Dynamics near the critical point: The hot renormalization group in quantum field theory

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The perturbative approach to the description of long-wavelength excitations at high temperature breaks down near the critical point of a second order phase transition. We study the dynamics of these excitations in a relativistic scalar field theory at and near the critical point via a renormalization group approach at high temperature and an ϵ expansion in $d=5-\epsilon$ space-time dimensions. The long-wavelength physics is determined by a nontrivial fixed point of the renormalization group. At the critical point we find that the dispersion relation and width of quasiparticles of momentum p are $\omega_p \sim p^z$ and $\Gamma_p \sim (z-1)\omega_p$, respectively, and the group velocity of quasiparticles $v_g \sim p^{z-1}$ vanishes in the long-wavelength limit at the critical point. Away from the critical point for $T \gtrsim T_c$ we find $\omega_p \sim \xi^{-z}[1+(p\xi)^{2z}]^{1/2}$ and $\Gamma_p \sim (z-1)\omega_p (p\xi)^{2z}/[1+(p\xi)^{2z}]$ with ξ the finite temperature correlation length $\xi \propto |T-T_c|^{-\nu}$. The new dynamical exponent z results from anisotropic renormalization in the spatial and time directions. For a theory with $O(N)$ symmetry we find $z=1+\epsilon(N+2)/(N+8)^2+\mathcal{O}(\epsilon^2)$. This dynamical critical exponent describes a new universality class for dynamical critical phenomena in quantum field theory. Critical slowing down, i.e., a vanishing width in the long-wavelength limit, and the validity of the quasiparticle picture emerge naturally from this analysis.

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I. INTRODUCTION

The experimental possibility of studying the phase transitions of QCD via ultrarelativistic heavy ion collisions with the current effort at the BNL Relativistic Heavy Ion Collider (RHIC) and the forthcoming program at the CERN Large Hardron Collider (LHC) motivates a theoretical effort to understand the dynamical aspects of phase transitions at high temperature. QCD is conjectured to feature two phase transitions, the confinement-deconfinement (or hadronization) and the chiral phase transitions. Detailed lattice studies [1] seem to predict that both transitions occur at about the same temperature $T_c \sim 170$ MeV.

While lattice gauge theories furnish a nonperturbative tool to study the thermodynamic equilibrium aspects of the transition the *dynamical* aspects cannot be accessed with this approach.

In a condensed matter experiment the temperature is typically a control parameter and it can be varied sufficiently slowly so as to ensure that a phase transition occurs in local thermodynamic equilibrium. In an ultrarelativistic heavy ion collision the current theoretical understanding suggests that a thermalized quark-gluon plasma may be formed at a time scale of order 1 fm/c with a temperature larger than critical. This quark-gluon plasma then expands hydrodynamically and cools almost adiabatically, the temperature falling off as a power of time $T(t) \sim T_i(t_i/t)^{1/3}$ until the transition temperature is reached at a time scale $\sim$10–50 fm/c depending on the initial temperature [2].

Whether the phase transition occurs in local thermodynamic equilibrium or not depends on the ratio of the cooling time scale $t_{cool} \sim T(t)/\dot{T}(t)$ to the relaxation or thermalization time scale of a fluctuation of a given wavelength p^{-1}, $t_{rel}(p)$. If $t_{cool} \gg t_{rel}$ then the fluctuation relaxes on time scales much shorter than that of the temperature variation and reaches local thermodynamic equilibrium. If, on the other hand, $t_{cool} \ll t_{rel}$ the fluctuation does not have time to relax to local thermodynamic equilibrium and freezes out. For these fluctuations the phase transition occurs very fast and out of equilibrium. Thus an important dynamical aspect is to understand the relaxation time scales for fluctuations.

A large body of theoretical, experimental, and numerical work in condensed matter physics reveals that while typically short-wavelength $(p \gg T)$ fluctuations reach local thermal equilibrium, near a critical point long-wavelength fluctuations relax very slowly, and undergo *critical slowing down* [3,4]. A phenomenological description of the dynamics near a phase transition typically hinges on the time-dependent Landau-Ginzburg equation which is generalized to include conservation laws [3,4]. In the simplest case of a nonconserved order parameter, such as in a scalar field theory with discrete (Ising-like) symmetry, the time-dependent Landau-Ginzburg equation is purely dissipative.

While phenomenological, this approach has proved very successful in a variety of experimental situations and is likely to provide a suitable description of the dynamics for macroscopic, coarse-grained systems such as binary mixtures, etc. [3]. The phenomenological approach based on the time-dependent Landau-Ginzburg equations, which are first order in time derivatives, seems to provide a suitable description of coarse-grained macroscopic dynamics in *nonrelativistic systems*. However, it is clear that this approach is not

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justified in a relativistic quantum field theory, since the underlying equations of motion are second order and time reversal invariant.

In particular, in the case of a purely dissipative time-dependent Landau-Ginzburg (phenomenological) description [3,4], frequency and momenta enter with different powers in the propagators, and at the mean field (or tree) level this results in a dynamical scaling exponent $z=2$. This situation must be contrasted with that of a relativistic quantum field theory where at tree level (mean field) frequencies and momenta enter with the same power in the propagator, leading to a dynamical scaling exponent $z=1$. Furthermore, critical slowing down is automatically built into the phenomenological description, even at tree level, as a consequence of the dissipative equations of motion [5]. Clearly, this is not the case in a relativistic quantum field theory. For a detailed discussion of the differences of nonequilibrium dynamical aspects between the time-dependent Landau-Ginzburg approach and quantum field theory, see Ref. [5].

There are important nonequilibrium consequences of slow dynamics near critical points. If the cooling time scale is much shorter than the relaxation time scale of long-wavelength fluctuations, these freeze out and undergo spinodal instabilities when the temperature falls below critical [3,5] during continuous (no metastability) phase transitions. These instabilities result in the formation of correlated domains that grow in time [3,5] with a law that in general depends on the cooling rate [6].

In ultrarelativistic heavy ion collisions during the expansion of the quark-gluon plasma the critical point for the chiral phase transition may be reached. If long-wavelength fluctuations freeze out shortly before the transition, the ensuing instabilities may lead to distinct event by event observables [7] in the pion distribution as well as in the photon spectrum at low energies [8].

Thus an important aspect of the chiral phase transition is to establish the relaxation time scales of long-wavelength fluctuations, and whether critical slowing down and freeze-out of long-wavelength fluctuations can ensue.

In the strict chiral limit with massless up and down quarks, QCD has a $SU(2)_R \otimes SU(2)_L$ symmetry which is spontaneously broken to $SU(2)_{R+L}$ at the chiral phase transition, the three pions being the Goldstone bosons associated with the broken symmetry. It has been argued that the low energy theory that describes the chiral phase transition is in the same universality class as the Heisenberg ferromagnet, i.e., the $O(4)$ linear sigma model [9]. This argument has been used [9] to provide an assessment of the dynamical aspects of low energy QCD based on the phenomenological time-dependent Landau-Ginzburg approach to dynamical critical phenomena in condensed matter [3]. While the universality arguments are appealing, a more microscopic understanding of dynamical critical phenomena in quantum field theory is needed and has begun to emerge only recently [10,11].

In Ref. [10] a Wilsonian renormalization group extended to finite temperature was implemented in a scalar quartic field theory. In this approach only one-loop diagrams enter in the computation of the beta functions, and the imaginary part of the self-energy, which arises first at two-loop order for $T>T_c$, is accounted for by an imaginary part in the effective quartic coupling [10]. There it is found that the relaxation rate of zero momentum fluctuations γ reveals critical slowing down in the form $\gamma \sim |T-T_c|^{\nu} \ln|T-T_c|$ with $\nu \sim 0.53$ being the critical exponent for the correlation length [10].

In Ref. [11] the width of quasiparticles near the critical point was studied via the large N approximation. This study revealed that at high temperature the effective coupling is driven to a (Wilson-Fischer) fixed point, a result that is in agreement with the numerical evidence presented in Ref. [10]. While the results in leading order in the large N limit found in [11] hinted at critical slowing down, albeit in a manner different from the numerical evidence of Ref. [10], they also hinted at the breakdown of the quasiparticle picture. A conclusion in [11] is that, while the large N limit provides a partial resummation of the perturbative expansion, further resummation is needed to fully address the relaxation of quasiparticles.

The large N limit in *static* critical phenomena presents a similar situation: while it sums the series of bubbles replacing the bare vertex by the effective coupling that is driven to the fixed point in the infrared, the self-energy still features infrared logarithms that require further resummation [4]. Such a resummation is provided by the renormalization group [4].

While our motivation for studying dynamical critical phenomena near critical points is driven by the experimental program in ultrarelativistic heavy ion collisions to study the QCD phase transitions, the underlying questions are more overarching and of a truly interdisciplinary nature. In particular, we mention an impressive body of work on aspects of *quantum phase transitions* in condensed matter systems [12] that addresses very similar questions. The work in Ref. [12] focuses on understanding the static, dynamical, and transport properties of low dimensional systems in the quantum regime, in which the frequency and momentum of excitations is $\omega;p \gg T$.

Our study in this article is complementary to that program in that we focus on the dynamical aspects of long-wavelength quasiparticles with $\omega;p \ll T$. As discussed in [12,13] and in detail below, this is closer to the *classical* regime.

A. The goals

In this article we study the *dynamical* aspects of quasiparticles near the critical point in a scalar quartic field theory by implementing a renormalization group program at high temperature. While the renormalization group has been generalized to finite temperature in various formulations [14,15], mainly to study critical phenomena associated with finite temperature phase transitions in field theory, only *static* aspects were studied with these approaches.

Instead we focus on dynamical aspects, in particular the dispersion relations and relaxation rates of long-wavelength excitations at and near the critical point. Already at the technical level one can see the differences: to understand dynamical aspects, in particular relaxation, a consistent treatment of the absorptive parts of the self-energies is required. This aspect is notoriously difficult to implement in a Wilsonian approach in Euclidean field theory [14]. Reference [10] proposes a method to circumvent this problem but a complete treatment that manifestly includes the absorptive parts of the self-energy contributions is still lacking in this approach. Other approaches using the Euclidean version of the renormalization group adapted to finite temperature field theory were restricted to static quantities [15] and, in fact, as will be seen in detail below, miss important phenomena that will be at the heart of the results presented here.

B. Brief summary of results

Long-wavelength phenomena at high temperature T imply a dimensional reduction from the decoupling of Matsubara modes with nonzero frequency [16,17]. The coupling in the dimensionally reduced theory is λT, where λ is the quartic coupling. For dimensional reasons, the perturbative expansion in four space-time dimensions is in terms of the dimensionless ratio $\lambda T/\mu$ with μ the typical momentum scale, which is strongly relevant in the infrared. As a result, a perturbative approach to studying long-wavelength phenomena breaks down. This is manifest in the breakdown of the quasiparticle picture in naive perturbation theory (see [11] and below).

In $5-\epsilon$ space-time dimensions, the effective coupling in the high temperature, long-wavelength limit is $g(\mu)=\lambda T\mu^{-\epsilon}$. We implement an ϵ expansion around *five* space-time dimensions and a renormalization group resummation program at high temperature with $T\gg s,\mu$ near the critical point, with s,μ the typical frequency and momentum scales. We analyze the high temperature behavior of the relevant graphs and find that it is dominated for $\epsilon>0$ by the zero Matsubara mode, while the sum of the nonzero modes gives subdominant contributions. The effective renormalized coupling is driven to an infrared stable fixed point $g^*=\mathcal{O}(\epsilon)$, which for small ϵ allows a consistent perturbative expansion near the fixed point.

An important feature that emerges clearly in this approach, and that has been missed in most other treatments of renormalization group at finite temperature, is the *anisotropic* scaling between spatial and time directions, which is manifest in a nontrivial renormalization of the speed of light. This is a consequence of the fact that in the Euclidean formulation at finite temperature time is compatified to $0\leq\tau\leq 1/T$; thus space and time or momentum and frequency play different roles. This results in a novel dynamical critical exponent z, which determines the anisotropic scaling. The renormalization group leads to scaling in the infrared region in terms of anomalous dimensions which can be computed systematically in the ϵ expansion. In particular, to lowest order in ϵ we find for a scalar theory with discrete symmetry $z=1+\epsilon/27+\mathcal{O}(\epsilon^2)$, which describes a new universality class for dynamical critical phenomena in quantum field theory. All dynamical aspects, such as the relaxation rates and dispersion relations, depend on this critical exponent, while the static aspects are completely described by the usual critical exponents.

We provide a renormalization group analysis of the quasiparticle properties near the critical point, such as their dispersion relation and width, complemented by an explicit evaluation to lowest order in the ϵ expansion. The main results obtained in this article are the following. At $T=T_c$ we find that the dispersion relation and width of quasiparticles of momentum p are $\omega_p\sim p^z$ and $\Gamma_p\sim(z-1)\,\omega_p$, respectively, with a vanishing group velocity of quasiparticles in the long-wavelength limit highlighting the collective nature of the quasiparticle excitations. For $T>T_c$ but $|T-T_c|\ll T$ we find $\omega_p\sim\xi^{-z}[1+(p\xi)^{2z}]^{1/2}$ and $\Gamma_p\sim(z-1)\,\omega_p(p\xi)^{2z}/[1+(p\xi)^{2z}]$ with ξ the finite temperature correlation length $\xi\propto|T-T_c|^{-\nu}$. In the case of $O(N)$ symmetry we find to lowest order in the epsilon expansion that the dynamical exponent $z=1+\epsilon(N+2)/(N+8)^2+\mathcal{O}(\epsilon^2)$.

Critical slowing down emerges near the critical point and in the ϵ expansion $\Gamma_p/\omega_p\ll 1$, confirming the quasiparticle picture.

We discuss some relevant cases of threshold singularities in which the usual (Breit-Wigner) parametrization of the quasiparticle propagator is not available since the real part of the inverse Green's function vanishes at the quasiparticle frequency with an anomalous power law.

In Sec. II we introduce the model and discuss the breakdown of naive perturbation theory. In Sec. III we introduce the ϵ expansion and analyze the static case. In Sec. IV the renormalization aspects and the anisotropic scaling are analyzed in detail. Section V presents the renormalization group in the effective, dimensionally reduced theory both at and near the critical point. This section contains the bulk of our results, which are summarized in Sec. VI. Our conclusions and a discussion of potential implications are presented in Sec. VII. The high temperature behavior of the relevant diagrams is computed in the Appendixes.

II. THE THEORY AND THE NECESSITY FOR RESUMMATION

The low energy sector of QCD with two massless (up and down) quarks is conjectured to be in the same universality class as the $O(4)$ Heisenberg ferromagnet [9] described by the $O(4)$ linear sigma model. Furthermore, since we are interested in describing the dynamical aspects associated with critical slowing down and freeze-out of long-wavelength fluctuations just before the chiral phase transition, we focus on $T\to T_c^+$.

While our motivation for studying critical slowing down stems from the experimental program in ultrarelativistic heavy ion collisions, the questions are of a fundamental nature.

To understand the dynamical aspects near the critical point, we focus on the simpler case of a single scalar field theory, and we will recover the case of $O(N)$ symmetry at the end of the discussion. We thus focus on the theory described by the Lagrangian density

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\Phi)^2+\frac{1}{2}m_0^2\Phi^2-\frac{\lambda_0}{4!}\Phi^4 \tag{2.1}$$

where the subscripts in the mass and coupling refer to bare quantities. The case of N components in the unbroken phase $T \geqslant T_c$ differs from the single scalar field by combinatoric factors that change the critical exponents quantitatively. These factors will be included at the end of the calculation to obtain an estimate of the critical exponents for the $O(N)$ theory in Sec. VI.

We are interested in obtaining the relaxation properties of long-wavelength excitations near the critical temperature, which in this scalar theory $T_c \sim |m_R|/\sqrt{\lambda_R}$ with m_R, λ_R being the renormalized mass and coupling. Thus the regime of interest for this work is $p, \omega \ll T \sim T_c$ with p, ω being the momentum and frequency of the long-wavelength excitation. As will become clear below it is convenient to work in the Matsubara representation of finite temperature field theory, which is more amenable to the implementation of the (Euclidean) renormalization group.

In the Matsubara formulation Euclidean time τ is compactified in the interval $0 \leqslant \tau \leqslant \beta = 1/T$ whereas space is infinite; bosonic fields are periodic in Euclidean time and can be expanded as [18–20]

$$\Phi(\vec{x},\tau) = \frac{1}{\sqrt{\beta V}} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \phi(\vec{p},\omega_n) e^{-i\omega_n \tau + i\vec{p}\cdot\vec{x}} \tag{2.2}$$

$$\omega_n = 2\pi n T, \quad n = 0, \pm 1, \pm 2, \ldots . \tag{2.3}$$

Thus we see that, while the spatial momentum is a continuum variable, the Matsubara frequencies are discrete as a consequence of the compactification of Euclidean time. This feature of Euclidean field theory at finite temperature will be seen to lead to *anisotropic* rescaling between space and time and therefore, as will be clear below, new dynamical critical exponents. Anticipating anisotropic rescaling, we then introduce the bare speed of propagation v_0 of excitations in the medium by writing the Euclidean Lagrangian in the form

$$\mathcal{L}_E = \frac{1}{2}\frac{(\partial_\tau \Phi)^2}{v_0^2} + \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}M^2(T)\Phi^2 + \frac{\lambda_R}{4!}\Phi^4 + \frac{1}{2}\delta m^2(T)\Phi^2 + \frac{\delta\lambda}{4!}\Phi^4 \tag{2.4}$$

where we have introduced the effective renormalized *temperature-dependent* mass $M(T)$ and the counterterms, in particular the mass counterterm $\delta m^2(T) = -M^2(T) - m_0^2$, are adjusted order by order in perturbation theory so that the inverse two-point function obeys

$$\Gamma^{(2)}(\vec{p}=0;\ \omega_n = 0) = M^2(T). \tag{2.5}$$

The critical point is defined with (the inverse susceptibility) $M^2(T) \equiv 0$. We will begin our study by focusing our attention on the critical theory for which $M(T) = 0$. We will later consider the theory near the critical point but in a regime in which $M(T) \ll T \sim T_c$. Thus, the general regime to be studied is $\vec{p}, \omega, M(T) \ll T \sim T_c$.

