Exact results in 5D from instantons and deconstruction

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We consider nonperturbative effects in theories with extra dimensions and the deconstructed versions of these theories. We establish the rules for instanton calculations in 5D theories on the circle, and use them for an explicit one-instanton calculation in a supersymmetric gauge theory. The results are then compared to the known exact Seiberg-Witten type solution for this theory, confirming the validity both of the exact results and of the rules for instanton calculus for extra dimensions introduced here. Next we consider the nonperturbative results from the perspective of deconstructed extra dimensions. We show that the nonperturbative results of the deconstructed theory do indeed reproduce the known results for the continuum extra dimensional theory, thus providing the first nonperturbative evidence in favor of deconstruction. This way deconstruction also allows us to make exact predictions in higher dimensional theories which agree with earlier results, and helps to clarify the interpretation of 5D instantons.

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I. INTRODUCTION

Theories with extra dimensions might play an important role in resolving a variety of outstanding issues in particle physics: they might resolve the hierarchy problem [1], give new mechanisms for communicating supersymmetry breaking [2], or yield new insights into the flavor problem and proton stability [3]. In many of the interesting applications [2,3] the gauge sector of the SM propagates in the extra dimension (though not in the models of [1] which aim to solve the hierarchy problem). If the gauge fields do propagate along the extra dimension, then nonperturbative effects in the low-energy effective theory may differ significantly from those in ordinary 4D theories. The reason is that once the extra dimension is compactified, the instanton can wrap the compact extra dimension. Therefore, the presence of the extra dimension itself will modify the rules for instanton calculus and influence the resulting nonperturbative effects.

In this paper, we initiate the study of nonperturbative effects for extra dimensional model building, using explicit instanton calculations, existing exact results in higher dimensional gauge theories [4–11], and deconstruction [12]. We will concentrate on a single extra dimension compactified on a circle. In 5D with all dimensions non-compact there are no known finite action instanton configurations that would contribute to the semiclassical expression for the path integral. Ordinary 4D instantons would give a diverging action once integrated over the fifth coordinate (assuming that the 4D instanton is independent of the fifth coordinate), and no fully localized 5D instanton solutions are known to exist. This situation changes drastically once the fifth coordinate is compactified. In this case the ordinary 4D instanton does give a finite contribution. In addition there is a tower of instantons that contribute, due to the fact that the 4D instanton can wrap the extra dimension. In order to gain control over the nonperturbative effects in a strongly interacting theory we will be considering supersymmetric extra dimensional theories. The simplest such theory is an $SU(2)$ gauge theory with 8 supercharges in 5D (which corresponds to $\mathcal{N}=2$ supersymmetry in 4D). The reason behind the doubling of the minimal number of supercharges is that in 5D the Dirac spinor is irreducible. The aim of considering this model is not to build a realistic theory with extra dimensions, but rather to establish the rules for instanton calculations in the presence of extra dimensions, which can later be applied to more realistic models. Since in this toy model the effective 4D theory is an $\mathcal{N}=2$ theory, it can be exactly solved in terms of a Seiberg-Witten curve [13,14]. This solution was first proposed by Nekrasov in [4].

We begin the first part of this paper by reviewing Nekrasov's solution, and slightly modify it to account for an ambiguity in the Seiberg-Witten curve. This ambiguity is analogous to those appearing in the ordinary 4D Seiberg-Witten results discussed in [15]. We then turn to an explicit instanton calculation to verify the exact results of the curve. During the course of this calculation we show that there are two towers of instantons that contribute to the effective action. One of these towers is comprised by the large gauge transformed versions of the ordinary 4D instanton wrapping the extra dimension n times. The second tower is obtained by applying an ''improper'' gauge transformation on the instanton solution, and the corresponding large gauge transformed versions of the solution obtained this way. This improper gauge transformation is not among the allowed gauge trans-

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formations of the theory, since it does not obey the necessary boundary condition. Nevertheless, the transformed instanton solution obtained this way does obey all the conditions for a proper semiclassical solution. A sum over these two towers of instantons does indeed reproduce the exact results. Thus, our calculation confirms and improves the exact results, and more importantly it establishes the rules for instanton calculus in 5D theories. The agreement of the two calculations confirms that there are no fully localized 5D instantons, and that the full semiclassical results can be obtained by the sum over the two instanton towers.

A recent major development in the field of extra dimensions is the construction of 4D gauge theories which reproduce the effects of extra dimensions. The "deconstructed" theory [12] is based on a product gauge group theory in 4D, and in fact provides a latticized version of the extra dimensional theory. This has several interesting applications for model building in four dimensions [16–22]. So far, the equivalence between the deconstructed theory and the higher dimensional models has been purely based on perturbative arguments, like matching of the perturbative mass spectra of the two theories. In the second part of this paper, we provide the first evidence that deconstruction captures the nonperturbative effects as well. Deconstruction of the simplest 5D supersymmetric theory was done in [19].[1] The deconstructed version of the theory turns out to be the $\mathcal{N}=1$, 4D product group theory considered in [24], where some nonperturbative results for this theory were obtained. Since the deconstructed theory only has $\mathcal{N}=1$ supersymmetry, one cannot provide a full solution to the low-energy effective theory, like the Seiberg-Witten solutions; exact results are restricted to the holomorphic quantities in the theory—in this case, the gauge kinetic term which includes the gauge coupling. We will show that the nonperturbative information that can be easily extracted from the deconstructed theory agrees with results from the continuum theory. This then serves partly as an independent derivation of the nonperturbative results for the 5D theory, which have previously been obtained from symmetry and consistency requirements, and also shows that the deconstructed theory does indeed capture the nonperturbative effects of the higher dimensional theory.

This paper is organized in two major parts: Sec. II, devoted to an analysis of the 5D theory on the circle and its low-energy nonperturbative dynamics, and Sec. III, containing the corresponding analysis of the deconstructed theory and a comparison with the compactified continuous theory.

We begin, in Sec. II A, with a review of the 4D Seiberg-Witten setup and of the curve describing the low-energy dynamics of the 5D theory on the circle due to Nekrasov (Sec. II B). In Sec. II C we derive the rules for instanton calculations in the compactified supersymmetric 5D theory. We show that a summation over two infinite towers of instantons is required to restore invariance under the proper and improper large gauge transformations. We perform a one-instanton calculation of the contribution to the low-energy τ parameter of the theory and show that the result is in agreement with the improved Nekrasov curve.

We begin the study of the dynamics of the deconstructed theory in Sec. III. We review the deconstructed theory and its Seiberg-Witten curve in Sec. II A. The matching of the perturbative mass spectra between the deconstructed and continuous 5D theories is reviewed in Sec. III B. After that, in Sec. III C, we study the correspondence between continuum and deconstructed instantons. We show, using the brane picture of the deconstructed theory, how the proper and improper large gauge transformations arise in deconstruction, and argue that the contribution of the diagonal $(1,1,1,\ldots,1)$-instantons of the deconstructed theory match those of the two infinite towers of instantons of the continuum theory. In Sec. III D we derive the continuum Seiberg-Witten curve from the deconstructed theory and show that it matches the curve of the continuous theory. Section III E is devoted to a detailed discussion of the matching of moduli between the continuous and deconstructed theories. This is an important issue, somewhat complicated by the fact that relations between moduli receive corrections from the quantum modification of the moduli space of the individual gauge groups of the deconstructed theory. In Sec. III E 1, we give a heuristic argument in favor of the correct matching. We strengthen this argument by an explicit instanton calculation (Sec. III E 2) showing that the modulus, which is to be identified with the continuum theory modulus, does not receive corrections from instantons in the broken gauge groups. In Sec. III E 3, motivated by the brane picture, we point out the existence of a special flat direction where corrections to the holomorphic deconstructed theory moduli from instantons in the broken gauge groups vanish. Finally, in Sec. III F, we show that the large radius limit of the low-energy τ parameter has the behavior required by 5D nonrenormalization theorems.

II. 5D $SU(2)$ CURVE AND EXPLICIT INSTANTON CALCULATIONS

In this section, we first review the solution of the 5D, $\mathcal{N}=2$ pure $SU(2)$ gauge theory, in terms of a Seiberg-Witten type curve, and then show how to perform an explicit instanton calculation in the theory. We will explain how to obtain the relevant instanton contributions from the ordinary 4D instanton, and find that the result of the explicit calculation agrees with the curve prediction.

We will perform a 5D calculation of the path-integral contributions of 4D instantons, summing over two infinite towers of instanton solutions. Every solution we sum over is obtained as an x_5-dependent large gauge transformation of the usual 4D x_5-independent instanton, giving it a nontrivial dependence on the compactified coordinate. These instantons have precisely the same number of bosonic and fermionic zero modes as the conventional 4D instantons. In addition, due to supersymmetry and self-duality of the instanton background, all contributions of non-zero modes to the determinants and higher loops in the instanton background cancel, as in the 4D case. The dependence of the instanton amplitude on the instanton size is determined entirely by the number of bosonic and fermionic zero modes and is the same as in the

[1]Very recently deconstruction of 6D supersymmetric theories has been considered in [23].

4D instanton calculation (in particular, the dependence on the instanton size in the compactified supersymmetric theory is controlled by the 4D $\mathcal{N}=2$ beta function). Thus the instanton effects in the supersymmetric 5D theory turn out to be renormalizable in the 4D sense. Therefore, it is meaningful to compare instanton-induced nonperturbative effects in the continuum 5D and deconstructed large-N 4D theories.

