Precision determination of the pion form factor and calculation of the muon $g-2$

J. F. de Trocóniz and F. J. Ynduráin
Departamento de Física Teórica, C-XI, Universidad Autónoma de Madrid, Canto Blanco, E-28049, Madrid, Spain
(Received 29 June 2001; published 23 April 2002)

We perform a new calculation of the hadronic contributions, a(Hadronic), to the anomalous magnetic moment of the muon, a_μ. For the low-energy contributions of order α^2 we carry over an analysis of the pion form factor $F_\pi(t)$ using recent data both on $e^+e^-\to\pi^+\pi^-$ and $\tau^+\to\overline{\nu}_\tau\pi^+\pi^0$. In this analysis we take into account that the phase of the form factor is equal to that of $\pi\pi$ scattering. This allows us to profit fully from analyticity properties so we can also use experimental information on $F_\pi(t)$ at spacelike t. At higher energy we use QCD to supplement experimental data, including the recent measurements of $e^+e^-\to$hadrons both around 1 GeV and near the $\bar{c}c$ threshold. This yields a precise determination of the $O(\alpha^2)$ and $O(\alpha^2)+O(\alpha^3)$ hadronic part of the photon vacuum polarization $10^{11}\times a^{(2)}(\text{h.v.p.})=6909\pm64$; $10^{11}\times a^{(2+3)}(\text{h.v.p.})=7002\pm66$. As by-products we also get the masses and widths of the ρ^0, ρ^+, and very accurate values for the charge radius and second coefficient of the pion. Adding the remaining order α^3 hadronic contributions we find $10^{11}\times a^{\text{theory}}(\text{hadronic})=6993\pm69$ $(e^+e^-+\tau+\text{spacel})$. The error above includes statistical, systematic, and estimated theoretical errors. The figures given are obtained including τ decay data; if we restrict ourselves to e^+e^- data, slightly lower values and somewhat higher errors are found. This is to be compared with the figure obtained by subtracting pure electroweak contributions from the recent experimental value, obtained from measurements of the muon gyromagnetic ratio $(g-2)$, which reads $10^{11}\times a^{\text{expt}}(\text{hadronic})=7174\pm150$.

DOI: 10.1103/PhysRevD.65.093001 PACS number(s): 12.15.Lk, 12.20.Ds

I. INTRODUCTION

The appearance of a new, very precise measurement of the muon magnetic moment [1] has triggered the interest in *theoretical* calculations of this quantity. Particularly, because the experimental figure (we give the result for the anomaly, averaged with older determinations [2])

$$10^{11}\times a_\mu(\text{expt})=116\,592\,030\pm150 \tag{1.1}$$

lies slightly above theoretical evaluations based on the standard model, as much as 2.6σ in some cases.

It should be noted that all modern[1] theoretical determinations [3–7] are compatible among themselves within errors (of order 100×10^{-11}) and that, with few exceptions, they are also compatible with the experimental result, Eq. (1.1), at the level of 1.5σ or less. Because of this, it is our feeling that a new, *complete* evaluation would be welcome since, in fact, there exists as yet no calculation that takes fully into account all theoretical constraints and all the new experimental data. These experimental data allow an improved evaluation of the low-energy hadronic contributions to a_μ, both directly from e^+e^- annihilations (in the ρ region [8] and around the ϕ resonance [9]) and, indirectly, from τ decays [10] and, also indirectly, from measurements of the pion form factor in the spacelike region [11]. Moreover, the BES [12] data, covering e^+e^- annihilations in the vicinity of the $\bar{c}c$ threshold, permit a reliable evaluation of the corresponding hadronic pieces. In fact, the main improvements of the present paper are the calculation of the two pion contribution to the hadronic part of a_μ, using all available experimental information and fulfilling compatibility with all our theoretical knowledge, and the pinning down of the multipion, KK, and $\bar{c}c$ contributions. This we do in Secs. III and IV (in Sec. II we formulate the problem). In Sec. V we discuss other hadronic corrections, including one that, as far as we know, has been hitherto neglected, and which, though small ($\sim46\times10^{-11}$) is relevant at the level of accuracy for which we are striving. Finally, in Sec. VI we discuss our results and compare them with experiment.

The main outcome of our analysis is an accurate and reliable determination of the hadronic contributions to a_μ at order α^2. In fact, in all regions where there are difficulties we perform at least two evaluations, and take into account their consistency (or lack thereof). Furthermore, we discuss in some detail (including ambiguities) the $O(\alpha^3)$ hadronic contributions.

As a by-product of the low-energy calculations we can also give precise values for the ρ^0,ρ^+ masses and widths,

$$m_{\rho^0}=772.6\pm0.5\ \text{MeV},\quad \Gamma_{\rho^0}=147.4\pm0.8\ \text{MeV},$$
$$m_{\rho^+}=773.8\pm0.6\ \text{MeV},\quad \Gamma_{\rho^+}=147.3\pm0.9\ \text{MeV}, \tag{1.2}$$

for the P-wave $\pi\pi$ scattering length,

$$a_1^1=(41\pm2)\times10^{-3}m_\pi^{-3}, \tag{1.3}$$

and for the pion mean squared charge radius and second coefficient:

$$\langle r_\pi^2\rangle=0.435\pm0.002\ \ \text{fm}^2,\quad c_\pi=3.60\pm0.03\ \text{GeV}^{-4}$$
$$(e^+e^-+\tau+\text{spacelike}),$$

[1] By modern here we mean somewhat arbitrarily, those obtained since 1985. A more complete list of references, including earlier work, may be found in Ref. [7].

$$\langle r_\pi^2\rangle = 0.433 \pm 0.002\ \text{fm}^2,\quad c_\pi = 3.58 \pm 0.04\ \text{GeV}^{-4}\quad (e^+e^- + \text{spacelike}). \tag{1.4}$$

We give results both using only direct data on F_π, from e^+e^- annihilations, or involving also the decay $\tau^+ \to \overline{\nu}_\tau \pi^+\pi^0$, which last we consider to be our best estimates. So we write

$$10^{11}\times a(\text{hadronic}) = \begin{cases} 6993\pm 69 & (e^+e^- + \tau + \text{spacel.}), \\ 6973 \pm 99 & (e^+e^- + \text{spacel.}). \end{cases} \tag{1.5}$$

Note that in a(hadronic) we include *all* hadronic contributions, $O(\alpha^3)$ as well as $O(\alpha^2)$. The errors include the statistical errors, as well as the estimated systematic and theoretical ones. This is to be compared with the value deduced from Eq. (1.1) and electroweak corrections

$$10^{11}\times a^{\text{expt}}(\text{hadronic}) = 7174 \pm 150,$$

from which Eq. (1.5) differs by 1.1σ.

II. CONTRIBUTIONS TO a_μ

We divide the various contributions to a_μ as follows:

$$a_\mu = a(\text{QED}) + a(\text{weak}) + a(\text{hadronic}). \tag{2.1}$$

Here $a(\text{QED})$ denote the pure quantum electrodynamics corrections, and a(weak) are the ones due to W, Z, and Higgs exchange. The hadronic contributions can, in turn, be split as

$$a(\text{hadronic}) = a^{(2)}(\text{h.v.p.}) + a[\text{other hadronic},\ O(\alpha^3)]. \tag{2.2}$$

$a^{(2)}$(h.v.p.) are the corrections due to the hadronic photon vacuum polarization contributions (Fig. 1), nominally of order α^2 (see Secs. III C and V B for a qualification of this statement). We will discuss in detail the "other hadronic, $O(\alpha^3)$" in Sec. V.

According to the review of Hughes and Kinoshita [13] one has

$$\begin{aligned} 10^{11}\times a(\text{QED}) &= 116\,584\,705 \pm 1.8, \\ 10^{11}\times a(\text{weak}) &= 151 \pm 4. \end{aligned} \tag{2.3}$$

There is no dispute about these numbers. If we combine them with Eq. (1.1), we can convert this into a measurement of the hadronic part of the anomaly:

$$10^{11}\times a^{\text{expt}}(\text{hadronic}) = 7174 \pm 150. \tag{2.4}$$

Our task in the present paper is the evaluation of this quantity.

We now say a few words about the piece $a^{(2)}$(h.v.p.), which is the most important component of a(hadronic). As Brodsky and de Rafael [14] have shown, it can be written as

$$\begin{aligned} a^{(2)}(\text{h.v.p.}) &= 12\pi \int_{4m_\pi^2}^{\infty} dt\, K(t)\,\text{Im}\,\Pi(t), \\ K(t) &= \frac{\alpha^2}{3\pi^2 t}\hat{K}(t); \\ \hat{K}(t) &= \int_0^1 dx\, \frac{x^2(1-x)}{x^2 + (1-x)t/m_\mu^2}. \end{aligned} \tag{2.4a}$$

Here Π is the hadronic part of the photon vacuum polarization function. An alternate formula is obtained by expressing $\text{Im}\Pi$ in terms of the ratio

$$\begin{aligned} R(t) &= \frac{\sigma^{(0)}(e^+e^- \to \text{hadrons})}{\sigma^{(0)}(e^+e^- \to \mu^+\mu^-)}, \\ \sigma^{(0)}(e^+e^- \to \mu^+\mu^-) &\equiv \frac{4\pi\alpha^2}{3t}: \\ a^{(2)}(\text{h.v.p.}) &= \int_{4m_\pi^2}^{\infty} dt\, K(t) R(t). \end{aligned} \tag{2.4b}$$

The superindex (0) here means "lowest order in the electromagnetic interactions."

At low energy ($t \leqslant 0.8\ \text{GeV}^2$) we can separate the contribution from three pion states and that from two pions. The first will be discussed in Sec. IV. The two pion contribution in turn can be expressed in terms of the pion form factor F_π,

$$\text{Im}\,\Pi_{2\pi}(t) = \frac{1}{48\pi}\left(1 - \frac{4m_\pi^2}{t}\right)^{3/2} |F_\pi(t)|^2, \tag{2.5}$$

so that, for the two-pion contribution up to energy squared t_0,

$$a_\mu(2\pi; t_0) = \frac{1}{4}\int_{4m_\pi^2}^{t_0} dt \left(1 - \frac{4m_\pi^2}{t}\right)^{3/2} K(t)|F_\pi(t)|^2. \tag{2.6}$$

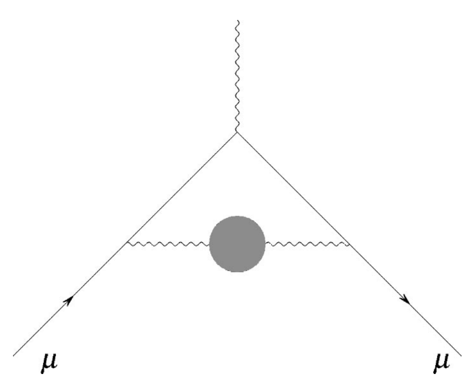

FIG. 1. The order α^2 hadronic contributions to the muon magnetic moment. The blob represents an arbitrary hadronic state.

III. THE PION FORM FACTOR

A. Theory

The evaluation of the pion form factor is slightly complicated by the phenomenon of ω-ρ interference. This can be solved by considering only the isospin $I=1$ component, and adding later the $\omega\to 2\pi$ and interference in the standard Gounaris-Sakurai way. This is equivalent to neglecting, in a first approximation, the breaking of isospin invariance. We will also neglect for now electromagnetic corrections. In this approximation the properties of $F_\pi(t)$ are the following: (i) $F_\pi(t)$ is an analytic function of t, with a cut from $4m_\pi^2$ to infinity; (ii) on the cut, the phase of $F_\pi(t)$ is, because of unitarity, identical to that of the P wave, $I=1$, $\pi\pi$ scattering, $\delta_1^1(t)$, and this equality holds until the opening of the inelastic threshold at $t=t_0$ (Fermi-Watson final-state interaction theorem); (iii) for large t, $F_\pi(t)\simeq 1/t$. Actually, one knows the coefficient of this behavior, but we will not need it here; (iv) $F(0)=1$.

The inelastic threshold occurs, rigorously speaking, at $t=16m_\pi^2$. However, it is an experimental fact that inelasticity is negligible until the quasi-two-body channels $\omega\pi$, $a_1\pi\cdots$ are open. In practice we will take

$$t_0\simeq 1\ \mathrm{GeV}^2,$$

and fix the best value for t_0 empirically. It will be $t_0=1.1\ \mathrm{GeV}^2$, and we will see that, if we keep close to this value, the dependence on t_0 is very slight.

The properties (i)–(iv) can be taken into account with the well-known Omnès-Muskhelishvili method. We construct a function $J(t)$ with the proper phase by defining

$$J(t)=\exp\left\{\frac{t}{\pi}\int_{4m_\pi^2}^{t_0} ds\,\frac{\delta_1^1(s)}{s(s-t)}+\frac{t}{\pi}\int_{t_0}^{\infty} ds\,\frac{\overline{\delta}_1^1(s)}{s(s-t)}\right\}. \tag{3.1a}$$

We have written the dispersion relation with one subtraction to ensure that $J(0)=1$. The singular integrals are understood to be calculated replacing $t\to t+i\epsilon$, $\epsilon>0$, and letting then $\epsilon\to 0$. In particular, we have

$$|J(t)|=\exp\left\{\frac{t}{\pi}\mathrm{P.P.}\int_{4m_\pi^2}^{t_0} ds\,\frac{\delta_1^1(s)}{s(s-t)}+\frac{t}{\pi}\int_{t_0}^{\infty} ds\,\frac{\overline{\delta}_1^1(s)}{s(s-t)}\right\},\quad 4m_\pi^2\leqslant t\leqslant t_0. \tag{3.1b}$$

Defining then the function G by

$$F_\pi(t)=G(t)J(t), \tag{3.2}$$

it follows from properties (i) and (ii) that $G(t)$ is analytic with only the exception of a cut from t_0 to infinity, as we have already extracted the correct phase below $t=t_0$.

We can, in Eq. (3.1a), take any value we like for the phase $\overline{\delta}_1^1(t)$, as a change of it only results in a redefinition of G; but it is convenient to choose $\overline{\delta}_1^1(t)$ so that it joins smoothly $\delta_1^1(t)$ at $t=t_0$ to avoid spurious singularities that would deteriorate the convergence, and so that J has the correct behavior at infinity. Both properties are ensured if we take, simply,

$$\overline{\delta}_1^1(t)=\pi+[\delta_1^1(t_0)-\pi]\frac{t_0}{t}$$

so that $\overline{\delta}_1^1(t_0)=\delta_1^1(t_0)$ and, for large t, $\overline{\delta}_1^1(t)\to\pi$ and we recover the behavior $1/t$ of the form factor. Then we can rewrite more explicitly Eq. (3.1) by integrating the piece with $\overline{\delta}_1^1$:

$$J(t)=e^{1-\delta_1^1(t_0)/\pi}\left(1-\frac{t}{t_0}\right)^{[1-\delta_1^1(t_0)/\pi]t_0/t}\times\left(1-\frac{t}{t_0}\right)^{-1}\exp\left\{\frac{t}{\pi}\int_{4m_\pi^2}^{t_0} ds\,\frac{\delta_1^1(s)}{s(s-t)}\right\}. \tag{3.3}$$

1. The phase δ_1^1

We can apply the effective range theory to the phase δ_1^1. According to this, the function

$$\psi(t)\equiv\frac{2k^3}{t^{1/2}}\cot\delta_1^1(t),\quad k=\frac{\sqrt{t-4m_\pi^2}}{2} \tag{3.4a}$$

is analytic in the variable t except for two cuts: a cut from $-\infty$ to 0, and a cut from $t=t_0$ to $+\infty$. To profit from the analyticity properties of ψ we will make a conformal transformation.[2] We define

$$w=\frac{\sqrt{t}-\sqrt{t_0-t}}{\sqrt{t}+\sqrt{t_0-t}}. \tag{3.4b}$$

When t runs the cuts, w goes around the unit circle. We may therefore expand ψ in a power series convergent inside the unit disc. However, the existence of the ρ resonance implies that we must have $\cot\delta_1^1(m_\rho^2)=0$. It is therefore convenient to incorporate this piece of knowledge and expand not ψ itself but the ratio $\psi(t)/(m_\rho^2-t)\equiv\hat{\psi}(t)$: so we write

$$\psi(t)=(m_\rho^2-t)\hat{\psi}(t)=(m_\rho^2-t)\{b_0+b_1w+b_2w^2+\cdots\}. \tag{3.4c}$$

[2] The method of conformal transformations is rigorous, simpler and produces better results than that employed in Ref. [4].