A. Infrared behavior of the critical theory: Static limit in three space dimensions

In order to highlight the nature of the infrared behavior when $\vec{p}, \omega \ll T$ we focus first on the critical theory in the static limit, when the Matsubara frequencies of all external legs in the n-point functions vanish. For the purpose of understanding the nature of the infrared physics in the static limit we will set $v_0 = 1$ in this and the next section, and we will recover this variable, the speed of light, when we study the dynamics in Sec. IV.

1. The scattering amplitude in D=3

We consider first the $2 \to 2$ scattering amplitude, or four-point function, to one-loop order in three spatial dimensions. The full expression is given by

$$\begin{aligned}\Gamma^{(4)}(\vec{p}_1,s_1,\vec{p}_2,s_2,\vec{p}_3,s_3,\vec{p}_4,s_4) &= -\lambda_0 + \lambda_0^2[H(\vec{p}_1+\vec{p}_2, s_1+s_2) + H(\vec{p}_1+\vec{p}_3, s_1+s_3) \\ &\quad + H(\vec{p}_1+\vec{p}_4, s_1+s_4)] + \mathcal{O}(\lambda_0^3),\end{aligned} \tag{2.6}$$

where $s_i = 2\pi T m_i$, $1 \leqslant i \leqslant 4$, and $m_i \in \mathcal{Z}$,

$$H(p,s) = \frac{T}{2}\sum_{n\in\mathcal{Z}} \int \frac{d^3q}{(2\pi)^3} \times \frac{1}{[q^2 + (2\pi T n)^2][(\vec{q}+\vec{p})^2 + (2\pi T)^2(n+m)^2]}, \tag{2.7}$$

$s \equiv 2\pi T m$. Since the external momentum $p \ll T$ it is clear from the above expression that the dominant infrared behavior of $H(p,0)$ is determined by the zero Matsubara frequency in the sum. As will be explicitly shown below, the contribution from the nonzero Matsubara frequencies will introduce a renormalization of the bare coupling which in the limit $T \gg p$ is independent of the external momentum (this will be seen explicitly in the next section). Keeping only the zero internal Matsubara frequency and carrying out the three-dimensional integral explicitly, we find

$$H_{ir}(p,0) = \frac{T}{16p}. \tag{2.8}$$

Thus, defining the effective coupling constant at the symmetric point $\vec{p}_i = \overline{P}_i$ where

$$\overline{P}_i \cdot \overline{P}_j = (4\delta_{ij} - 1)\frac{\mu^2}{4} \tag{2.9}$$

in the static limit, one finds that in the infrared limit $\mu/T \ll 1$

$$\lambda_{eff}(\mu) = \lambda_0\left[1 - \frac{3\lambda_0 T}{16\mu}\right] + \mathcal{O}(\lambda_0^3). \tag{2.10}$$

Two important features transpire from this expression.

(i) the factor T/μ can be explained by dimensional arguments: in the Matsubara formulation for each loop there is a factor T from the sum over internal Matsubara frequencies. The infrared behavior for $\mu \ll T$ is obtained by considering only the zero internal Matsubara frequency in the loop. This integral has only one scale and since λ is dimensionless in three spatial dimensions the one-loop contribution must be proportional to T/μ. A similar argument shows that for a diagram with m internal loops and transferred momentum scale μ there will be a power T^m from the Matsubara sums; the infrared behavior is obtained by the contribution with *all* the internal Matsubara frequencies equal to zero, which by dimensional power counting must be of the form $(T/\mu)^m$. Therefore a diagram with m internal loops will contribute to the scattering amplitude by $\lambda(\lambda T/\mu)^m$. In taking only the zero internal Matsubara frequency we are assuming that the internal loop momenta are cut off at a scale below T.

Thus, at the critical point the most important infrared behavior is that of the *dimensionally reduced* three-dimensional theory [16,17]. The reason for this dimensional reduction is clear: at finite temperature T the Euclidean time is compactified to a cylinder of radius $L=1/T$ for transferred momenta μ and the spatial resolution is on distances $d\sim 1/\mu$. Therefore for $\mu \ll T \to d \gg L$; thus the compactification radius is effectively zero insofar as the long distance (infrared) physics is concerned.

We will study below the contribution from the nonzero Matsubara frequencies.

(ii) For a transferred momentum scale μ perturbation theory breaks down for $\mu \ll \lambda T$ since the contribution from higher orders is of the form $\lambda(\lambda T/\mu)^m$. This suggests that a resummation scheme is needed to study the infrared limit. This situation is similar to that in critical phenomena, where infrared divergences must be summed and the renormalization group provides a consistent and systematic resummation procedure. We can obtain a hint of how to implement the renormalization group in finite temperature field theory in the limit when T is much larger than any other scale (masses, momenta, and frequencies) by realizing that, from the argument presented above, the perturbative expansion is actually in terms of the dimensionless coupling $g_0=\lambda T/\mu$. Therefore from Eq. (2.10) we can write

$$g_{eff}(\mu)=g_0\left[1-\frac{3}{16}g_0\right]+\mathcal{O}(g_0^3). \tag{2.11}$$

We can improve the scattering amplitude via the renormalization group (RG) by considering the RG β function

$$\beta_g=\mu\frac{\partial g_{eff}(\mu)}{\partial\mu}\bigg|_{\lambda_0,T}=-g_{eff}+\frac{3}{8}g_{eff}^2+\mathcal{O}(g_{eff}^3). \tag{2.12}$$

The first term (with the minus sign) just displays the scaling dimension (for fixed λT) of the effective coupling; that this dimension is -1 is a consequence of the dimensional reduction since λT is the effective dimensionful coupling of the three-dimensional theory.

We thus see that the renormalization group improved coupling runs to the infrared fixed point

$$g^*=\frac{8}{3} \tag{2.13}$$

as the momentum scale $\mu\to 0$. Comparing with the renormalization group beta function of critical phenomena [4,21–25] we see that this is the Wilson-Fischer fixed point in three dimensions, again revealing the dimensional reduction of the low energy theory. The resummation of the effective coupling and the fixed point structure can also be understood in the large N limit [11]. As described in [11] the large N limit can be obtained by replacing the interaction in the Lagrangian density by [26]

$$\mathcal{L}_{int}=\frac{\bar{\lambda}}{2N}(\vec{\Phi}\cdot\vec{\Phi})^2 \tag{2.14}$$

with $\vec{\Phi}=(\phi_1,\cdots\phi_N)$, and the form of the quartic coupling has been chosen for consistency with the notation of Ref. [11]. The leading order in the large N limit for the scattering amplitude is obtained by summing the geometric series of one-loop bubbles in the s channel (only this channel out of the three contributes to leading order in the large N limit), each one proportional to N, which is the number of fields in the loop. As a result one finds that the effective scattering amplitude at a momentum transfer μ is given by [11]

$$\bar{\lambda}_{eff}(\mu)=\frac{\bar{\lambda}}{1+\bar{\lambda}T/4\mu}. \tag{2.15}$$

Thus, introducing the dimensionless effective coupling $\bar{g}_{eff}(\mu)=\bar{\lambda}_{eff}(\mu)(T/\mu)$ one finds that

$$\lim_{\mu\to 0}\bar{g}_{eff}(\mu)=4, \tag{2.16}$$

i.e., the effective coupling constant goes to the three-dimensional fixed point [11].

2. The two-point function in D=3

The two-point function in the static limit is given by

$$\Gamma^{(2)}(p,0)=p^2+\delta m^2(T)-\Sigma(p,0)+\mathcal{O}(\lambda^3) \tag{2.17}$$

where $\Sigma(p,0)$ stands for the two-loop sunset diagram at zero external Matsubara frequency and the counterterm $\delta m^2(T)$ will cancel the momentum-independent but temperature-dependent parts of the self-energy. The two-loop self-energy for external momentum $\vec{p}$ and Matsubara frequency $\omega_m=2\pi Tm$ is given by

$$\Sigma(p,\omega_m)=\frac{\lambda^2T^2}{6}\sum_{l,j\in\mathcal{Z}}\int\frac{d^3q}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{[q^2+\omega_l^2][k^2+\omega_j^2][(\vec{p}+\vec{k}+\vec{q})^2+(\omega_l+\omega_j+\omega_m)^2]} \tag{2.18}$$

with $\omega_j=2\pi jT$. The static limit is obtained by setting $\omega_m=0$ $(m=0)$. In this limit the dominant contribution in the infrared for $T\gg p$ arises from the term $l=j=0$ in the sum. The terms $l\neq 0$, $j\neq 0$ for which we can take $p=0$ (since $p\ll T$) will be canceled by the counterterm. A straightforward calculation leads to

$$\Gamma^{(2)}(p,0)|_{ir}=p^2\left[1+\frac{\lambda^2T^2}{12(4\pi)^2p^2}\ln\left(\frac{p^2}{\mu^2}\right)\right]+\mathcal{O}(\lambda^3) \tag{2.19}$$

where μ is a renormalization scale. This expression clearly reveals the effective coupling $\lambda T/p$ which becomes very large in the limit $p\ll\lambda T$. Clearly, we need to implement a resummation scheme that will effectively replace the bare dimensionless coupling constant by an effective coupling that goes to a fixed point in the long-wavelength limit, and also ensure that at this fixed point the effective coupling is *small* so that perturbation theory near this fixed point is reliable. This is precisely what the renormalization group combined with the ϵ expansion achieves in critical phenomena [4,21–25].

B. Dynamics in $D=3$

The two-loop contribution to the self-energy for $\omega\neq 0$ is obtained from a dispersive representation of the self-energy in terms of the spectral density,

$$\Sigma(p,\omega)=\int\frac{d\nu}{\pi}\frac{\rho(p,\nu)}{\nu-\omega-i0^+}. \tag{2.20}$$

The spectral density $\rho(p,\nu)$ was obtained in Ref. [11] in the high temperature limit [27] in $D=3$. Using the expression given in [11] for the spectral density at two loops in the high temperature limit, and after some lengthy but straightforward algebra, we find

$$\rho(p,\nu)=\frac{\pi}{12}\left(\frac{\lambda T}{4\pi}\right)^2\mathrm{sgn}(\nu)\left[\Theta(|\nu|-p)+\frac{|\nu|}{p}\Theta(p-|\nu|)\right]. \tag{2.21}$$

Carrying out the dispersive integral (2.20) and subtracting off the terms that are independent of p,ω which are absorbed by the counterterm, we find

$$\Sigma(p,\omega)=-\frac{1}{12}\left(\frac{\lambda T}{4\pi}\right)^2\left[\ln\left(\frac{p^2-(\omega+i0^+)^2}{\mu^2}\right)-\frac{\omega}{p}\ln\left(\frac{\omega+i0^+-p}{\omega+i0^++p}\right)\right]. \tag{2.22}$$

Clearly, the static limit $\omega\to 0$ of the self-energy coincides with Eq. (2.19). The two-point function is therefore given by

$$\Gamma^{(2)}(p,\omega)=p^2-\omega^2+\frac{1}{12}\left(\frac{\lambda T}{4\pi}\right)^2\left[\ln\left|\frac{p^2-\omega^2}{\mu^2}\right|-\frac{\omega}{p}\ln\left|\frac{\omega-p}{\omega+p}\right|\right]-i\rho(p,\omega) \tag{2.23}$$

with $\rho(p,\omega)$ given by Eq. (2.21).

There are several features of this expression that are noteworthy.

It is clear that for $\lambda T\gg p,\omega$ the-two loop contribution is much larger than the tree level term $p^2-\omega^2$. This already signals the breakdown of perturbation theory in the high temperature regime when $\lambda T\gg p,\omega$ in the dynamical case.

Consider the *real* part of the two-point function as $\omega\to p$, i.e., near the mass shell:

$$\begin{aligned}\mathrm{Re}\,\Gamma^{(2)}(p,\omega\approx p)&\approx 2p^2\left\{\left(1-\frac{\omega}{p}\right)\left[1+\frac{1}{24}\left(\frac{\lambda T}{4\pi p}\right)^2\ln\left(\frac{|\omega-p|}{\mu}\right)\right]\right.\\&\left.+\frac{1}{12}\left(\frac{\lambda T}{4\pi p}\right)^2\ln\left(\frac{2p}{\mu}\right)\right\}.\end{aligned} \tag{2.24}$$

This expression reveals that $\omega=p$ is *not* the position of the mass shell of the (quasi)particle. The coefficient of $(1-\omega/p)$ hints at wave function renormalization but the fact that the two-point function does not vanish at this point prevents such identification. Furthermore, we see that the term that does not vanish at $\omega=p$ hints at a momentum-dependent shift of the position of the pole, i.e., a correction to the dispersion relation. However, for $\lambda T/p\gg 1$ both contributions are nonperturbatively large and the analysis is untrustworthy.

Now consider the width of the quasiparticle,

$$\gamma_p=-\frac{\mathrm{Im}\,\Sigma(p,\omega=p)}{2p}=\frac{\pi p}{12}\left(\frac{\lambda T}{4\pi p}\right)^2 \tag{2.25}$$

so that $\gamma_p/p\gg 1$ for $\lambda T/p\gg 1$. This signals the breakdown of the quasiparticle picture.

A similar analysis reveals the breakdown of perturbation theory away from but near the critical point with $|T-T_c|\ll T$. The imaginary part of the two-loop self-energy at $\vec{p}=\vec{0}$ and in terms of the temperature-dependent mass $m_R(T)$ can be obtained straighforwardly in *three spatial dimensions* [11,13] in the limit $T\gg m(T)$. It is found to be [11,13]

$$\mathrm{Im}\,\Sigma^{(2)}(\vec{p}=0,\omega=m_R(T))\propto\lambda^2T^2. \tag{2.26}$$

Consequently, at two-loop order the width of the zero momentum quasiparticle in three spatial dimensions is given by [11,13]

$$\Gamma \propto \frac{\lambda^2 T^2}{m_R(T)} \gg m_R(T). \tag{2.27}$$

This behavior is different from that of gauge theories at high temperature where low order fermion or gauge boson loops are infrared safe and determined by the hard thermal loop contributions [28]. This is so because for fermions there is no zero Matsubara frequency, while in the case of gauge bosons the vertices are momentum dependent. While the second term inside the square brackets in expression (2.23) determined by Landau damping is infrared finite and is similar to the leading contribution in the hard thermal loop program [19,28], the first term arises from the three-particle cut. The dependence of this term on the renormalization scale μ arises from the subtraction of the mass term at the critical point and reveals the infrared behavior. Furthermore, in the hard thermal loop program [19,28] one finds thermal masses of order gT with g the gauge coupling, and widths of order g^2T (up to logarithms), so that $\Gamma_p/\omega_p \ll 1$ in the weak coupling limit, while in the scalar theory under consideration Eq. (2.27) suggests that $\Gamma_p/\omega_p \gg 1$ in naive perturbation theory.

We note at this stage that the high temperature limit $p,\omega \ll T$ of the self-energy calculated from the spectral representation (2.20) can be directly obtained by computing $\Sigma(p,\omega_m)$ in the Matsubara representation given by Eq. (2.18) by setting the internal Matsubara frequencies $\omega_l = \omega_j = 0$ and analytically continuing $\omega_m \to -i\omega + 0^+$.

We highlight this observation since it will be the basis of further analysis in what follows: *the high temperature limit of the self-energy $p,\omega \ll T$ can be obtained by setting the internal Matsubara frequencies to zero and analytically continuing in the external Matsubara frequency*, i.e.,

$$\Sigma(p,\omega)|_{p,\omega\ll T} \equiv \frac{\lambda^2 T^2}{6} \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \times \frac{1}{q^2k^2[(\vec{p}+\vec{k}+\vec{q})^2+\omega_m^2]}\Bigg|_{\omega_m \to -i\omega+0^+}. \tag{2.28}$$

We provide one- and two-loop examples of this statement in Appendixes A and B and formal proof of this statement to one-loop order in Appendix C.

The result for the width of the quasiparticle at two loops was anticipated in Refs. [11,13]. This width is purely classical since the product λT is independent of $\hbar$ [11]. This result for the damping rate of long-wavelength quasiparticles. in the critical theory is in striking contrast with that for $T \gg T_c$ which has been studied in detail in [29,30]. For $T \gg T_c$ the thermal mass is $m_{th} \propto \sqrt{\lambda} T$ [29] while the two-loop contribution to the imaginary part of the self-energy for $T \gg T_c, p$ is still proportional to $\lambda^2 T^2$. Thus, the damping rate of long-wavelength excitations is $\gamma \propto \lambda^{3/2} T \ll \lambda^{1/2} T$ in the weak coupling limit. Therefore for $T \gg T_c$ long-wavelength excitations are true weakly coupled quasiparticles with narrow widths.