A. 4D Seiberg-Witten setup

First, let us introduce standard notation for the ordinary Seiberg-Witten case [13]. Consider pure $\mathcal{N}=2$ SU(2) theory in 4D. On the Coulomb branch the adjoint scalar field of the $\mathcal{N}=2$ vector superfield develops a vacuum expectation value (VEV)

$$\langle\phi\rangle = a\frac{\sigma^3}{2}, \tag{2.1}$$

and the gauge-invariant modulus u is defined via

$$u = \langle \mathrm{Tr}\,\phi^2\rangle. \tag{2.2}$$

In the weak coupling regime u is given by

$$u = \frac{a^2}{2} + \sum_{k=1}^{\infty} G_k \frac{\Lambda^{4k}}{a^{4k-2}}, \tag{2.3}$$

where the infinite sum represents instanton contributions. Here k is the instanton number and Λ is the dynamical scale of the theory. The complexified coupling τ_{SW} is given by the second derivative of the holomorphic prepotential $\mathcal{F}_{\mathrm{SW}}$:

$$\tau^{\mathrm{SW}}(a) = \frac{\partial^2 \mathcal{F}_{\mathrm{SW}}(a)}{\partial a^2} = \frac{4\pi i}{g^2(a)} + \frac{\vartheta(a)}{2\pi}. \tag{2.4}$$

In the weak coupling regime it receives contributions in perturbation theory at one loop and from all orders in instantons

$$\tau_{\mathrm{SW}}(a) = \frac{i}{\pi}\log\frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \tau_k \frac{\Lambda^{4k}}{a^{4k}}. \tag{2.5}$$

The low-energy dynamics of the theory can be determined from a genus one auxiliary Riemann surface described by an elliptic Seiberg-Witten curve. The curve is given by

$$y^2 = (x^2 - \Lambda^4)(x-u), \tag{2.6}$$

where x and y parametrize the surface. The first step toward obtaining the exact low-energy effective action for the Seiberg-Witten theory is to define the meromorphic differential λ:

$$\lambda = \frac{y\,dx}{\Lambda^4 - x^2}. \tag{2.7}$$

Then the VEVs of the scalar, a, and of the dual scalar, a_D, are determined as functions of the modulus u by integrating the meromorphic form λ over the appropriately chosen cycles γ_a and γ_{a_D} of the Riemann surface (2.6):

$$a(u) = \frac{\sqrt{2}}{2\pi}\oint_{\gamma_a}\lambda = \frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}\lambda, \tag{2.8}$$

$$a_D(u) \equiv \frac{\partial\mathcal{F}(a)}{\partial a} = \frac{\sqrt{2}}{2\pi}\oint_{\gamma_{a_D}}\lambda = \frac{\sqrt{2}}{\pi}\int_{\Lambda^2}^{u}\lambda. \tag{2.9}$$

Combining these expressions one can obtain $\mathcal{F}(a)$, which in turn determines the complete low-energy effective action for an $\mathcal{N}=2$ theory. A few comments are in order. First, the dynamical scale of the theory is defined in the so-called Seiberg-Witten scheme. It is related [25] to the Pauli-Villars (PV) or dimensional reduction ($\overline{\mathrm{DR}}$) scheme (which are used for explicit perturbative and instanton calculations) via the one-loop exact expression,[2] $\Lambda^2 = 2\Lambda^2_{\mathrm{PV}} = 2\Lambda^2_{\overline{\mathrm{DR}}}$. The integrals in the expressions for $a(u)$ and $a_D(u)$ can be easily evaluated. In particular, in the weak-coupling regime, $a \gg \Lambda$, the expression for $a(u)$ can be inverted giving the modulus (2.3), and then the expression for $a_D(u(a))$ can be differentiated with respect to a to determine the coupling (2.5). All the coefficients of these expansions can be obtained from the exact solution above. In particular, in the Seiberg-Witten scheme the one-instanton coefficients are

$$G_1 = \frac{1}{4}, \quad \tau_1 = -\frac{i}{\pi}\frac{3}{4}. \tag{2.10}$$

In fact, for all instanton numbers the instanton contributions to τ_{SW} and u are related via the Matone relation [26,27]

$$\tau_k = -\frac{i}{\pi}\frac{(4k-2)(4k-1)}{2k}G_k. \tag{2.11}$$

Alternatively, these coefficients for $k=1,2$ can be derived [28] via direct multi-instanton calculation of the effective action.

Now, following Nekrasov [4] (and keeping all the numerical factors in place) we make a change of variables:

$$y = i\frac{p}{\sqrt{2}}\Lambda^2\sinh(q), \tag{2.12}$$

$$x = \Lambda^2\cosh(q). \tag{2.13}$$

The Seiberg-Witten curve becomes

$$u = \frac{p^2}{2} + \Lambda^2\cosh(q), \tag{2.14}$$

and the meromorphic differential is now

$$\lambda = -\frac{i}{\sqrt{2}}p\,dq. \tag{2.15}$$

[2] In this section we will use the Seiberg-Witten scheme, while in Sec. III, we use the $\overline{\mathrm{DR}}$ scales. This difference will only be important for our comparison of τ parameters and is trivial to account for.

The VEVs $a(u)$ and $a_D(u)$ are given by

$$\vec{a}(u)=(a_D(u),a(u))=-\frac{i}{2\pi}\oint_{\vec{\gamma}}pdq. \tag{2.16}$$

The cycles $\vec{\gamma}$ are chosen in such a way that the correct asymptotic behavior of $a(u)$ and $a_D(u)$ as $u\to\infty$ is reproduced.

In particular we have

$$a(u)=-\frac{i}{2\pi}\int_{-i\pi}^{i\pi}pdq\to\sqrt{2u}, \tag{2.17}$$

in agreement with Eq. (2.3). For future use we introduce two new parameters,

$$w\equiv\sqrt{2u},\quad \nu_4\equiv\frac{\Lambda^2}{u}, \tag{2.18}$$

and rewrite Eq. (2.16) in the convenient form:

$$\frac{\partial\vec{a}(u)}{\partial w}=-\frac{i}{2\pi}\oint_{\vec{\gamma}}\frac{dq}{\sqrt{1-\nu_4\cosh(q)}}. \tag{2.19}$$

B. The improved 5D $SU(2)$ Seiberg-Witten curve

The $\mathcal{N}=1$ 5D SU(2) theory on $\mathbf{R}^4\times S^1$ will be viewed from the perspective of the low-energy effective 4D theory, i.e., all the 5D fields are represented as infinite sets of KK modes which are functions of the $\mathbf{R}^4$ variables.

There is a complex scalar $\Phi=\phi+iA_5$, which develops the VEV

$$\langle\Phi\rangle=A\frac{\sigma^3}{2}, \tag{2.20}$$

and the gauge-invariant modulus U is now defined as [4]

$$U=\frac{1}{2}\left\langle \mathrm{Tr}\frac{\cosh(2\pi\Phi R)}{\pi^2R^2}\right\rangle, \tag{2.21}$$

which has the weak-coupling expansion

$$U=\frac{\cosh(\pi AR)}{\pi^2R^2}+\text{instantons}. \tag{2.22}$$

We claim that the curve describing the low-energy dynamics of the theory is given by

$$U=\frac{1}{\pi^2R^2}\cosh(\pi Rp)\sqrt{1+2(\pi R\Lambda)^2f(\pi R\Lambda)\cosh(q)}. \tag{2.23}$$

Here Λ is exactly the same dynamical scale as before in Eq. (2.6). Notice that the curve (2.23) is slightly different from Nekrasov's relativistic generalization of Toda's chain [4]: The expression on the right hand side of Eq. (2.23) contains an *a priori* unknown function $f(\pi R\Lambda)$, which cannot be determined based on symmetry arguments only. This function is just 1 classically, but it can get instanton corrections at every level k:

$$f(\pi R\Lambda)=1+\sum_{k=1}^{\infty}f_k(\pi R\Lambda)^{4k}, \tag{2.24}$$

where each coefficient f_k has to be determined from an explicit k-instanton calculation. We will see below that the function f will be needed to remove certain constant (i.e. VEV independent) contributions from the complexified coupling $\tau(A)$. The ambiguity in the curve predictions introduced by f is similar in spirit to the ambiguities [15] of the Seiberg-Witten curves in the presence of matter.

In terms of this curve, the vevs $A(U)$ and $A_D(U)$ are determined in exactly the same way as in Eq. (2.16):

$$\vec{A}(U)=(A_D(U),A(U))=-\frac{i}{2\pi}\oint_{\vec{\gamma}}pdq. \tag{2.25}$$

The cycles $\vec{\gamma}$ are the same as in Eq. (2.16) such that

$$A(U)=-\frac{i}{2\pi}\int_{-i\pi}^{i\pi}pdq\to\frac{1}{\pi R}\cosh^{-1}(\pi^2R^2U), \tag{2.26}$$

in agreement with Eq. (2.22). In terms of the new parameters,

$$\cosh(\alpha)\equiv\pi^2R^2U,\quad \nu_5\equiv\frac{2f(\pi R\Lambda)(\pi R\Lambda)^2}{\sinh^2(\alpha)}, \tag{2.27}$$

Eqs. (2.25) can be expressed [4] in the form of Eq. (2.19)

$$\frac{\partial\vec{A}(U)}{\partial ga}=-\frac{i}{2\pi}\oint_{\vec{\gamma}}\frac{1}{\pi R}\frac{dq}{\sqrt{1-\nu_5\cosh(q)}}. \tag{2.28}$$

Hence, when $\nu_4=\nu_5$, i.e.,

$$u=\tilde{U}\equiv\frac{\Lambda^2}{2f(\pi R\Lambda)(\pi R\Lambda)^2}(\pi^4R^4U^2-1), \tag{2.29}$$

the VEVs of the two theories are simply related to each other,

$$\frac{\partial\vec{A}}{\partial\alpha}=\frac{1}{\pi R}\frac{\partial\vec{a}}{\partial w}\bigg|_{u=\tilde{U}}. \tag{2.30}$$

From this we can instantly calculate τ as a function of the modulus U of the 5D theory,

$$\tau(U)=\frac{\partial A_D}{\partial A}(U)=\frac{\partial a_D}{\partial a}(u=\tilde{U})=\tau_{\mathrm{SW}}(u=\tilde{U}). \tag{2.31}$$

Here on the left hand side we have the coupling τ of the 5D theory and on the right hand side we have the coupling τ_{SW} of the 4D Seiberg-Witten theory.