The P wave, $I=1\,\pi\pi$ scattering length,[3] a_1^1, is related to ψ by

$$a_1^1=\frac{1}{m_\pi\psi(4m_\pi^2)}. \tag{3.5a}$$

Likewise, from the relation

$$\frac{1}{\cot\delta_1^1(t)-i}\underset{t\simeq m_\rho^2}{\simeq}\frac{\text{const}}{m_\rho^2-t-2k^3 i/t^{1/2}\hat{\psi}(t)}$$

we find the expression for the rho width:

$$\Gamma_\rho=\frac{2k_\rho^3}{m_\rho^2\hat{\psi}(m_\rho^2)},\quad k_\rho=\frac{1}{2}\sqrt{m_\rho^2-4m_\pi^2}. \tag{3.5b}$$

Experimentally [15], $a_1^1\simeq(0.038\pm0.003)m_\pi^{-3}$, and, according to the Particle Data Group [16], $m_\rho=769.3\pm0.8$; $\Gamma_\rho=150.2\pm0.8$ MeV. Note, however, that we do *not* assume the values of m_ρ, Γ_ρ. We only require that ψ has a zero, and will let the fits fix its location and residue.

It turns out that, to reproduce the width and scattering length, and to fit the pion form factor as well (see below), only two terms in the expansion are needed, so we approximate

$$\delta_1^1(t)=\operatorname{arc\,cot}\left\{\frac{t^{1/2}}{2k^3}(m_\rho^2-t)\left[b_0+b_1\frac{\sqrt{t}-\sqrt{t_0-t}}{\sqrt{t}+\sqrt{t_0-t}}\right]\right\}; \tag{3.6}$$

m_ρ, b_0, b_1 are free parameters in our fits.

2. The function $G(t)$

Because we have already extracted the correct phase up to $t=t_0$, it follows that the function $G(t)$ is analytic except for a cut from $t=t_0$ to $+\infty$. The conformal transformation

$$z=\frac{\frac{1}{2}\sqrt{t_0}-\sqrt{t_0-t}}{\frac{1}{2}\sqrt{t_0}+\sqrt{t_0-t}} \tag{3.7a}$$

maps this cut plane into the unit circle. So we may write the expansion

$$G(t)=1+A_0+c_1z+c_2z^2+c_3z^3+\cdots \tag{3.7b}$$

that will be convergent for all t inside the cut plane. We can implement the condition $G(0)=1$, necessary to ensure $F_\pi(0)=1$ to each order, by writing

$$A_0=-[c_1z_0+c_2z_0^2+c_3z_0^3+\cdots],\quad z_0\equiv z|_{t=0}=-1/3.$$

[3] For details on $\pi\pi$ scattering, including analyticity properties and the Fermi-Watson theorem, see, e.g., Ref. [15]. More details on the solution of the Omnès-Muskhelishvili equation can be found in N. I. Muskhelishvili, *Singular Integral Equations* (Nordhoof, 1958).

The expansion then reads

$$G(t)=1+c_1(z+1/3)+c_2(z^2-1/9)+c_3(z^3+1/27)+\ \ldots\ . \tag{3.8}$$

We will need two-three terms in the expansion, so we will approximate

$$G(t)=1+c_1\left[\frac{\frac{1}{2}\sqrt{t_0}-\sqrt{t_0-t}}{\frac{1}{2}\sqrt{t_0}+\sqrt{t_0-t}}+\frac{1}{3}\right]+c_2\left[\left(\frac{\frac{1}{2}\sqrt{t_0}-\sqrt{t_0-t}}{\frac{1}{2}\sqrt{t_0}+\sqrt{t_0-t}}\right)^2-\frac{1}{9}\right],$$

c_1, c_2 free parameters.

An interesting feature of our method is that, even if we only kept *one* term in each of the expansions (3.6) and (3.8), that is to say, if we took $b_1=c_1=c_2=0$, we could reproduce the experimental data with only a 15% error; so we expect (and this is the case) fast convergence of the series. It is important also that our expression for $F_\pi(t)$ is valid in the spacelike as well as in the timelike region, provided only $t<t_0$. What is more, Eqs. (3.6) and (3.8) represent the more general expressions compatible with analyticity, the Fermi-Watson theorem and the effective range theory, which follow only from the requirements of unitarity and causality. Therefore, by employing our expansions, we do not introduce uncontrolled biases in the analysis, and hence we minimize the model dependent errors.[4]

B. Fits

In order to fit F_π, and hence get the 2π low-energy $(4m_\pi^2\leqslant t\leqslant 0.8\ \text{GeV}^2)$ contribution to $a^{(2)}(\text{h.v.p.})$, we have available three sets of data: $e^+e^-\to\pi^+\pi^-$, t timelike (Novosibirsk, Ref. [8]); $F_\pi(t)$, t spacelike (NA7, Ref. [11]); in addition, one can use data from the decay $\tau^+\to\overline{\nu}_\tau\pi^+\pi^0$ (Aleph and Opal, Ref. [10]).

For this last we have to assume isospin invariance, *and* neglect the isospin $I=2$ component of $\pi^+\pi^0$, to write the form factor v_1 for τ decay in terms of F_π:

$$v_1=\frac{1}{12}\left(1-\frac{4m_\pi^2}{t}\right)^{3/2}|F_\pi(t)|^2, \tag{3.9a}$$

[4] The remaining approximations are neglect of the inelasticity between $16m_\pi^2$ and t_0, experimentally known to be at the 10^{-3} level or below, and we have the errors due to the truncation of the expansions; we will also check that they are small. By contrast, other functional forms used in the literature are either incompatible with the phase of F_π, or with its analyticity properties (or both), which causes biases in the fits. The errors due to breaking of isospin and electromagnetic corrections will be discussed below.

where, in terms of the weak vector current $V_\mu = \bar{u}\gamma_\mu d$, and in the exact isospin approximation,

$$\Pi^V_{\mu\nu} = (-p^2 g_{\mu\nu} + p_\mu p_\nu)\Pi^V(t) = i\int d^4x\, e^{ip\cdot x}\langle 0|TV^+_\mu(x)V_\nu(0)|0\rangle; \quad v_1 = 2\pi\,\mathrm{Im}\,\Pi^V. \tag{3.9b}$$

Before presenting the results of the fits a few matters have to be discussed. A first point to clarify is that we will *not* include in the fits the old data on F_π in the spacelike or timelike regions, or on pion-pion phase shifts [17]. We have checked that, if we add the first two sets, the results of the fit vary very little (see below); but they cause a bias. This is so because there is incompatibility[5] between old spacelike and timelike data, and also with data on $\pi\pi$ phase shifts, already noticed by Casas, López, and Yndurain (CLY) [14]. Doubtless, this is due to the fact that most old data for spacelike momentum were extracted from processes with one pion off its mass shell so that models were necessary to extrapolate to the physical form factor. In fact, a very important feature of the NA7 [11] data is that they are obtained from scattering of real pions off electrons, hence we do not require models to extract F_π from data.

The reason for the model dependence of $\pi\pi$ phase shift analyses is that these are extracted from fits to $\pi N \to \pi\pi N$ scattering and thus require a model for the pseudoscalar form factor of the nucleon, a model for the interactions of the nucleon and the final state pions, and a model for the dependence of $\pi\pi$ scattering on the mass of an external pion. Indeed, different methods of extrapolation result in different sets of phase shifts, as can be seen in the experimental papers of Hyams *et al.* and Protopopescu *et al.*, Ref. [17], where five different determinations are given. However, we will use the scattering length a_1^1 and employ the $\pi\pi$ phase shifts as a very important *a posteriori* test of our results.

We could consider, besides this information, to include as input the values of several quantities that can be estimated with chiral perturbation theory methods, such as $\langle r_\pi^2\rangle$ and a_1^1. We do not do so because the problem with these calculations is the estimate of their errors, a difficult matter; so we have preferred to avoid possible biases and instead *obtain* these quantities as by-product of our calculations. Then we check that the results we get for all of them are in agreement, within errors, with the chiral perturbation theory results; see below. With respect to a_1^1 we actually constrain it to the region obtained from $\pi\pi$ scattering experimental data only; its error is chosen such that it encompasses the various values given in the different experimental determinations (Ref. [17]). We take

$$a_1^1 = (38\pm 3)\times 10^{-3} m_\pi^{-3};$$

we will see that the value our best fit returns for this quantity is satisfactorily close to this, as indeed we get $(41\pm 2)\times 10^{-3} m_\pi^{-3}$.

Another remark concerns the matter of isospin breaking, due to electromagnetic interactions or the mass difference between u, d quarks, that would spoil the equality (3.9a). It is not easy to estimate this. A large part of the breaking, the $\omega\to 2\pi$ contribution and ω-ρ mixing, are taken into account by hand, but this does not exhaust the effects. For example, merely changing the quark masses from $m_{\pi^+}+m_{\pi^-}$ to $m_{\pi^0}+m_{\pi^0}$, in a Breit-Wigner model for the ρ, shifts $a^{(2)}$(h.v.p.) by $\sim 50\times 10^{-11}$, so a deviation of this order should not be surprising.[6]

As stated above, Eqs. (3.9) were obtained neglecting the mass difference $m_u - m_d$ and electromagnetic corrections, in particular the π^0-π^+ mass difference. We can take the last partially into account by distinguishing between the pion masses in the phase space factor in Eq. (3.9a). To do so, write now Eq. (3.9b) as

$$\Pi^V_{\mu\nu} = i\int d^4x\, e^{ip\cdot x}\langle 0|TV^+_\mu(x)V_\nu(0)|0\rangle = (-p^2 g_{\mu\nu} + p_\mu p_\nu)\Pi^V(t) + p_\mu p_\nu \Pi^S; \quad v_1 \equiv 2\pi\,\mathrm{Im}\,\Pi^V. \tag{3.10a}$$

We get

$$v_1 = \frac{1}{12}\left\{\left[1 - \frac{(m_{\pi^+}-m_{\pi^0})^2}{t}\right] \times \left[1 - \frac{(m_{\pi^+}+m_{\pi^0})^2}{t}\right]\right\}^{3/2} |F_\pi(t)|^2. \tag{3.10b}$$

To compare with the experimentally measured quantity, which involves all of $\mathrm{Im}\,\Pi^V_{\mu\nu}$, we have to neglect the scalar component Π^S, which is proportional to $(m_d - m_u)^2$, and thus very small.

To understand the situation we will proceed by steps. First of all, we start by fitting *separately* e^+e^- and τ data, in the timelike region, using Eq. (3.9a) (we remark that although in $a(2\pi; t\leq 0.8\ \mathrm{GeV}^2)$ only enter the values of $F_\pi(t)$ for $4m_\pi^2$ to $0.8\ \mathrm{GeV}^2$, we fit the whole range up to $t = t_0 = 1.1\ \mathrm{GeV}^2$). Then, we get quite different results:

$$a(2\pi; t\leq 0.8\ \mathrm{GeV}^2) = \begin{cases} 4715\pm 67 & (e^+e^-;\ \chi^2/\mathrm{d.o.f.} = 106/109 = 0.96), \\ 4814\pm 26 & (\tau;\ \chi^2/\mathrm{d.o.f.} = 52/48 = 1.09). \end{cases} \tag{3.11a}$$

This takes into account statistical errors only for e^+e^-, but includes systematic ones for τ decay as these are incorporated in the available data.

The slight advantage of the first figure in Eq. (3.11a) in

[5] At the level of 1.5 to 2σ.

[6] The relevance of isospin breaking in this context was pointed out by V. Cirigliano, G. Ecker, and H. Neufeld, hep-ph/0104267, 2001.

what regards the χ^2/d.o.f. makes one wonder that the difference is really caused by isospin breaking (in which case the value obtained from τ decay should be rejected) or is due to random fluctuations of the data, as well as to the systematics of the experiments. The second explanation has in its favor that, if we include the *spacelike* data into the fit [but still use Eq. (3.9a)] the discrepancy is softened, and we get compatible results:

$$a(2\pi;t\leqslant 0.8\ \mathrm{GeV}^2)=\begin{cases}4754\pm 55 \ \ (e^+e^- + \text{spacelike};\ \ \chi^2/\text{d.o.f.}=179/154),\\ 4826\pm 23 \ \ (\tau + \text{spacelike};\ \ \chi^2/\text{d.o.f.}=112/93).\end{cases} \tag{3.11b}$$

This last result allows us to draw the following conclusion: that part of the discrepancy between results obtained with e^+e^- and τ decay is still of statistical origin, but also it would seem that part is genuine.

In an attempt to take into account at least some of the isospin breaking effects, we have fitted simultaneously e^+e^-, τ decay, both including spacelike data, allowing for different values of the mass and width of the rho (but keeping other parameters, in particular c_1, c_2, common for both e^+e^- and τ fits). We, however, still use Eq. (3.9a). In this case we find convergence of the results; we have[7]

$$a(2\pi;t\leqslant 0.8\ \mathrm{GeV}^2)=4779\pm 30, \quad \chi^2/\text{d.o.f.}=248/204 \quad (e^+e^- + \tau + \text{spacelike}), \tag{3.12}$$

which is compatible (within errors) with both numbers in Eq. (3.11b).

It is to be noted that, if we had not allowed for different masses and widths for the neutral and charged rho, we would have obtained, in this common fit,

$$10^{11}\times a(2\pi;t\leqslant 0.8\ \mathrm{GeV}^2)=4822\pm 30,$$
$$\chi^2/\text{d.o.f.}=264/206 \ \ (e^+e^- + \tau + \text{spacelike}),$$

i.e., a larger χ^2/d.o.f. and a value quite different from that obtained with only e^+e^- and spacelike data. So it would appear that allowing for different parameters for the neutral and charged rho really takes into account a good part of the isospin breaking effects.

Finally, we take into account the kinematical effects of the $\pi^\pm$, π^0 mass difference repeating the fit using Eq. (3.10b) now.[8] The result of the fit with e^+e^- data only is of course unchanged, but we reproduce it to facilitate the comparison and for ease of reference. We find what we consider our best results:

$$\begin{aligned}&10^{11}\times a(2\pi;t\leqslant 0.8\ \mathrm{GeV}^2)=4774\pm 31,\\ &\chi^2/\text{d.o.f.}=246/204 \ \ (e^+e^- + \tau + \text{spacelike}),\\ &10^{11}\times a(2\pi;t\leqslant 0.8\ \mathrm{GeV}^2)=4754\pm 55,\\ &\chi^2/\text{d.o.f.}=179/154 \ \ (e^+e^- + \text{spacelike}).\end{aligned} \tag{3.13}$$

We remark that the results for the evaluation including τ decays are rather insensitive to the use of Eq. (3.10b), but what change there is, it goes in the right direction: the χ^2/d.o.f. has improved slightly, and the values for the anomaly including the τ have become slightly more compatible with the figure obtained using e^+e^- data only. This makes us confident that most of the effects due to isospin breaking, both from *u, d* mass differences and from electromagnetic effects (about which we will say more in Secs. III C and V B) have already been incorporated in our calculation. The fit may be seen depicted in Fig. 2 for $|F_\pi|^2$, with timelike and spacelike data, and in Fig. 3 for the quantity v_1 in τ decay.

The χ^2/d.o.f. of the fits is slightly above unity; in next subsection we will see that including *systematic* errors cures the problem. For example, just adding the systematic normalization error for the spacelike data [11] gives a shift of the central value of 31×10^{-11} and the χ^2/d.o.f. decreases to 152/153 for the evaluation with e^+e^- data only. The quality of the fit to the spacelike data is shown in Fig. 4, which is a blowup of the corresponding part of Fig. 2.