III. THE ϵ EXPANSION: STATIC CASE

The analysis of the previous section points out that naive perturbation theory at finite temperature breaks down at the critical point for momenta $< \lambda T$. The reason for this breakdown, as revealed by the analysis of the previous section, is the following. In four space-time dimensions the quartic coupling is dimensionless; however, each loop diagram in the perturbative expansion has a factor T from the sum over the Matsubara frequencies. After performing the renormalization of the mass including the finite temperature corrections and setting the theory at (or near) the critical point, the effective expansion parameter for long-wavelength correlation functions is λT, which has dimensions of momentum. If a given diagram has a momentum transfer scale μ, the effective *dimensionless* expansion parameter is therefore $\lambda T/\mu$, which becomes very large for $\mu \ll \lambda T$, i.e., the effective coupling is strongly relevant in the infrared. The analysis based on the RG beta function (2.12),(2.13) suggests that the effective coupling $g = \lambda T/\mu$ is driven to the three-dimensional (Wilson-Fischer) fixed point in the infrared, obviously a consequence of the dimensional reduction in the high temperature limit. This is confirmed by the large N resummation of the scattering amplitude (2.15),(2.16). *If* the value of the coupling at the fixed point is $\ll 1$ then a perturbative expansion *near the fixed point* be reliable; however, the value of the coupling at the fixed point is $g^* \sim \mathcal{O}(1)$, which of course is a consequence of the fact that for fixed λT the effective coupling scales with dimension of inverse momentum in the infrared. This situation is the same as in critical phenomena for theories that are superrenormalizable, in which the infrared divergences are severe.

The remedy in critical phenomena is to study the perturbative series via the ϵ expansion, wherein the value of the coupling at the fixed point is $\mathcal{O}(\epsilon)$ [4,21–23,25] and sum the perturbative series via the renormalization group. We now implement this program in the high temperature limit.

In five space-time dimensions the quartic coupling λ has the canonical dimension of inverse momentum, therefore the product λT that occurs in the perturbative expansion in the dimensionally reduced low energy theory is *dimensionless*. Then in a perturbative expansion at (or very near) the critical point we expect that infrared divergences will be manifest in the form of logarithms of the momentum scale in the loop. This implies that the effective coupling is marginal. Considering the theory in $4-\epsilon$ spatial dimensions and one Euclidean (compactified) time dimension, the effective coupling of the dimensionally reduced theory, λT, has dimensions of μ^{ϵ} with μ being a momentum scale. Therefore the effective dimensionless coupling for diagrams with a transferred momentum scale μ is $g(\mu) = \lambda T \mu^{-\epsilon}$. Thus, for fixed T the scaling dimension of this effective coupling is $-\epsilon$; hence we expect a nontrivial fixed point at which the coupling $g^* \sim \mathcal{O}(\epsilon)$.

Therefore for $\epsilon \ll 1$ we can perform a systematic perturbative expansion near the fixed point. This is the spirit of the

ϵ expansion in critical phenomena which, when combined with a resummation of the perturbative series via the renormalization group, has provided a spectacular quantitative and qualitative understanding of critical phenomena [4,21–25].

While dimensional regularization and the ϵ expansion have been used to study the dimensionally reduced high temperature theory insofar as thermodynamic quantities are concerned, i.e., static phenomena [15–17], we emphasize that our focus is to study *dynamics* at and near the critical point, which is fundamentally different from the studies of static phenomena in these references.

As a prelude to the study of the dynamics, we now reconsider the scattering amplitude at one-loop level and the self-energy at two-loop level in $4-\epsilon$ *spatial* dimensions at high temperature in the static limit. The one-loop self-energy is momentum independent and is absorbed in the definition of the thermal mass [29], which is set to zero at the critical point. There are two main purposes of this exercise: the first is to quantify the role of the higher Matsubara modes and the second to obtain a guide for the infrared running of the coupling constant.

A. Scattering amplitude

The one-loop contribution in the static limit, i.e., when the external Matsubara frequencies are zero, is given by

$$H(p,0)=\frac{T}{2}\sum_{n\in\mathcal{Z}}\int\frac{d^dq}{(4\pi)^d}\times\frac{1}{[q^2+(2\pi Tn)^2][(\vec{q}+\vec{p})^2+(2\pi Tn)^2]},\quad d=4-\epsilon,\quad (3.1)$$

In the high temperature limit the nonzero Matsubara terms give subdominant contributions. This property can be argued in different ways. For example, the $l\neq 0$ terms in Eq. (3.1) can be interpreted as Feynman diagrams in $d=4-\epsilon$ dimensions with mass $(2\pi Tl)^2$. Such contributions are negligible for $p\ll T$ [31,32].

We find, for high temperatures (see Appendix A),

$$H(p,0)\overset{T\gg p,s}{=}H_{asi}(p,0)-T^{1-\epsilon}\frac{\Gamma(1+\epsilon/2)\zeta(2+\epsilon)}{192\pi^{4+\epsilon/2}}\frac{p^2}{T^2}\left[1+\mathcal{O}\left(\frac{p^2}{T^2}\right)\right],\quad (3.2)$$

where $H_{asi}(p,)$ stands for the $l=0$ contribution to the sum (3.1) plus the dominant high temperature limit of the sum over the $l\neq 0$ terms, which is obtained by setting $p=0$. Separating the $l=0$ mode and setting $p=0$ in the contribution from the sum over $l\neq 0$, we find

$$H_{asi}(p,0)=\frac{Tp^{-\epsilon}}{2(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\left[\frac{\Gamma^2(1-\epsilon/2)}{\Gamma(2-\epsilon)}+2\left(\frac{4\pi^2T^2}{p^2}\right)^{-\epsilon/2}\zeta(\epsilon)\right]\quad (3.3)$$

with ζ Riemann's zeta function, which has the following properties [33]:

$$\zeta(0)=-\frac{1}{2},\quad \lim_{\epsilon\to 1}\zeta(\epsilon)=\frac{1}{\epsilon-1}+\gamma,\quad (3.4)$$

where $\gamma=0.577216\ldots$ stands for the Euler-Mascheroni constant.

Near four space dimensions $\epsilon\to 0$ we find

$$H_{asi}(p,0)=-\frac{T}{2(4\pi)^2}\left[\log\left(\frac{p^2}{4T^2}\right)-2\right]+\mathcal{O}(\epsilon).\quad (3.5)$$

There is *no pole* in ϵ and the argument of the logarithm reveals that T acts as an ultraviolet cutoff. The reason that there is no pole in epsilon as $\epsilon\to 0$ is that the poles in ϵ should be independent of temperature and should be those of the zero temperature theory. However, in dimensional regularization one-loop integrals have no poles in odd space-time dimensions [34]. On the other hand, near three space dimensions $\epsilon\to 1$ we find

$$H_{asi}(p,0)=\frac{T}{16p}\left[1-\frac{1}{\pi^2}\frac{p}{T}\ln\frac{T}{\mu}\right]+\frac{1}{(4\pi)^2(\epsilon-1)},\quad (3.6)$$

where the pole term at $\epsilon=1$ corresponds to the usual coupling constant renormalization. This divergent term is temperature independent, as expected; the $\ln(T)$ is reminiscent of an upper momentum cutoff for the high temperature limit. The first term is precisely what we obtained in Eq. (2.8) by setting the internal Matsubara frequency to zero, i.e., the result of the dimensionally reduced theory. After subtracting the pole near three space dimensions, the first term gives the leading infrared contribution in the limit $T/p\gg 1$, whereas the logarithm is subleading.

This expression coincides with that given in Refs. [16,17]. In these references the four-dimensional high temperature static theory was studied and a systematic analysis of Feynman diagrams in the dimensionally reduced theory (three dimensions) was performed. The $\epsilon-1$ in our expressions should be mapped onto 2ϵ for comparison with the results in these references.

For $\epsilon>0$ we can neglect the terms of the form $(T/p)^{-\epsilon}$ in Eq. (3.3) in the limit $T/p\gg 1$. And for $1\gg\epsilon>0$ we find that the static scattering amplitude at a symmetric point (2.9) in the limit in which the temperature is much larger than the external momentum scales is given by

$$\Gamma^{(4)}(p_i=\overline{P_i},0)=-\lambda_{eff}(\mu,T)=-\lambda\left[1-\frac{3}{(4\pi)^{d/2}}\frac{\lambda T\mu^{-\epsilon}}{\epsilon}\right].\quad (3.7)$$

The factor T arising from the Matsubara sum is such that λT has dimensions of (momemtum)$^{\epsilon}$ so that in $5-\epsilon$ space time dimensions $\lambda T\mu^{-\epsilon}$ is *dimensionless*. Thus, introducing the dimensionless renormalized coupling

$$g_R(\mu)=\frac{\lambda_{eff}(\mu,T)T\mu^{-\epsilon}}{(4\pi)^{d/2}},\quad d=4-\epsilon, \tag{3.8}$$

we find

$$\beta_g=\mu\frac{\partial g}{\partial\mu}\bigg|_{\lambda,T}=-\epsilon g+3g^2+\mathcal{O}(g^3). \tag{3.9}$$

Therefore this effective coupling in the infrared limit is driven to a nontrivial fixed point

$$g^*=\frac{\epsilon}{3}. \tag{3.10}$$

Hence for $\epsilon\ll 1$ the fixed point theory can be studied perturbatively. This of course is the basis of the ϵ expansion in critical phenomena [4,21–25] and will be the important point upon which our analysis will hinge.

B. Two-loop self-energy

As mentioned above, the one-loop contribution to the self-energy is momentum independent and absorbed into the definition of the thermal mass. The two-loop contribution in the static limit in d spatial dimensions is given by

$$\Sigma^{(2)}(k,0)=\frac{\lambda^2T^2}{6}\sum_l\sum_m\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{[q^2+\omega_l^2][p^2+\omega_m^2][(p+q+k)^2+(\omega_l+\omega_m)^2]}. \tag{3.11}$$

We now introduce two Feynman parameters, separate the $l=m=0$ term from the Matsubara sums, and take $T\gg p$ in the sums with $l,m\neq 0$, to find

$$\Sigma^{(2)}(k,0)=\frac{g^2}{6}\frac{\Gamma(-1+\epsilon)\Gamma^3(1-\epsilon/2)}{\Gamma(3-3\epsilon/2)}k^2\left[\frac{k^2}{\mu^2}\right]^{-\epsilon}+g^2T^2\left[\frac{T^2}{\mu^2}\right]^{-\epsilon}\mathcal{C}(d) \tag{3.12}$$

with $\mathcal{C}(d)$ depending only on the dimensionality. The second term proportional to $T^{4-2\epsilon}$ does not depend on the momentum and is therefore canceled by the mass counterterm that defines the critical theory. Therefore for $\epsilon>0$ but small we find

$$\Gamma^{(2)}(k,0)=p^2-[\Sigma(k,0)-\Sigma(0,0)]=k^2\left[1+\frac{g^2}{12\epsilon}-\frac{g^2}{12}\ln\left(\frac{k^2}{\mu^2}\right)\right]+\mathcal{O}(g^3). \tag{3.13}$$

We introduce the wave function renormalization in dimensional regularization by the usual relation

$$\Gamma_R^{(2)}(k,0)=Z_\phi\Gamma^{(2)}(p,0) \tag{3.14}$$

and choose

$$Z_\phi=1-\frac{g^2}{12\epsilon}+\mathcal{O}(g^3). \tag{3.15}$$

Therefore the renormalized two-point function in the static, high temperature limit is given by

$$\Gamma_R^{(2)}(k,0)=k^2\left[1-\frac{g}{12}\ln\left(\frac{k^2}{\mu^2}\right)\right]+\mathcal{O}(g^3). \tag{3.16}$$

The infrared behavior is obtained by resumming the perturbative series via the renormalization group [4,21–25] which leads to the scaling form of the two-point function in the infrared limit,

$$\Gamma_R^{(2)}(k)\propto k^{2-\eta}, \tag{3.17}$$

with

$$\eta=\frac{g^{*2}}{6}=\frac{\epsilon^2}{54}+\mathcal{O}(\epsilon^3). \tag{3.18}$$

This is the anomalous dimension to lowest order in ϵ [4,21–25].

C. The strategy

The analysis of the static case above has highlighted several important features of the infrared behavior near the critical point, which determine the strategy for studying the dynamical case. (a) The infrared behavior in the limit when $T\gg p$ with p the typical momentum of the Feynman diagram is determined by the dimensionally reduced theory obtained by setting the *internal* Matsubara frequencies to zero. (b) Naive perturbation theory breaks down in three space dimensions because the dimensionless coupling is $\lambda T/\mu$ with μ the external momentum scale in the Feynman diagram, while a large N or renormalization group resummation suggests a nontrivial infrared stable fixed point. The coupling at this fixed point is of $\mathcal{O}(1)$. (c) Just as in critical phenomena the perturbative expansion can be systematically controlled in an ϵ expansion around *four spatial dimensions* corresponding to a theory dimensionally reduced from $5-\epsilon$ space time dimensions. The effective dimensionless coupling of the dimensionally reduced theory (four dimensional) is λT, which is marginal. This combination is independent of $\hbar$—this can be seen by restoring powers of $\hbar \lambda \to \hbar \lambda$, $T \to T/\hbar$—so that the low energy, dimensionally reduced theory is classical. In $4-\epsilon$ spatial dimensions the effective dimensionless coupling $g=\lambda T \mu^{-\epsilon}$ is driven to the infrared stable Wilson-Fischer fixed point of $\mathcal{O}(\epsilon)$ by the renormalization group trajectories. Thus, the strategy to follow becomes clear: we will now study the dynamics by including the contribution from the external Matsubara frequency, focusing on the infrared behavior for $p,\omega \ll T$ near the critical point in a systematic ϵ expansion around *four* spatial dimensions.

We note that the theory in $5-\epsilon$ dimensions is formally nonrenormalizable in the ultraviolet; however, this is irrelevant for the infrared which is the region of interest here. The analysis provided above in the limit $\epsilon \to 0$ clearly shows that near five space-time dimensions there are no poles in dimensional regularization in one-loop diagrams as expected [31]. The potential poles are replaced by ln(T). The low energy theory must be understood with a cutoff of $\mathcal{O}(T)$ and the dimensionally regularized integrals in five space-time dimensions clearly display this cutoff in the arguments of logarithms. The long-wavelength $\mu/T \ll 1$ and the $\epsilon \to 0$ limits do not commute: keeping the subleading terms in the high temperature limit and taking $\epsilon \to 0$ results in poles in ϵ actually translating into logarithms of the cutoff T. On the other hand, keeping $\epsilon > 0$ and small, the $T/\mu \to \infty$ limit can be taken and the subleading high temperature corrections vanish. Clearly, it is the latter limit that has physical relevance, since eventually we are interested in studying the infrared behavior of the physical theory in three space dimensions. Hence in what follows we consider the long-wavelength limit for $\epsilon > 0$ but small and approach the physical dimensionality $\epsilon \to 1$ in a consistent ϵ expansion improved via the renormalization group. This is the strategy in classical critical phenomena as well, where for $\epsilon > 0$ and small the ultraviolet cutoff can be taken to infinity.

At this stage it is important to highlight the difference between the main focus of this work and that in Refs. [16,17]. The work of Refs. [16,17] studies the *static* limit of the dimensionally reduced theory near *three* spatial dimensions arising from the high temperature limit of a four-dimensional Euclidean theory compactified in the time direction. In contrast, we here focus on studying the *dynamics* in the limit when $p,\omega \ll T$, which as emphasized by the analysis above will be studied in an ϵ expansion in a dimensionally reduced theory near *four* space dimensions.

The limit of physical interest $\epsilon \to 1$ must be studied by improving the perturbative expansion via the renormalization group [4,21–25] and eventually by other nonperturbative resummation methods, such as Padé approximants or Borel resummation, that will extend the regime of validity of the ϵ expansion [23].

IV. DYNAMICS NEAR THE CRITICAL POINT

We now turn to the dynamics. Our main goal is to study the feasibility of a quasiparticle description of low energy excitations at and near the critical point. Of particular interest is the dispersion relation as well as the damping rates of these excitations. This information is contained in the two-point function $\Gamma^{(2)}(p,\omega_n)$ which is the inverse propagator, analytically continued to $\omega_n \to -i\omega + 0^+$. The region of interest is $p,\omega \ll T$ and if the theory is (slightly) away from the critical point $M(T) \ll T$ as well. In principle for a fixed ω_n or fixed external Matsubara frequencies in the external legs of n-point functions, one must perform the sum over the internal Matsubara frequencies first and then take the analytic continuation. However, as was shown above in detail in the static case, the largest infrared singular contribution arises from setting the internal Matsubara frequencies to zero. That this is also the case in the dynamics can be seen by considering a diagram with m internal lines, rerouting the external Matsubara frequency through one of the lines. All of the lines are equivalent since rerouting the external Matsubara frequency corresponds to a shift in one of the sums. The other $m-1$ lines contain propagators in which the internal Matsubara frequency acts as a mass of $O(2\pi l T)$. These are the superheavy modes in the description of Refs. [16,17,31,32]. The contribution that is dominant in the infrared is from the region of loop momenta $\ll T$ which is largest when the mass of the propagator is zero, i.e., the zero Matsubara frequency. Keeping nonzero Matsubara frequencies in any of the $m-1$ legs will lead to subleading contributions in the limit $p,\omega \ll T$.