From Eqs. (2.31), (2.30) and (2.5) it is easy to determine the 5D coupling at one-loop order,

$$\tau^{\rm pert}(U)=\frac{i}{\pi}\log\left(\frac{\sinh^2(\pi AR)}{\pi^2R^2\Lambda^2}\right). \tag{2.32}$$

We will discuss the interpretation of the perturbative part of τ in Sec. III F.

Now we will determine $\tau(A)$ in the 5D theory at the 1-instanton level. In order to do this we will

(1) determine $A=A(U)$ from Eq. (2.30),

(2) invert it as $U=U(A)$ and express it as $\tilde{U}=\tilde{U}(A)$ using Eq. (2.29),

(3) calculate $a^2(A)$ via $a^2=a^2(u=\tilde{U}(A))$,

(4) finally obtain $\tau(A)=\tau_{\rm SW}(a^2(A))$.

Each of these steps is explained in detail below:

(1) At the 1-instanton level equation (2.3) can be inverted:

$$a^2=2u-G_1\frac{\Lambda^4}{u}. \tag{2.33}$$

Substituting this to the right hand side of Eq. (2.30) we get

$$\frac{\partial A}{\partial\alpha}=\frac{1}{\pi R}\left.\frac{\partial a}{\partial\omega}\right|_{u=\bar{U}}=\frac{1}{\pi R}\left(1+\frac{3G_1(\Lambda\pi R)^4}{\sinh^4(\alpha)}\right). \tag{2.34}$$

Integrating this with respect to α we obtain

$$\pi RA=\alpha+G_1(\Lambda\pi R)^4\frac{\cosh(\alpha)}{\sinh(\alpha)}\left(2-\frac{1}{\sinh^2(\alpha)}\right). \tag{2.35}$$

(2) Evaluating cosh of both sides of Eq. (2.35) and using the definition of α (2.27),

$$U=\frac{\cosh(\pi RA)}{\pi^2R^2}\left(1-G_1(\Lambda\pi R)^4\left(2-\frac{1}{\sinh^2(\pi RA)}\right)\right). \tag{2.36}$$

By Eq. (2.29) we then determine $\tilde{U}(A)$:

$$\tilde{U}=\frac{\sinh^2(\pi RA)}{2\pi^2R^2}\left(1-(\Lambda\pi R)^4\left(f_1+4G_1\frac{\cosh^2(\pi RA)}{\sinh^2(\pi RA)}-2G_1\frac{\cosh^2(\pi RA)}{\sinh^4(\pi RA)}\right)\right). \tag{2.37}$$

In deriving the last expression we used the definition of f, Eq. (2.24), in the 1-instanton approximation.

(3) From Eq. (2.23) we determine $a^2(A)$ as

$$a^2=2\tilde{U}(A)-G_1\frac{\Lambda^4}{\tilde{U}(A)}. \tag{2.38}$$

(4) Finally, we can write down the expression for $\tau(A)=\tau_{\rm SW}(a^2(A))$ via Eq. (2.5),

$$\tau=\frac{i}{\pi}\log\left(\frac{\sinh^2(\pi AR)}{\pi^2R^2\Lambda^2}\right)-(\Lambda\pi R)^4\left(f_1\frac{i}{\pi}+4G_1\frac{i}{\pi}+2G_1\frac{i}{\pi}\frac{1}{\sinh^2(\pi AR)}+\tau_1\frac{1}{\sinh^4(\pi AR)}\right) \tag{2.39}$$

This expression together with Eq. (2.10) constitutes the curve-prediction for the coupling of the 5D theory. Now we will verify this prediction with an explicit 1-instanton calculation. As a by-product of this comparison we will also determine $f_1=-4G_1$.

C. Rules for instanton calculations and results

In order to carry out the explicit instanton calculation we first need to identify the classical instanton solutions in this theory. As mentioned before, there are no known instantons in a 5D theory with all dimensions infinitely large; that is, there are no fully localized 5D instanton solutions. Once one of the dimensions is compactified, the action of an ordinary 4D instanton (which is assumed to be independent of the coordinate of the extra dimension) will become finite. However, it turns out that this is not the only finite action solution that exists in this theory. In fact, the finite action solutions of the 5D SU(2) theory on $\mathbf{R}^4\times S^1$ are given by two infinite towers obtained from the ordinary instantons on $\mathbf{R}^4$. The analysis of these solutions is a generalization to 5 dimensions of the $\mathbf{R}^3\times S^1$ analysis carried out in [29]. There the role of the 3D instantons was played by the Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles.

The first infinite tower of instanton configurations, labeled by $n\in\mathbf{Z}$, is obtained from the ordinary $\mathbf{R}^4$ instanton by applying periodic gauge transformations

$$U_n=\exp\left(in\frac{x_5}{R}\sigma^3\right). \tag{2.40}$$

As a result of these gauge transformations, $\Phi\to U^\dagger\Phi U+U^\dagger\partial_5U$, the large-distance asymptotics of the Φ-component of the instanton becomes

$$\Phi\to\sigma^3\left(\frac{A}{2}+i\frac{n}{R}\right). \tag{2.41}$$

The existence of this tower represents the fact that the ordinary instanton can wrap the extra dimension an arbitrary number of times. It is also related to the fact that once an extra compact dimension is added to the ordinary 4D theory, there will be additional gauge transformations related to the existence of the extra dimension. A summation over the whole instanton tower generated as above will ensure that the final result is gauge invariant under the full 5D gauge transformations, and not only under the subgroup generated by 4D transformations.

The second tower is obtained from the first tower by applying an antiperiodic gauge transformation,

$$U_{\rm special}=\exp\left(i\frac{x_5}{2R}\sigma^3\right). \tag{2.42}$$

This ''improper'' gauge transformation is not among the usual gauge transformations of the theory, since it obeys an antiperiodic boundary condition instead of being periodic. However, since all the fields of the model are in the adjoint representation of SU(2), this gauge transformation does not change the periodicity of the field configurations. Therefore the instanton solution generated this way still obeys periodic boundary conditions, and has to be considered as an ordinary instanton solution. The large-distance asymptotics of the Φ-component of the second instanton tower is

$$\Phi \to \sigma^3\left(\frac{A}{2}+i\,\frac{n+1/2}{R}\right). \tag{2.43}$$

In order to derive the instanton contribution to τ of the 5D theory we simply need to sum over the contributions to $\tau_{\rm SW}$ of all the instanton configurations in each tower. Since the contribution of a single instanton is given by $\tau_1\Lambda^4/a^4$ as in Eq. (2.5), the sum over the two instanton towers is

$$\begin{aligned}\pi^{1-\rm inst}(A) &= \frac{\tau_1\Lambda^4}{2^4}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\left(\frac{A}{2}+i\,\frac{n}{R}\right)^4}+\frac{1}{\left(\frac{A}{2}+i\,\frac{n+1/2}{R}\right)^4}\right)\\ &= \frac{\tau_1\Lambda^4R^4}{2^4}\,\frac{1}{6}\,\frac{\partial^2}{\partial x^2}\\ &\quad\times\sum_{n=-\infty}^{\infty}\left(\frac{1}{(x+in)^2}+\frac{1}{(x+i(n+1/2))^2}\right)\\ &= \frac{\tau_1\Lambda^4R^4}{2^4}\,\frac{1}{6}\,\frac{\partial^2}{\partial x^2}\,\frac{4\pi^2}{\sinh^2(2\pi x)},\end{aligned} \tag{2.44}$$

where we have introduced the notation $x=AR/2$. Combining with the perturbative expression for τ we obtain the final result:

$$\begin{aligned}\tau = \frac{i}{\pi}\log\left(\frac{\sinh^2(\pi AR)}{\pi^2R^2\Lambda^2}\right)+(\Lambda\pi R)^4\tau_1\bigg(\frac{1}{\sinh^4(\pi AR)}\\ +\frac{2}{3}\,\frac{1}{\sinh^2(\pi AR)}\bigg).\end{aligned} \tag{2.45}$$

Comparing this to Eq. (2.39) and using $\tau_1=-(i/\pi)3G_1$ we confirm the prediction of the 5D curve and in addition fix the 1-instanton coefficient in the function f

$$f_1=-4G_1=-1. \tag{2.46}$$

The consistency of the exact result with the explicit instanton calculation is strong evidence for the absence of fully localized 5D instantons with finite action. Such instantons would give additional contributions to τ, which we do not see. Furthermore, the agreement between the curve prediction and our instanton calculation confirms the rules for explicit 5D instanton calculations detailed above.

III. NONPERTURBATIVE RESULTS FROM DECONSTRUCTION

In this section we will study the 5D theory using deconstruction. A deconstructed version of a 5D theory is a 4D gauge theory. For an appropriate choice of VEV of its fields, this 4D theory gives a latticized version of the original 5D theory [12]. The deconstructed version of the theory discussed here was proposed in [19], and we refer the reader there for a demonstration of the perturbative agreement of the deconstructed and continuum theories. In what follows we demonstrate exact nonperturbative agreement of the gauge coupling functions in the deconstructed and continuum theories. The comparison between the deconstructed and the continuum theories has to be done in the (infinitely) strong coupling regime of the deconstructed theory. However, the quantities that we are going to calculate are protected by holomorphy, and thus our results remain reliable. In addition, the deconstructed theory provides a more precise understanding of the meaning of instanton effects in five dimensions.