The parameters of the fits are also compatible. We have

$$\begin{aligned}&c_1=0.23\pm 0.02,\ \ c_2=-0.15\pm 0.03;\ \ b_0=1.062\pm 0.005,\\ &b_1=0.25\pm 0.04 \ \ (e^+e^- + \tau + \text{spacelike});\\ &c_1=0.19\pm 0.04,\ \ c_2=-0.15\pm 0.10;\\ &b_0=1.070\pm 0.006,\\ &b_1=0.28\pm 0.06 \ \ (e^+e^- + \text{spacelike}).\end{aligned} \tag{3.14}$$

In the first line the parameters c_1, c_2 are common for ρ^0, ρ^+. b_0 and b_1 vary very little; the ones shown correspond to the values of m_{ρ^0}, Γ_{ρ^0} as given below in Eq. (3.15). The values $b_0=\text{const}$, $b_1=0$ would correspond to a perfect Breit-Wigner shape for the ρ. Another fact to be mentioned is that including the corrected phase space factor (3.10b) helps a little to make compatible the parameters for both fits; if we had used Eq. (3.9a) we would have obtained

[7]When evaluating $a(2\pi;t\leqslant 0.8\ \mathrm{GeV}^2)$ we of course use the parameters m_ρ, b_0, b_1 corresponding to ρ^0; see below.

[8]For consistency we should also have taken the expression $k=(1/2)\{[t-(m_{\pi^+}-m_{\pi^0})^2][t-(m_{\pi^+}+m_{\pi^0})^2]\}^{1/2}$, altered the threshold to $t=(m_{\pi^+}+m_{\pi^0})^2$ for tau decay and allowed for different scattering lengths. We have checked that the effect of this on the contribution to a leaves it well inside our error bars; we will discuss the results one gets in a separate paper. Note that it makes sense to still consider the same c_1, c_2 for e^+e^- and tau decay as these parameters are associated with G whose imaginary part vanishes below $t=s_0\sim 1\ \mathrm{GeV}^2$ where the effects of isospin breaking should be negligible.

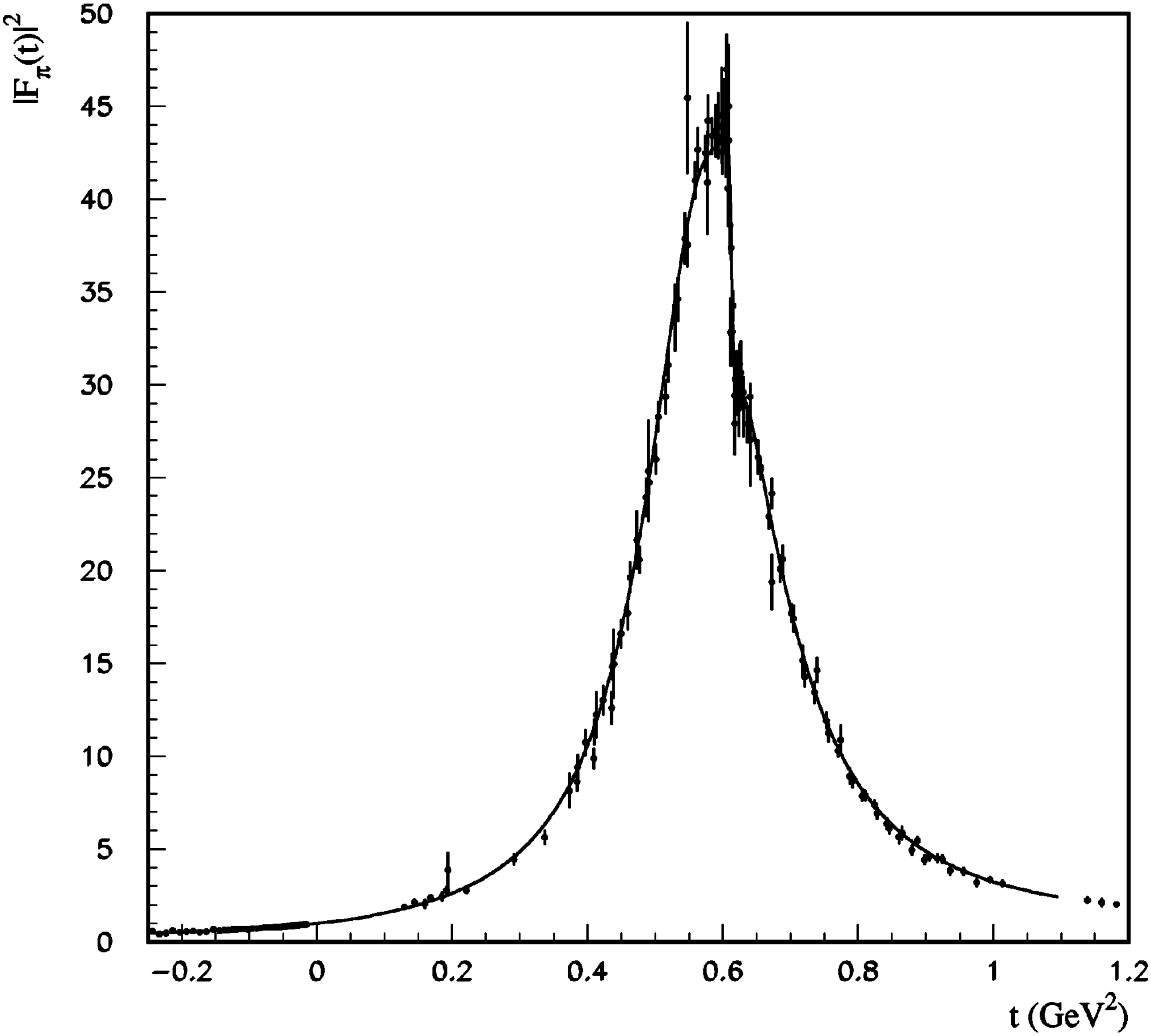

FIG. 2. Plot of the fit to $|F_\pi(t)|^2$, timelike (Ref. [8]) and spacelike (Ref. [11]) data. The theoretical curve actually drawn is that obtained by fitting also τ data, but the curve obtained fitting only e^+e^- could not be distinguished from that drawn if we plotted it. A blowup of the fit in the spacelike region may be seen in Fig. 4.

$$c_1=0.23\pm0.01,\;\; c_2=-0.16\pm0.03;\;\; b_0=1.060\pm0.005,$$

$$b_1=0.24\pm0.04\;\; (e^+e^-+\tau+\text{spacelike}).$$

An important feature of our fit is that the coefficients decrease with increasing order. This, together with the fact that the conformal variables *w*, *z* are of modulus well below unity in the regions of interest ($4m_\pi^2\leqslant t\leqslant 0.8\ \mathrm{GeV}^2$ for *w*, $-0.25\ \mathrm{GeV}^2\leqslant t\leqslant 0.8\ \mathrm{GeV}^2$ for *z*):

$$-0.57\leqslant w\leqslant 0.24,\;\; -0.38\leqslant z\leqslant -0.02,$$

ensures good convergence of the expansions. We have also checked that including extra terms in the expansions does not improve the quality of the fits significantly.

Besides the results for the anomaly we obtain reliable determination of a set of parameters. We have those pertaining to the rho,

$$m_{\rho^0}=772.6\pm0.5\ \mathrm{MeV},\;\; \Gamma_{\rho^0}=147.4\pm0.8\ \mathrm{MeV}; \tag{3.15}$$

$$m_{\rho^+}=773.8\pm0.6\ \mathrm{MeV},\;\; \Gamma_{\rho^+}=147.3\pm0.9\ \mathrm{MeV}.$$

The figures are in reasonable agreement with the Particle Data Group values[9] given before.

The value for the scattering length the fit returns is comfortably close to the one obtained from $\pi\pi$ phase shifts; we get

$$a_1^1=(41\pm2)\times10^{-3}m_\pi^{-3}.$$

[9] It should be noted that the various determinations for m_ρ reported by the PDG [16] actually cluster around several different values.

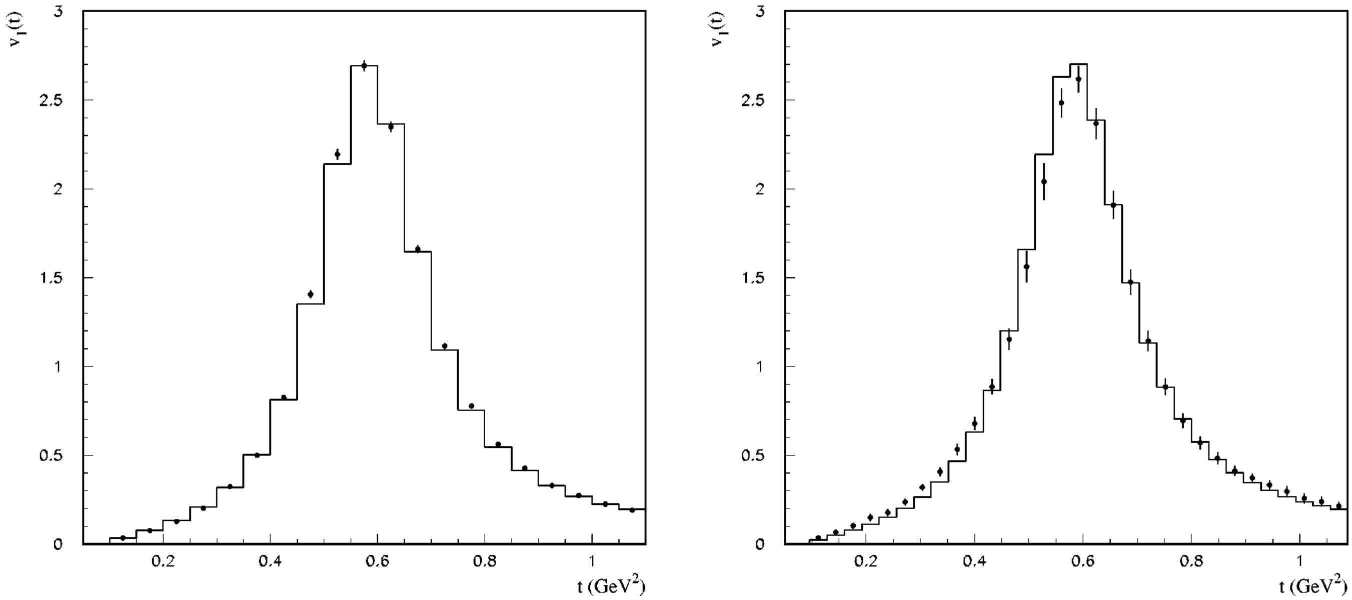

FIG. 3. Plot of the fits to $v_1(t)$ (histograms), and data from τ decay (black dots). Left: Aleph data. Right: Opal data. (Ref. [10]). Note that the theoretical values (histograms) are results of the same calculation, with the same parameters, so the differences between the two fits merely reflect the slight variations between the two experimental determinations.

This value of a_1^1 is slightly larger, but compatible with recent determinations based on an analysis of $\pi\pi$ scattering [Ananthanarayan-Colangelo-Gasser-Leutwyler (ACGL)] or chiral perturbation theory [CGL, Amorós-Bijnens-Talavera (ABT)] that give (Ref. [18])

$$a_1^1 = (37.9 \pm 0.5) \times 10^{-3} m_\pi^{-3} (\mathrm{CGL});$$

$$a_1^1 = (37 \pm 2) \times 10^{-3} m_\pi^{-3} (\mathrm{ACGL});$$

$$a_1^1 = (38 \pm 2) \times 10^{-3} m_\pi^{-3} (\mathrm{ABT}).$$

Also from our fits we obtain the low-energy coefficients of the pion form factor,

$$F_\pi^2(t) \underset{t\to 0}{\simeq} 1 + \frac{1}{6}\langle r_\pi^2 \rangle t + c_\pi t^2:$$

$$\langle r_\pi^2 \rangle = 0.435 \pm 0.002 \ \ \mathrm{fm}^2, \ \ c_\pi = 3.60 \pm 0.03 \ \mathrm{GeV}^{-4} \quad (e^+e^- + \tau + \mathrm{spacelike}); \tag{3.16}$$

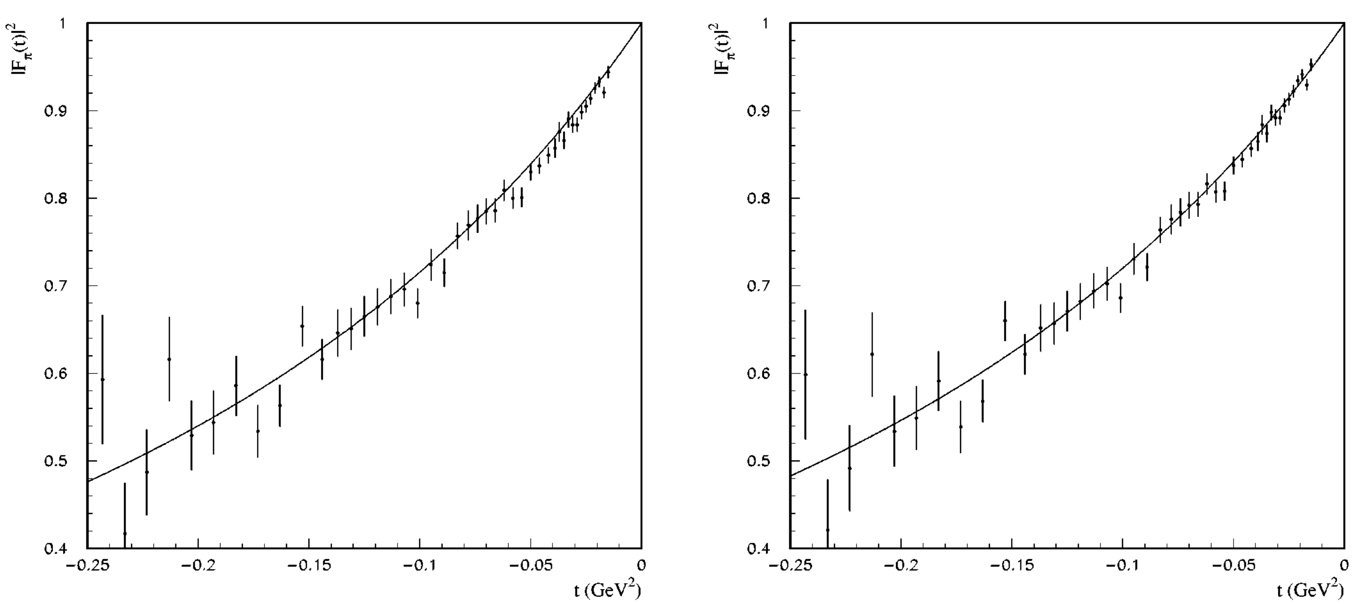

FIG. 4. Plot of the fit to $|F_\pi(t)|^2$ in the spacelike region. With only statistical errors (left) and including systematic experimental errors (right).

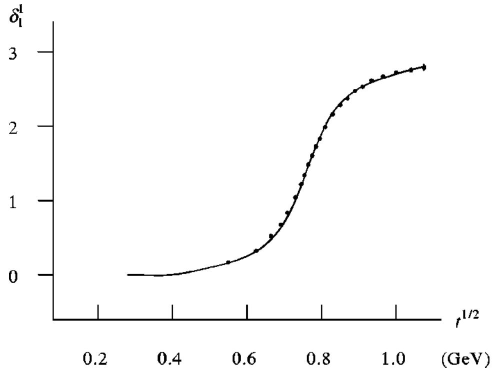

FIG. 5. Our predicted phase $\pi\pi$ shift δ_1^1 (in radians), compared to the experimental values for the same (solution 1 from Protopopescu *et al.*, Ref. [17]). The experimental errors are of the order of the size of the black dots.

$$\langle r_\pi^2 \rangle = 0.433 \pm 0.002 \ \ \mathrm{fm}^2, \ \ c_\pi = 3.58 \pm 0.04 \ \mathrm{GeV}^{-4}$$

$$(e^+e^- + \text{spacelike}).$$

These figures are also compatible with, but much more precise than, the current estimates [18]:

$$\langle r_\pi^2 \rangle = 0.431 \pm 0.026 \ \ \mathrm{fm}^2, \ \ c_\pi = 3.2 \pm 1.0 \ \mathrm{GeV}^{-4}.$$

Another remark is that in all these fits we took $t_0 = 1.1\ \mathrm{GeV}^2$. The dependence of the results on this parameter t_0 is very slight, provided we remain around this value. Thus, for example, if we take $t_0 = 1.2\ \mathrm{GeV}^2$ the value of $a(2\pi; t \leqslant 0.8\ \mathrm{GeV}^2)$ only increases by 4×10^{-11}, and the global χ^2 only varies by one unit.