Once the internal Matsubara frequencies have been set to zero we can analytically continue the external Matsubara frequency to a continuous Euclidean variable $\omega_n \to s$ to obtain the Euclidean two-point function. The dispersion relation and damping rate are obtained by further analytical continuation $s \to -i\omega + 0^+$.

As anticipated in Sec. (II), because Euclidean time is compactified and plays a different role from the spatial dimensions, we must consider the anisotropic Lagrangian density (2.4), which includes the velocity of light multiplying the derivatives with respect to Euclidean time. If this velocity of light is simply a constant it can be reabsorbed into a trivial redefinition of the time variable. However, as it will become clear below, this velocity of light acquires a nontrivial renormalization as a consequence of the anisotropy between space and time directions at finite temperature and will run with the renormalization group. Thus, the Euclidean propagator is generalized to

$$G(k,\omega_m) = \frac{1}{k^2 + \omega_m^2/v_0^2}. \tag{4.1}$$

A. The scattering amplitude

We begin by studying the scattering amplitude, now as a function of external momenta and frequencies. The one-loop contribution is determined by the function $H(p,s)$ given by Eq. (2.7), which for $p,s \ll T$ is given by (see Appendix A)

$$H(p,s) \overset{T\gg p,s}{=} H_{asi}(p,s) - \frac{\Gamma(1+\epsilon/2)\zeta(2+\epsilon)}{192\pi^{4+\epsilon/2}T^{1+\epsilon}} \times \left[p^2+\frac{s^2}{v_0^2}(1-\epsilon)\right]\left[1+\mathcal{O}\left(\frac{p^2,s^2/v_0^2}{T^2}\right)\right], \tag{4.2}$$

where $H_{asi}(p,s)$ stands for the $l=0$ contribution to the sum (2.7) plus the dominant high temperature limit of the sum over the $l\neq 0$ terms, which is obtained by setting $p,s=0$,

$$H_{asi}(p,s) = \frac{T}{2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2[(\vec{q}+\vec{p})^2+s^2/v_0^2]} + \frac{\Gamma(1+\epsilon/2)\zeta(\epsilon)}{8\pi^{2+\epsilon/2}\epsilon}T^{1-\epsilon}, \tag{4.3}$$

This integral is computed in Appendix A with the result

$$H_{asi}(p,s) = -\frac{\Gamma(\epsilon/2-1)}{2(4\pi)^{2-\epsilon/2}}T\left(\frac{s^2}{v_0^2}+p^2\right)^{-\epsilon/2} \times F\left(\frac{\epsilon}{2},1-\frac{\epsilon}{2};2-\frac{\epsilon}{2};\frac{p^2}{p^2+s^2/v_0^2}\right) + \frac{\Gamma(1+\epsilon/2)\zeta(\epsilon)}{8\pi^{2+\epsilon/2}\epsilon}T^{1-\epsilon}, \tag{4.4}$$

where $F(a,b;c;z)$ stands for the hypergeometric function. For $\epsilon>0$ and $p,s/v_0 \ll T$ we can neglect the second term, since it is proportional to $T^{1-\epsilon}\ll T(p^2+s^2/v_0^2)^{-\epsilon/2}$. We note that the infrared dominant contribution can be written in the form $Tp^{-\epsilon}\mathcal{F}(s^2/v^2p^2)$ the factor T thus combines with the coupling λ to give the effective coupling of dimension μ^ϵ in $d=4-\epsilon$ spatial dimensions.

For $\epsilon>0$ but small and neglecting the second term, Eq. (4.4) can be expanded in ϵ, leading to

$$\lambda_0 H_{asi}(p,s) = \frac{g(\mu)}{2}\left[\frac{2}{\epsilon}-\left(1+\frac{s^2}{v_0^2p^2}\right)\ln\left(\frac{s^2+p^2v_0^2}{\mu^2v_0^2}\right) + \frac{s^2}{v_0^2p^2}\ln\left(\frac{s^2}{v_0^2\mu^2}\right)+\ln 4\pi+2-\gamma+\mathcal{O}(\epsilon)\right] \tag{4.5}$$

where we introduced the dimensionless bare coupling

$$g(\mu) = \frac{\lambda_0 T\mu^{-\epsilon}}{(4\pi)^{d/2}}. \tag{4.6}$$

We remark that one cannot take $\epsilon\to 0$ in this expression since in this limit the pole is actually canceled by the second term in Eq. (4.4) above. As emphasized above, this expression must be understood for $\epsilon>0$ but small so that the contributions of the form $(T/s,T/p)^{-\epsilon}\to 0$ for $T\gg s,p$. Therefore, the expression above must be understood in the sense that (i) $T\gg s,p$ with fixed $\epsilon>0$ and (ii) $\epsilon\ll 1$, and the resulting expressions have a Laurent expansion for small ϵ.

B. The self-energy at two loops

Neglecting the contribution from the nonzero Matsubara frequencies which will be absorbed by the mass counterterm in the definition of the critical temperature [or $M^2(T)$ away from the critical point] and also neglecting terms that vanish in the limit $T\gg p,s$, the dominant contribution in the infrared to the two-loop self-energy is

$$\Sigma(p,s) = \frac{\lambda_0^2T^2}{6}\int\frac{d^dq}{(2\pi)^d} \times \int \frac{d^dk}{(2\pi)^d}\frac{1}{q^2k^2[(q+k+p)^2+s^2/v_0^2]} = \frac{\lambda_0^2T^2}{6(4\pi)^d}\frac{\Gamma^2(d/2-1)\Gamma(3-d)}{\Gamma(d-2)} \times \int_0^1 dx(1-x)^{d-3}x^{1-d/2}\left[xp^2+\frac{s^2}{v_0^2}\right]^{d-3}. \tag{4.7}$$

While this expression can be written in terms of hypergeometric functions, it is more convenient to expand it in ϵ with the result that

$$\Gamma^{(2)}(p,s) = p^2+\frac{s^2}{v_0^2}+\frac{g^2(\mu)}{6\epsilon}\left[\frac{p^2}{2}+\frac{2}{\epsilon}\frac{s^2}{v_0^2}-\epsilon\frac{p^2}{2}\ln\left(\frac{p^2}{\mu^2}\right) - 2\frac{s^2}{v_0^2}\ln\left(\frac{s^2}{v_0^2\mu^2}\right)\right]+\mathcal{O}(g^2\epsilon^0,g^2\epsilon,g^3) \tag{4.8}$$

with $g(\mu)$ given by Eq. (4.6).

C. Renormalization

The forms of the two- and four-point functions immediately suggest a renormalization scheme akin to the familiar one used in critical phenomena [21–23,25] with *one important difference*: we see from Eq. (4.8) that the velocity of light v_0 must also be renormalized. The wave function renormalization is introduced as usual via

$$\Gamma_R^{(2)}(p,s,v_R) = Z_\phi\Gamma^{(2)}(p,s,v_0). \tag{4.9}$$

The renormalized mass as a function of temperature is defined as

$$\Gamma_R^{(2)}(0,0) = M^2(T). \tag{4.10}$$

This definition, however, defines the inverse susceptibility or correlation length, rather than the pole mass; the critical theory is defined by $M^2(T)=0$. Coupling constant and velocity of light renormalization are achieved by

$$\lambda_R = Z_\phi^2 Z_\lambda \lambda_0,$$

$$\Gamma_R^{(4)}(p_i=\overline{P_i},\ \ s=0) = -\lambda_R, \tag{4.11}$$

$$v_R^2 = v_0^2 \frac{Z_v}{Z_\phi}. \tag{4.12}$$

The renormalization conditions that determine the constants Z_ϕ, Z_λ, Z_v are

$$\left.\frac{\partial \Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=\mu^2,\ s^2/v_R^2=\mu^2} = 1,$$

$$\left.\frac{\partial \Gamma_R^{(2)}}{\partial s^2}\right|_{p^2=\mu^2,\ s^2/v_R^2=\mu^2} = \frac{1}{v_R^2},$$

$$\Gamma^{(4)}(p_i=\overline{P_i};\ \ s=0) = -\lambda_R. \tag{4.13}$$

Consistently with the ϵ expansion, we choose the renormalization constants Z_ϕ, Z_λ, Z_v in the minimal subtraction scheme to lowest order, since keeping higher powers of the coupling or ϵ results in higher order corrections in the ϵ expansion.

To lowest order, one loop for the four-point function and two loops for the two-point function, we find from the results (4.5) and (4.8)

$$Z_\lambda = 1 - \frac{3g(\mu)}{\epsilon} \Rightarrow g_R = g(\mu) - \frac{3g^2(\mu)}{\epsilon},$$

$$g_R = \frac{\lambda_R T \mu^{-\epsilon}}{(4\pi)^{d/2}}, \quad Z_\phi = 1 - \frac{g^2}{12\epsilon}, \tag{4.14}$$

$$Z_v = 1 - \frac{g^2}{3\epsilon^2}. \tag{4.15}$$

Thus, the renormalized two-point function reads

$$\Gamma_R^{(2)}(p,s) = p^2\left[1 - \frac{g^2}{12}\ln\left(\frac{p^2}{\mu^2}\right)\right] + \frac{s^2}{v_R^2}\left[1 - \frac{g^2}{3\epsilon}\ln\left(\frac{s^2}{v_R^2\mu^2}\right)\right] + \mathcal{O}(g^3, g^2\epsilon). \tag{4.16}$$

V. THE RENORMALIZATION GROUP

Before we embark on the resummation program via the renormalization group, it is important to highlight two important features.

The contributions that are dominant in the infrared in the limit $T \gg p, s$ correspond to the terms with internal Matsubara frequencies equal to zero; the nonzero Matsubara frequencies give subleading contributions for $\epsilon > 0$. This in turn results in the dependence on temperature being solely through the effective coupling $g = \lambda T \mu^{-\epsilon}$. This can be seen from the fact that each loop has a factor T from the Matsubara sum over the internal loop frequencies as well as one power of the coupling constant λ; for dimensional reasons the dimensionless coupling is obtained by multiplying by $\mu^{-\epsilon}$.

The velocity of light v always enters in the form s/v since this is the form that enters in the propagators and the renormalization conditions above.

A. The critical point

The bare n-point functions are independent of the renormalization scale μ, and this independence leads to the renormalization group equations (we now suppress the subscript R, understanding that all quantities are renormalized)

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g} + \beta_v\frac{\partial}{\partial v} - \frac{N}{2}\gamma\right] \times \Gamma^{(N)}\left(p_1, \frac{s_1}{v}; p_2, \frac{s_2}{v}; \ldots, p_N; \frac{s_N}{v}; g, \mu\right) = 0 \tag{5.1}$$

with

$$\beta_g = \mu\left.\frac{\partial g}{\partial\mu}\right|_{\lambda_0, T, v_0}, \tag{5.2}$$

$$\beta_v = \mu\left.\frac{\partial v}{\partial\mu}\right|_{\lambda_0, T, v_0}, \tag{5.3}$$

$$\gamma = \mu\left.\frac{\partial \ln Z_\phi}{\partial\mu}\right|_{\lambda_0, T, v_0}. \tag{5.4}$$

To lowest order we find

$$\beta_g = -\epsilon g + 3g^2 + \mathcal{O}(g^3, g^2\epsilon), \tag{5.5}$$

$$\beta_v = \frac{1}{2}\left[\frac{2g^2}{3\epsilon} - \gamma\right] v + \mathcal{O}(g^3, g^2\epsilon), \tag{5.6}$$

$$\gamma = \frac{g^2}{6} + \mathcal{O}(g^3, g^2\epsilon). \tag{5.7}$$

While we can write down the general solution of the RG equation (5.1) for an arbitrary N-point function, our focus is to understand the quasiparticle structure, which is obtained from $\Gamma^{(2)}$.

Since $\Gamma^{(2)}$ has dimension 2, it follows that

$$\Gamma^{(2)}\left(p, \frac{s}{v}, g, \mu\right) = \mu^2 \Phi\left(\frac{p}{\mu}, \frac{s}{v\mu}, g\right); \tag{5.8}$$

therefore

$$\Gamma^{(2)}\left(e^t p, \frac{e^t s}{v}, g, \mu\right) = e^{2t}\Gamma^{(2)}\left(p, \frac{s}{v}, g, \mu e^{-t}\right). \tag{5.9}$$

This scaling property then leads to

$$\left[\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial\mu} - 2\right]\Gamma^{(2)}\left(e^t p, \frac{e^t s}{v}, g, \mu\right) = 0, \tag{5.10}$$

which combined with the RG equation (5.1) leads to the following equation that determines the scaling properties of the two-point function:

$$\left[-\frac{\partial}{\partial t}+\beta_g\frac{\partial}{\partial g}+\beta_v\frac{\partial}{\partial v}-(\gamma-2)\right]\Gamma^{(2)}\left(e^t p,\frac{e^t s}{v},g,\mu\right)=0. \tag{5.11}$$

The solution of this equation is standard [21–23]:

$$\Gamma^{(2)}\left(e^t p,\frac{e^t s}{v},g,\mu\right)=e^{\int_0^t dt'[2-\gamma(t')]}\Gamma^{(2)}\left(p,\frac{s}{v(t)},g(t),\mu\right) \tag{5.12}$$

with

$$\frac{\partial g(t)}{\partial t}=\beta_g(g(t)),\quad g(0)=g_R(\mu),$$

$$\frac{\partial v(t)}{\partial t}=\beta_v(v(t),g(t)),\quad v(0)=v_R(\mu),$$

$$\gamma(t)=\gamma(g(t),v(t)). \tag{5.13}$$

As $t\to-\infty$, i.e., the momentum and frequency are scaled toward the infrared, we see from the RG β function (5.5) that the coupling is driven to its fixed point

$$\lim_{t\to-\infty} g(t)=g^*=\frac{\epsilon}{3}+\mathcal{O}(\epsilon^2), \tag{5.14}$$

which in turn implies that

$$\lim_{t\to-\infty}\gamma(t)=\eta=\frac{\epsilon^2}{54}+\mathcal{O}(\epsilon^3), \tag{5.15}$$

$$\lim_{t\to-\infty} v(t)=v(0)\,e^{(z-1)t}, \tag{5.16}$$

where we introduced the new *dynamical* critical exponent

$$z=1+\frac{1}{2}(\eta_t-\eta)+\mathcal{O}(\epsilon^2),$$

$$\eta_t=\frac{2g^{*2}}{3\epsilon}=\frac{2\epsilon}{27}+\mathcal{O}(\epsilon^2). \tag{5.17}$$

Therefore, in the asymptotic infrared limit we find that

$$\Gamma^{(2)}\left(e^t p,\frac{e^t s}{v},g,\mu\right)=e^{t(2-\eta)}\Gamma^{(2)}\left(p,\frac{s e^{(1-z)t}}{v(0)},g^*,\mu\right). \tag{5.18}$$

It is convenient to redefine $pe^t=P$, $se^t=S$, to find

$$\Gamma^{(2)}\left(P,\frac{S}{v},g,\mu\right)=e^{t(2-\eta)}\Gamma^{(2)}\left(Pe^{-t},\frac{S}{v(0)}e^{-t}e^{(1-z)t},g^*,\mu\right), \tag{5.19}$$

and finally writing $P=\mu e^t$ and using the property (5.8) we find the scaling form in the infrared limit

$$\Gamma^{(2)}\left(P,\frac{S}{v},g,\mu\right)=\mu^2\left[\frac{P}{\mu}\right]^{2-\eta}\Phi(\vartheta),$$

$$\vartheta=\left(\frac{S}{v(\mu)\mu^{1-z}P^z}\right)^2. \tag{5.20}$$

The solution of the RG equation clearly shows that the two-point function in the infrared limit is a *scaling* function of the ratio $s/v(\mu)\mu^{1-z}p^z$ highlighting the role of the new dynamical exponents z given by Eq. (5.17) with η_t to lowest order given by Eq. (5.17).

The emergence of the new dynamical exponent z is a consequence of the anisotropic renormalization between momentum and frequency, or space and time, manifest in the renormalization of the speed of light. This novel phenomenon can be traced back to the different role played by time (compactified) and space in the Euclidean formulation at finite temperature. A similar anisotropic rescaling emerges in a different context, a Heisenberg ferromagnet with correlated impurities [35] with similar renormalization group results.