A. Review of the deconstructed theory and its Seiberg-Witten curve

Consider the 4D $\mathcal{N}=1$ $SU(2)^N$ theory with bifundamental chiral multiplets as in [24]. This is the deconstructed version of the 5D $\mathcal{N}=1$ SU(2) theory, as described in [19]. To be explicit, the deconstructed theory is given by $\mathcal{N}=1$ vector multiplets for each of the SU(2) gauge groups, and chiral multiplets Q_i transforming as summarized in the following table:

	$SU(2)_1$	$SU(2)_2$	$SU(2)_3$	$\cdots$	$SU(2)_N$	
Q_1	$\square$	$\bar{\square}$	1	$\cdots$	1	
Q_2	1	$\square$	$\bar{\square}$	$\cdots$	1	(3.1)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
Q_N	$\bar{\square}$	1	1	$\cdots$	$\square$	

The gauge invariant operators (whose VEVs parametrize the moduli space) are $B_i=\det Q_i$, $i=1,\ldots,N$ and $T=\mathrm{Tr}(Q_1\cdots Q_N)$. The Seiberg-Witten curve for the product group theory is most easily expressed [24] in terms of a composite field which transforms as an adjoint under one of the SU(2)'s; namely,

$$\Phi=Q_1Q_2\cdots Q_N-\frac{1}{2}\mathrm{Tr}(Q_1Q_2\cdots Q_N). \tag{3.2}$$

From this adjoint we form the usual SU(2) invariant VEV, $\tilde{u}=\langle\mathrm{Tr}\,\Phi^2\rangle$, which is then re-expressed in terms of the gauge invariants T and B_i, taking into consideration the quantum modified constraints among gauge invariants. The Seiberg-Witten curve is then given by [24]

$$y^2=(x^2-\tilde{u})^2-4\prod_{j=1}^{N}\Lambda_i^4. \tag{3.3}$$

This has the form of the 4D $\mathcal{N}=2$ Seiberg-Witten curve in terms of the modulus $\tilde{u}$. This curve was shown to agree with a brane picture of the theory in [30]. To compare with the 5D theory we first give identical VEVs $v\mathbf{1}$ to the Q_i, and we assume all the couplings and Λ's are equal. The VEVs break the $SU(2)^N$ theory to a diagonal $SU(2)$. The corresponding 5D theory (classically) has a lattice spacing $l=1/gv$ and a radius $R=N/(2\pi gv)$, where g is the gauge coupling of the individual SU(2) factors. This identification is most easily determined by comparing the spectra of the deconstructed and continuum theories [12]. However, the exact Seiberg-Witten results are most easily written in terms of holomorphic quantities. In particular, it is the holomorphic Novikov-Shifman-Vainstein-Zakharov (NSVZ) gauge coupling [31] that is relevant here. This requires that the normalization of the fields be changed from the one conventionally used in deconstructed models, and should instead coincide with the normalization used in the preceding sections. We can accomplish this by redefining the gauge fields as $A'_\mu = gA_\mu$, so that the gauge kinetic terms in the new variables become $-(1/4g^2)F_{\mu\nu}F^{\mu\nu}$. Since in the deconstructed theory in the limit $N\to\infty$ one expects to recover $\mathcal{N}=2$ supersymmetry in 4D [19], one needs to rescale the scalar fields and the fermions as well, such that, for example, the bosonic kinetic term becomes

$$\mathcal{L}_{kin} = -\frac{1}{4g^2}F^i_{\mu\nu}F^{\mu\nu,i} + \frac{1}{g^2}D_\mu Q_i^\dagger D^\mu Q_i, \tag{3.4}$$

where the covariant derivative is now given by $D_\mu\varphi = (\partial_\mu - iA^a_\mu T^a)\varphi$. In fact from the derivation in [24,32] of the Seiberg-Witten curve (3.3) it is easy to see that even for finite N the moduli in the curve are implicitly defined in terms of the rescaled fields with the kinetic term given by Eq. (3.4).

In this normalization we then obtain the holomorphic gauge coupling. However, the usual formula for the radius of the deconstructed extra dimension has to be modified. The reason is that in this normalization the physical masses of the gauge bosons are changed to $4v^2\sin^2(n\pi/N)$, where v is the vev of the rescaled scalar bifundamentals. Therefore the lattice spacing is given by $l=1/v$, and the radius of the extra dimension is $R=N/2\pi v$. One can see that this radius is holomorphic in the fields, as required from a quantity that we expect to appear in the SW curve. We will refer to this radius as the holomorphic radius. Notice that at this point the radius is defined perturbatively. In particular, the spectra through which the radius is defined are expected to receive nonperturbative corrections. By studying the Seiberg-Witten curve and explicit instanton contributions to the moduli of the deconstructed theory we will be able to make a precise nonperturbative definition of the radius of the 5D theory.

B. Matching of the perturbative mass spectra

Once we higgs the theory down to the diagonal subgroup with a VEV proportional to the identity for each of the bifundamentals Q_i, we can shift the VEVs of Q_i by an amount proportional to σ_3 in order to give a VEV to the adjoint of the 5D theory. The shifted VEVs break the gauge group to a single $U(1)$. Furthermore, notice that giving the same diagonal VEV to all the Q_i also satisfies the D-flatness constraints,

$$Q_iQ_i^\dagger - Q_{i+1}^\dagger Q_{i+1} \propto \mathbf{1}. \tag{3.5}$$

Hence, we have

$$Q_i = \begin{pmatrix} v_+ & \\ & v_- \end{pmatrix}. \tag{3.6}$$

Let us first match the perturbative mass spectrum of the gauge bosons of the deconstructed theory to that of the 5D theory. This is obtained by analyzing the kinetic terms for the bifundamental scalars. The covariant derivative on the bifundamental will be given by

$$D_\mu Q_i = \partial_\mu Q_i - \frac{i}{2}\begin{pmatrix} A_\mu^{(i)} & \sqrt{2}W_\mu^{(i)-} \\ \sqrt{2}W_\mu^{(i)+} & -A_\mu^{(i)} \end{pmatrix} Q_i + \frac{i}{2} Q_i \begin{pmatrix} A_\mu^{(i+1)} & \sqrt{2}W_\mu^{(i+1)-} \\ \sqrt{2}W_\mu^{(i+1)+} & -A_\mu^{(i+1)} \end{pmatrix}, \tag{3.7}$$

where $A^{(i)}$ denotes the third gauge boson of the ith gauge group, while $W^{(i)\pm} = (A^{(i),1} \pm A^{(i),2})/\sqrt{2}$. Substituting the VEV of Q_i into the kinetic terms we obtain a mass term for the gauge bosons of the form

$$\frac{1}{4}\sum_i \left[((A^{(i+1)} - A^{(i)})^2 + 4|W^{(i)}|^2)(|v_+|^2 + |v_-|^2) - 4(W^{(i+1)+}W^{(i)-}v_+v_-^* + \text{H.c.}) \right]. \tag{3.8}$$

This will give rise to a mass matrix for the A bosons of the form

$$\frac{(|v_+|^2+|v_-|^2)}{2}\begin{pmatrix} 2 & -1 & & -1 \\ -1 & 2 & & \\ & & \ddots & -1 \\ -1 & & -1 & 2 \end{pmatrix}. \tag{3.9}$$

The mass eigenvalues are then given by

$$m_n^2 = 2(|v_+|^2+|v_-|^2)\sin^2\frac{\pi n}{N}, \tag{3.10}$$

from which the radius of the extra dimension in the large N limit is read off to be $R = N/\pi\sqrt{2(v_+^2+v_-^2)}$, and the corresponding lattice spacing is given by $a^{-1} = (v_+^2+v_-^2)/2^{1/2}$. The masses of the W bosons are given by the matrix

$$\frac{1}{2}\begin{pmatrix} C & -B & & -B^* \\ -B^* & C & & \\ & & \ddots & -B \\ -B & & -B^* & C \end{pmatrix}, \tag{3.11}$$

with $C = 2(|v_+|^2+|v_-|^2)$ and $B = 2v_+^*v_-$. The mass eigenvalues of the W bosons are then given by

$$\tilde{m}_n^2 = |v_+|^2 + |v_-|^2 - v_+^* v_- e^{2\pi i n/N} - v_+ v_-^* e^{-2\pi i n/N}$$
$$= |v_+ - v_- e^{2\pi i n/N}|^2$$
$$= m_n^2 + |v_+ - v_-|^2 \cos\frac{2\pi n}{N} + i(v_+ v_-^* - v_+^* v_-)\sin\frac{2\pi n}{N}. \tag{3.12}$$

In the large N limit this reduces to $n^2/R^2 + |v_+ - v_-|^2$, which has to match the expression in the continuum limit in order to match the expectation value of the 5D adjoint field correctly. The corresponding expression for the mass of the KK modes in the continuum theory in terms of the adjoint VEV A is $\tilde{m}_n^2 = n^2/R^2 + A^2$. From this we obtain that $v_+ - v_- = A$.

We should comment on the fact that the large-N perturbative spectrum agrees with that of the 5D theory for fixed values of the N extra moduli (one linear combination of the $N+1$ moduli T, $B_1, \ldots, B_N$ is the $SU(2)_D$ modulus). There are several possible ways to deal with the extra N moduli. For example, in the brane construction reviewed in the next section (III C), the $N-1$ anomalous $U(1)$ symmetries are gauged (anomalies are cancelled via Green-Schwarz mechanism at the cutoff scale). Their D-flat conditions now leave only 2 moduli, T and $B_1 \ldots B_N$. One combination of the two is then the $SU(2)_D$ modulus. The real part of the remaining modulus can be interpreted as the radion of the compactified continuum 5D theory, while its imaginary part can be identified with the Wilson line of the graviphoton B_5. It is possible to stabilize the remaining modulus by adding a Lagrange multiplier term for $B_1 \ldots B_N$ to the superpotential. In the continuum theory, this term would have the interpretation as arising due to some (unspecified) radion stabilization mechanism. Alternatively, without employing anomalous $U(1)$s, one could stabilize all baryons via Lagrange multipliers L_i, e.g., by adding a superpotential of the form $W = L_i(B_i - v^2)$.