As further checks of the stability and reliability of our results we mention the following two. First, we could, as discussed above, have imposed the more stringent values for a_1^1 as given in Ref. [18]. Now, if for example we take, in accordance with ACGL in this reference, the value $a_1^1 = (37 \pm 2) \times 10^{-3} m_\pi^{-3}$, instead of the value $a_1^1 = (38 \pm 3) \times 10^{-3} m_\pi^{-3}$ that follows from only *experimental* $\pi\pi$ data, the fit deteriorates. The fit returns the value $a_1^1 = (39 \pm 1) \times 10^{-3} m_\pi^{-3}$ for the scattering length, in (slightly) better agreement with the input; but we do not consider this an improvement as the global χ^2 increases by two units.

The corresponding value for the contribution to the anomaly changes very little, from the value $(4774 \pm 31) \times 10^{-11}$ [Eq. (3.13)] to $(4768 \pm 32) \times 10^{-11}$ now, i.e., a shift of only 6×10^{-11} with a small increase of the error. Thus the results are insensitive to a more stringent input for a_1^1 but, because the quality of the fit deteriorates, we still consider the result with the more relaxed input $a_1^1 = (38 \pm 3) \times 10^{-3} m_\pi^{-3}$ to be less biased.

Second, we have *not* used the experimental phase shifts as input (except for the values of the scattering length). So, the values that follow from our expression (3.6), with the parameters given in Eq. (3.14), constitute really a *prediction* for $\delta_1^1(t)$. This can be compared with the existing experimental values for this quantity [17], a comparison that may be found in Fig. 5. The agreement is remarkable. The result one would have obtained if *including* the phase shifts in the fit will be given at the end of this subsection.

TABLE I. Comparison of evaluations of $10^{11} \times a(2\pi; t \leqslant 0.8\ \mathrm{GeV}^2)$. N1; N2 are in Ref. [7]. TY denotes our result here (statistical errors only for the e^+e^- and spacelike data).

4795 ± 61	N1	$\tau + e^+e^-$
4730 ± 100	N2	e^+e^-
4846 ± 50	CLY, AY [4]	e^+e^-
4794 ± 50	CLY-II [4]	$e^+e^- + \pi\pi$ ph. shifts
4774 ± 31	TY1	$\tau + e^+e^-$
4754 ± 55	TY2	only e^+e^-

Before finishing this section we have to clarify the matter of the ω and ω-ρ contribution to $a(2\pi; t \leqslant 0.8\ \mathrm{GeV}^2)$. Our fits to e^+e^- data have actually been made including in the function F_π as given above, Eq. (3.2), a coefficient to take into account the $\omega \to 2\pi$ contribution. To be precise, we have used the expression

$$F_\pi^{\mathrm{all}}(t) = F_\pi(t) \times \frac{1 + \sigma \dfrac{m_\omega^2}{m_\omega^2 - t - i m_\omega \Gamma_\omega}}{1 + \sigma}, \qquad (3.17)$$

where the notation is obvious. We take from the PDG [16] the values for the mass and width of the ω,

$$m_\omega = 782.6 \pm 0.1\ \mathrm{MeV}, \ \ \Gamma_\omega = 8.4 \pm 0.9\ \mathrm{MeV}, \qquad (3.18)$$

and the fit gives a mixing parameter $\sigma = (16 \pm 1) \times 10^{-4}$.

As is known, this Gounnaris-Sakurai [19] parametrization is only valid for $t \simeq m_{\omega,\rho}^2$ and, in particular, its extrapolation to $t \sim 0$ is not acceptable. This effect is very small, less than one part in a thousand. However, to play it safe, we have also adopted the following alternate procedure: we have obtained a first approximation to F_π by fitting the experimental data *excluding* the region $0.55 \leqslant t \leqslant 0.65\ \mathrm{GeV}^2$. Then we have fitted only this region adding there also the ω piece, as in Eq. (3.17). The resulting value for $a(2\pi; t \leqslant 0.8\ \mathrm{GeV}^2)$ varies very little; it decreases by something between 2 and 12 $\times 10^{-11}$, depending on the fit. We may consider this as part of the theoretical error of our calculation, to be discussed in next subsection.

To finish this subsection, we present in Table I a comparison both with old results that also use analyticity properties, and a recent one (which does not).[10]

The difference between the old CLY, AY, and the new determinations is due to a large extent to the influence of the new Novosibirsk and NA7 data which allow us in particular to obtain a robust result: the CLY evaluation used only 18 data points! In this respect, we note that, if we had included the $\pi\pi$ phase shifts in the fit (with also τ decay data) we would have obtained 4781 ± 29 for this 2π contribution, i.e., a shift of only seven units (as compared with a shift of 48 in the CLY-II evaluation). The value of the scattering length would be $a_1^1=(43\pm3)\times10^{-3}m_\pi^{-3}$ now. The corresponding χ^2/d.o.f., 276/227 with only statistical errors, is also good. This is an important proof of the stability of our results against introduction of extra data. (However, as explained above, we prefer the result without fitting phase shifts because of the model dependence of the last.)

The difference between the results of Narison (N), who does not take into account the Fermi-Watson theorem or the spacelike data and TY, who do, is due in good part to, precisely, the influence of the spacelike data which also help reduce the errors.

C. Systematic and theoretical errors for the pion form factor contribution

Errors included in this work are divided into statistical and systematic. Evaluation of the statistical errors is standard: the fit procedure (using the program MINUIT) provides the full error (correlation) matrix at the χ^2 minimum. This matrix is used when calculating the corresponding integral for a_μ, therefore incorporating automatically all the correlations among the various fit parameters.

In addition, for every energy region, we have considered the errors that stem from experimental systematics, as well as those originating from deficiencies of the theoretical analysis. The experimental systematics covers the errors given by the individual experiments included in the fits. Also, when conflicting sets of data exist, the calculation has been repeated, and the given systematic error bar enlarged to encompass all the possibilities.

In general, errors (considered as uncorrelated) have been added in quadrature. The exceptions are explicitly discussed along the text.

We next discuss the errors that stem from experimental systematics, as well as those originating from deficiencies of the theoretical analysis for the 2π contribution, in the low-energy region $4m_\pi^2\leq t\leq0.8\ \mathrm{GeV}^2$. We start with the systematic errors of the data. They are evaluated by taking them into account in a new fit. In this way we find, in units of 10^{-11}, and neglecting the mass differences corrections [i.e., using Eq. (3.9a) for tau data]

$$\text{expt sys.}=\pm40\quad(e^+e^-+\tau),$$

$$\text{expt sys.}=\pm66\quad(e^+e^-).$$

To estimate the degree of correlation of the systematic errors pertaining to several experiments is a difficult task; we choose to consider the full range from 0 to 1. The error bars given cover all the possibilities. The χ^2/d.o.f. of the fit improves when taking these systematic errors into account to

$$\chi^2/\text{d.o.f.}=214/204\quad(e^+e^-+\tau),$$

$$\chi^2/\text{d.o.f.}=145/154\quad(e^+e^-).$$

The error given for the case in which we include the decays of the tau would be smaller, and the χ^2/d.o.f. would improve, if we used the correct kinematical formula for phase space, Eq. (3.10b); we would have obtained

$$\text{expt sys.}=\pm30;\quad \chi^2/\text{d.o.f.}=213/204\quad(e^+e^-+\tau).$$

In spite of this, we choose to accept the larger error (±40) as we feel that it includes residual effects of isospin breaking and electromagnetic corrections, different for the tau and e^+e^- cases, that we will discuss at the end of this subsection.

We discuss the systematic and theoretical errors in the higher energy regions in next section, but we mention here that the systematic error (4×10^{-11}) for 2π between $t=0.8$ and $1.2\ \mathrm{GeV}^2$ is added coherently to the lower energy 2π piece.

In addition to this we have several theoretical sources of error. First, that originating in the approximate character of

[10]A different case is the analysis of A. Pich and J. Portolés, Phys. Rev. D **63**, 093005 (2001), which also uses the Omnès equation method. The aim of this paper is *not* to give a precise determination of the two pion contribution to the anomaly, but to ascertain to what extent a number of theoretical considerations, especially chiral perturbation theory, can lead to a reasonable approximation. The authors use an expression for δ_1^1 [their equations (8) and (A2)], without left-hand cut or inelasticity cut; they also employ a mere Breit-Wigner to describe the phase in the rho region, where it is known that the rho deviates from this (e.g., our term b_1). They also do not include the cut at high energy in their equivalent of our G function. In what respects their results, the situation is as follows. The value Pich and Portolés gives (in units of 10^{-11} and for the contribution of 2π at energies below 1.2 GeV^2) is $5110\pm60(PP)$, with $\chi^2/\text{d.o.f.}=33.8/21$. This is substantially higher than other results: for example, we have 5044 ± 67 (our result, only timelike τ data), $\chi^2/\text{d.o.f.}=53/48$, and 5004 ± 51 (our best result, including e^+e^- and spacelike data), $\chi^2/\text{d.o.f.}=213/204$. (We have taken the piece $0.8\leq t\leq1.2\ \mathrm{GeV}^2$, equal to 230 ± 5, from e^+e^- data. The results of Narison (Ref. [7]) or of Davier and Höcker (Ref. [5]), using tau data only, are essentially like ours. No doubt the bias introduced by the simplified parametrization used in the paper of Pich and Portolés is responsible for this discrepancy. We are grateful to Pich and Portolés for discussions concerning their work.

the Gounnaris-Sakurai method for including the ω. This we estimate as discussed at the end of Sec. III B, getting on the average $\pm 7\times 10^{-11}$. Then, the dependence of our results on t_0 can be interpreted as a theoretical uncertainty, that we take equal to 4×10^{-11}. Composing these errors quadratically, we can complete Eq. (3.13) to

$$a(2\pi;\ t\leqslant 0.8\ \mathrm{GeV}^2)=\begin{cases}4774\pm 31(\mathrm{stat})\pm 41(\mathrm{sys.+th.})=4774\pm 51 & (e^+e^-+\tau+\mathrm{spacelike}),\\ 4754\pm 55\ (\mathrm{stat})\pm 66(\mathrm{sys.+th.})=4754\pm 86 & (e^+e^-+\mathrm{spacelike}).\end{cases}\tag{3.19}$$

To finish this subsection we will discuss in some detail some matters concerning to radiative corrections and isospin breaking in as much as they affect the error analysis. We will start with the analysis based on e^+e^- data. When evaluating the pion form factor we have used formulas, deduced in particular from unitarity and analyticity, that only hold if we neglect electromagnetic (e.m.) interactions. However, experimentalists measure the pion form factor in the real world, where the $\pi^+\pi^-$ certainly interact electromagnetically. Not only this, but the initial particles (e^+e^-) also interact between themselves, and with the pions.

These last electromagnetic interactions, however, can be evaluated and they are indeed subtracted when presenting experimental data on F_π; the uncertainties this process generates are estimated and included in the errors provided with the data. We will thus only discuss the uncertainties associated to e.m. interactions of the $\pi^+\pi^-$ alone. These particles may exchange a photon, or radiate a soft photon that is not detected (see the corresponding figures in Sec. V B). We may then define two quantities: $F_\pi^{(0)}$, which is the form factor we would have if there were no e.m. interactions; and $F_\pi^{(\mathrm{real})}$, which is the measured quantity, even after removal of radiative corrections for initial states or mixed ones. Actually, $F_\pi^{(\mathrm{real})}$ depends on the experimental setup through the cuts applied to ensure that no (*hard*) photon is emitted.

Our formalism, as developed in Sec. III A, applies to $F_\pi^{(0)}$, but we fit $F_\pi^{(\mathrm{real})}$. Therefore we are introducing an ambiguity

$$F_\pi^{(\mathrm{real})}-F_\pi^{(0)}$$

which is of order α.

The error induced by this ambiguity should be small. In fact, what enters into a_μ is the sum of the contribution of $F_\pi^{(\mathrm{real})}$, which is what we actually fit, and that of the state $\pi^+\pi^-\gamma$, which can be obtained from the process

$$e^+e^-\to(\gamma)\to\pi^+\pi^-\gamma.$$

For this reason, we believe that the error due to the mismatch of $F_\pi^{(\mathrm{real})}$ and $F^{(0)}$ is included in the errors to our fits here;[11] the estimated error for the $\pi^+\pi^-\gamma$ contribution, 9×10^{-11}, will be evaluated in Sec. V B.

Tau decay presents the same difficulties, and we expect a similar uncertainty as for e^+e^- collisions. But apart from the effect $F_\pi^{(\mathrm{real})}\neq F_\pi^{(0)}$ discussed, it poses extra problems when relating it to F_π. To discuss this, we take first $m_u\neq m_d$, $\alpha=0$; then we choose $\alpha\neq 0$, but $m_u=m_d$. Higher effects, proportional to $\alpha(m_u-m_d)$, α^2 and $(m_u-m_d)^2$ shall be neglected.

For $\alpha=0$, the masses of π^+ and π^0 become equal, but isospin invariance is still broken. This means that, in particular, the quantity Π^S in Eq. (3.10a) is nonzero and therefore the experimentally measured v_1 does not coincide with that in Eq. (3.10b). We expect this effect to be small, since it is of second order, $(m_d-m_u)^2$. If the scale is the QCD parameter Λ, then this will be of relative size 10^{-4}; but other effects need not be so small. We have tried to take them into account by allowing for different parameters for ρ^+, ρ^0; this produced a substantial shift, of about 40×10^{-11} for a_μ.

Then we set $m_u=m_d$ and take e.m. interactions to be nonzero. Apart from the effects already discussed, this produces the mass difference between charged and neutral pions. This we took (partially) into account when using the modified phase space of Eq. (3.10b). The ensuing effect for a_μ turned out to be small, $\sim 4\times 10^{-11}$.

Remnants of $\alpha\neq 0$ and $m_u\neq m_d$ will likely still affect the determination of F_π from e^+e^- and τ decay data; but we feel that accepting the systematic/theoretical error of 40×10^{-11} covers the related uncertainty.

IV. CONTRIBUTIONS TO $a^{(2)}$(h.v.p.) FROM $t>0.8$ GeV^2. THE FULL $a^{(2)}$(h.v.p.)

A. The higher energy contributions, and the 3π contribution

At higher energies we will get a substantial improvement over determinations based on old data [20] because of the existence of very precise measurements from Novosibirsk [9] and Beijing [12], gathered in the last two to three years, which will help remove a large part of the existing errors. This is particularly true of the region up to $t=3$ GeV^2 which caused an important part of the total errors in pre-1998 calculations of $a^{(2)}$(h.v.p.). We turn to it next.

1. The region up to $t=3$ GeV^2

We consider first the contribution of two, three, four pion, . . . , and *KK* intermediate states for $0.8\leqslant t\leqslant 1.2$ GeV^2. In what follows n.w.a. will mean *narrow width approximation*, r.d.a. *resonance dominance approximation* (but not narrow approximation), and s.o.i.c. *sum over individual channels*. For the n.w.a. we use the standard formula. Denoting by $\Gamma_{ee}(V)$ to the width into e^+e^- of a vector resonance *V* with mass *M*, its contribution to $a^{(2)}$(h.v.p.) is given in this approximation by

[11] In particular for the evaluation including τ decay data; see below.

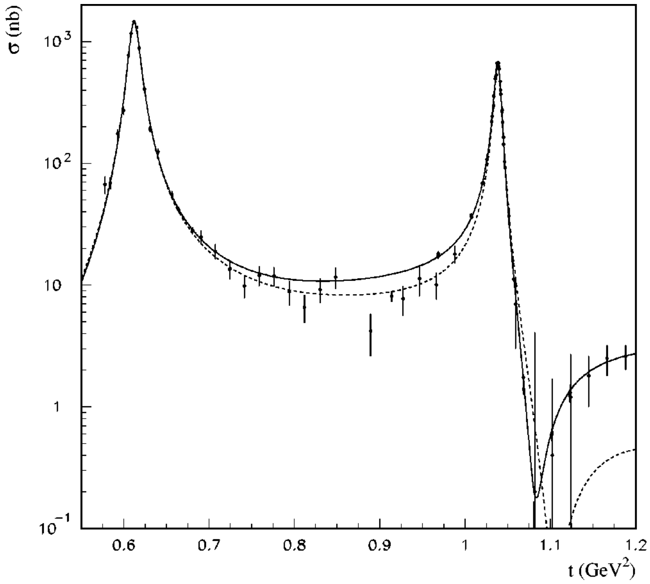

FIG. 6. Plot of the fit to the $e^+e^-\to 3\pi$ cross section up to $t=1.2\ \mathrm{GeV}^2$, with data from Ref. [9]. Continuous line: fit to CMD2 and SND data. Dashed line: fit to CMD2 and ND.