While the formal solution does not yield the function Φ, we can find it by matching to the lowest order perturbative expansion (4.16) when the coupling is at the nontrivial fixed point. From the form of the perturbative renormalized two-point function given by Eq. (4.16) and assuming the exponentiation of the leading logarithms via the renormalization group near the nontrivial fixed point,

$$\Gamma_R^{(2)}(p,s;g^*)=p^2\left[1-\eta\ln\left(\frac{p}{\mu}\right)\right]+\frac{s^2}{v_R^2}\left[1-\eta_t\ln\left(\frac{s}{v_R\mu}\right)\right]$$

$$\approx p^{2-\eta}\mu^{\eta}+\left(\frac{s}{v_R}\right)^{2-\eta_t}\mu^{\eta_t}, \tag{5.21}$$

which can immediately be written in the scaling form

$$\Gamma_R^{(2)}(p,s;g^*)\sim p^{2-\eta}\mu^{\eta}\left[1+\left(\frac{s}{v(\mu)\mu^{1-z}p^z}\right)^{2-\eta_t}\right]$$

$$z=\frac{2-\eta}{2-\eta_t}\simeq 1+\frac{1}{2}(\eta_t-\eta). \tag{5.22}$$

Clearly, this form coincides with the scaling solution of the renormalization group and the perturbative expansion in the regime in which it is valid. We note, however, that in the computation of η_t we have neglected contributions to the renormalization of the velocity v of $\mathcal{O}(g^3/\epsilon)$, which would appear at next order and would lead to an $\mathcal{O}(\epsilon^2)$ contribution to η_t, which is of the same order as η in z. Thus, consistently we must neglect the contribution of η to the dynamical exponent z, which to lowest order is therefore

$$z=1+\frac{\eta_t}{2}+\mathcal{O}(\epsilon^2)=1+\frac{\epsilon}{27}+\mathcal{O}(\epsilon^2). \tag{5.23}$$

Quasiparticles and critical slowing down

The quasiparticle structure of the theory is obtained from the Green's function $G^{-1}(p,\omega)=\Gamma^{(2)}\quad(p,s=-i\omega+0^+,g^*)$. In particular, the dispersion relation and the width of the quasiparticle are obtained from the real and imaginary parts, respectively.

While the general solution of the RG equation does not determine the scaling function Φ in Eq. (5.20), the fact that it is only a function of the scaling ratio ϑ allows us to extract the quasiparticle structure. The analytic continuation $s\rightarrow -i\omega+0^+$ leads to the analytic continuation of the scaling variable $\vartheta\rightarrow -\varpi^2-i\,\mathrm{sgn}(\varpi)0^+$ with

$$\varpi=\frac{\omega}{v(\mu)\mu^{1-z}p^z}. \tag{5.24}$$

Writing the scaling function Φ analytically continued in terms of the real and imaginary parts $\Phi(\vartheta=-\varpi^2-i\,\mathrm{sgn}(\varpi)0^+)=\Phi_R(\varpi)+i\Phi_I(\varpi)$, the position of the quasiparticle pole corresponds to the value of ϖ for which the real part vanishes. We call this dimensionless real number ϖ^*; hence it is clear that the dispersion relation for the quasiparticles obeys

$$\omega_p=\varpi^* v(\mu)\mu^{1-z}p^z. \tag{5.25}$$

Furthermore, assuming that Φ_R vanishes linearly at ϖ^* we can write the Green's function near the position of the pole in the form

$$G(\omega,p)\simeq\frac{1}{\mu^2[p/\mu]^{2-\eta}}\;\frac{1}{(\varpi-\varpi^*)\Phi_R'(\varpi^*)+i\Phi_I(\varpi^*)}. \tag{5.26}$$

Alternatively, we can write the RG improved propagator near the quasiparticle pole in the Breit-Wigner form

$$G_{BW}(\omega,p)\sim\frac{\mathcal{Z}_p}{\omega-\omega_p+i\Gamma_p} \tag{5.27}$$

with the dispersion relation, residue at the quasiparticle pole, and quasiparticle width given by

$$\omega_p=\varpi^* v(\mu)\mu^{1-z}p^z, \tag{5.28}$$

$$v_g=(z)\varpi^* v(\mu)\mu^{1-z}p^{z-1}, \tag{5.29}$$

$$\mathcal{Z}_p=\frac{v(\mu)\mu^{1-z}p^z}{\mu^2[p/\mu]^{2-\eta}\Phi_R'(\varpi^*)}, \tag{5.30}$$

$$\Gamma_p=\frac{\Phi_I(\varpi^*)v(\mu)\mu^{1-z}p^z}{\Phi_R'(\varpi^*)}\equiv\frac{\omega_p\Phi_I(\varpi^*)}{\varpi^*\Phi_R'(\varpi^*)}. \tag{5.31}$$

The imaginary part $\Phi_I(\varpi)$ must be proportional to the anomalous dimensions and hence perturbatively small in the ϵ expansion (this will be seen explicitly below to lowest order).

The definite values for ϖ^*, $\Phi_R(\varpi^*)$, and $\Phi_I(\varpi^*)$ must be found by an explicit calculation. However, the above quasiparticle properties, such as the position of the pole, group velocity, residue, and width, are *universal* in the sense that they depend only on the fixed point theory. For a positive dynamical exponent z the above analysis reveals a vanishing group velocity and width for long-wavelength quasiparticles at the critical point.

Furthermore, the expression for the width given by Eq. (5.31) displays not only the phenomenon of critical slowing down, i.e., the width of the quasiparticle vanishes in the long-wavelength limit, but also the validity of the quasiparticle picture, since $\Gamma_p/\omega_p\ll 1$ in the ϵ expansion.

Threshold singularities. While we have assumed above that the real part of the scaling function vanishes linearly at the quasiparticle pole, this need not be the general situation. It is possible that the real part vanishes with an anomalous power law, i.e.,

$$\Phi_R(\Omega)\sim|\Omega-\Omega^*|^{1+\chi},\quad \chi=\chi^{(1)}\epsilon+\chi^{(2)}\epsilon^2+\cdots. \tag{5.32}$$

In this case a quasiparticle width cannot be defined as the residue will either vanish or diverge depending on the sign of χ. It is also possible that $\Phi_I(\Omega)$ also vanishes with an anomalous power at Ω^*. We refer to these cases as threshold singularities and we will find below an example of this case. Another example of this situation has been found in dense QCD as a result of the breakdown of Fermi liquid theory in the normal phase [36]. Clearly, only a detailed calculation of the scaling functions can reveal whether the real part of the scaling function vanishes linearly or with an anomalous power law at Ω^*. The set of quasiparticle properties given above (5.28)–(5.31) is valid only provided the real part vanishes *linearly*.

We can go further and find the explicit form of the scaling function by focusing on the renormalization group improved propagator obtained in lowest order in the ϵ expansion given by Eq. (5.22). The analytic continuation to real frequencies of the RG improved two-point function to lowest order given by Eq. (5.22) leads to

$$G^{-1}(p,\omega)=p^{2-\eta}\mu^{\eta}\left[1-\left(\frac{|\omega|}{v(\mu)\mu^{1-z}p^z}\right)^{2-\eta_t}\times\left(1+i\frac{\pi\eta_t}{2}\mathrm{sgn}(\omega)\right)\right] \tag{5.33}$$

where we have approximated $\cos(\pi\eta_t/2)\approx 1$, $\sin(\pi\eta_t/2)\approx\pi\eta_t/2$ to lowest order.

From this expression we see that the dispersion relation ω_p, group velocity $v_g(p)$, width $\Gamma(p)$, and quasiparticle residue $\mathcal{Z}_p$ are of the form given by Eqs. (5.28)–(5.31) with $\varpi^*=1$, and with the following explicit expressions to leading order in the ϵ expansion:

$$|\omega_p|=v(\mu)\mu^{1-z}p^z, \tag{5.34}$$

$$v_g(p)=zv(\mu)\mu^{1-z}p^{z-1}, \tag{5.35}$$

$$\Gamma(p)=\frac{\pi\eta_t}{4}v(\mu)\mu^{1-z}p^z\equiv\frac{\pi\eta_t}{4}|\omega_p|, \tag{5.36}$$

$$\mathcal{Z}_p=\frac{1}{2}v(\mu)\mu^{1-z}p^{z-1}, \tag{5.37}$$

with z given by Eq. (5.23) above.

Several important features of these expressions must be highlighted.

The dispersion relation (5.25) features an *anomalous dimension* given by the dynamical exponent $z\approx 1+\eta_t/2=1+\epsilon/27$. The product $v(\mu)\mu^{1-z}$ is a renormalization group invariant as can be seen from Eqs. (5.3) and (5.6) evaluated at the fixed point. Thus, all of the above quantities that describe the physical quasiparticle properties are manifestly renormalization group invariant.

The group velocity (5.35) *vanishes* in the long-wavelength limit as a power law completely determined by the dynamical anomalous dimension z. This feature highlights the collective aspects of the long-wavelength excitations.

Critical slowing down is explicitly manifest in the width $\Gamma(p)$ since $\Gamma(p)\to 0$ as $p\to 0$. Furthermore, we also emphasize the validity of the quasiparticle picture, the ratio $\Gamma(p)/\omega_p\approx\pi\eta_t/2\sim\mathcal{O}(\epsilon)\ll 1$. Thus, the quasiparticles are narrow in the sense that their width is much smaller than the position of the pole. Even considering $\epsilon=1$, corresponding to dynamical critical phenomena in three spatial dimensions, $\Gamma_p/\omega_p\sim 0.1$.

Thus, we see that the renormalization group resummation has led to a consistent quasiparticle picture, but in terms of a dispersion relation that features an anomalous dimension and a group velocity that vanishes in the long-wavelength limit. Obviously these features of the quasiparticles cannot be extracted from a naive perturbative expansion.

B. Away from the critical point: $T\gtrsim T_c$

Having studied the quasiparticle aspects at the critical point, we now turn our attention to their study slightly away from the critical point. The critical region of interest is $|T-T_c|\ll T_c$. Critical behavior in the broken symmetry phase near the critical point with $T\lesssim T_c$ will be studied elsewhere with particular attention to the critical dynamics of Goldstone bosons. In this article we restrict our attention to the normal phase near the critical point.

In the Lagrangian density (2.4) the term $M^2(T)$ is the *exact* mass (rather than the exact inverse susceptibility) determined by the condition (2.5) and the counterterm $\delta m^2(T)$ is adjusted consistently in perturbation theory to fulfill this condition. However, to relate the mass to the departure away from the critical temperature it is more convenient to rearrange the perturbative expansion in a manner that explicitly displays the departure from T_c. This is achieved as follows. Consider the one-loop contribution to the self-energy in the *massless* theory in d spatial dimensions:

$$\Sigma^{(1)}=-\frac{\lambda T}{2}\sum_m\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+\omega_m^2}=-\lambda T^{d-1}\mathcal{A}(d)+\text{z.t.t.} \tag{5.38}$$

where $\mathcal{A}(d)$ depends only on the spatial dimensionality and z.t.t. stands for zero temperature terms. It is convenient to group this contribution with the bare mass term in the Lagrangian in the form

$$m^2(T)=-m_0^2-\Sigma^{(1)}=\lambda T^{d-1}\mathcal{A}(d)-m_R^2(0)$$
$$\approx a(T-T_c)\quad\text{for}\quad T\sim T_c=\left[\frac{m_R^2}{\lambda\mathcal{A}(d)}\right]^{1/(d-1)} \tag{5.39}$$

where the zero temperature contributions [denoted by z.t.t. in Eq. (5.38)] have been absorbed in $m_R^2(0)$. We reorganize the perturbative expansion by rewriting the Euclidean Lagrangian in the form

$$\mathcal{L}_E=\frac{1}{2}\frac{(\partial_\tau\Phi)^2}{v_0^2}+\frac{1}{2}(\nabla\Phi)^2+m^2(T)\Phi^2+\frac{\lambda_R}{4!}\Phi^4$$
$$+\frac{1}{2}\delta m^2(T)\Phi^2+\frac{\delta\lambda}{4!}\Phi^4 \tag{5.40}$$

where now the counterterm $\delta m^2(T)$ is simply the one-loop tadpole diagram evaluated for zero mass given by Eq. (5.38), it is of $\mathcal{O}(\lambda)$ and is included in the perturbative expansion consistently.

To one-loop order the two-point function is now given by

$$\Gamma^{(2)}(p,s)=p^2+\frac{s^2}{v_0^2}+m^2(T)-\Sigma_S,$$

$$\Sigma_S=[\Sigma-\delta m^2(T)]$$
$$=\frac{\lambda T}{2}\sum_m\int\frac{d^dq}{(2\pi)^d}\frac{m^2(T)}{(q^2+\omega_m^2)[q^2+\omega_m^2+m^2(T)]}. \tag{5.41}$$

The integral above is of the typical form are those studied in the previous sections [see Eq. (3.3)]. Separating the $m=0$ Matsubara contribution from the $m\neq 0$ for which we can set $m^2(T)\propto(T-T_c)\approx 0$ for $T-T_c\ll T_c$, we obtain

$$\sum_m \int \frac{d^d q}{(2\pi)^d} \frac{m^2(T)}{(q^2+\omega_m^2)[q^2+\omega_m^2+m^2(T)]} = m^2(T)\left[\frac{\Gamma(\epsilon/2)\Gamma(1-\epsilon/2)}{(4\pi)^{d/2}\Gamma(d/2)} m^{-\epsilon}(T) + \mathcal{C}(d) T^{-\epsilon}\right]. \tag{5.42}$$

Again, for $\epsilon > 0$ we can neglect the second term in the square brackets in the limit $T \gg m(T)$. Expanding in ϵ to obtain the lowest order contribution consistently in the ϵ expansion, we obtain the two-point function to one-loop order:

$$\Gamma^{(2)}(p,s) \approx p^2 + \frac{s^2}{v_0^2} + m^2(T)\left[1 - \frac{g(\mu)}{\epsilon} + \frac{g(\mu)}{2}\ln\left(\frac{m^2(T)}{\mu^2}\right)\right]. \tag{5.43}$$

The renormalized mass parameter $m_R(T)$ is defined by

$$Z_\phi m^2(T) = Z_m m_R^2(T) \tag{5.44}$$

and Z_m is fixed by the renormalization condition

$$\Gamma^{(2)}(p=0,\ s=0,\ m_R^2 = \mu^2) = \mu^2. \tag{5.45}$$

Since Z_ϕ receives corrections at $\mathcal{O}(g^2)$ we choose Z_m to lowest order in the ϵ expansion to be

$$Z_m = 1 + \frac{g(\mu)}{\epsilon} + \mathcal{O}(g^2, g\epsilon). \tag{5.46}$$

1. Static aspects

Before we embark on a full discussion of the dynamics away from the critical point, it proves convenient and illuminating to discuss the static aspects first. In particular, since we will study the $p=0$ case but $T \neq T_c$ a relevant quantity is the inverse susceptibility $\chi^{-1}(T)$, which is defined as

$$\chi^{-1}(T) = M^2(T) = \Gamma_R^2(p=0;\ s=0), \tag{5.47}$$

which near the critical point and the nontrivial fixed point g^* given by Eq. (5.14) is given by

$$M^2(T \sim T_c) \approx m_R^2(T)\left[1 + \frac{g^*}{2}\ln\left(\frac{m_R^2(T)}{\mu^2}\right)\right] \approx [m_R^2(T)]^{1+g^*/2}, \tag{5.48}$$

where we anticipated an exponentiation of the leading logarithms via the renormalization group, which will be borne out by the renormalization group analysis below. Recalling that $m^2(T) \propto |T - T_c|$ by Eq. (5.39), we find

$$\chi^{-1}(T \sim T_c) \propto |T - T_c|^\gamma, \quad \gamma = 1 + \frac{\epsilon}{6} + \cdots. \tag{5.49}$$

The critical exponent γ is seen to be the correct one [4,21–23].

Just as in the case of the theory at the critical point studied above, we now study the dynamics of the theory in an ϵ expansion and implement a resummation of the leading infrared divergences via the renormalization group.