C. Correspondence between continuum and deconstructed instantons

We showed that the perturbative spectra of the compactified continuous and large-N deconstructed theories agree. The next step towards demonstrating the equivalence of the two theories is to find a map between the (semiclassical) nonperturbative effects. In this section, we will discuss in some detail the map between instanton contributions to the low-energy τ parameters in the two theories.

On the compactified 5D theory side, the semiclassical calculation of the instanton corrections to the "photon" τ parameter involves a sum over two towers of instantons. These two towers of instanton solutions are obtained from the four-dimensional BPST instanton by applying the "proper" periodic (2.40) and "improper," i.e., antiperiodic (2.42), large gauge transformations. These transformations only exist in the unbroken $U(1)$ subgroup of the $SU(2)$ theory on S^1 since $\pi_1(U(1)) = Z$, while $\pi_1(SU(2)/U(1)) = \pi_1(SU(2)) = 0$. The summation over these towers of instantons ensures that the instanton amplitude is gauge invariant. In other words, the full gauge invariance of the 5D theory is recovered only after all the semiclassical configurations in each instanton tower are taken into account.

Now let us consider instanton configurations in the deconstructed theory. This is a four-dimensional product group theory and its instanton solutions are given by the complete set of instantons in each of the $SU(2)$ gauge factors. The general instanton solution of this theory is a $(k_1, k_2, \ldots, k_N)$ instanton, where k_i stands for an instanton charge in the ith $SU(2)$ gauge factor.

In order to establish the correspondence between instantons in the two theories, we have to identify the contributions of the two instanton towers of charge k in the continuum 5D theory, with the contributions of the diagonal $(k,k,\ldots,k)$-instanton in the product group theory in the large N limit. At the same time, the off-diagonal or so-called fractional instantons, $(k_1, k_2, \ldots, k_N)$, with $k_i \neq k_j$ have no semiclassical analogues in the continuum 5D theory. The argument in favor of such an identification is as follows:

(1) In the following section we will derive the matching of the dynamical scales of two theories, Eq. (3.26), which identifies an instanton charge k in 5D with $N^{-1}\Sigma_{i=1}^N k_i$ in 4D.

(2) The instanton in the deconstructed theory should break the diagonal $SU(2)_D$ subgroup in order to be compared to the instanton in the continuum 5D theory in the Coulomb phase. This requirement together with **1** singles out the (1, 1, . . . ,1)-instanton as the counterpart of the $k=1$ instanton in 5D.

We now discuss the analogs of the large gauge transformations (2.40) and (2.42) in the deconstructed theory and their relation to the instanton calculus. An instructive way to find the large gauge transformations is via the brane construction of the four dimensional $SU(2)^N$ theory [30]. An added bonus of the brane picture is the simple geometric interpretation of the deconstructed KK mass spectrum.

The brane-engineered deconstructed theory is a C^2/Z_N orbifold of the type-IIA construction of pure $\mathcal{N}=2$ $SU(2N)$ theory of Ref. [33]. It involves $2N$ D4-branes, with world volumes in $x^0 \ldots x^3$ and x^6, suspended between two parallel Neveu-Schwarz 5-branes (NS5-branes) with world volumes in $x^0 \ldots x^5$ and separated along x^6. The orbifold acts on the $x^4 + ix^5$ (as well as on $x^6 + ix^7$) coordinates; the details are given in [30].

What is important for us is the description of the classical moduli space of the orbifold theory. The $2N$ D4 branes are only allowed to move in the $x^4 + ix^5$ plane, in a Z_N symmetric manner, as shown in Fig. 1. The most general configuration is that of two branes in each Z_N wedge, away from each other and from the origin. As indicated in the figure, one can identify the positions of the two branes with the parameters $v_\pm$ of Eq. (3.6). The center of mass of the two branes in a given Z_N wedge is identified with the VEV v, breaking $SU(2)^N$ to the diagonal group, while the relative displacement is the expectation value of the diagonal-$SU(2)$ adjoint field, i.e., $2a = A$. In particular, the mass spectrum given in Eqs. (3.10),(3.12) can be easily derived from the picture. The KK masses in the deconstructed theory are given by the

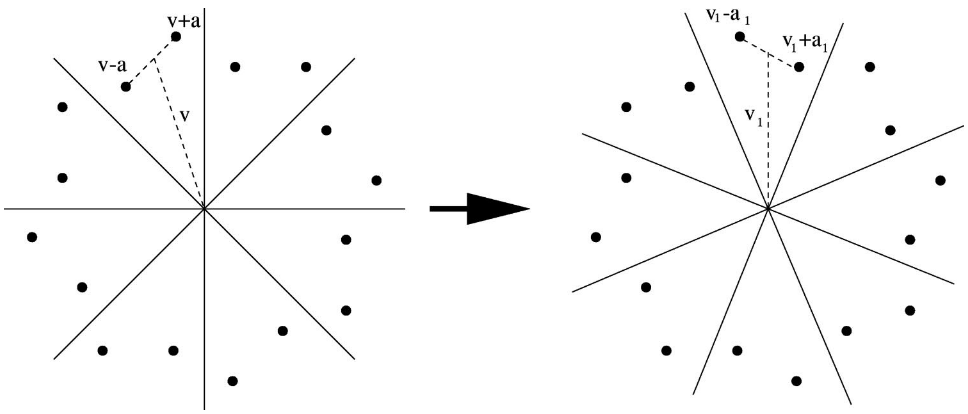

FIG. 1. The classical moduli space of the $SU(2)^N$ theory (shown for $N=8$) and the $k=1$ large gauge transformation in the brane construction.

lengths of the strings stretched between the branes in a given Z_N wedge and their images. For example, in the simplest case of unbroken $SU(2)_D$, $v_+=v_-=v$, the length of a string stretched between a brane a distance v from the origin and its kth image is $m_k=2v\,\sin(k\pi/N)$, as in Eq. (3.10); the masses in the broken $SU(2)_D$ vacuum (3.10),(3.12) can also be easily derived from the geometry of the brane construction. In a picture where all Z_N wedges are identified, the deconstructed KK modes correspond to open strings winding around the cone.[3]

It is important to note that there is a discrete arbitrariness in the assignment of pairs of branes to Z_N wedges in this picture. As we will see, one can regroup the branes into pairs in N different Z_N invariant ways, one of which is shown on Fig. 1. One can pair a brane in a given wedge with the image of the other brane in the neighboring wedge and then redraw the Z_N wedges to pass between the original pair. The ''old'' and ''new'' wedges are shown on the left and right in Fig. 1, respectively. The resulting world volume theory is, of course, identical to the original one in all aspects, including masses and interactions.

It is easy to work out the transformation corresponding to the regrouping shown on the figure in terms of $v_\pm$: from the picture one can immediately see that the relation between $v_\pm$ (the VEVs in the ''old'' wedge) and $v_{1,\pm}$ (the VEVs in the ''new'' wedge) is

$$v_{1,+}=\alpha^{-1}v_-,\ \ v_{1,-}=v_+, \tag{3.13}$$

where $\alpha=e^{i2\pi/N}$. Clearly, one can generalize this regrouping in N different Z_N symmetric ways, by combining one of the branes in the 1st wedge with the image of the other brane in the kth (counting clockwise) wedge. The resulting transformation is

$$v_+\rightarrow\alpha^{-k}v_-,\ \ v_-\rightarrow v_+, \tag{3.14}$$

with $k=1,\dots,N$; the transformation with $k=N$ gives, of course, the original pair.

It is clear from the mass formulas (3.10),(3.12) that the mass spectrum is invariant under the transformations (3.14): the masses of the KK tower of the vector supermultiplet, neutral under the diagonal $U(1)$, are invariant, while the transformations with $k\neq N$ shift the KK number of the $W^\pm$ vector supermultiplets by k units. It is easy to see that, in the large-N limit, the action of the transformation (3.14) on the spectrum is exactly that of the continuous large gauge transformations (2.40),(2.42). At large N and fixed R, recalling $v=N/(2\pi R)$, Eq. (3.14) reduces to

$$v\rightarrow v,\quad a\rightarrow -a-\frac{ik}{2R}. \tag{3.15}$$

The minus sign can be undone by a transformation in the Weyl group, $a\rightarrow -a$ (or equivalently, by accompanying Eq. (3.14) with an interchange of v_+ and v_-). Hence, recalling the identification $a=A/2$, we see that the action of both the proper and improper (2.40),(2.42) continuum large gauge transformations is reproduced by the deconstructed theory, for even and odd k, respectively.

It is possible to construct the discrete transformations giving rise to Eq. (3.14) directly in the field theory. The ones with even k correspond then to gauge transformations, while those with odd k are ''improper'' gauge transformations, in one to one correspondence with the continuum theory. It is easy to check that both types of large gauge transformation are symmetries of the deconstructed theory action.

Instantons can now be easily added into the brane picture of the deconstructed theory. In fact, an instanton of the type (1,1, . . . ,1) corresponds to a D0-brane in the vicinity of each of the N pairs of D4-branes. In other words, there is a D0-brane in each of the N wedges depicted in Fig. 1. Now, we can redraw the wedges in exactly the same way as above and discover that there is still precisely one D0 brane inside each new wedge. Of course its position inside the wedge has changed, but we need to integrate over the D0-brane posi-

[3]It is interesting to note that the brane picture suggests that string theory T duality may be underlying deconstruction, at least in the supersymmetric cases. To see this, note that the large-N limit of Fig. 1 looks like a continuous distribution of branes on a circle of radius v (in string units; recall that $1/v$ is the size of the UV cutoff in the deconstructed 5D theory). The distance between two neighboring branes is $\simeq 2\pi v/N$ (in string units and at large N). T duality relates a straight infinite periodic chain of Dp branes, with period $2\pi v/N$, to a D$(p+1)$ brane with world volume wrapped on a circle of circumference $2\pi R=N/v$. The worldvolume theory of the latter is a compactified $(p+1)$-dimensional Yang-Mills theory (the use of T duality to the construction on Fig. 1 can be strictly justified only in the $v\rightarrow\infty$ limit).

tions when we calculate instanton partition functions. Integrations over instanton collective coordinates (bosonic and fermionic) in field theory correspond to integrations over the D0-brane positions in each wedge. This means that the integral over the (1,1, . . . ,1)-instanton measure is automatically invariant under Eq. (3.14). This transformation is a symmetry not only of the microscopic action, but also of the D-instanton theory. In the deconstructed theory there is no need to sum over the instanton images under Eq. (3.14).