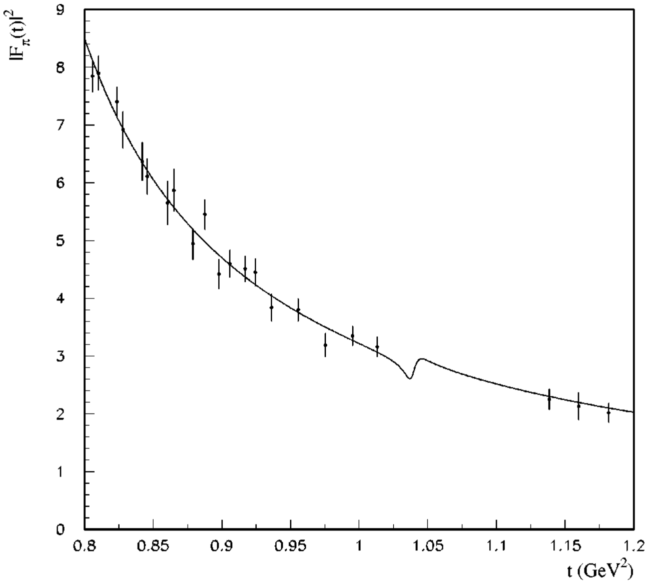

FIG. 7. Plot of the fit to $|F_\pi(t)|^2$ in the region $0.8\leqslant t\leqslant 1.2\ \mathrm{GeV}^2$.

$$a(V)=\frac{3\Gamma_{ee}(V)\hat{K}(M^2)}{\pi M}. \tag{4.1}$$

The uncertainty on $a(V)$ is calculated by Gaussian error propagation for the parameters in Eq. (4.1). In practice, it is dominated by the experimental error of the electronic width.

3π states, $9m_\pi^2\leqslant t\leqslant 1.2\ \mathrm{GeV}^2$. In the narrow width approximation one gets the ω, ϕ contributions:

$$\begin{aligned}10^{11}\times a(3\pi;\omega)&=348\pm 13,\\ 10^{11}\times a(3\pi;\phi)&=62\pm 3,\end{aligned} \tag{4.2}$$

but this misses the region between ω and ϕ, and the interference effect just above the last. So we will use experimental data [9]. This gives

$$10^{11}\times a(3\pi;t\leqslant 1.2\ \mathrm{GeV}^2)=438\pm 4(\mathrm{stat.})\pm 11(\mathrm{sys.}). \tag{4.3}$$

The fit to the 3π experimental cross section, with data from Ref. [9], may be found in Fig. 6. The upper curve (continuous line in Fig. 6) is a fit to the CMD2 and SND data. We have used a Breit-Wigner parametrization for the ω and ϕ resonances, plus a constant term and the exact threshold factor for 3π states. The χ^2/d.o.f. is 63/60; we consider this our central result here. The dashed curve fits instead the data from CMD2 and ND (Dolinsky *et al.*, Ref. [20]); the quality of the fit is poorer (χ^2/d.o.f.$=52/37$). It fits better the region between the ω and ϕ, but fails to reproduce the data beyond 1.06 GeV^2. In fact, we include the second fit to estimate the corresponding systematic uncertainty; the small difference in terms of the integrals between the two fits, 8×10^{-11}, is included into the systematic error.

2π states, $0.8\leqslant t\leqslant 1.2\ \mathrm{GeV}^2$. This 2π state contribution is

$$10^{11}\times a(2\pi;t\leqslant 1.2\ \mathrm{GeV}^2)=230\pm 3\pm 4. \tag{4.4}$$

The evaluation of the contribution of the 2π state has greatly improved (with respect to older calculations) because of the information from recent Novosibirsk [8] data on $e^+e^-\to 2\pi$. We have fitted the experimental value of $|F_\pi|^2$ with an expression $1/(bt+a),a,b$ completely free parameters; the result of this fit may be seen depicted in Fig. 7. [A similar result is obtained if we extended our earlier calculation of $F_\pi(t)$ to $t\sim 1.2\ \mathrm{GeV}^2$ by setting $t_0=1.2$; but we prefer the result based only on experimental data.] Of the two errors given for the 2π contribution the first is statistical and the second, systematic, has been added *coherently* to the systematic error on the low-energy 2π contribution, as discussed in Sec. III C.

The wiggle in Fig. 7 for $t\sim m_\phi^2$ is due to the interference of the decay $\phi\to 2\pi$. This is similar to the ω-ρ effect, and has been treated in a similar manner; we have incorporated it using the formulas and parameters given by Achasov *et al.* [9]. The influence of this effect on a_μ is minute.

KK states, $0.8\leqslant t\leqslant 1.2\ \mathrm{GeV}^2$. An important contributions is that of KK states. In the n.w.a., this is given by the ϕ:

$$10^{11}\times a(KK;\phi)=332\pm 9, \tag{4.5}$$

but this is a dangerous procedure here; the vicinity of the KK threshold distorts the shape of the resonance. We thus have to calculate this KK contribution directly from experiment. We have used two fitting procedures. In the first, we fit simultaneously the K^+K^- and K_LK_S data of Achasov *et al.* [9], with the same parameters for the ϕ. We get

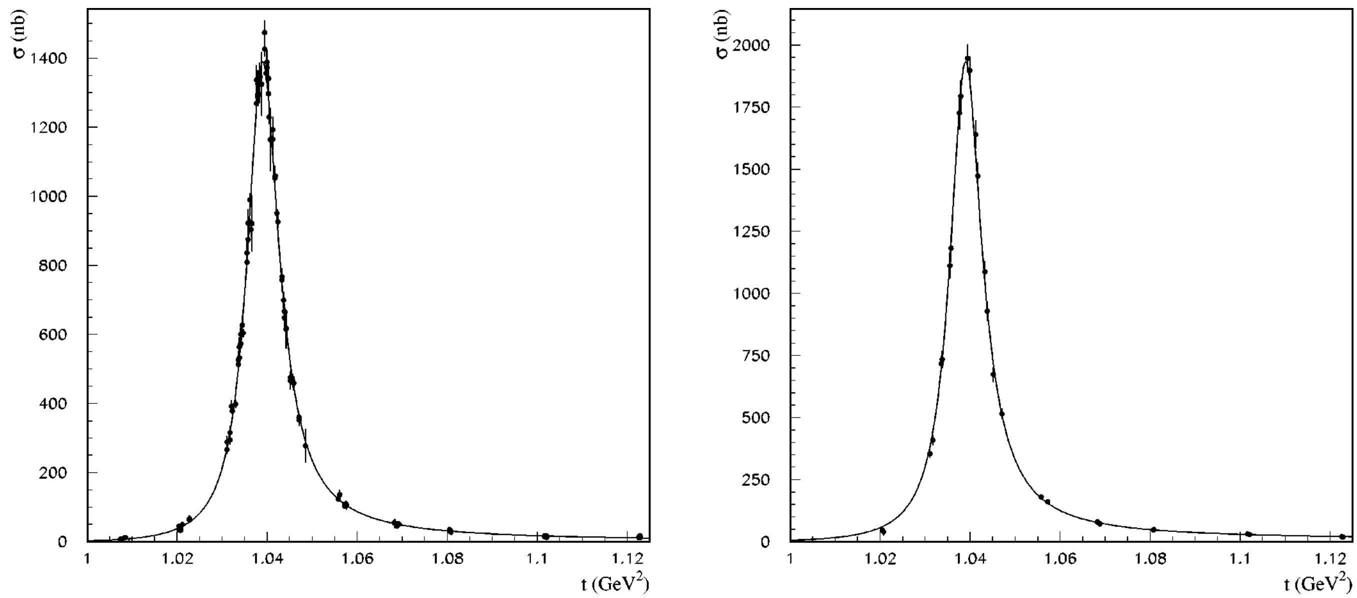

FIG. 8. Plot of the fit to the cross section $e^+e^-\to K_LK_S$ (left), and to $e^+e^-\to K^+K^-$ (right). Data are from Ref. [9].

$$10^{11}\times a(K^+K^-;t\leqslant 1.2\ \mathrm{GeV}^2)$$
$$=185.5\pm 1.5(\mathrm{stat.})\pm 13(\mathrm{sys.})$$

and

$$10^{11}\times a(K_LK_S;t\leqslant 1.2\ \mathrm{GeV}^2)=129.5\pm 0.7.$$

The quality of the fit, shown in Fig. 8, is good (χ^2/d.o.f. $=84/82$).

In the second fitting procedure, we add the K_LK_S data of Akhmetshin *et al.* [9], obtaining the result

$$10^{11}\times a(K_LK_S;t\leqslant 1.2\ \mathrm{GeV}^2)$$
$$=128.4\pm 0.5(\mathrm{stat.})+2.6(\mathrm{sys.}).$$

The fit is now less good, but the integrals are essentially identical for both fits. Adding the *KK* results together we find

$$10^{11}\times a(KK;t\leqslant 1.2\ \mathrm{GeV}^2)=314\pm 2(\mathrm{stat.})\pm 13(\mathrm{sys.}). \tag{4.6}$$

The systematic errors have been obtained repeating the fits, including now the systematic errors given by the experiments.

We mention in passing that the ratio of contributions of K^+K^- and K_LK_S, 1.44, agrees well with the ratio [16]

$$\frac{\Gamma(\phi\to K^+K^-)}{\Gamma(\phi\to K_LK_S)}=1.46\pm 0.03.$$

Other states: 4π, 5π, $\eta\pi^0\pi^0$ . . . ; $0.8\leqslant t\leqslant 1.2$ GeV^2.

The four pion contribution, including the quasi-two-body state $\omega\pi$, may be obtained from recent Novosibirsk data [9], or from the compilation of Dolinsky *et al.* [20]. If we use the last we get

$$10^{11}\times a(4\pi;t\leqslant 1.2\ \mathrm{GeV}^2)=25\pm 4;$$

if we fit the data of Akhmetshin *et al.* [9] we find

$$10^{11}\times a(4\pi;t\leqslant 1.2\ \mathrm{GeV}^2)=18\pm 3.$$

Of the 4π contribution most is due to the $\omega\pi^0$ channel; only a small fraction (2.4×10^{-11}) comes from the $\pi^+\pi^-\pi^+\pi^-$ states. We take, for this 4π contribution, the figure

$$10^{11}\times a(4\pi;t\leqslant 1.2\ \mathrm{GeV}^2)=20\pm 5, \tag{4.7}$$

which covers all possibilities.

The five, six, . . . , pions as well as $\omega\to\eta+2\pi^0$ contributions are very small [16,20]. Altogether, they give

$$10^{11}\times a(5\pi,6\pi,\eta\pi^0\pi^0,\ \ldots,\ \ t\leqslant 1.2\ \mathrm{GeV}^2)=4\pm 2. \tag{4.8}$$

We present the summary of our results in the important region $0.8\leqslant t\leqslant 1.2$ GeV^2 plus the 3π contribution below 1.2 GeV^2 in Table II.

$1.2\leqslant t\leqslant 2$ GeV^2. We consider three determinations:

$$\begin{aligned} &270\pm 27\ \ (\mathrm{here}),\\ &278\pm 25\ \ (\mathrm{s.o.i.c.,\ \ CLY\ [4]}),\\ &265\pm 22\ \ (\mathrm{VMD+QCD;AY}).\end{aligned} \tag{4.9}$$

The first is obtained from a numerical integration of the data [20], with a parabolic fit. The method referred to as "VMD+QCD;AY," details of which can be found in the AY [4] paper, consists in interpolating between a vector meson dominance (VMD) calculation for quasi-two-body processes ($\omega\pi$, $\rho\pi$, . . .), plus a Breit-Wigner expression for two-body channels ($\pi\pi$,KK, . . .) at the lower end, and perturbative QCD at the upper end, the interpolation being obtained by

TABLE II. Contribution to $a^{(2)}$ of various channels up to $t=1.2\ \mathrm{GeV}^2$ (2π below $0.8\ \mathrm{GeV}^2$ *not* included).

Channels		Comments
$\pi^+\pi^-$	$230\pm3\pm4$	$0.8\leqslant t\leqslant 1.2\ \mathrm{GeV}^2$
3π	$438\pm4\pm11$	$9m_\pi^2\leqslant t\leqslant 1.2\ \mathrm{GeV}^2$
K^+K^-	$186\pm2\pm13$	
K_LK_S	$128\pm1\pm2$	
4π	20 ± 5	including $\omega\pi^0$
Multipion, $\eta+2\pi$, ...	4 ± 2	
Total	1006 ± 19	syst. error for 2π not included

fitting experimental data (see Fig. 9). Because we want to present a result as model independent as possible, we will take as our preferred figure that obtained here from experimental data:

$$10^{11}\times a(1.2\leqslant t\leqslant 2\ \mathrm{GeV}^2)=270\pm27. \tag{4.10}$$

$2\leqslant t\leqslant 3\ \mathrm{GeV}^2$.

$$\begin{aligned}&240\pm3(\Lambda)\pm3(\mathrm{cond.})\quad(\mathrm{QCD}),\\ &250\pm19\quad(\mathrm{N}[7]\ \text{r.d.a.};\ \text{only}\ e^+e^-),\\ &276\pm36\quad(\mathrm{N}\ [7],\ \text{r.d.a.};\ e^+e^-+\tau),\\ &222\pm5\ \ (\mathrm{stat})\pm15\ \ (\mathrm{sys.})\ \ (\mathrm{J,expt\ data}).\end{aligned} \tag{4.11}$$

J here denotes an evaluation, integrating with the trapezoidal rule, of a compilation of experimental data supplied by F. Jegerlehner.

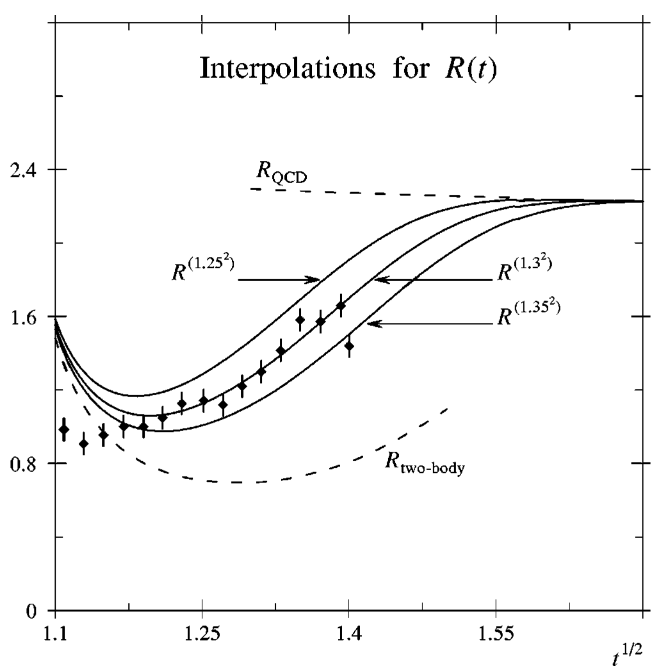

FIG. 9. Experimental data and various interpolations between the VMD calculation, for small t, and QCD for larger t. From AY, Ref. [4]; data from Ref. [20].