2. Dynamics away from the critical point

As argued above the leading infrared behavior is obtained by setting the internal Matsubara frequencies to zero in the two-loop self-energy. In $d = 4 - \epsilon$ spatial dimensions, the self-energy at two loops is

$$\Sigma^{(2)}(p;s;m(T)) = \frac{\lambda^2 T^2}{6} \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \times \frac{1}{[q^2+m^2(T)][k^2+m^2(T)][(q+k+p)^2 + s^2/v_0^2 + m^2(T)]}. \tag{5.50}$$

The loop integrals are evaluated by introducing two Feynman parameters leading to

$$\Sigma^{(2)}(p;s;m(T)) = g^2(\mu)\Gamma(-1+\epsilon)\mu^{2\epsilon}\int_0^1 dx \int_0^1 dy [x(1-x)]^{\epsilon/2-1} y^{\epsilon/2-1} \times \left[x(1-x)y(1-y)p^2 + (1-y)x(1-x)m^2(T) + y m^2(T) + yx\frac{s^2}{v_0^2}\right]^{1-\epsilon}. \tag{5.51}$$

It is convenient to separate the static contribution from the dynamical part by writing

$$\Sigma^{(2)}(p;s;m(T))=\Sigma^{(2)}(p;0;m(T))+\tilde{\Sigma}^{(2)}(p;s;m(T)),$$
$$\tilde{\Sigma}^{(2)}(p;s;m(T))\equiv\Sigma^{(2)}(p;s;m(T))-\Sigma^{(2)}(p;0;m(T)). \tag{5.52}$$

The static contribution $\Sigma^{(2)}(p;0;m(T))$ leads to wave function renormalization, a renormalization of the mass, and $\mathcal{O}(\epsilon^2)$ corrections to the static anomalous dimensions, which will be neglected to leading order in the ϵ expansion. The second, dynamical contribution is obtained consistently in an ϵ expansion: the regions of the integrals in the Feynman parameters that lead to inverse powers of ϵ in an ϵ expansion are $x\sim 0,1$ and $y\sim 0$. The contributions of these regions can be isolated by partial integration, and after some straightforward algebra we find

$$\tilde{\Sigma}^{(2)}(p;s;m(T))=\frac{g^2(\mu)}{6\epsilon}\left[-\frac{2}{\epsilon}\frac{s^2}{v_0^2}+2\frac{s^2}{v_0^2}\ln\left(\frac{m^2(T)}{\mu^2}\right)+2\left(m^2(T)+\frac{s^2}{v_0^2}\right)\ln\left(1+\frac{s^2}{v_0^2m^2(T)}\right)\right]+\mathcal{O}(\epsilon^0,\epsilon). \tag{5.53}$$

Thus, putting together the one-loop contribution found previously and the two-loop contribution found above, the two-point function at zero spatial momentum but away from the critical point is found to be

$$\Gamma^{(2)}(p;s;m(T))=p^2\left[1+\frac{g^2(\mu)}{12\epsilon}+\mathcal{O}(g^2\epsilon^0)\right]+\frac{s^2}{v_0^2}+m^2(T)\left[1-\frac{g(\mu)}{\epsilon}+\frac{g(\mu)}{2}\ln\left(\frac{m^2(T)}{\mu^2}\right)\right]+\frac{g^2(\mu)}{6\epsilon}\left[\frac{2}{\epsilon}\frac{s^2}{v_0^2}-2\frac{s^2}{v_0^2}\ln\left(\frac{m^2(T)}{\mu^2}\right)-2\left(m^2(T)+\frac{s^2}{v_0^2}\right)\ln\left(1+\frac{s^2}{v_0^2m^2(T)}\right)\right]+\mathcal{O}(g^2,g^2\epsilon), \tag{5.54}$$

where we have neglected logarithmic corrections that will exponentiate to anomalous dimensions of $\mathcal{O}(\epsilon^2)$ for momentum-dependent terms. We have displayed only the contribution that will be canceled by wave function renormalization just as in the critical case.

Obviously the $m^2(T)=0$ limit coincides with the two-point function at the critical point (4.8) to leading order $\mathcal{O}(\epsilon)$. In the above expression we have not included the two-loop contribution to the static $s=0$ part, since it will lead to an $\mathcal{O}(\epsilon^2)$ correction to the critical exponent for the correlation length (inverse susceptibility).

The renormalization conditions for the two-point function away from the critical point are now summarized as follows:

$$\Gamma_R^{(2)}(p,s,v_R;m_R(T))=Z_\phi\Gamma^{(2)}(p,s,v_0;m(T)),$$
$$v_R^2=v_0^2\frac{Z_v}{Z_\phi},\quad Z_\phi m^2(T)=m_R^2(T)Z_m,$$
$$\left.\frac{\partial\Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=\mu^2,\ s^2/v_R^2=\mu^2}=1,$$
$$\left.\frac{\partial\Gamma_R^{(2)}}{\partial s^2}\right|_{p^2=\mu^2,\ s^2/v_R^2=\mu^2}=\frac{1}{v_R^2}, \tag{5.55}$$

$$\Gamma_R^{(2)}(p=0,\ s=0;\ m_R^2(T)=\mu^2)=\mu^2, \tag{5.56}$$

along with the renormalization conditions on the four point function (4.11). To leading order in the ϵ expansion the renormalization constants Z_ϕ, Z_v, and Z_m are given by Eqs. (4.14), (4.15), and (5.46), respectively.

Thus, we find the renormalized two-point function at two-loop order and to leading order in the ϵ expansion (since $g^*\sim\epsilon$):

$$\Gamma_R^{(2)}(p;s;m_R(T))=p^2+m^2(T)\left[1+\frac{g(\mu)}{2}\ln\left(\frac{m_R^2(T)}{\mu^2}\right)\right]+\frac{s^2}{v_R^2}\left[1-\frac{g^2(\mu)}{3\epsilon}\ln\left(\frac{m_R^2(T)}{\mu^2}\right)\right]-\frac{g^2(\mu)}{3\epsilon}\left(m_R^2(T)+\frac{s^2}{v_0^2}\right)\times\ln\left(1+\frac{s^2}{v_0^2m_R^2(T)}\right)+\mathcal{O}(g^2,g^2\epsilon) \tag{5.57}$$

Since $m_R^2(T)$ has dimension it is convenient to introduce the dimensionless quantity

$$\overline{m}^2=\frac{m_R^2(T)}{\mu^2} \tag{5.58}$$

and the corresponding renormalization group beta function

$$\beta_{\overline{m}}=\mu\left.\frac{\partial\overline{m}^2}{\partial\mu}\right|_{m_0,T,\lambda_0}=(\gamma_{\overline{m}}-2)\,\overline{m}^2, \tag{5.59}$$

$$\gamma_{\overline{m}}=g+\mathcal{O}(g^2,g\epsilon), \tag{5.60}$$

where we have used Eqs. (5.44) and (5.46).

The renormalization group equation for the N-point function away from the critical point is now given by

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_g\frac{\partial}{\partial g}+\beta_v\frac{\partial}{\partial v}+\beta_{\bar{m}}\frac{\partial}{\partial\bar{m}^2}-\frac{N}{2}\gamma\right]\times\Gamma^{(N)}\left(p_1,\frac{s_1}{v};p_2,\frac{s_2}{v};\ldots,p_N;\frac{s_N}{v};g,\bar{m},\mu\right)=0. \tag{5.61}$$

The new ingredient as compared to the critical case (5.1) is the dependence on $\bar{m}$. Following the same steps as for the critical case, we now find that the solution of the renormalization group equation for the two-point function obeys

$$\Gamma^{(2)}\left(e^t p,\frac{e^t s}{v},g,\bar{m},\mu\right)=e^{\int_0^t dt'[2-\gamma(t')]}\Gamma^{(2)}\left(p,\frac{s}{v(t)},g(t),\bar{m}^2(t),\mu\right) \tag{5.62}$$

with $\bar{m}(t)$ the solution of the differential equation

$$\frac{\partial\bar{m}^2(t)}{\partial t}=\beta_{\bar{m}}(g(t),v(t),\bar{m}(t)) \tag{5.63}$$

with the initial condition

$$\bar{m}^2(0)=\frac{m_R^2(T,\mu)}{\mu^2}\propto|T-T_c(\mu)|. \tag{5.64}$$

In the infrared the coupling is driven to the nontrivial fixed point $g^*=\epsilon/3$ and

$$\bar{m}^2(t)\to\bar{m}^2(0)e^{(\gamma^*_m-2)t},\quad \gamma^*_m=\frac{\epsilon}{3}. \tag{5.65}$$

Just as in the solution of the renormalization group equation at criticality near the fixed point (5.18),(5.19), introducing $pe^t\equiv P$, $se^t\equiv S$ we now find

$$\Gamma^{(2)}\left(P,\frac{S}{v},g,\mu\right)=e^{t(2-\eta)}\Gamma^{(2)}\times\left(Pe^{-t},\frac{S}{v(0)}e^{-t}e^{(1-z)t},g^*,\bar{m}^2(0)e^{(\gamma^*_m-2)t},\mu\right). \tag{5.66}$$

Following the analysis of the critical case, and the scaling property (5.8) and writing $P=\mu e^t$ we find the following scaling form:

$$\Gamma^{(2)}\left(P,\frac{S}{v},g,\mu\right)=\mu^2\left[\frac{P}{\mu}\right]^{2-\eta}\Phi\left(\frac{S}{v(\mu)\mu^{1-z}P^z};P\xi\right),$$

$$\xi=\frac{1}{\mu}\left[\frac{m_R^2(T,\mu)}{\mu^2}\right]^{1/(\gamma^*_m-2)}. \tag{5.67}$$

ξ is therefore identified with the *correlation length* [4,21–23]

$$\xi\sim|T-T_c|^{-\nu},\quad \nu=\frac{1}{2-\gamma^*_m}\sim\frac{1}{2}+\frac{\epsilon}{12}+\cdots. \tag{5.68}$$

It is important to note at this stage that the correlation length ξ is a renormalization group invariant, as can be easily checked by using Eq. (5.58) with the renormalization group beta function (5.59).

To study the limit of zero spatial momentum it is more convenient to rewrite the above scaling solution in the following form:

$$\Gamma^{(2)}\left(P,\frac{S}{v},g,\mu\right)=\mu^2\left[\frac{\xi}{\mu}\right]^{-(2-\eta)}\Psi\left(\frac{S\xi^z}{v(\mu)\mu^{1-z}};P\xi\right). \tag{5.69}$$

From the definition of the inverse susceptibility $M^2(T)=\chi^{-1}(T)=\Gamma^{(2)}$ $(p=0,\ s=0)$ we find the known result [4,21–23,25]

$$\chi^{-1}(T)\propto|T-T_c|^{-\gamma},\quad \gamma=\frac{2-\eta}{2-\gamma^*_m}=\nu(2-\eta). \tag{5.70}$$

Furthermore, the two-point function is a function of two renormalization group invariant, dimensionless scaling variables

$$\Gamma^{(2)}(p,s,m_R^2(T,\mu))=\mu^2\left[\frac{m_R^2(T,\mu)}{\mu^2}\right]^{(2-\eta)/(2-\gamma^*_m)}\Psi(\varphi,\delta) \tag{5.71}$$

with

$$\varphi=\left[\frac{s}{v(\mu)\mu}\right]^2\left[\frac{m_R^2(T)}{\mu^2}\right]^{2z/(\gamma^*_m-2)}\equiv\left[\frac{s\ \xi^z}{v(\mu)\mu^{1-z}}\right]^2, \tag{5.72}$$

$$\delta=\frac{p^2}{\mu^2}\left[\frac{m_R^2(T,\mu)}{\mu^2}\right]^{2/(\gamma^*_m-2)}\equiv(p\ \xi)^2. \tag{5.73}$$

The renormalization condition (5.56) determines that $\Psi(0,0)=1$.

We can now follow the arguments provided in the previous subsection for the critical case. Under the analytic continuation $s^2\to-\omega^2-i\ \mathrm{sgn}(\omega)0^+$:

$$\varphi \to -\Omega^2 - i\,\mathrm{sgn}(\Omega)\,0^+, \quad \Omega = \frac{\omega\,\xi^z}{v(\mu)\mu^{1-z}}, \tag{5.74}$$

$$\Psi(\varphi = -\Omega^2 - i\,\mathrm{sgn}(\Omega)0^+, \delta) = \Psi_R(\Omega,\delta) + i\Psi_I(\Omega,\delta). \tag{5.75}$$

The position of the quasiparticle pole in the two-point Green's function corresponds to the value of $\Omega = \Omega^*(\delta)$ for which $\Psi_R(\Omega^*(\delta),\delta) = 0$. This condition determines the dispersion relation of the quasiparticle and is given by

$$\omega_p = \Omega^*(\delta) v(\mu)\mu^{1-z}\xi^{-z}. \tag{5.76}$$

This expression emphasizes that the dispersion relation depends on p through the scaling variable $\delta = (p\,\xi)^2$.

Assuming that near the quasiparticle pole Ψ_R vanishes *linearly*, the Green's function can be approximated by

$$G(p,\omega,m_R(T)) \sim \frac{(\mu\xi)^{2-\eta}}{\mu^2} \frac{1}{(\Omega-\Omega^*)\Psi'_R(\Omega^*,\delta) + i\Psi_I(\Omega^*,\delta)} \tag{5.77}$$

where $\Psi'_R(\Omega^*,\delta) = \partial\Psi_R(\Omega,\delta)/\partial\Omega|_{\Omega=\Omega^*}$. Near the quasiparticle pole we can further write the above expression in the Breit-Wigner form

$$G_{BW}(p,\omega,m_R(T)) \sim \frac{\mathcal{Z}_p}{\omega - \omega_p + i\,\Gamma_p} \tag{5.78}$$

with

$$\omega_p = \Omega^*(\delta)\,\frac{v(\mu)\mu^{1-z}}{\xi^z}, \tag{5.79}$$

$$\mathcal{Z}_p = \Psi'_R(\Omega^*)\frac{v(\mu)}{\mu}(\mu\xi)^{2-z-\eta}, \tag{5.80}$$

$$\Gamma_p = \frac{\Psi_I(\Omega^*)}{\Psi'_R(\Omega^*)}\,\frac{v(\mu)\mu^{1-z}}{\xi^z}$$
$$\equiv \frac{\omega_p\Psi_I(\Omega^*)}{\Omega^*(\delta)\Psi'_R(\Omega^*)} \propto |T-T_c|^{z\nu}, \tag{5.81}$$

where we have suppressed the dependence on the scaling variable δ in the arguments of the real and imaginary parts to avoid cluttering of notation. Furthermore, we have made explicit the combination of *static and dynamic* critical exponents using the expression given in Eq. (5.68) for the static critical exponent ν, and the dependence on the momentum is implicit through the dependence on the scaling variable δ of Ω^* as well as the explicit dependence of the real and imaginary parts.

Again, the imaginary part must be proportional to the anomalous dimensions, and hence perturbatively small in the ϵ expansion. Therefore the expression for the width (5.81) reveals both critical slowing down, since $\Gamma_p \sim |T-T_c|^{z\nu}$ vanishing at $T = T_c$, and the validity of the quasiparticle picture, since $\Gamma_p/\omega_p \ll 1$ in the ϵ expansion.

At this point we recognize a fundamental difference from the Wilsonian results of Ref. [10]. While in Ref. [10] the width was found to be proportional to $|T-T_c|^\nu$ up to logarithms, we see from Eq. (5.81) that the quasiparticle width actually involves the new dynamical anomalous exponent z. The difference can be traced to the fact that the Wilsonian approach advocated in Ref. [10] does not include two-loop diagrams, which are necessary to reveal the anisotropic renormalization through the renormalization of the speed of light and are directly responsible for the new dynamical anomalous exponent z.

We emphasize that the above Breit-Wigner form as well as the quasiparticle properties rely on the assumption that the real part of the scaling function vanishes linearly near the quasiparticle pole. As emphasized before in the critical case this need not be the general situation, and anomalous power laws can lead to threshold singularities as discussed above.

While the solution of the renormalization group leads to a scaling form of the two-point correlation function, it does *not* explicitly specify the scaling function Ψ. However, we can obtain the function Ψ by matching the leading logarithms to those of the perturbative expression (5.57) evaluated at the fixed point $g^* = \epsilon/3$ to lowest order in the ϵ expansion. Matching the leading logarithms and assuming their exponentiation via the renormalization group it is straightforward to see that the two-point function is given by

$$\Gamma^{(2)}(p,s,m_R^2(T,\mu)) \sim \mu^2\left[\frac{m_R^2(T,\mu)}{\mu^2}\right]^{(2-\eta)/(2-\gamma_m^*)} \{\delta + [1+\varphi]^{2-z}\} \tag{5.82}$$

where we have used the lowest order results in the ϵ expansion:

$$\gamma_m^* = \frac{\epsilon}{3}, \quad z = 1 + \frac{\epsilon}{27}, \quad \eta = \mathcal{O}(\epsilon^2), \tag{5.83}$$

and kept consistently the lowest $\mathcal{O}(\epsilon)$ in the exponentiation of the leading logarithms leading to Eq. (5.82). Thus, we obtain the lowest order result for the scaling function:

$$\Psi(\varphi,\delta) = \delta + [1+\varphi]^{2-z}. \tag{5.84}$$

We can now obtain an explicit form of the real and imaginary parts of the scaling function that enter into the quasiparticle parameters. This is achieved by performing the analytic continuation (5.74), which leads to

$$\Psi_R(\Omega) + i\Psi_I(\Omega) = \delta + [1 - \Omega^2 - i\,\mathrm{sgn}(\Omega)\,0^+]^{2-z}$$
$$= \delta - |\Omega^2 - 1|^{2-z}[1 + i\pi(z-1)$$
$$\times \mathrm{sgn}(\Omega)\Theta(\Omega^2-1)]. \tag{5.85}$$

For $p=0$ i.e., $\delta = 0$, we see that both the real and imaginary parts of the scaling function vanish at $\Omega^* = 1$ *with an* *anomalous power law*, providing an explicit example of the case of threshold singularities mentioned above.

For $p\neq 0$ and $T\neq T_c$ we find a quasiparticle pole at

$$\Omega^{*2}=1+\delta^{1/(2-z)}\sim 1+(p\,\xi)^{2z}, \tag{5.86}$$

where we have approximated the anomalous dimension by its leading order in ϵ using $z=1+\epsilon/27$. From this expression for Ω^* we obtain the dispersion relation for quasiparticles

$$\omega_p^2=v^2(\mu)\mu^2\left\{\left[\frac{m_R^2(T,\mu)}{\mu^2}\right]^{2z\nu}+\left[\frac{p^2}{\mu^2}\right]^z\right\} \tag{5.87}$$

with ν given by Eq. (5.68). In particular, we find that the frequency of zero momentum quasiparticles $\omega_{p=0}\propto|T-T_c|^{z\nu}$. Obviously at $T-T_c$ $[m_R(T)=0]$ the dispersion relation coincides with that of the critical case given by Eq. (5.25). For $p\neq 0$, i.e., $\delta\neq 0$, the real part of the scaling function vanishes linearly, the Breit-Wigner approximation (5.78) near the quasiparticle pole is valid, and the relations (5.79)–(5.81) describe the properties of the quasiparticles. To lowest order in the ϵ expansion we find, using Eq. (5.23), that

$$\frac{\Psi_I(\Omega^*)}{\Omega^*(\delta)\Psi_R'(\Omega^*)}=\frac{\pi\eta_t}{4}\,\frac{(p\xi)^{2z}}{1+(p\xi)^{2z}}. \tag{5.88}$$

For $p=0$, i.e., $\delta=0$, this ratio vanishes and $\Psi_R'(\Omega^*)\propto|\Omega^*-1|^{1-z}_{\Omega^*=1}$ diverges, displaying the phenomenon of threshold singularity with a divergent residue $\mathcal{Z}_p$.