We can understand the difference between the continuum and the deconstructed case in more detail by considering what the gauge transformations in these two theories are. In the continuum calculation we have viewed the theory from the effective 4D theory's point of view. This means that all information about the 5th coordinate in that theory was lost, all we kept was a tower of 4D KK modes. Then we have considered the 1-instanton in this effective theory. Since we omitted the x_5 dependent gauge transformations from the effective theory, the instanton measure and action will not be invariant under the large gauge transformations. In order to reproduce the correct 5D answer, this additional symmetry has to be imposed by hand, which is achieved by the summation over the two towers of the gauge-transformed instantons. The analog of this procedure in the deconstructed theory would be to take the 1-instanton in the unbroken diagonal $SU(2)_D$ gauge group. This instanton (and its measure) would not be invariant under all the broken $SU(2)$ gauge groups, and a way to restore the full gauge invariance would be to sum over the discretized versions of the large gauge transformations described above. However, a more natural way to proceed in the deconstructed theory is to consider the effect of the (1,1, . . . ,1) instanton. In this case, the situation is very different from before. The main difference is that, as explained above, the discretized version of the x_5 dependent gauge transformations are themselves part of the gauge symmetries of the theory, they are simply given by i dependent gauge transformations in the $SU(2)_i$ factors. Also, as explained above, instead of considering a single instanton, one would have to look at the (1,1, . . . ,1) instanton calculation, and thus in effect calculate an N instanton amplitude. However, the N instanton measure must be constructed in a way that it is completely gauge invariant. Thus, there would be no need for additional summation over the images of the (1,1, . . . ,1) instanton, that sum is implicitly performed by using the correct N-instanton measure for the theory. Hence we conclude that the contribution of the (1,1, . . . ,1)-instanton in the large N-limit must match the contribution of the two 1-instanton towers in the continuum theory.

This argument applies directly to all diagonal (1,1, . . . ,1)-instanton effects. We have thus constructed a dictionary relating the $SU(2)_D$ instantons, contributing to the τ parameter in the deconstructed theory to those in the continuum theory.

D. Deriving the continuum Seiberg-Witten curve from deconstruction

Given the identification of the instantons in the continuum and deconstructed theories, we are now ready to compare the Seiberg-Witten curves for the two theories. We should stress again that the deconstructed theory has only four supercharges, while the continuum theory has eight. Therefore, a priori, the curve obtained through deconstruction contains less information than the original Seiberg-Witten curve (or Nekrasov's curve). With eight supercharges one can exactly solve both for the Kähler potential and the gauge kinetic function, while in this case only the gauge kinetic function can be obtained.[4]

As explained before, in order to obtain the Seiberg-Witten curve for the deconstructed theory one needs to evaluate $\tilde{u} = \langle \mathrm{Tr}\,\Phi^2 \rangle$, with Φ given in Eq. (3.2). Using $Q_i = \mathrm{diag}(v_+, v_-) = (v + A/2, v - A/2)$ we can now write Φ classically as

$$\Phi = [v_+^N - v_-^N]\frac{\sigma_3}{2} = v^N\left[\left(1+\frac{A}{2v}\right)^N - \left(1-\frac{A}{2v}\right)^N\right]\frac{\sigma_3}{2} \tag{3.16}$$

$$= v^N\left[\left(1+\frac{\pi R A}{N}\right)^N - \left(1-\frac{\pi R A}{N}\right)^N\right]\frac{\sigma_3}{2} \tag{3.17}$$

$$\to v^N \sinh(\pi R A)\,\sigma_3. \tag{3.18}$$

Here we have used the holomorphic radius $R = N/2\pi v$. This corresponds to the radius that appears in Nekrasov's curve (2.23), since this is the correct holomorphic variable. We also have

$$\tilde{u} = \langle \mathrm{Tr}\,\Phi^2 \rangle \to \langle 2 v^{2N} \sinh^2(\pi R A)\rangle. \tag{3.19}$$

Thus we can see that $\tilde{u}$ includes the correct variable of the 5D curve in the continuum limit. The appearance of the gauge invariant $\sinh^2(\pi RA)$ in the 5D curve is predicted from the deconstructed theory.

In order to actually match the deconstructed curve to the 5D curve obtained above, we have to first calculate the relation between the scale Λ appearing in the deconstructed curve (3.3) and the low-energy scale Λ_D which appears in the 5D curve. The matching is slightly non-trivial due to the presence of the KK modes, whose effects on the running of the coupling have to be taken into account. The matching of the holomorphic gauge couplings at the scale of the highest KK mode $m_{KK} = 2v$ is given by

$$\frac{1}{g_D^2} = \frac{N}{g^2}. \tag{3.20}$$

We now want to run the diagonal coupling down to a scale μ which is below the mass of the lowest KK mode. The renormalization group evolution equation is given by

$$\frac{1}{\alpha_D(\mu)} = \frac{N}{\alpha(m_{KK})} - \frac{2}{\pi}\log\frac{m_{KK}}{\mu} - \frac{2}{\pi}\sum_{n=1}^{N}\log\frac{m_{KK}}{m_n}, \tag{3.21}$$

[4]See, however, the discussion at the end of Sec. III E 3.

where the first logarithm is the effect of the zero modes, while the sum gives the contribution of the KK modes, and $\alpha = g^2/4\pi$. The mass ratio in the logarithm is just given by $m_{KK}/m_n = 1/\sin(n\pi/N)$. Using the relation [21]

$$\prod_{n=1}^{N-1} \sin^2 \frac{n\pi}{N} = \frac{4N^2}{2^{2N}}, \tag{3.22}$$

we obtain the expression for the low-energy gauge coupling

$$\frac{1}{\alpha_D(\mu)} = \frac{N}{\alpha(m_{KK})} - \frac{2}{\pi}\log\frac{m_{KK}}{\mu} + \frac{1}{\pi}\log\frac{4N^2}{2^{2N}}. \tag{3.23}$$

Using the definitions of the scales

$$\Lambda_D^4 = \mu^4 e^{-8\pi^2/g_D^2(\mu)}, \quad \Lambda^4 = m_{KK}^4 e^{-8\pi^2/g^2(m_{KK})}, \tag{3.24}$$

we obtain the scale matching relation

$$\Lambda_D^4 = \frac{\Lambda^{4N}}{m_{KK}^{4N-4}} \frac{2^{4N}}{16N^4}. \tag{3.25}$$

Using $m_{KK} = 2v$ and $2\pi R = N/v$ this can be rewritten as

$$\Lambda_D^4 = \frac{\Lambda^{4N}}{v^{4N}} \frac{1}{(2\pi R)^4}. \tag{3.26}$$

There may *a priori* be instanton corrections to these matching relations, but we can make precise the correspondence between the parameters of the deconstructed and continuum Seiberg-Witten curves as follows.

First, we define a Z_N symmetric gauge invariant radius (along the branch of moduli space where this identification makes sense) via

$$\left(\frac{N}{2\pi R}\right)^{2N} = \prod_{i=1}^{N} B_i. \tag{3.27}$$

In the continuum limit along the branch of moduli space we are considering[5] $B_i \to v^2$. For simplicity we define $B = (\Pi_i B_i)^{1/N}$. Let us now rescale the curve in Eq. (3.3) by $x^2 \to x^2 B^N (2\pi R)^2$ and $y^2 \to y^2 B^{2N}((2\pi R)^2)^2$, and rescale the modulus by

$$\bar{U} \equiv \frac{\tilde{u}}{B^N (2\pi R)^2} \to \frac{\langle 2\sinh^2(\pi R A)\rangle}{(2\pi R)^2}. \tag{3.28}$$

The last relation in Eq. (3.28) deserves some comment. It is obtained by identifying $\langle v^{2N}\rangle \sim \langle v^2\rangle^N$. We will demonstrate in the next section that there are no corrections to Eq. (3.28) from instantons in the broken gauge groups. There may be diagonal instanton corrections to this relation (which we do not calculate), which may be related to the function $f(\pi R\Lambda)$ in Eq. (2.24). In what follows the first relation in Eq. (3.28) should serve as the definition of $\bar{U}$, which is then unambiguous. In the large N limit we can also rewrite Eq. (3.26) in using the gauge invariant definition of the radius

$$\Lambda_D^4 = \frac{\Lambda^{4N}}{B^{2N}(2\pi R)^4}. \tag{3.29}$$

The curve we obtain then is given by

$$y^2 = (x^2 - \bar{U})^2 - 4\Lambda_D^4. \tag{3.30}$$

Finally, we note that in the continuum limit $\bar{U}$ is related via Eq. (3.28) to the modulus that appears in the continuum curve (2.23), and Λ_D is the dynamical scale in that theory; hence, we exactly reproduce the expected gauge coupling $\tau(\bar{U})$ in the continuum theory. In fact, to be more precise the modulus that appears in the continuum theory in [4] involves $\langle \cosh(\pi RA)\rangle^2 - 1$ and in the deconstructed theory it is $\langle v^{2N}\sinh(\pi AR)^2\rangle/\langle v^2\rangle^N$. Hence, deconstruction leads us to suspect that the origin of the function $f(\pi R\Lambda)$ in Eq. (2.24) are the diagonal instantons that relate these moduli. Note that this function cannot be fixed by symmetry arguments, but an explicit instanton calculation of the sort we have performed is necessary to determine it at every instanton level. However, this possibility implies that matching of additional operators between deconstructed and continuous theories may be rather nontrivial. In Sec. III E 3 we will argue that the correspondence between deconstructed and continuum models may be more direct along certain special flat directions of the deconstructed theory.