For the QCD calculations we take the following approximation: for n_f massless quark flavors, with charges Q_f, we write

$$\begin{aligned}R^{(0)}(t)=3\sum_f Q_f^2\Bigg\{&1+\frac{\alpha_s}{\pi}+(1.986-0.115n_f)\left(\frac{\alpha_s}{\pi}\right)^2\\ &+\left(-6.64-1.20n_f-0.005n_f^2-1.240\frac{(\Sigma_fQ_f)^2}{3(\Sigma_fQ_f^2)}\right)\\ &\times\left(\frac{\alpha_s}{\pi}\right)^3\Bigg\}.\end{aligned}$$

To this one adds mass and nonperturbative corrections. We take into account the $O(m^2)$ effect for quarks with running mass $\overline{m}_i(t)$, which correct $R^{(0)}$ by the amount[12]

$$-3\sum_i Q_i^2\overline{m}_i^2(t)\left\{6+28\frac{\alpha_s}{\pi}+(294.8-12.3n_f)\left(\frac{\alpha_s}{\pi}\right)^2\right\}t^{-1}.$$

Finally, for the condensates we add

$$\frac{2\pi}{3}\left(1-\frac{11}{18}\frac{\alpha_s}{\pi}\right)\langle\alpha_sG^2\rangle\sum_f Q_f^2$$

and

$$24\pi^2\left[1-\frac{23}{27}\frac{\alpha_s}{\pi}\right]m_i\langle\overline{\psi}_i\psi_i\rangle.$$

We neglect the condensates corresponding to heavy quarks (c, b) and express those for u, d, s in terms of $f_\pi^2m_\pi^2$, $f_K^2m_K^2$ using the well-known PCAC (partial conservation of axial vector current) relations.

In the QCD calculation, the error labeled "cond." is found by inserting the variation obtained setting quark and gluon condensates to zero, and that labeled Λ by varying the QCD parameter. For this parameter we take the recent determinations [21] that correspond to the value

$$\alpha_s(M_Z^2)=0.1172\pm0.003;$$

[12]The corrections of order m^4 may be found in the paper of Narison [7], together with references. We have checked that the effect of this correction is smaller than the errors of the leading terms.

to be precise, we have taken (in MeV, and to four loops),

$$\Lambda = 373 \pm 80,\ t \leqslant m_c^2;\ \ \Lambda = 283 \pm 50,\ m_c^2 \leqslant t \leqslant m_b^2;$$

$$\Lambda = 199 \pm 30,\ t \geqslant m_b^2.$$

For the gluon condensate we take $\langle \alpha_s G^2 \rangle = 0.07\ \mathrm{GeV}^4$.

The four evaluations give comparable results, with the r.d.a. ones larger, and presenting also larger errors. As proved by the reliability of QCD calculations of semileptonic τ decays, a similar process in a similar energy range, we think perturbative QCD can be trusted here, so we select the corresponding value as our best result. We write thus

$$10^{11} \times a(2 \leqslant t \leqslant 3\ \mathrm{GeV}^2) = 240 \pm 6, \tag{4.12}$$

where we have added linearly the errors due to Λ and the condensates.

As a verification of the reliability of the calculation, as well as the improvement it presents when compared with earlier determinations, in the rather involved energy range $0.8 \leqslant t \leqslant 3\ \mathrm{GeV}^2$ (including here the full 3π contribution), we compare our value here (adding, for the occasion, the channels ω, $\phi \to \pi^0 \gamma$, $\eta\gamma$, see Sec. V B) with that obtained by Narison [7] who uses resonance dominance and s.o.i.c., and to the old CLY [4] evaluation, with s.o.i.c. and QCD:

$$\begin{aligned} &10^{11} \times a(0.8 \leqslant t \leqslant 3\ \mathrm{GeV}^2 + \omega \to 3\pi) = 1559 \pm 34\ \ (\text{here}),\\ &10^{11} \times a(0.8 \leqslant t \leqslant 3\ \mathrm{GeV}^2 + \omega \to 3\pi) = 1631 \pm 46\\ &\quad (\text{Narison},\ \ \tau + e^+e^-),\\ &10^{11} \times a(0.8 \leqslant t \leqslant 3\ \mathrm{GeV}^2 + \omega \to 3\pi) = 1675 \pm 65\\ &\quad (\text{Narison},\ \ e^+e^-),\\ &10^{11} \times a(0.8 \leqslant t \leqslant 3\ \mathrm{GeV}^2 + \omega \to 3\pi) = 1618 \pm 97\\ &\quad (\text{CLY},\ \ e^+e^-). \end{aligned} \tag{4.13}$$

The compatibility between the results, using different methods of evaluation for many pieces, is reasonable.

2. The region $3 \leqslant t \leqslant 4.6^2\ \mathrm{GeV}^2$

This is another region where the availability of recent precise data [12] in the neighborhood of the $\bar{c}c$ threshold, previously poorly known, permits a reliable evaluation. As a by-product, we get an experimental validation of QCD calculations.

$3 \leqslant t \leqslant 2^2\ \mathrm{GeV}^2$. We use perturbative QCD here and get

$$10^{11} \times a(3 - 2^2\ \mathrm{GeV}^2) = 120 \pm 0.8(\Lambda) \pm 0.8(\text{cond.}).$$

$2^2 \leqslant t \leqslant 3^2\ \mathrm{GeV}^2$. We have now very good recent experimental data. So we present two determinations:

$$10^{11} \times a(2^2 - 3^2\ \mathrm{GeV}^2) = 200 \pm 1(\Lambda)\ \ (\text{QCD}),$$

$$10^{11} \times a(2^2 - 3^2\ \mathrm{GeV}^2) = 210 \pm 3(\text{stat}) \pm 14(\text{sys.})\quad (\text{expt BES}).$$

We only give the error due to Λ here because that due to the condensates is negligible. When integrating the BES data we have used the trapezoidal rule. If instead we fitted a horizontal line, we would have obtained

$$10^{11} \times a(2^2 - 3^2\ \mathrm{GeV}^2) = 207 \pm 2(\text{stat}) \pm 13(\text{sys.})\quad (\text{expt BES}).$$

The BES [12] purely experimental result and the QCD calculation are compatible, but one has to take into account the *systematic* errors of the first. This shows clearly the importance of systematic variations in e^+e^- annihilations data. We take as our preferred value for the sum of the two intervals that obtained from the QCD calculations:

$$10^{11} \times a(3 \leqslant t \leqslant 3^2\ \mathrm{GeV}^2) = 320 \pm 2 \pm 1 = 320 \pm 3. \tag{4.14}$$

$3^2 \leqslant t \leqslant 4.6^2\ \mathrm{GeV}^2$. We give here the results in units of 10^{-11}. We separate the contribution of the J/ψ, ψ', that we calculate in the n.w.a., and the rest. For the first we have

$$62.0 \pm 4.0\ \ J/\psi,$$

$$14.8 \pm 1.3\ \ \psi'.$$

For the remainder we have the following possibilities:

$$91 \pm 0.4(\Lambda)\ \ uds;\ \ (\text{QCD}; 3^2 \leqslant t \leqslant 4.6^2\ \mathrm{GeV}^2),$$

$$4.0 \pm 0.4\ \ \psi'',\ \ \psi''',\ \ \psi^{\mathrm{IV}}\ \ (\text{N},\ \ \text{r.d.a.}),$$

total: 172 ± 4 (N; QCD+r.d.a.),

$$91 \pm 0.4(\Lambda)\ \ \text{uds};\ (\text{QCD}; 3^2 \leqslant t \leqslant 4.6^2\ \mathrm{GeV}^2),$$

$$46.8 - 28.6 \to 38 \pm 10\ \ \bar{c}c.\ \ (\text{AY},\ \ \text{NRQCD}),$$

total: 206 ± 11 (AY; QCD+NRQCD)

$$54.7 \pm 0.3(\Lambda)\ \ (\text{QCD}): 3.0^2 \leqslant t \leqslant 3.7^2,$$

$$56 \pm 0.3 \pm 3\ \ (\text{expt, BES});\ \ 3.7^2 \leqslant t \leqslant 4.6^2,$$

total: $188 \pm 4 \pm 3$ (expt, BES+QCD).

Here N refers to the paper of Narison [7], and AY is in Ref. [4]. BES are the experimental data from Ref. [12]. The first error for them is the statistical, the second the systematic one.

This region merits a somewhat detailed discussion, as there is a certain controversy about it. We have made the calculation in three different manners. First, we separate the u, d, s quarks contribution, that can be evaluated using perturbative QCD. The contribution of the $\bar{c}c$ states is then evaluated saturating it by the ψ resonances, in the r.d.a.; this

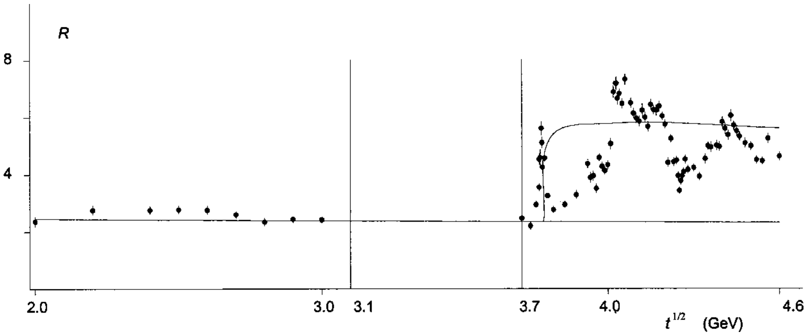

FIG. 10. Plot of BES experimental data and QCD for the *u, d, s* quarks (lower *t*) and the same plus NRQCD for the *c* quark contribution, from $t=3.74^2\ \mathrm{GeV}^2$ to $4.6^2\ \mathrm{GeV}^2$ at the right. Only systematic errors shown for experimental data. Statistical ones are even smaller.

is the result labeled (N, r.d.a.). This saturation procedure does not produce a good description.

In a second method one separates also the *u, d, s* contribution; but the $\bar{c}c$ one is treated differently. If a resonance is below the channel for open charm production, which is set at $t=4m_c^2$ (with *c* the pole mass of the *c* quark), then it is treated as a bound state, in the n.w.a. Above $\bar{c}c$ threshold, one uses nonrelativistic QCD (see Adel and Yndurain (AY), Refs. [4] and [22]). The two values reported above for such a calculation [AY, nonrelativistic QCD (NRQCD)] are for two values of the *c* quark mass: $m_c=1.750$ GeV, in which case only the J/ψ should be taken to be below threshold, and $m_c=1.866$ GeV and then both J/ψ and ψ' are to be added below threshold. This last gives the smallest number (28.6). The result of the calculation is taken as the average of both numbers, with half the difference as the estimated error. In Fig. 10 one can see the BES [12] data and the predictions of QCD and NRQCD, the last for $m_c=1.87$ GeV.

The third method, which is the one that yields our preferred number,

$$10^{11}\times a(3^2\leqslant t\leqslant 4.6^2\ \mathrm{GeV}^2)=188\pm 4\pm 3, \quad (4.15)$$

is obtained by using QCD for *u, d, s* quarks *plus* J/ψ, ψ' below $t=3.7^2\ \mathrm{GeV}^2$, and experimental data (BES [12]) above that energy.

All three methods give overlapping results, within errors, with the r.d.a. below experiment, and with an underestimated error, and with the NRQCD calculation reproducing better the data. This NRQCD calculation depends strongly on the mass of the *c* quark and, in fact, one can turn the argument backwards and *predict* m_c by requiring equality with the experimental figure. If we do so, we find

$$m_c\simeq 1.89\ \mathrm{GeV},$$

a very reasonable estimate consistent with the two loop result [23], correct to $O(\alpha_s^4)$, which gives $m_c=1.866\pm 0.20$ GeV.

3. The region $t\geqslant 4.6^2\ \mathrm{GeV}^2$

The results from this region have not changed noticeably, but we give them for completeness.

$4.6^2\leqslant t\leqslant 11.2^2\ \mathrm{GeV}^2$. For the first Υ resonances, and in units of 10^{-11},

$$0.55\pm 0.03\ \ \Upsilon,$$

$$0.18\pm 0.01\ \ \Upsilon'.$$

Then,

$$88.8\pm 1.0(\Lambda)\ \ udsc:\ (\mathrm{QCD}),\ \ 4.6^2\leqslant t\leqslant 11.2^2\ \mathrm{GeV}^2,$$

$$0.22\pm 0.04\ \ \bar{b}b:\ (\mathrm{N},\ \mathrm{n.w.a.}),\Upsilon'',\Upsilon''',\ldots,$$

$$0.53\pm 0.08\ \ \bar{b}b:\ (\mathrm{AY},\ \mathrm{NRQCD}).$$

Adding this, we get
total: 90 ± 1 (N; QCD+n.w.a.),
total: 90 ± 1 (AY; QCD+NRQCD).

The notation is like for the *c* threshold region. The error in the (AY, NRQCD) evaluation is due to the error in the QCD parameter, Λ, and the *b* quark pole mass, for which we have taken [23] $m_b=5.00\pm 0.10$ GeV. Both figures are essentially identical and we thus take

$$10^{11}\times a(4.6^2\leqslant t\leqslant 11.2^2\ \mathrm{GeV}^2)=90\pm 1. \quad (4.16)$$

$11.2^2\ \mathrm{GeV}^2\leqslant t\to\infty\ \mathrm{GeV}^2$. The use of QCD is mandatory here. The contribution above $\bar{t}t$ threshold is negligible, so we calculate with $n_f=5$ and get

$$10^{11}\times a(11.2^2\ \mathrm{GeV}^2\leqslant t\to\infty)=21\pm 0.1(\Lambda). \quad (4.17)$$

B. The whole $a^{(2)}$ (h.v.p.)

Our final result for the $O(\alpha^2)$ hadronic contribution to a_μ is then

$$10^{11}\times a^{(2)}(\mathrm{h.v.p.})=\begin{cases}6909\pm 64 & (e^+e^-+\tau+\mathrm{spacel.}),\\ 6889\pm 96 & (e^+e^-+\mathrm{spacel.}).\end{cases} \quad (4.18)$$

To compare with other evaluations we have to add the contribution $(43\pm 4)\times 10^{-11}$ of some of the radiative decays of the ρ, ω, ϕ (see Sec. V B) that other authors include. This comparison is shown, for a few representative calculations,[13] in Table III.

[13] A more complete list of evaluations, including some of the very earliest ones, may be found in the paper of Narison, Ref. [7].

TABLE III. KNO: Ref. [3]; BW: Ref. [3]; J: Ref. [6]; CLY, CLY-II, AY: Ref. [4]; N: Ref. [7]; DH, ADH: Ref. [5].

Authors	$10^{11}\times a$(h.v.p.)	Comments
KNO	7068 ± 174	e^+e^- only
CLY	7100 ± 116	e^+e^-+spacel.
CLY-II	7070 ± 116	e^+e^-+sp.$+\pi\pi$ ph. shifts
ADH	7011 ± 94	$e^+e^-+\tau$
BW	7026 ± 160	e^+e^-
AY	7113 ± 103	e^+e^-+spacel.
DH	6924 ± 62	$e^+e^-+\tau$
J	6974 ± 105	
N1	7031 ± 77	$e^+e^-+\tau$
N2	7011 ± 117	e^+e^- only
TY1	6952 ± 64	$e^+e^-+\tau+$spacel.
TY2	6932 ± 96	e^+e^-+spacel. only

If we had added also the other radiative contributions ($\pi\pi\gamma$, and the continuum hadron $+\gamma$, cf. Sec. V B) we would have found the hadronic vacuum polarization piece, correct to order α^2 and α^3,

$$10^{11}\times a^{(2+3)}(\text{h.v.p.})=\begin{cases}7002\pm66 & (e^+e^-+\tau+\text{spacel.})\\ 6982\pm97 & (e^+e^-+\text{spacel.}).\end{cases} \tag{4.19}$$

V. HIGHER-ORDER HADRONIC CONTRIBUTIONS

A. Hadronic light-by-light contributions

A contribution in a class by itself is the hadronic light-by-light one. So we split

$$\begin{aligned}a[\text{other hadronic},\ O(\alpha_3)]\\ =a[\text{``one blob'' hadronic},\ O(\alpha^3)]\\ +a(\text{hadronic light by light}).\end{aligned} \tag{5.1}$$

We will start by considering the last, given diagrammatically by the graph of Fig. 11. This can be evaluated only using *models*. One can make a chiral model calculation, in the Nambu-Jona-Lasinio version or the chiral perturbation theory variety, with a cutoff, or one can use a constituent quark model in which we replace the blob in Fig. 11 by a quark loop (Fig. 12). The result depends on the cutoff (for the chiral calculation) or on the constituent mass chosen for the quarks. After the correction of a sign error in the evaluations of Ref. [24] by Knecht and Nyffeler [25], confirmed in Hayakawa and Kinoshita [26]

$$10^{11}\times a(\text{hadronic light by light})=86\pm25,\quad \text{chiral calculation; BPP, HKS.} \tag{5.2a}$$

Earlier calculations with the chiral model, using VMD to cure its divergence, gave (HKS, Ref. [24])

$$10^{11}\times a(\text{hadronic light by light})=52\pm20,\quad \text{chiral calculation (HKS).} \tag{5.2b}$$

For the constituent quark model we use the results of Laporta and Remiddi [27]. The contribution to a_μ of light by light scattering, with a loop with a fermion of charge Q_i, and mass m_i larger than the muon mass, is

$$a_{l\times l;i}=Q_i^4\left(\frac{\alpha}{\pi}\right)^3 c_{l\times l,i},$$

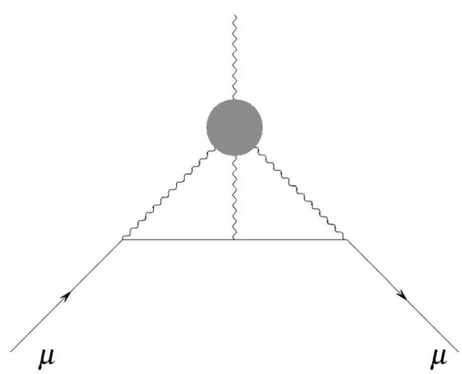

FIG. 11. The hadronic light-by-light contributions to the muon magnetic moment. (Only one graph is shown.)

