D(\alpha_G)\equiv N[(d-1)(k^2)^{(2+\alpha_D)}\Gamma^2(1+\alpha_G)]^{-1} \times\int_0^\infty dx\int_0^\infty dy(xy)^{\alpha_G}J, \tag{A2}$$

where

[10]The "unphysical propagator" that is infrared enhanced may be either the ghost propagator in the nonperturbative Landau-gauge Faddeev-Popov theory, or the 44-component of the gluon propagator in Coulomb gauge. The ghost propagator coincides, approximately, with the remnant of the longitudinal gluon propagator that survives the Landau-gauge limit in stochastic quantization.

$$J\equiv(2\pi)^{-d}\int d^dp[p^2k^2-(p\cdot k)^2]\times\exp[-x(p-k)^2-yp^2]=(d-1)k^2[2(4\pi)^{d/2}(x+y)^{1+d/2}]^{-1}\times\exp[-(x+y)^{-1}xyk^2]. \tag{A3}$$

We introduce the identity $1=\int d\gamma\delta(x+y-\gamma)$ and change variable according to $x=\gamma x'$ and $y=\gamma y'$. This gives, after dropping primes,

$$I_D(\alpha_G)=N(k^2)^{-(1+\alpha_D)}\times[2(4\pi)^{d/2}\Gamma^2(1+\alpha_G)]^{-1}K, \tag{A4}$$

$$K\equiv\int_0^\infty dx\int_0^\infty dy\int_0^\infty d\gamma\,\delta(x+y-1)(xy)^{\alpha_G}\times\gamma^{2\alpha_G-d/2}\exp[-\gamma xyk^2]=(k^2)^{-2\alpha_G-1+d/2}\times\frac{\Gamma(2\alpha_G+1-d/2)\Gamma^2(-\alpha_G+d/2)}{\Gamma(d-2\alpha_G)}. \tag{A5}$$

This gives

$$I_D(\alpha_G)=\frac{N}{2(4\pi)^{d/2}}\times\frac{\Gamma(2\alpha_G+1-d/2)\Gamma^2(-\alpha_G+d/2)}{\Gamma^2(1+\alpha_G)\Gamma(d-2\alpha_G)}, \tag{A6}$$

where we used $\alpha_D=-2\alpha_G-(4-d)/2$.

To evaluate the ghost self-energy, $I_G(\alpha_G)$, Eq. (6.15), we use the identities

$$\frac{1}{[(k)^2]^{2+\alpha_D}}=\frac{1}{\Gamma(2+\alpha_D)}\int_0^\infty dx\,x^{\alpha_D+1}\times\exp(-xk^2), \tag{A7}$$

$$\frac{1}{[(k)^2]^{1+\alpha_G}}-\frac{1}{[(k-p)^2]^{1+\alpha_G}}=\frac{1}{\Gamma(1+\alpha_G)}\int_0^\infty dy\,y^{\alpha_G+1}[(k-p)^2-k^2]\times\int_0^1 dz\exp\{-y[z(k-p)^2+(1-z)k^2]\}, \tag{A8}$$

which allow us to cancel the leading power of k explicitly. This gives

$$I_G(\alpha_G)=\frac{N}{(p^2)^{1+\alpha_G}\Gamma(2+\alpha_D)\Gamma(1+\alpha_G)}\times\int_0^\infty dx\int_0^\infty dy\int_0^1 dz\,x^{\alpha_D+1}y^{\alpha_G+1}L, \tag{A9}$$

where

$$L\equiv(2\pi)^{-d}\int d^dk[p^2k^2-(p\cdot k)^2](p^2-2p\cdot k)\times\exp\{-[x+y(1-z)]k^2-yz(k-p)^2\}, \tag{A10}$$

$$L=\frac{(p^2)^2(d-1)}{2(4\pi)^{d/2}}\frac{x+y-2yz}{(x+y)^{2+d/2}}\times\exp\left(-\frac{yz[x+y(1-z)]p^2}{x+y}\right). \tag{A11}$$

This gives

$$I_G(\alpha_G)=\frac{N(d-1)(p^2)^{1-\alpha_G}}{2(4\pi)^{d/2}\Gamma(2+\alpha_D)\Gamma(1+\alpha_G)}J, \tag{A12}$$

$$J\equiv\int_0^\infty dx\int_0^\infty dy\int_0^1 dz\,x^{\alpha_D+1}y^{\alpha_G+1}\frac{x+y-2yz}{(x+y)^{2+d/2}}\times\exp\left(-\frac{yz[x+y(1-z)]p^2}{x+y}\right). \tag{A13}$$

We again introduce the identity $1=\int d\gamma\delta(x+y-\gamma)$ and change variables according to $x=\gamma x'$ and $y=\gamma y'$. This gives, after dropping primes,

$$J=\int_0^\infty d\gamma\int_0^\infty dx\int_0^\infty dy\int_0^1 dz\,\delta(x+y-1)\times x^{\alpha_D+1}y^{\alpha_G+1}(1-2yz)\times\gamma^{-\alpha_G}\exp\{-yz[x+y(1-z)]\gamma p^2\}, \tag{A14}$$

$$J=(p^2)^{\alpha_G-1}\Gamma(1-\alpha_G)\times\int_0^1 dy\int_0^1 dz\,y^{\alpha_G+1}(1-y)^{\alpha_D+1}(1-2yz)\times[yz(1-yz)]^{\alpha_G-1}, \tag{A15}$$

where we again used $\alpha_D=-2\alpha_G-(4-d)/2$. We change the variable of integration to $u=yz(1-yz)$, with $du=y(1-2yz)dz$, and obtain

$$J=(p^2)^{\alpha_G-1}\alpha_G^{-1}\Gamma(1-\alpha_G)\times\int_0^1 dy\, y^{2\alpha_G}(1-y)^{-\alpha_G-1+d/2}=\frac{(p^2)^{\alpha_G-1}\Gamma(1-\alpha_G)}{\alpha_G}\times\frac{\Gamma(1+2\alpha_G)\Gamma(-\alpha_G+d/2)}{\Gamma(\alpha_G+1+d/2)},\qquad (A16)$$

where we have again used $\alpha_D=-2\alpha_G-(4-d)/2$. This gives

$$I_G(\alpha_G)=\frac{N(d-1)}{2(4\pi)^{d/2}}\frac{\pi}{\sin(\pi\alpha_G)}\times\frac{\Gamma(2\alpha_G+1)\Gamma(-\alpha_G+d/2)}{\Gamma^2(\alpha_G+1)\Gamma(-2\alpha_G+d/2)\Gamma(\alpha_G+1+d/2)},\qquad (A17)$$

where we have used $\Gamma(\alpha_G)\Gamma(1-\alpha_G)=\pi/\sin(\pi\alpha_G)$. This integral is positive for $\alpha_G<d/4$.

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