E. The role of instantons in the broken groups and of the quantum modified constraints

In the following we clarify one subtlety: the role of quantum modified constraints in the relation between moduli of the deconstructed and continuum theories.

The modulus $\bar{U}_{cl}$ defined in terms of the moduli T and B_i via classical constraints, and the modulus $\bar{U}$ that becomes the modulus of the continuum theory in the appropriate limit, differ by instanton contributions even though they have the same classical limit. So the question is which modulus to equate with the continuum modulus in the continuum limit. We first answer this by a physical argument, and then demonstrate that it is correct by a technical one.

1. Relations between moduli

The continuum variable U in Eq. (2.21) was defined in the low-energy effective 4D theory, where the only instantons that exist are the usual 4D $SU(2)$ instantons. However, in the deconstructed theory there is more than just one kind of instanton. Before breaking the diagonal $SU(2)$ group to $U(1)$ there are two types of instantons: the instantons in the diagonal unbroken $SU(2)$, which will be mapped to the instantons that remain in the effective 4D theory, but there are also instantons in the broken $SU(2)$ factors. We can denote these as $(1,0,\ldots,0),(0,1,0,\ldots,0)$ instantons, while the instanton in the diagonal $SU(2)$ factor is the $(1,1,\ldots,1)$ instanton [34]. Since the instantons in the broken gauge groups

[5] Recall that in this limit, $A/v \to 0$, so that $v_{\pm} \to v$.

have no analogs in the effective 4D theory, the definitions of the two variables $\bar{U}$ and $\bar{U}_{cl}$ may differ by the effects of these instantons. To highlight the issue, we write the deconstructed curve in terms of the moduli T and B_i as in [24] (along the flat direction (3.6)):

$$y^2=\left[x^2-\bar{U}_{cl}(T,B_i)+\sum_{j=1}^{N}\frac{\Lambda_j^4}{(2\pi R)^2B^2}\right]^2-4\Lambda_D^4. \tag{3.31}$$

So it is important that it is $\bar{U}$ and not $\bar{U}_{cl}$ that corresponds to the modulus in the continuum curve (2.23). We can understand why this is the case as follows.

For the purpose of demonstration we study the simple case of $N=2$, with the discussion easily extended to higher N. For $N=2$, the theory is given by

	$SU(2)$	$SU(2)$	$SU(2)$
Q_{aAf}	$\square$	$\square$	$\square$

(3.32)

where one has an additional $SU(2)$ global symmetry in the special case $N=2$, which is the last $SU(2)$ factor in Eq. (3.32). This is the theory considered by Intriligator and Seiberg in [32], and the derivation of the relation between moduli for this case is basically already contained in [32]. Here we repeat it in order to make the argument complete, and also to give a more physical explanation for the origin of these extra terms in Eq. (3.31). The argument (which in fact is the essence of the whole derivation of the curves in [32] and [24]) is as follows. Consider the case when the first gauge group is much stronger than the second one, $\Lambda_1 \gg \Lambda_2$. Then the second gauge group can be neglected and the first gauge group is simply an $SU(2)$ theory with two flavors (four fundamentals). This theory was described in [35] (see also [36]). At low energies it is described by the confined mesons

$$M_{AfBg}=Q_{aAf}Q_{bBg}\epsilon^{ab}. \tag{3.33}$$

This meson contains three singlets and an adjoint **3** under the weakly gauged second gauge group. This adjoint is formed by the field

$$\Phi_A^B=\frac{1}{2\Lambda_1}M_{AfCg}\epsilon^{fg}\epsilon^{CB}. \tag{3.34}$$

In terms of this adjoint field the theory is simply described by an ordinary $\mathcal{N}=2$ $SU(2)$ Seiberg-Witten curve

$$y^2=(x^2+\tilde{u})^2-4\Lambda_2^4. \tag{3.35}$$

Here $\tilde{u}$ is the invariant formed from the composite adjoint field Φ_A^B

$$\tilde{u}=\frac{1}{2}\mathrm{Tr}\,\Phi^2=\frac{1}{8\Lambda_1^2}M_{AfCg}M_{BhDi}\epsilon^{fg}\epsilon^{CB}\epsilon^{hi}\epsilon^{DA}. \tag{3.36}$$

Notice that this agrees with our earlier definition of $\tilde{u}$ for the general $SU(2)^N$ theory up to a dimensionful constant. However, we would like to express the curve in terms of the natural variable u', which is defined as the invariant

$$u'=\det\tilde{M}, \tag{3.37}$$

where $\tilde{M}_{fg}=f\frac{1}{2}Q_{aAf}Q_{bBg}\epsilon^{ab}\epsilon^{AB}$. We can now express the variable u' in terms of $\tilde{u}$. An explicit calculation shows that the relation between the two invariants is given by

$$\Lambda_1^2\tilde{u}+u'=\mathrm{Pf}\,M, \tag{3.38}$$

where the Pfaffian PfM is most easily expressed in terms of the $SU(4)$ symmetric meson matrix (obtained by ignoring the gauge interactions of the second gauge group since $\Lambda_2 \ll \Lambda_1$). One can translate between the two sets of indices of M_{AfBg} and the $SU(4)$ notation $M_{\alpha\beta}$ by the assignment $(11)\to 1$, $(12)\to 2$, $(21)\to 3$, $(22)\to 4$. With this translation $\mathrm{Pf}\,M=\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}M_{\alpha\beta}M_{\gamma\delta}$. However, the Pf$M$ is exactly the quantity which classically vanishes (once expressed in terms of the underlying quark fields), but receives a one-instanton correction quantum mechanically and yields the quantum modified constraint

$$\mathrm{Pf}\,M=\Lambda_1^4. \tag{3.39}$$

The coefficient of the one-instanton contribution was fixed by Seiberg [35] by matching to the ADS superpotential [37] after integrating out one flavor, and by Finnell and Pouliot [25] by a direct instanton calculation. Using this relation we obtain

$$\Lambda_1^2\tilde{u}+u'=\Lambda_1^4. \tag{3.40}$$

The curve (3.35) is now rewritten (after rescaling x and y) as

$$y^2=(x^2-(u'-\Lambda_1^4))^2-4\Lambda_1^4\Lambda_2^4. \tag{3.41}$$

This explains the extra shift in the curve due to instantons in the first gauge group, and there is a similar shift due to instantons in the second gauge group, and the final curve becomes

$$y^2=(x^2-(u'-\Lambda_1^4-\Lambda_2^4))^2-4\Lambda_1^4\Lambda_2^4. \tag{3.42}$$

This derivation of the $SU(2)\times SU(2)$ curve teaches us that the variable $\tilde{u}$ obtains a correction from its classical value in terms of the fundamental moduli M_{ij} due to the instantons in the individual gauge groups. These are the instantons which after the breaking to the diagonal gauge group become instantons in the broken gauge group. The extra instanton terms in Eq. (3.31) arise due to the fact that the curve has been expressed in terms of a variable which obtains a correction from these instantons. We have used this expression for the curve since these are the variables that are natural for the deconstructed theory. However, in the continuum limit it is more convenient to work with the variable $\tilde{u}$, in terms of which instantons in the broken group never appear. This modulus is directly related to the modulus of the continuum theory because the low energy continuum theory simply does not have such instantons in it.

To stress the point, the deconstructed analog of the continuum modulus proportional to $\langle \sinh(\pi RA)^2\rangle$ is related to $\tilde{u}\propto\langle v^N[(1+\pi RA/N)^N-(1-\pi RA/N)^N]\rangle$. This gauge invariant VEV, not being directly related to the fundamental gauge invariants M_{ij} (or T and B_i in the general case), is subject to quantum modified constraints among the moduli. When expressed in terms of the ''fundamental'' gauge invariants T and B_i there appears to be a superfluous term in the Seiberg-Witten curve, but this is only because of the choice of gauge invariants in terms of which we expressed the curve, and is not relevant for comparison with the 5D theory. It remains to be proven that $\tilde{u}(A)$ does not receive broken instanton corrections, and we will demonstrate this (at least for one-instanton corrections) in the next section.

2. Explicit instanton calculation of $\tilde{u}(A)$

In the following, we perform an explicit one-instanton calculation to confirm that the modulus $\tilde{u}$ does not receive any contribution from instantons in the broken groups. *A priori*, a zero mode counting would allow such a term, but an exact cancellation demonstrates that such terms are absent at the $(1,0,\ldots,0)$-instanton level. This verifies the identification of $\tilde{u}(A)$ as the modulus of the continuum theory.

Let us consider a single instanton in the second $SU(2)$ factor of the deconstructed theory (3.1)—the $(0,1,0,\ldots,0)$-instanton.[6] The field components of this instanton are the $SU(2)_2$ gauge field and gaugions, and the (anti)-fundamental flavors Q_1 and Q_2 comprising fermions and scalars. Instanton components of all other fields are trivial. Thus, from the perspective of the $(0,1,0,\ldots,0)$-instanton, the product group theory (3.1) is equivalent to the ordinary $SU(2)$ supersymmetric QCD with $N_{\mathrm{f}}=2$ real flavors: Q_{1f} with $f=1,2$, play the role of the antifundamental chiral flavors $\tilde{Q}_f$, and Q_{2f} are the fundamental chiral flavors Q_f.

We can now apply the standard rules of instanton calculus to the case at hand. For calculating instanton contributions to $\tilde{u}$ we need three ingredients: the instanton action, the instanton components of the (anti)-fundamental scalars, and the instanton measure.