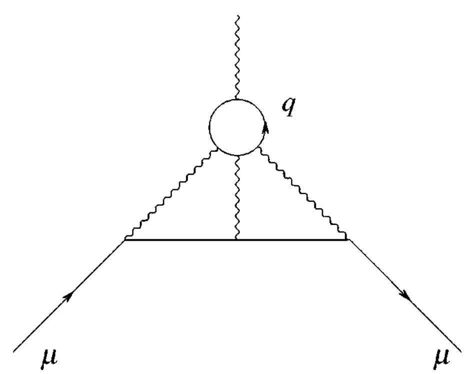

FIG. 12. The light-by-light hadronic correction in the constituent quark model.

where, to order $(m_\mu/m_i)^4$,

$$c_{l\times l,i}=\left(\frac{m_\mu}{m_i}\right)^2\left[\frac{3}{2}\zeta(3)-\frac{19}{16}\right]+\left(\frac{m_\mu}{m_i}\right)^4\left[-\frac{161}{810}\log^2\frac{m_i}{m_\mu}-\frac{16189}{48600}\log\frac{m_i}{m_\mu}+\frac{13}{18}\zeta(3)-\frac{161}{9720}\pi^2-\frac{831931}{972000}\right]+\cdots.$$

Taking constituent masses,

$$m_{u,d}=0.33,\quad m_s=0.50,\quad m_c=1.87\text{ GeV},$$

we find

$$10^{11}\times a(\text{hadronic light by light})=46\pm 10 \quad (\text{quark const model}) \tag{5.2c}$$

and the error is estimated by varying $m_{u,d}$ by 10%.

One could also take the estimate of the π^0 pole from Hayakawa and Kinoshita [24] *and* add the constituent quark loop, in which case we get

$$10^{11}\times a(\text{hadronic light by light})\sim 98\pm 22 \quad (\text{quark const model}+\text{pion pole}). \tag{5.2d}$$

One expects the chiral calculation to be valid for small values of the virtual photon momenta, and the constituent model to hold for large values of the same.[14] Thus almost half of the contribution to a (hadronic light by light) in the chiral calculation comes from a region of momenta above 0.5 GeV, where the chiral perturbation theory starts to fail, while for this range of energies, and at least for the imaginary part of (diagonal) light by light scattering, the quark model reproduces reasonably well the experimental data (see, for example, Ref. [28] for a recent review of this).

We will take here the figure, which comprises the relevant determinations,

$$10^{11}\times a(\text{hadronic light by light})=92\pm 20. \tag{5.2e}$$

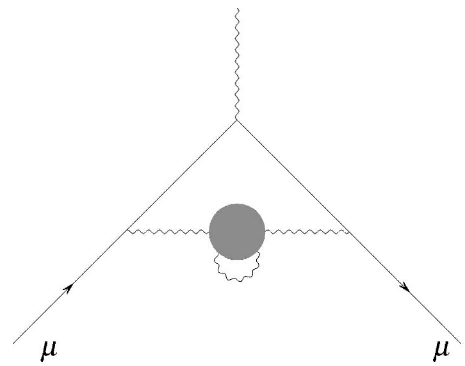

FIG. 13. The $O(\alpha^3)$ hadronic correction a (h.v.p., γ).

B. Photon radiation corrections to the hadronic vacuum polarization

The a [one blob hadronic, $O(\alpha^3)$] corrections are obtained by attaching a photon or fermion loop to the various lines in Fig. 1. They can be further split into two pieces: the piece where both ends of the photon line are attached to the hadron blob, a (h.v.p., γ), shown in Fig. 13, and the rest. So we write

$$a[\text{one blob hadronic, } O(\alpha_3)]=a(\text{h.v.p., }\gamma)+a(\text{one blob hadronic, rest}). \tag{5.3}$$

The last can be evaluated [29] in terms of the hadronic contributions to the photon vacuum polarization, finding

$$10^{11}\times a(\text{one blob hadronic, rest})=-101\pm 6 \tag{5.4}$$

(note, however, that this result has not, as far as we know, been checked by an independent calculation).

The only contribution that requires further discussion is that depicted in Fig. 13, a (h.v.p., γ). In principle, this contribution can be evaluated straightforwardly by a generalization of the Brodsky–de Rafael method. We can write

$$a^{(2)}(\text{h.v.p.})+a(\text{h.v.p., }\gamma)=\int_{4m_\pi^2}^{\infty}dt\,K(t)R^{(2)}(t), \tag{5.5}$$

where

$$R^{(2)}(t)=\frac{\sigma^{(0)}(e^+e^-\to\text{hadrons})+\sigma^{(2)}(e^+e^-\to\text{hadrons})+\sigma^{(0)}(e^+e^-\to\text{hadrons; }\gamma)}{\sigma^{(0)}(e^+e^-\to\mu^+\mu^-)}.$$

[14] Strictly speaking, one would also need large momentum of the external photon to get a really trustworthy evaluation with the constituent model.

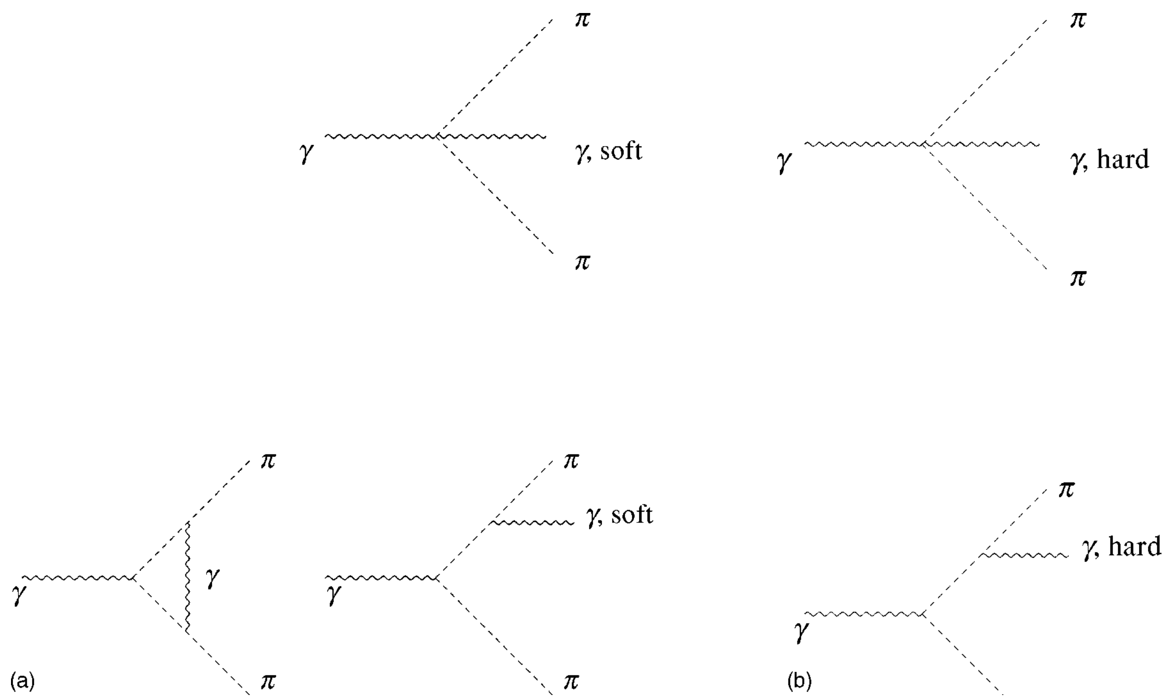

FIG. 14. (a) Diagrams included in the pion form factor. (b) Diagrams *not* included in the pion form factor.

The notation means that we evaluate the hadron annihilation cross section to second order in α, and we add to it the first order annihilation into hadrons plus a photon.

For energy (t) large enough, this can be calculated with the parton model, and leads to a correction $3/4(\Sigma_f Q_f^4/\Sigma_f Q_f^2)\alpha/\pi$ times the parton model evaluation. Taking then $t \geq 1.2\ \mathrm{GeV}^2$, this is $(0.76\pm 0.04)\times 10^{-11}$. The error is that due to Λ and the masses of c, b quarks. We have excluded the contribution of the radiative decays of the J/ψ, ψ', Υ, Υ' resonances since we have taken these into account already (we took the full e^+e^- width for them).

Then comes the contribution of small momenta, $t \leq 1.2\ \mathrm{GeV}^2$. We start by discussing the process involving two pions. In our determination in Secs. III and IV of $a^{(2)}$ (h.v.p.), we made calculations by fitting the experimental cross section $e^+e^- \to \pi^+\pi^-$, which specifically excludes radiation of *hard* photons (hard photons defined as those that are identified experimentally). Diagrammatically, this means that our evaluations of Secs. III and IV included the diagrams of Fig. 14(a) (where a soft photon is one that is not detected), but not those of Fig. 14(b) (radiation of a hard photon). So, we have to include this radiation into a (h.v.p., γ). This can be easily done if we consider this region to be dominated by the rho, hence we approximate

$$\sigma^{(0)}(e^+e^- \to \text{hadrons};\ \ \gamma) \simeq \sigma^{(0)}(e^+e^- \to (\rho) \to \pi^+\pi^-\gamma).$$

The last can be evaluated in terms of the branching ratio for the decay $\rho \to \pi^+\pi^-\gamma$, which is indeed measured experimentally (see the review of Dolinsky *et al.*, Ref. [20]) from the reaction $e^+e^- \to \rho \to \pi^+\pi^-\gamma$. In the narrow width approximation for the rho, the contribution to a_μ is

$$\frac{\Gamma(\rho \to \pi^+\pi^-\gamma)}{\Gamma_\rho}\,\frac{3\Gamma_{ee}(\rho)\overline{K}(m_\rho^2)}{\pi m_\rho}. \qquad (5.6a)$$

In this way, we find

$$10^{11}\times a(\text{h.v.p.}, \pi^+\pi^-\gamma) = 45 \pm 7 \quad (\text{n.w.a}), \qquad (5.6b)$$

and the error is that induced by the experimental error in the width $\Gamma(\rho \to \pi^+\pi^-\gamma)$.

We will elaborate a bit more on this contribution. The final state interaction of the $\pi^+\pi^-$ in the state $\pi^+\pi^-\gamma$ is very strong. The pions are produced in an S wave, which presents a wide enhancement [15] in the energy region $E_{\pi^+\pi^-} \simeq 0.6\pm 0.2$ GeV. However, this is only a small part of the contribution to the rate for $\pi\pi\gamma$. According to Dolinsky *et al.* [20], pp. 126 ff, most of the effect would be due to Bremsstrahlung by the pions. Above procedure to estimate this, in terms of the $\pi^+\pi^-\gamma$ decay of the ρ would be exact only if the experimental cuts made for identifying this decay, and to measure the pion form factor were the same. A more accurate procedure is as follows. We write

$$a(\pi^+\pi^-\gamma,\ \ t \leq 1.2) = \int_{4m_\pi^2}^{1.2} dt\, K(t) R_{\pi^+\pi^-\gamma}(t), \qquad (5.7a)$$

where

$$R_{\pi^+\pi^-\gamma}(t) = B(t, E_\gamma) R_{\pi^+\pi^-}(t)$$

and the Bremsstrahlung factor B is given by [30]

$$B(t, E_\gamma) = \frac{8t^{1/2}\alpha}{\pi(t-4m_\pi^2)^{3/2}} \int_{E_\gamma}^{k_m} \frac{dk}{k} I(k),$$

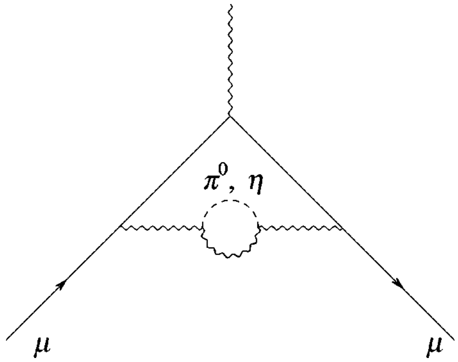

FIG. 15. The $\pi^0\gamma$, $\eta\gamma$ contribution to a (h.v.p., γ).

$$I(k)=k_m\left(\frac{t-2m_\pi^2}{2t^{1/2}}-k\right)\log\frac{1+\xi}{1-\xi}-\left[k_m\left(\frac{1}{2}t^{1/2}-k\right)-k^2\right]\xi. \tag{5.7b}$$

Here $\xi=[(k_m-k)/t^{1/2}/2-k)]^{1/2}$, $k_m=(t-4m_\pi^2)/2t^{1/2}$ is the maximum energy of the photon and, finally, E_γ is the minimum energy for photon detection.

To evaluate E_γ, we have to look at the setup of experiments measuring the pion form factor. Typically, one takes that no (hard) photon has been emitted when the angle between the pion momenta differs from π by less than a small given amount, η_0. The energy cut is, in this case,

$$E_\gamma=\frac{\sqrt{t-4m_\pi^2}}{2}\,\eta_0.$$

The effective η_0 depends on the specific cuts made in each experiment; those in Ref. [8] are covered if we take $\eta_0=0.15\pm0.05$. Using this we find the result, for this range of η_0, of

$$10^{11}\times a(\pi^+\pi^-\gamma,\ t\leqslant 1.2)=46\pm0.5\pm9. \tag{5.8}$$

The first error corresponds to the error in the integral of $|F_\pi|^2$ and the second is induced by the dispersion in the values of η_0 of the various experiments. Equation (5.8) is practically identical to the n.w.a. result, Eq. (5.6b) (which of course is a satisfactory result). The reason for this agreement is that, when *detecting* $\pi^+\pi^-\gamma$, the energy cut made, $E_\gamma=50$ MeV (Dolinsky *et al.*, Ref. [20]), turns out to be very similar to the average energy cut made when measuring the pion form factor.

A similar analysis ought to be made, in principle, for other radiative intermediate states like $3\pi+\gamma$ and $KK+\gamma$, which can be estimated in terms of the corresponding decays of the ω and ϕ, but they give a contribution below the 10^{-11} level and we neglect them.

The lowest energy contributions to $\sigma^{(0)}(e^+e^-\to\text{hadrons};\gamma)$ are those of the intermediate states $\pi^0\gamma$ and $\eta\gamma$ (Fig. 15). At energies below the rho mass, one can evaluate the first (the only one that gives a sizable contribution) by relating the process to the decay $\pi^0\to2\gamma$. We write an effective interaction, corresponding to the vertex factor in the Feynman rules of

$$\frac{iG_\pi}{2m_\pi}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta,$$

with

$$\Gamma(\pi^0\to2\gamma)=\frac{G_\pi^2m_\pi}{256\pi}$$

so that $G_\pi^2=4.6\times10^{-5}$. Then, with e the electron charge,

$$\operatorname{Im}\Pi^{\pi\gamma}(t)=\frac{G_\pi^2t}{384\pi e^2m_\pi^2}\left(1-\frac{m_\pi^2}{t}\right)^3.$$

This gives a very small contribution to a_μ, about 0.76×10^{-11} if we integrate up to $t^{1/2}\simeq0.7$ GeV, and 0.96×10^{-11} if we go to $t^{1/2}\simeq0.84$ GeV (the integral only grows logarithmically). We only integrate up to the rho, i.e., to $t^{1/2}=0.7$ GeV with this pointlike model.