For $p\neq 0$ but $T\to T_c$ ($\delta\to\infty$) this ratio equals that of the critical case [see Eq. (5.36)].

For $T\neq T_c$, $p\neq 0$ we finally find the width of the long-wavelength quasiparticles to be given to lowest order in the ϵ expansion by

$$\Gamma_p\sim\frac{\pi\eta_t}{4}\,\frac{(p\xi)^{2z}}{1+(p\xi)^{2z}}\,\frac{v(\mu)\mu^{1-z}}{\xi^z}[1+(p\xi)^{2z}]^{1/2} \tag{5.89}$$

with the following behavior to lowest order in the ϵ expansion:

$$\Gamma_p\sim\frac{\pi\eta_t}{4}v(\mu)\mu^{-z}\times\begin{cases}p^z & \text{for } p \text{ fixed, } T\to T_c,\\ p^{2z}\xi^z & \text{for } \xi \text{ fixed, } p\to 0.\end{cases} \tag{5.90}$$

Thus, critical slowing down emerges in both limits; furthermore, the validity of the quasiparticle picture is warranted in the ϵ expansion, since $\eta_t\simeq 2(z-1)=2\epsilon/27+\mathcal{O}(\epsilon^2)\ll 1$.

VI. SUMMARY OF RESULTS

Critical phenomena, both static and dynamic, in quantum field theory at finite temperature result in dimensional reduction since momenta and frequencies are $p,\omega\ll T$ and the correlation length is $\xi\gg 1/T$. The infrared physics is dominated by the contribution of the zero Matsubara frequency in internal loops, which in turn results in an effective coupling λT in the perturbative expansion. Naive perturbation theory at high temperature breaks down in four space-time dimensions because of the strong infrared behavior of loop diagrams near the critical point for long-wavelength phenomena.

We propose an implementation of the renormalization group to study *dynamical* critical phenomena which hinges upon two main ingredients.

The leading infrared behavior near the critical point is determined by keeping only the zero Matsubara internal frequency in the loops. To control the infrared consistently we implement an expansion in ϵ in $5-\epsilon$ space-time dimensions. Dimensional reduction for long-wavelength phenomena near the critical point results in the perturbative expansion being in terms of $g(\mu)\propto\lambda T\mu^{-\epsilon}$ where μ is the scale of external momenta and frequencies in the diagram. The renormalized effective coupling is driven to a fixed point in the infrared which is of $\mathcal{O}(\epsilon)$. Therefore long-wavelength phenomena can be studied in perturbation theory around this fixed point for $\epsilon\ll 1$. The perturbative expansion is improved by implementing a renormalization group resummation which reveals dynamical scaling phenomena with anomalous dimensions. Eventually the limit of physical interest $\epsilon\to 1$ must be studied by further Borel and/or Padé resummations.

The second important ingredient is the *anisotropic* scaling between space and time. While space is infinite, at finite temperature in the Euclidean formulation the time direction is compactified to the interval $[0,1/T]$. We introduce a new parameter, the effective speed of light in the medium, which is renormalized and runs with the renormalization transformations. The infrared renormalization of the speed of light results in a new *dynamical* anomalous exponent which determines the dispersion relation and all the quasiparticle properties. The ϵ expansion combined with the renormalization group leads to a consistent quasiparticle description of long-wavelength excitations near the critical point.

The critical exponents, both static and dynamic, are summarized for the critical case $T=T_c$ in Table I as well as for $T\neq T_c$ (Table II) but in the symmetric phase with $T\to T_c^+$.

TABLE I. Quasiparticles at $T=T_c$.

$\omega_p\propto$	p^z
$v_g\propto$	p^{z-1}
$\mathcal{Z}_p\propto$	$p^{z+\eta-2}$
$\Gamma_p\propto$	$\eta_t p^z$
$\eta=$	$\epsilon^2/54+\mathcal{O}(\epsilon^3)$ (static)
$\eta_t=$	$2\epsilon/27+\mathcal{O}(\epsilon^2)$ (dynamic)
$z=$	$1+(1/2)(\eta_t-\eta)\sim 1+\epsilon/27+\mathcal{O}(\epsilon^2)$ (dynamic)

TABLE II. Quasiparticles at $T\gtrsim T_c$.

$\omega_p^2\propto$	$[m_R^2(T,\mu)/\mu^2]^{2z\nu}+[p^2/\mu^2]^z$
$\Gamma_p\propto$	$\eta_t\omega_p(p\,\xi)^{2z}/[1+(p\xi)^{2z}]$
$\xi\propto$	$\lvert T-T_c\rvert^{-\nu}$
$m_R^2(T)\propto$	$\lvert T-T_c\rvert$
$\nu=$	$1/2+\epsilon/12+\mathcal{O}(\epsilon^2)$ (static)
$\eta_t=$	$2\epsilon/27+\mathcal{O}(\epsilon^2)$ (dynamic)
$z\ =$	$\simeq 1+(1/2)(\eta_t-\eta)=1+\epsilon/27+\mathcal{O}(\epsilon^2)$ (dynamic)

TABLE III. Critical exponents for $O(N)$.

$\nu=$	$1/2+(N+2)/4(N+8)\epsilon+\mathcal{O}(\epsilon^2)$	(static)
$\eta=$	$\epsilon^2(N+2)/2(N+8)^2+\mathcal{O}(\epsilon^3)$	(static)
$\eta_t=$	$\epsilon\;\,2(N+2)/(N+8)^2+\mathcal{O}(\epsilon^2)$	(dynamic)
$z=$	$1+\epsilon(N+2)/(N+8)^2+\mathcal{O}(\epsilon^2)$	(dynamic)

The new dynamical exponent z is missed by the Wilsonian approach advocated in Ref. [10] since two-loop diagrams are completely neglected in that approach, and anisotropic rescaling of frequency and momenta becomes manifest at two-loop order and beyond.

Critical exponents for $O(N)$ symmetry

At this stage we can generalize our results to the case of a scalar theory with $O(N)$ symmetry at or slightly above the critical point. While the static critical exponents for the $O(N)$ case are available in the literature [4,21–23,25], the dynamical critical exponent to lowest order in ϵ can be obtained simply by recognizing that the symmetry factors corresponding to the $O(N)$ theory multiply the two-loop expression for the self-energy by an overall factor. From the expression (4.8) we see that the coefficient of s^2/v^2 is a factor $4/\epsilon$ times the coefficient of p^2, which immediately leads to the result

$$\eta_t=\frac{4}{\epsilon}\,\eta. \tag{6.1}$$

Since for the $O(N)$ theory $\eta=\epsilon^2(N+2)/[2(N+8)^2]+\mathcal{O}(\epsilon^3)$, we find to lowest order in ϵ

$$\eta_t=\epsilon\;\frac{2(N+2)}{(N+8)^2}+\mathcal{O}(\epsilon^2).$$

In summary, the static and dynamic critical exponents to lowest order in the ϵ expansion for the $O(N)$ theory are given in Table III.

VII. CONCLUSIONS, DISCUSSION, AND IMPLICATIONS

We have studied the dynamical aspects of long-wavelength (collective) excitations at and near the critical point in scalar quantum field theories at high temperatures. After recognizing that naive perturbation theory breaks down at high temperature in the long-wavelength limit, we introduced an ϵ expansion around $5-\epsilon$ space-time dimensions combined with the renormalization group at high temperature to resum the perturbative series.

The effective long-wavelength theory at high temperature is described by a nontrivial fixed point at which the correlation functions feature scaling behavior. The anisotropy between spatial and time coordinates in Euclidean space-time at finite temperature leads us to consider the renormalization of the speed of light, which, in turn leads to a new dynamical exponent z. All dynamical quantities, such as the dispersion relation and widths of long-wavelength quasiparticle (collective) excitations, depend on this new dynamical exponent, as well as the static exponents.

Our results are summarized in Tables I–III in the previous section.

Two very important aspects emerge from this treatment: (i) critical slowing down, i.e., the relaxation rate of the quasiparticle, vanishes in the long-wavelength limit or at the critical point with definite anomalous dimensions determined by the new dynamical exponent z, and (ii) the quasiparticle picture, i.e., narrow widths $\Gamma_p \ll \omega_p$, is valid. The group velocity of quasiparticles vanishes at the critical point in the long-wavelength limit revealing the collective aspects of these excitations. The dynamical exponent $z=1+\epsilon(N+2)/(N+8)^2+\mathcal{O}(\epsilon^2)$ describes a *new universality class* for *dynamical* critical phenomena in quantum field theory.

As mentioned in the introduction these phenomena have phenomenological implications for the chiral phase transition in the quark-gluon plasma with potential observational consequences if long-wavelength pion fluctuations freeze out at the chiral phase transition. An important aspect revealed by this program is that the effective coupling $\lambda_{eff}T\mu^{-\epsilon}$ is driven to the Wilson-Fischer fixed point in the infrared; this in turn means that in this limit $\lambda_{eff}\rightarrow 0$. This may be important in the linear sigma model description of low energy QCD near the critical point and may give rise to interesting phenomenological consequences.

In this article we focused our attention on the approach to the critical temperature from above; therefore our results regarding the dynamical exponent z are valid in the symmetric phase. An important question that we are currently addressing [37] is the relaxation of pions slightly below T_c. Since the scattering amplitude of pions (at zero temperature) vanishes in the long-wavelength limit we expect novel behavior of critical slowing down for pion fluctuations below the critical temperature. We expect to report on our findings on these and other related issues soon [37].

While we have provided a quantitative implementation of the program of ϵ expansion with resummation via the renormalization group, the physical limit $\epsilon\rightarrow 1$ requires higher order calculations with Borel or Padé resummations in much the same way as for static critical phenomena. We have studied the dynamical aspects to lowest order in the ϵ expansion but clearly a formal proof of the consistency of the ϵ expansion to higher orders, just as in the usual critical phenomena, must be explored. While clearly such programs are beyond the scope and goals of this article, we here provided the first steps of a program whose potential phenomenological implications as well as intrinsic interest in finite temperature quantum field theory warrant further study.

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APPENDIX A: ONE-LOOP DIAGRAM AT HIGH TEMPERATURE

We derive in this appendix the behavior of the one-loop diagram $H(p,s)$ contributing to the four-point function for high temperatures $T \gg p,s$. Setting $v=1$ to avoid cluttering of notation (the velocity of light is not relevant for the discussion in this section), we have, from Eq. (2.7),

$$H(p,s)=\frac{T}{2}\sum_{l\in\mathcal{Z}}\int\frac{d^dq}{(2\pi)^d}\times\frac{1}{[q^2+(2\pi Tl)^2][(\vec{q}+\vec{p})^2+(2\pi T)^2(n+l)^2]}, \tag{A1}$$

where $d=4-\epsilon$ is the number of spatial dimensions. The denominators in Eq. (A1) can be combined using Feynman parameters with the result

$$\begin{aligned}H(p,s)&=\frac{T}{2}\sum_{l\in\mathcal{Z}}\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\frac{1}{[q^2+A_l(x,p,s)]^2}\\&=\frac{T}{2(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\\&\quad\times\sum_{l\in\mathcal{Z}}\int_0^1 dx[A_l(x,p,s)]^{d/2-2}\end{aligned} \tag{A2}$$

where we integrated over the spatial momenta and

$$A_l(x,p,s)=x(1-x)(p^2+s^2)+(\omega_l+xs)^2,$$

$$\omega_l=2\pi Tl,\quad s=2\pi Tn. \tag{A3}$$

We single out now the contribution from the $l=0$ mode and study the behavior of the sum over $l\neq 0$ for large $T\gg p,s$.

Let us first evaluate the $l=0$ term in the sum (A2):

$$\begin{aligned}&\frac{T}{2(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx[A_0(x,p,s)]^{-\epsilon/2}\\&=\frac{T}{2(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 x^{-\epsilon/2}dx[(1-x)p^2+s^2]^{-\epsilon/2}\\&=-\frac{T\mu^{-\epsilon}}{2}\frac{\Gamma(\epsilon/2-1)}{(4\pi)^{2-\epsilon/2}}\left(\frac{s^2+p^2}{\mu^2}\right)^{-\epsilon/2}\\&\quad\times F\left(\frac{\epsilon}{2},1-\frac{\epsilon}{2};2-\frac{\epsilon}{2};\frac{p^2}{p^2+s^2}\right),\end{aligned} \tag{A4}$$

where $F(a,b;c;z)$ stands for the hypergeometric function [33].

We have, for the $l\neq 0$ terms in the high temperature limit,

$$\begin{aligned}[A_l(x,p,s)]^{-\epsilon/2}&\overset{T\gg p,s}{=}(2\pi T|l|)^{-\epsilon}-\mathrm{sgn}(l)\frac{\epsilon x}{|2\pi Tl|^{1+\epsilon}}\\&\quad-\frac{\epsilon x}{2|2\pi Tl|^{2+\epsilon}}\\&\quad\times\{(1-x)p^2+s^2[1-(2+\epsilon)x]\}\\&\quad+\mathcal{O}(|Tl|^{-3-\epsilon}).\end{aligned} \tag{A5}$$

The sum over $l\neq 0$ then yields, in the high temperature limit,

$$\begin{aligned}&\frac{T}{2(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\sum_{l\neq 0}[A_l(x,p,s)]^{-\epsilon/2}\\&\overset{T\gg p,s}{=}\frac{(2\pi T)^{1-\epsilon}\Gamma(\epsilon/2)}{2(4\pi)^{2-\epsilon/2}}\\&\quad\times\left\{\frac{\zeta(\epsilon)}{\pi}-\frac{\epsilon\zeta(2+\epsilon)}{12\pi(2\pi T)^2}\right.\\&\quad\left.\times[p^2-s^2(1+2\epsilon)]+\mathcal{O}(T^{-3-\epsilon})\right\}.\end{aligned} \tag{A6}$$

We see that only the first term in the right-hand side is important for $0<\epsilon<1$ and high temperature. This term is the dominant high temperature limit of the sum of nonzero Matsubara modes. We then have

$$\begin{aligned}H(p,s)&\overset{T\gg p,s}{=}H_{asi}(p,s)-\frac{\Gamma(1+\epsilon/2)\zeta(2+\epsilon)}{384\pi^{4+\epsilon/2}T^{1+\epsilon}}[p^2-s^2(2-\epsilon)]\\&\quad\times\left[1+\mathcal{O}\left(\frac{p^2,s^2}{T^2}\right)\right],\end{aligned}$$

where

$$\begin{aligned}H_{asi}(p,s)\equiv&-\frac{T\mu^{-\epsilon}}{2}\frac{\Gamma(\epsilon/2-1)}{(4\pi)^{2-\epsilon/2}}\left(\frac{s^2+p^2}{\mu^2}\right)^{-\epsilon/2}\\&\times F\left(\frac{\epsilon}{2},1-\frac{\epsilon}{2};2-\frac{\epsilon}{2};\frac{p^2}{p^2+s^2}\right)\\&+\frac{\mu^{2\epsilon-2}\Gamma(1+\epsilon/2)\zeta(\epsilon)}{8\pi^{2+\epsilon/2}\epsilon}T^{1-\epsilon}.\end{aligned} \tag{A7}$$

APPENDIX B: TWO-LOOP DIAGRAM AT HIGH TEMPERATURE

The renormalized two-point function is given by (all quantities are renormalized below)

$$\Gamma^{(2)}(p,s)=Z_\phi p^2+\frac{Z_v}{v^2}s^2-\Sigma^{(2)}(p,s)+\mathcal{O}(\lambda^3) \tag{B1}$$

where $\Sigma^{(2)}(p,s)$ is the two-loop self-energy. Using the renormalization conditions (4.13) we find for the wave function and velocity of light renormalization

$$Z_\phi = 1 + \left.\frac{\partial \Sigma^{(2)}(p,s)}{\partial p^2}\right|_{p=\mu,\ s=\mu v} + \mathcal{O}(\lambda^3),$$

$$Z_v = 1 + v^2 \left.\frac{\partial \Sigma^{(2)}(p,s)}{\partial s^2}\right|_{p=\mu,\ s=\mu v} + \mathcal{O}(\lambda^3). \tag{B2}$$

The two-loop contribution to the self-energy $\Sigma^{(2)}(p,s)$ is given by

$$\Sigma^{(2)}(p,s) = \frac{\lambda^2 T^2}{6} \sum_{l,j\in\mathcal{Z}} \int \frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon}} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{[q^2+\omega_l^2/v^2][k^2+\omega_j^2/v^2][(\vec{p}+\vec{k}+\vec{q})^2+(\omega_l+\omega_j+s)^2/v^2]}, \tag{B3}$$

where $\omega_j = 2\pi jT,\ \ s = 2\pi Tn$.