Around the ρ region we have to take into account the excitation of this resonance, which produces the corresponding enhancement. This piece can be obtained in terms of the radiative width $\rho\to\pi^0\gamma$. More important is the $\omega\to\pi^0\gamma$ process which gives $(33\pm2)\times10^{-11}$. Likewise, the contribution of the $\eta\gamma$ state is evaluated in terms of the decay $\phi\to\eta\gamma$. Finally, the contribution from $\pi^0\pi^0\gamma$ is taken from Ref. [31]. Collecting all of this, we get

$$\begin{aligned}
&10^{11}\times a(\text{h.v.p.},\ \rho\to\pi^0\gamma)=4\pm1,\\
&10^{11}\times a(\text{h.v.p.},\ \rho\to\eta\gamma)=1.1\pm0.4,\\
&10^{11}\times(\text{h.v.p.},\ \omega\to\pi^0\gamma)=33\pm2,\\
&10^{11}\times a(\text{h.v.p.},\ \phi\to\eta\gamma)=5\pm1;\\
&\text{total: } 43\pm4.
\end{aligned} \tag{5.9a}$$

We have included the lower energy contribution of $\pi^0\gamma$ into a (h.v.p., $\rho\to\pi^0\gamma$) and, because we are relying on models, we added the errors *linearly*. For the $\pi\pi\gamma$ states,

$$\begin{aligned}
&10^{11}\times a(\text{h.v.p.},\ \pi^+\pi^-\gamma)=46\pm9,\\
&10^{11}\times a(\text{h.v.p.},\pi^0\pi^0\gamma)=2\pm0.3,
\end{aligned} \tag{5.9b}$$

and, for the high energy piece,

$$10^{11}\times a(\text{hadrons}+\gamma,\ t\geqslant1.2\ \text{GeV}^2)=1\pm0.5. \tag{5.9c}$$

Adding other contributions that are below the 10^{-11} level [$\epsilon(700)\gamma\sim0.7\times10^{-11}$, etc.] we get the total effect of the states hadrons$+\gamma$,

$$10^{11}\times a(\text{hadrons}+\gamma)=93\pm11. \tag{5.10}$$

TABLE IV. Summary of contributions to $a^{(2+3)}$, with what we consider the more reliable methods, as used in the present work. ''B.-W.+const'' means a Breit-Wigner fit, including the correct phase space factors, plus a constant; note that only for the four narrow resonances J/ψ, ψ', Υ, Υ' we use the n.w.a. The errors are uncorrelated except those for QCD calculations (that have to be added linearly) and those for the 2π states, for whose treatment we refer to the text. The errors given include statistical, systematic and (estimated) theoretical errors. For the details of the final states $\gamma+$ hadrons we refer to Eqs. (5.9).

Channel	Energy range	Method of calculation	Contribution to $10^{11}\times a$(h.v.p.)
$\pi^+\pi^-$	$t\leqslant 0.8$ GeV^2	fit to $e^+e^-+\tau+$spacel. data	4774 ± 51
$\pi^+\pi^-$	$0.8\leqslant t\leqslant 1.2$ GeV^2	fit to expt e^+e^- data	230 ± 5
3π	$t\leqslant 1.2$ GeV^2	B.-W.+const fit to e^+e^- data	438 ± 12
$2K$	$t\leqslant 1.2$ GeV^2	B.-W.+const fit to e^+e^- data	314 ± 13
4π, 5π, $\eta\pi$, . . .	$t\leqslant 1.2$ GeV^2	fit to e^+e^- data	24 ± 5
Inclusive	$1.2\leqslant t\leqslant 2$ GeV^2	fit to e^+e^- data	270 ± 27
J/ψ, ψ'; Υ, Υ'		N.w.a.	77.5 ± 4.4
Inclusive	$3.7^2\leqslant t\leqslant 4.6^2$ GeV^2	fit to experimental e^+e^- data	56 ± 3
Inclusive; *uds*	$2\leqslant t\leqslant 3.7^2$ GeV^2	perturbative QCD	615 ± 9
Inclusive; *udsc*	$4.6^2\leqslant t\leqslant 11.2^2$ GeV^2	perturbative QCD	89 ± 1
b quark	$10.1^2\leqslant t\leqslant 11.2^2$ GeV^2	nonrelativistic QCD	0.5 ± 0.1
Inclusive	11.2^2 GeV$^2\leqslant t\leqslant\infty$	perturbative QCD	21 ± 0.1
$\gamma+$ hadrons	Full range	various methods	93 ± 11

VI. CONCLUSIONS

We present first, for ease of reference, Table IV with a summary of the results obtained for a (h.v.p.) in the previous sections.

Taking into account all contributions, and errors, we complete the best values for the h.v.p. piece, and the whole hadronic part of the anomaly:

$$10^{11}\times a^{(2+3)}(\text{h.v.p.})=\begin{cases}7002\pm 66 & (e^+e^-+\tau+\text{spacel.}),\\ 6982\pm 97 & (e^+e^-+\text{spacel.}),\end{cases}\tag{6.1}$$

and adding the other radiative and light-by-light corrections,

$$10^{11}\times a(\text{hadronic}) =\begin{cases}6993\pm 69 & (e^+e^-+\tau+\text{spacel.}),\\ 6973\pm 99 & (e^+e^-+\text{spacel.}).\end{cases}\tag{6.2}$$

Equation (6.2) is of course the main outcome of the present paper. Because, even after adding systematic and theoretical errors the evaluation including τ decay data is more precise, we may consider it to provide the best result for a(hadronic) available at present.

We can add to Eq. (6.2) the pure electroweak contributions and present the result as the standard model prediction for a_μ:

$$10^{11}\times a_\mu=116\,591\,849\pm 69 \quad (e^+e^-+\tau+\text{spacel.}).\tag{6.3}$$

We will next compare our results with other recent evaluations of the same quantities in Fig. 16, together with the experimental band. (They are shown incorporating the contribution of the $\pi^+\pi^-\gamma$ and $\pi^0\pi^0\gamma$ channels.)

A further point to emphasize is the importance of using, in the low-energy region, parametrizations of $F_\pi(t)$ compatible with unitarity and analyticity. Only in this way we can incorporate data on $F_\pi(t)$ for spacelike t into the fits. As discussed in Sec. III B, these data not only provide a substantial shift for a_μ (of 39×10^{-11} in the evaluation with e^+e^- data only) but, by so doing, allow compatibility of these with the results from τ decay, hence allowing a combination of the

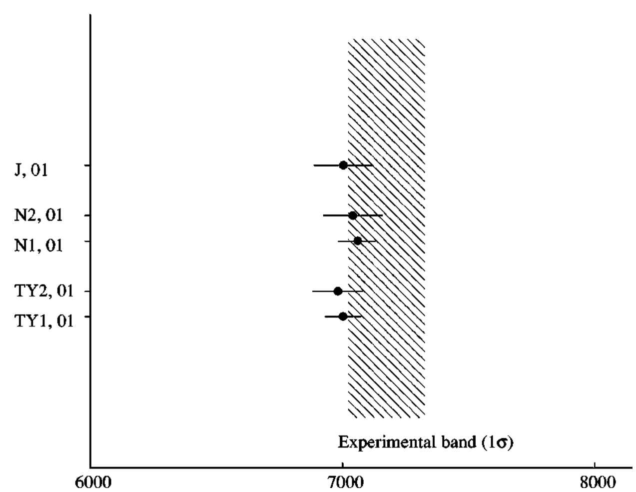

FIG. 16. Theoretical results on $a(\text{hadronic})\times 10^{-11}$, and experiment. J: Ref. [6]; N1: Ref. [7] data from $e^+e^-+\tau$. N2: id, e^+e^- only. T1, T2: this paper with data from $e^+e^-+\tau$ or data from e^+e^- only, respectively (including syst. and th. errors).

two in a meaningful way: this permits an important reduction of the errors of the calculation.[15]

To finish this section, we can add a few words on prospects for improvements. In our view they are rather dim in the sense that it is not easy to see how one could get an error estimate clearly below the 70×10^{-11} mark, when taking into account systematic and theoretical errors. In fact, the central values have moved little, and the errors have not improved much, since 1985. It is true that experiments planned or in progress can clear further the region between 1.2 and 3 GeV^2. However, a serious improvement of the very important low-energy region for $\pi\pi$ is unlikely: as our evaluations show, one can get a fit to *all* data relevant for the hadronic component of a_μ, with a χ^2/d.o.f. of unity and verifying *all* theoretical constraints, with an error of at least 51×10^{-11}, already for $t\leqslant 0.8$ GeV^2 [see Eq. (3.19)]. In this respect the improvement obtained by adding τ decay data, although not negligible, is minor: statistical errors are smaller, but theoretical ones are increased.

[15]Or, put conversely, not using data at spacelike t for F_π implies a hidden error of about 40×10^{-11}.

ACKNOWLEDGMENTS

The authors are indebted to G. Cvetic, F. Jegerlehner, and S. Narison for several illuminating discussions and, for the last two, for information concerning their work on the subject. Discussions with J. Vermaseren (who helped with a critical reading of the manuscript) are acknowledged for several matters; in particular, those concerning the $O(\alpha^3)$ hadronic contributions have been very helpful. It is also a pleasure to record our indebtedness to Achim Stahl (Opal) and Andreas Höcker (Aleph) for providing us with the corresponding data listings, and for very useful information about them. Thanks are also due to S. I. Eidelman for the same reason (4π data). The financial support of CICYT is gratefully acknowledged.

[1] H. N. Brown *et al.*, Phys. Rev. Lett. **86**, 2227 (2001).
[2] J. Bailey *et al.*, Nucl. Phys. **B150**, 1 (1979).
[3] (KNO): T. Kinoshita, B. Nizić, and Y. Okamoto, Phys. Rev. D **32**, 736 (1985); (DM): S. Dubnicka and L. Martinovic, *ibid.* **42**, 7884 (1990); (BW): D. H. Brown and W. A. Worstell, *ibid.* **54**, 3237 (1996).
[4] (CLY): A. Casas, C. López, and F. J. Ynduráin, Phys. Rev. D **32**, 736 (1985); (AY): K. Adel and F. J. Ynduráin, Rev. Acad. Ciencias (Esp.) **92**, 113 (1998).
[5] (ADH): R. Alemany, M. Davier, and A. Höcker, Eur. Phys. J. C **2**, 123 (1998); (DH): M. Davier and A. Höcker, Phys. Lett. B **435**, 419 (1998).
[6] (EJ): S. Eidelman and F. Jegerlehner, Z. Phys. C **67**, 585 (1995); (J): F. Jegerlehner, DESY 01-028 hep-ph/0104304, symposium in honor of A. Sirlin, New York University, 2000.
[7] (N): S. Narison, Phys. Lett. B **513**, 53 (2001).
[8] Novosibirsk, ρ region: L. M. Barkov *et al.*, Nucl. Phys. **B256**, 365 (1985); R. R. Akhmetshin *et al.* Budker INP 99-10, hep-ex/9904027.
[9] Novosibirsk, ω and ϕ region, KK and 3π: (CMD2) R. R. Akhmetshin *et al.*, Phys. Lett. B **466**, 385 (1999); **466**, 392 (1999); **434**, 426 (1998); **476**, 33 (2000); (SND) M. N. Achasov *et al.*, Phys. Rev. D **63**, 072002 (2001); Phys. Lett. B **462**, 365 (1999); $\phi\to2\pi$: M. N. Achasov *et al.*, *ibid.* **474**, 188 (2000).
[10] Aleph: R. Barate *et al.*, Z. Phys. C **76**, 15 (1997); Opal: K. Ackerstaff *et al.*, Eur. Phys. J. C **7**, 571 (1999).
[11] NA7: S. R. Amendolia *et al.*, Nucl. Phys. **B277**, 168 (1986).
[12] BES: J. Z. Bai *et al.*, hep-ex/0102003.
[13] V. W. Hughes and T. Kinoshita, Rev. Mod. Phys. **71**, S133 (1999).
[14] S. J. Brodsky and E. de Rafael, Phys. Rev. **168**, 1620 (1968).
[15] B. R. Martin, D. Morgan, and G. Shaw, *Pion-Pion Interactions in Particle Physics* (Academic Press, New York, 1976).
[16] Particle Data Group, D. E. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).
[17] F_π, spacelike data: C. J. Bebek *et al.*, Phys. Rev. D **17**, 1693 (1978); F_π, timelike data: D. Bollini *et al.*, Lett. Nuovo Cimento Soc. Ital. Fis. **14**, 418 (1977); A. Quenzer *et al.*, Phys. Lett. **76B**, 512 (1978); $\pi\pi$ phase shifts: B. Hyams *et al.*, Nucl. Phys. **B64**, 134 (1973); S. D. Protopopescu *et al.*, Phys. Rev. D **7**, 1279 (1973).
[18] a_1^1. (CGL): G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. **B603**, 125 (2001); (ACGL): B. Ananthanarayan, G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rep. **353**, 207 (2001); (ABT): G. Amorós, J. Bijnens, and P. Talavera, Nucl. Phys. **B585**, 329 (2000); Erratum, LU TP 00-11, 2001; $\langle r_\pi^2\rangle$, c_π: G. Colangelo, M. Finkelmeir, and R. Urech, Phys. Rev. D **54**, 4403 (1996).
[19] G. J. Gounnaris and J. J. Sakurai, Phys. Rev. Lett. **21**, 244 (1968).
[20] KK: P. M. Ivanov *et al.*, Phys. Lett. **107B**, 297 (1981); 3π: A. Cordier *et al.*, Nucl. Phys. **B172**, 13 (1980); 4π and more: G. Cosme *et al.*, *ibid.* **B152**, 215 (1979). For a review, see S. Dolinsky *et al.*, Phys. Rep. **C202**, 99 (1991), and work quoted there. All these references give results for energies below $t=2$ GeV^2. Between 2 and 9 GeV^2, see C. Bacci *et al.*, Phys. Lett. **86B**, 234 (1979).
[21] S. Bethke, J. Phys. G **26**, R27 (2000); J. Santiago and F. J. Yndurain, Nucl. Phys. **B611**, 497 (2001); D. Strom, ‘‘Electroweak measurements on the Z resonance,’’ talk presented at the 5th Int. Symposium on Radiative Corrections, RadCor2000, Carmel, California, September 2000.
[22] K. Adel and F. J. Ynduráin, Phys. Rev. **92**, 113 (1998). For a clear exposition of NRQCD calculations, see N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. **B566**, 275 (2000).
[23] S. Titard and F. J. Ynduráin, Phys. Rev. D **49**, 6007 (1994), to one loop; A. Pineda and F. J. Ynduráin, *ibid.* **58**, 094022 (1998); **61**, 077505 (2000), to two loops.
[24] (BPP): J. Bijnens, E. Pallante, and J. Prades, Nucl. Phys. **B474**, 379 (1996); (HKS, HK): M. Hayakawa, T. Kinoshita, and A. I. Sanda, Phys. Rev. D **54**, 3137 (1996); M. Hayakawa and T.

Kinoshita, *ibid.* **57**, 465 (1998).
[25] M. Knecht and A. Nyffeler, hep-ph/0111058.
[26] M. Hayakawa and T. Kinoshita, hep-ph/0112102.
[27] S. Laporta and E. Remiddi, Phys. Lett. B **301**, 440 (1993).
[28] F. Barreiro, *Photon Structure, Plenary talk, Proc. Int. Europhysics Conf. on High Energy Physics, Tampere, 1999*, edited by Huitu *et al.* (IOP, Bristol, U.K.).
[29] B. Krause, Phys. Lett. B **390**, 392 (1997).
[30] P. Singer, Phys. Rev. **130**, 2441 (1963); **161**, 1694(E) (1967).
[31] M. N. Achasov *et al.*, Phys. Lett. B **486**, 29 (2000).