We combine the propagators in Eq. (B3) using Feynman parameters and integrate over the momenta with the result

$$\Sigma^{(2)}(p,s) = \frac{\lambda^2 T^2 \Gamma(\epsilon-1)}{6(4\pi)^{4-\epsilon}} \sum_{l,j\in\mathcal{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\epsilon/2-1}}{[1-x+x\xi(1-\xi)]^{2-\epsilon/2}} \times \left\{ \frac{x(1-x)\xi(1-\xi)}{1-x+x\xi(1-\xi)} p^2 + \left(\frac{2\pi T}{v}\right)^2 [j^2 x\xi + l^2 x(1-\xi) + (j+l+n)^2(1-x)] \right\}^{1-\epsilon}. \tag{B4}$$

Using the definition (4.6) for the dimensionless coupling, to order g^2 we find from Eqs. (B2) and (B4)

$$Z_\phi = 1 - \frac{g^2}{6}\left(\frac{v\mu}{2\pi T}\right)^{2\epsilon} \Gamma(\epsilon) \sum_{l,j\in\mathcal{Z}} \int_0^1 dx \int_0^1 d\xi \frac{[x^{\epsilon/2}(1-x)\xi(1-\xi)]}{[1-x+x\xi(1-\xi)]^{3-\epsilon/2}} \times \left[\frac{x(1-x)\xi(1-\xi)}{1-x+x\xi(1-\xi)} \left(\frac{v\mu}{2\pi T}\right)^2 + j^2 x\xi + l^2 x(1-\xi) + \left(j+l+\frac{\mu}{2\pi T}\right)^2 (1-x) \right]^{-\epsilon},$$

$$Z_v = 1 - \frac{g^2}{3}\left(\frac{2\pi T}{v\mu}\right)^{1-2\epsilon} \Gamma(\epsilon) \sum_{l,j\in\mathcal{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\epsilon/2-1}(1-x)\,(j+l+v\mu/2\pi T)}{[1-x+x\xi(1-\xi)]^{2-\epsilon/2}} \times \left[\frac{x(1-x)\xi(1-\xi)}{1-x+x\xi(1-\xi)} \left(\frac{v\mu}{2\pi T}\right)^2 + j^2 x\xi + l^2 x(1-\xi) + \left(j+l+\frac{v\mu}{2\pi T}\right)^2 (1-x) \right]^{-\epsilon}. \tag{B5}$$

We find from the definition of the anomalous dimension (5.4) and (B5) for $\gamma(g,T/\mu,v)$,

$$\gamma\left(g,\frac{T}{\mu},v\right) = g^2 \frac{\Gamma(\epsilon+1)}{3} \sum_{l,j\in\mathcal{Z}} \int_0^1 dx \int_0^1 d\xi \frac{x^{\epsilon/2}(1-x)^2\xi(1-\xi)}{[1-x+x\xi(1-\xi)]^{4-\epsilon/2}} \times \left(1-x+2x\xi(1-\xi)\left[1+\frac{\pi T}{v\mu}(j+l)\right]\right) \times \left\{ \frac{x(1-x)\xi(1-\xi)}{1-x+x\xi(1-\xi)} + \left(\frac{2\pi T}{v\mu}\right)^2 \left[j^2 x\xi + l^2 x(1-\xi) + \left(j+l+\frac{v\mu}{2\pi T}\right)^2 (1-x) \right] \right\}^{-1-\epsilon} + \mathcal{O}(g^3). \tag{B6}$$

We split the expression for $\gamma(g,T/\mu,v)$ as follows:

$$\gamma\left(g,\frac{T}{\mu},v\right) = \gamma_0(g,v) + \gamma_{nz}\left(g,\frac{T}{\mu},v\right),$$

where $\gamma_0(g,v)$ is the contribution from the zero Matsubara mode in Eq. (B6)

$$\gamma_0(g,v) = g^2 \frac{\Gamma(\epsilon+1)}{3} \int_0^1 dx \int_0^1 d\xi \times \frac{x^{\epsilon/2}(1-x)^{1-\epsilon}\xi(1-\xi)}{[1-x+x\xi(1-\xi)]^{3-3\epsilon/2}[1-x+2x\xi(1-\xi)]^{\epsilon}} + \mathcal{O}(g^3),$$

and $\gamma_{nz}(g,T/\mu,v)$ stands for the contribution of the nonzero Matsubara modes.

For $T\gg\mu$, we see from Eq. (B6) that $\gamma_{nz}(g,T/\mu,v)$ decreases as $(T/\mu)^{-2-\epsilon}$. [Notice that the coefficient of $(T/\mu)^{-1-\epsilon}$ vanishes by symmetry when summing over $j+l$.]

Therefore, $\gamma_0(g,v)$ dominates for $T\gg\mu$. $\gamma_0(g,v)$ can easily be computed for small $\epsilon>0$ with the result

$$\gamma_0(g,v)=\frac{g^2}{3}\int_0^1 dx\int_0^1 d\xi\,\frac{(1-x)\xi(1-\xi)}{[1-x+x\xi(1-\xi)]^3}+\mathcal{O}(\epsilon g^2,g^3)$$

$$=\frac{g^2}{6}+\mathcal{O}(\epsilon g^2,g^3). \tag{B7}$$

Therefore,

$$\gamma\left(g,\frac{T}{\mu},v\right)=\frac{g^2}{6}+\mathcal{O}\left(\epsilon g^2,g^3,\frac{\mu^2}{T^2}\right). \tag{B8}$$

To the lowest nontrivial order in g, that is, g^2 (two loops), we find for the function $\beta_v(g,T/\mu,v)$

$$\beta_v\left(g,\frac{T}{\mu},v\right)=-\frac{v}{2}\gamma\left(g,\frac{T}{\mu},v\right)-\frac{v}{2}\left(\mu\frac{\partial}{\partial\mu}-2\epsilon\right)\log Z_v\,, \tag{B9}$$

where the derivatives are now at constant (bare) g. Using Eq. (B5) for $\log Z_v$ yields

$$W\equiv\left(\mu\frac{\partial}{\partial\mu}-2\epsilon\right)\log Z_v$$

$$=g^2\frac{\pi\Gamma(\epsilon)}{3}\sum_{l,j\in\mathcal{Z}}\int_0^1 dx\int_0^1 d\xi\,\frac{x^{\epsilon/2-1}(1-x)}{[1-x+x\xi(1-\xi)]^{2-\epsilon/2}}\times Q_{j,l}(x,\xi)^{-\epsilon}\left\{\left[\frac{x\xi(1-\xi)}{1-x+x\xi(1-\xi)}+1\right]\frac{(1-x)\epsilon}{\pi Q_{j,l}(x,\xi)}+\left[1+\frac{2\epsilon(1-x)}{Q_{j,l}(x,\xi)}\right](j+l)\frac{T}{v\mu}\right\} \tag{B10}$$

where

$$Q_{j,l}(x,\xi)\equiv\frac{x(1-x)\xi(1-\xi)}{1-x+x\xi(1-\xi)}+\left(\frac{2\pi T}{v\mu}\right)^2[j^2x\xi+l^2x(1-\xi)]+(1-x)\left[\frac{2\pi T}{v\mu}(j+l)+1\right]^2.$$

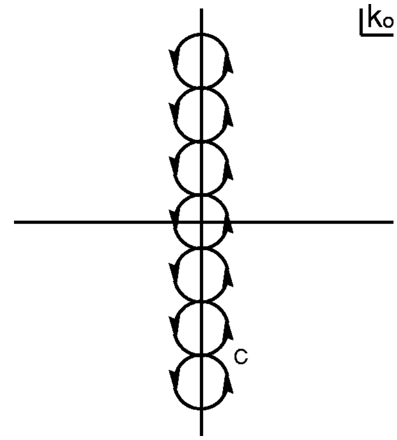

FIG. 1. Contour in the complex k_0 plane.

We find in the high temperature limit $T\gg\mu$ that this expression is dominated by its zero mode contribution W_0 (corresponding to $j=l=0$),

$$W_0=g^2\frac{\Gamma(1+\epsilon)}{3}\int_0^1 dx\int_0^1 d\xi\,\frac{x^{\epsilon/2-1}(1-x)^{1-\epsilon}}{[1-x+x\xi(1-\xi)]^{2-\epsilon/2}}\times\left[\frac{x\xi(1-\xi)}{1-x+x\xi(1-\xi)}+1\right]^{-\epsilon},$$

which turns out to be T independent.

The sum of nonzero terms gives a subdominant contribution for $T\gg\mu$ and ϵ strictly positive. We find from Eq. (B10) after calculation

$$W_{nz}\overset{T\gg\mu}{=}g^2\left(\frac{v\mu}{2\pi T}\right)^{2\epsilon}\frac{\Gamma(1+\epsilon)}{3}\int_0^1 dx\int_0^1 d\xi\times\frac{x^{\epsilon/2-1}(1-x)^2}{[1-x+x\xi(1-\xi)]^{2-\epsilon/2}}\times\sum_{l,j\in\mathcal{Z}}\frac{j^2}{[j^2(1-x+x\xi)+l^2x-2jlx\xi]^{1+\epsilon}}\times\left[1+\mathcal{O}\left(\frac{\mu^2}{T^2},g\right)\right]. \tag{B11}$$

For $0<\epsilon\ll 1$ and for $T\gg\mu$, W_0 and therefore W are dominated by the pole of W_0 at $\epsilon=0$. That is,

$$W\underset{T\gg\mu,0<\epsilon\ll 1}{=}\frac{2g^2}{3\epsilon}+\mathcal{O}\left[\left(\frac{\mu}{T}\right)^{2\epsilon},\epsilon^0\right], \tag{B12}$$

where we used that $x^{\epsilon/2-1}=_{\epsilon\to 0}(2/\epsilon)\delta(x)$.

Therefore, we find for $\beta_v(g,T/\mu,v)$ from Eqs. (B9), (B10), and (B12)

$$\beta_v\left(g,\frac{T}{\mu},v\right)\underset{=}{T \gg \mu, 0 < \epsilon \ll 1}\ \frac{vg^2}{3}\left[\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right]+\mathcal{O}\left[\left(\frac{\mu}{T}\right)^{2\epsilon}\right].$$

APPENDIX C: FORMAL PROOF

The formal proof to one-loop order begins with the expression (A1) from Appendix A above. We now use the identity [18,19]

$$I=T\sum_{l=-\infty}^{l=\infty}[A_l(x,p,s)]^{-\epsilon/2}$$

$$=\int_C\frac{dk_0}{4\pi i}[A(k_0,x,p,s)]^{-\epsilon/2}\coth\left[\frac{k_0}{2T}\right] \qquad \text{(C1)}$$

with $A(k_0,x,p,s)=A_l(x,p,s;w_l=-ik_0)$ and the contour C displayed in Fig. 1. The function $[A(k_0,x,p,s)]^{-\epsilon/2}$ has a cut running parallel to the real axis which for $p\neq 0$ or in the massive case for arbitrary p begins away from the imaginary axis and the contour C. The contour can now be deformed to wrap around the cut and the analytic continuation $s\to -i\omega+0^+$ can be performed. For $p,\omega \ll T$ the infrared behavior is dominated by $k_0 \ll T$ for which $\coth[k_0/2T]\sim 2T/k_0$, and the resulting expression features a pole at $k_0=0$ while the cut begins away from the origin. The cut can be deformed again to circle the origin and the integral is simply the residue at the pole $k_0=0$. Therefore the infrared dominant term is given by $I_{ir}=[A_0(x,p,s=-i\omega+0^+)]^{-\epsilon/2}$, a result that coincides with the analysis in terms of the Matsubara sums provided in Appendix A.

[1] See, for example, F. Karsch, "Lattice QCD at High Temperature and Density," lectures given at the 40th Internationale Universitatswochen fuer Theoretische Physik: Dense Matter (IUKT 40), Schladming, Styria, Austria, 2001, hep-lat/0106019, and references therein.

[2] H. Meyer-Ortmanns, Rev. Mod. Phys. **68**, 473 (1996); B. Muller, *The Physics of the Quark-Gluon Plasma*, Lecture Notes in Physics Vol. 225 (Springer-Verlag, Berlin, 1985); L.P. Csernai, *Introduction to Relativistic Heavy Ion Collisions* (John Wiley and Sons, Chichester, England, 1994); C.Y. Wong, *Introduction to High-Energy Heavy Ion Collisions* (World Scientific, Singapore, 1994).

[3] See, for example, A.J. Bray, Adv. Phys. **43**, 357 (1994); P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[4] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin Cummings, Reading, MA, 1976).

[5] D. Boyanovsky, D.-S. Lee and A. Singh, Phys. Rev. D **48**, 800 (1993); D. Boyanovsky, H.J. de Vega, and R. Holman, in *Topological Defects and the Non-Equilibrium Dynamics of Symmetry Breaking Phase Transitions*, edited by Y.M. Bunkov and H. Godfrin, Nato Science Series (Kluwer, Dordrecht, 2000), hep-ph/9903534; Phys. Rev. D **51**, 734 (1995).

[6] M. Bowick and A. Momen, Phys. Rev. D **58**, 085014 (1998); M. Bowick, A. Cacciuto, and A. Travesset, Phys. Rev. E **65**, 026133 (2002).

[7] K. Rajagopal, Nucl. Phys. **A680**, 211 (2000); B. Berdnikov and K. Rajagopal, Phys. Rev. D **61**, 105017 (2000); M. Stephanov, K. Rajagopal, and E. Shuryak, *ibid.* **60**, 114028 (1999).

[8] D. Boyanovsky, H.J. de Vega, R. Holman and S. Prem Kumar, Phys. Rev. D **56**, 3929 (1997).

[9] K. Rajagopal and F. Wilczek, Nucl. Phys. **B399**, 395 (1993); **B404**, 577 (1993); K. Rajagopal, in *Quark Gluon Plasma 2*, edited by R.C. Hwa (World Scientific, Singapore, 1995).

[10] M. Pietroni, Phys. Rev. Lett. **81**, 2424 (1998).

[11] D. Boyanovsky, H.J. de Vega, and M. Simionato, Phys. Rev. D **63**, 045007 (2001).

[12] A.V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B **49**, 11919 (1994); S. Sachdev, *ibid.* **59**, 14054 (1999); *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).

[13] G. Aarts and J. Smit, Phys. Lett. B **393**, 395 (1997); Nucl. Phys. **B511**, 451 (1998); W. Buchmuller and A. Jakovac, *ibid.* **B521**, 219 (1998); Phys. Lett. B **407**, 39 (1997).

[14] S.-B. Liao and M. Strickland, Phys. Rev. D **52**, 3653 (1995); Nucl. Phys. **B497**, 611 (1997).

[15] D. O'Connor, C.R. Stephens, and F. Freire, Mod. Phys. Lett. A **19**, 1779 (1993); M.A. van Eijck, D. O'Connor, and C.R. Stephens, hep-th/9406218; C.R. Stephens, Int. J. Mod. Phys. B **12**, 1379 (1998).

[16] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B458**, 90 (1996).

[17] P. Arnold and C. Zhai, Phys. Rev. D **50**, 7603 (1994); P. Arnold and O. Espinoza, *ibid.* **47**, 3546 (1993); P. Arnold and L. Yaffe, *ibid.* **55**, 7760 (1997); **49**, 3003 (1994).

[18] J.I. Kapusta, *Finite Temperature Field Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1989).

[19] M. Le Bellac, *Thermal Field Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1996).

[20] A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997).

[21] D. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984).

[22] J.L. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, Cambridge, England, 1996).

[23] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1996).

[24] K.G. Wilson and J.B. Kogut, Phys. Rep., Phys. Lett. **12**, 75 (1974).

[25] *Phase Transitions and Critical Phenomena*, edited by C. Domb and M.S. Green (Academic Press, New York, 1976), Vol. 6.

[26] Note that the quartic self-coupling in Ref. [11] has a different normalization; it differs from the one used here by a factor of 12.

[27] It is given by $\tilde{\rho}_{cl}$ right before Eq. (3.8) in [11].

[28] E. Braaten and R.D. Pisarski, Nucl. Phys. **B337**, 569 (1990); **B339**, 310 (1990).

[29] R.R. Parwani, Phys. Rev. D **45**, 4695 (1992).

[30] E. Wang and U. Heinz, Phys. Rev. D **53**, 899 (1996).
[31] V.A. Alessandrini, H.J. de Vega, and F.A. Schaposnik, Phys. Rev. B **10**, 3906 (1974); **12**, 5034 (1975).
[32] T. Appelquist and J. Carazzone, Phys. Rev. D **11**, 2856 (1975).
[33] I.S. Gradshteyn and I.M. Ryshik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).
[34] C.G. Bollini and J.J. Giambiagi, Phys. Lett. **40B**, 566 (1972); Nuovo Cimento Soc. Ital. Fis., B **12**, 20 (1972); H.J. de Vega and F.A. Schaposnik, J. Math. Phys. **15**, 1998 (1974).
[35] D. Boyanovsky and J.L. Cardy, Phys. Rev. B **26**, 154 (1982).
[36] D. Boyanovsky and H.J. de Vega, Phys. Rev. D **63**, 114028 (2001); **63**, 034016 (2001).
[37] D. Boyanovsky, H.J. de Vega, and M. Simionato (in preparation).