Using conventions of [15,28], the instanton action is given by

$$S=\frac{8\pi^2}{g^2}+2\pi^2\rho^2(|v_+|^2+|v_-|^2)-\frac{i}{\sqrt{2}}\begin{pmatrix}\bar{v}_+ & \\ & \bar{v}_-\end{pmatrix}_f^{\dot\beta}\times\mu_{\dot\beta}(\mathcal{K}_f+\tilde{\mathcal{K}}_f),\tag{3.43}$$

where ρ is the instanton size, $\mu_{\dot\beta}=\{\mu_1,\mu_2\}$ are the Grassmann collective coordinates of superconformal fermion zero modes and $\mathcal{K}_f$ and $\tilde{\mathcal{K}}_f$ are the Grassmann collective coordinates of fundamental and antifundamental fermion zero modes. The (anti)-fundamental scalar components of the instanton read [15]

$$q_f^{\dot\beta}=\sqrt{\frac{x^2}{x^2+\rho^2}}\begin{pmatrix}\bar{v}_+ & \\ & \bar{v}_-\end{pmatrix}_f^{\dot\beta}+\frac{i}{2\sqrt{2}}\frac{|x|}{(x^2+\rho^2)^{3/2}}\mu^{\dot\beta}\mathcal{K}_f-\frac{i}{2\sqrt{2}}\frac{\rho}{|x|}\frac{1}{(x^2+\rho^2)^{3/2}}\bar{x}^{\dot\beta\beta}M_\beta\mathcal{K}_f,\tag{3.44}$$

$$\tilde{q}_{f\dot\beta}=\sqrt{\frac{x^2}{x^2+\rho^2}}\begin{pmatrix}\bar{v}_+ & \\ & \bar{v}_-\end{pmatrix}_{f\dot\beta}-\frac{i}{2\sqrt{2}}\frac{|x|}{(x^2+\rho^2)^{3/2}}\bar{\mathcal{K}}_f\mu_{\dot\beta}-\frac{i}{2\sqrt{2}}\frac{\rho}{|x|}\frac{1}{(x^2+\rho^2)^{3/2}}\tilde{\mathcal{K}}_fM^\beta x_{\beta\dot\beta}.\tag{3.45}$$

Here $M^\beta=\{M^1,M^2\}$ denote supersymmetric fermion zero modes, and the Weyl indices $\dot\beta$ and β are raised and lowered with the ε-symbols. The fermion-bilinear terms in the scalar components above arise from the Yukawa sources in the corresponding Euler-Lagrange equations.

Finally, the instanton measure of the $SU(2)$ $N=1$ supersymmetric QCD with $N_f=2$ flavors is given by (cf. [15])

$$\int d\mu_{\mathrm{inst}}=\frac{2^9}{\pi^2}\frac{\mu_{\mathrm{PV}}^4}{g^4}\int d^4x_0\rho^3d\rho d^2Md^2\mu d\mathcal{K}_1d\tilde{\mathcal{K}}_1d\mathcal{K}_2d\tilde{\mathcal{K}}_2\times\exp[-S],\tag{3.46}$$

where x_0 is the instanton position and μ_{PV} is the Pauli-Villars renormalization scale,

$$\mu_{\mathrm{PV}}^4\exp\left(-\frac{8\pi^2}{g^2(\mu_{\mathrm{PV}})}\right)=\Lambda_{\mathrm{PV}}^4.\tag{3.47}$$

The instanton contribution to $\tilde{u}$ is given by

$$\tilde{u}=\langle\mathrm{Tr}\,\Phi^2\rangle=\int d\mu_{\mathrm{inst}}\mathrm{Tr}\,\Phi^2,\tag{3.48}$$

where the instanton component of Φ can be found from Eq. (3.2) and Eqs. (3.44), (3.45).

To simplify things a little we will now take the large N limit and hence set $v_+=v_-\equiv v$. Then the expression for Φ takes form

$$\Phi_{fh}=v^{N-2}\left(\tilde{q}_{f\dot\beta}q_h^{\dot\beta}-\frac{1}{2}\delta_{fh}\mathrm{Tr}(\tilde{q}q)\right),\tag{3.49}$$

and $\tilde{u}$ is

$$\tilde{u}=v^{2N-4}\left\langle\mathrm{Tr}(\tilde{q}q\tilde{q}q)-\frac{1}{2}\mathrm{Tr}(\tilde{q}q)\mathrm{Tr}(\tilde{q}q)\right\rangle.\tag{3.50}$$

The instanton solution for $\tilde{q}q$ can be schematically written as

[6] The contributions to $\tilde{u}$ of an instanton in the nth $SU(2)$ factor does not depend on the value of n since $\tilde{u}$ involves a trace over all bifundamentals.

$$\tilde{q}q=v^2+v\mu\mathcal{K}+\tilde{\mathcal{K}}\mu v+vM\mathcal{K}+\tilde{\mathcal{K}}Mv+\tilde{\mathcal{K}}\mu^2\mathcal{K}+\tilde{\mathcal{K}}M^2\mathcal{K}+\tilde{\mathcal{K}}\mu M\mathcal{K}. \tag{3.51}$$

Here we made explicit only the Grassmann collective coordinates and the VEVS. Notice that the first term on the right hand side of Eq. (3.51) is proportional to the unit matrix, $v^2\propto\mathbf{1}$, and can be dropped as it does not contribute to either Φ (which is traceless) or $\tilde{u}$.

Accordingly, the contributions to $\tilde{u}$ take form [cf. Eq. (3.50)]

$$\tilde{u}=\int d^2Md^2\mu d\mathcal{K}_1 d\tilde{\mathcal{K}}_1 d\mathcal{K}_2 d\tilde{\mathcal{K}}_2[(v\mu\mathcal{K})(\tilde{\mathcal{K}}M^2\mathcal{K})e^{v\mu\tilde{\mathcal{K}}}+(\tilde{\mathcal{K}}\mu v)(\tilde{\mathcal{K}}M^2\mathcal{K})e^{v\mu\mathcal{K}}+(v\mu\mathcal{K})(\tilde{\mathcal{K}}\mu v)e^{v\mu\mathcal{K}+v\mu\tilde{\mathcal{K}}}+(\tilde{\mathcal{K}}\mu^2\mathcal{K})(\tilde{\mathcal{K}}M^2\mathcal{K})+(\tilde{\mathcal{K}}\mu M\mathcal{K})(\tilde{\mathcal{K}}\mu M\mathcal{K})]. \tag{3.52}$$

Performing the integrations over Grassmanian collective coordinates and keeping careful track of the raised and lowered indices of the supersymmetric and superconformal zero modes[7] in Eqs. (3.44)–(3.52) one discovers that the first term on the right hand side of Eq. (3.52) cancels against the second term, the third term is vanishing and the fourth term cancels against the fifth term. Thus we conclude that the total contribution of single instantons of the (1,0, . . . ,0)-type to the modulus $\tilde{u}$ vanishes. This fact is in agreement with our identification of $\tilde{u}$ with the modulus of the continuum theory which can receive instanton corrections only of the type $(k,k,\ldots,k)$.

We conclude this discussion with an observation that such cancellation of the instanton contributions is specific to $\tilde{u}$. A modulus defined in a different way would not enjoy these cancellations. To illustrate this point one can consider a slightly different quantity

$$\langle \mathrm{Tr}(\tilde{q}q\tilde{q}q)-\det\tilde{q}\,\det q\rangle. \tag{3.53}$$

Classically this is equal to $\langle \mathrm{Tr}(\tilde{q}q\tilde{q}q)-1/2\,\mathrm{Tr}(\tilde{q}q)\mathrm{Tr}(\tilde{q}q)\rangle \propto\tilde{u}$, but there are quantum (1-instanton) corrections. In fact, it is well known [32] that there is a quantum-modified constraint in the $N=N_f$ supersymmetric QCD, $\det M-\tilde{B}B=\Lambda^{2N}$. For our case of $N=2=N_f$, the meson determinant is $\det M=\det\tilde{q}\det q$, and the baryons are $\tilde{B}=\tilde{q}_1\tilde{q}_2$ and $B=q_1q_2$, where 1, 2 denote flavor indices and the color indices are summed over. The quantum-modified constraint is

$$\langle \det\tilde{q}\,\det q\rangle=\langle\tilde{q}_1q_1\tilde{q}_2q_2\rangle+\Lambda^4, \tag{3.54}$$

and Eq. (3.53) can be written as

$$\langle \mathrm{Tr}(\tilde{q}q\tilde{q}q)-\det\tilde{q}\,\det q\rangle=\langle \mathrm{Tr}(\tilde{q}q\tilde{q}q)\rangle-\langle\tilde{q}_1q_1\tilde{q}_2q_2\rangle+\Lambda^4. \tag{3.55}$$

Repeating the same (0,1,0, . . . ,0)-instanton calculation as above one concludes that the first and the second term on the right-hand side of Eq. (3.55) cancel each other. But the last term, Λ^4, remains, giving a nonvanishing single-instanton contribution to Eq. (3.55).

3. A special flat direction

In this section, we show the existence of a flat direction for which the partially broken instantons do not contribute to the curve, even when the curve is expressed in terms of the modulus $\overline{U_{cl}}(T,B_i)$ of Eq. (3.31) (which, along a generic direction, does receive corrections from the instantons in the partially broken gauge groups).

This flat direction is easiest to infer from Fig. 1. Recall that in the brane picture, the positions of the center of mass of the branes in the kth Z_N wedge correspond to the expectation values $\langle Q_k\rangle=v_k\sigma^0$, where[8]

$$v_k=\alpha^k v. \tag{3.56}$$

The D-flat conditions and mass matrices are invariant under the replacement of the expectation values (3.6) with (3.56). There are a few points to make about the relevance of this phase choice, which might appear arbitrary in the deconstructed $SU(2)^N$ field theory, but is a consequence of the Z_N symmetry of the brane configuration (it can, of course, also be imposed on the field theory). The most important point is that, along the flat direction (3.56), the baryon expectation values obey $B_k=\alpha^{2k}v^2$. Recall now that the term in the curve due to the instantons in the partially broken $SU(2)$ groups has the form (see Ref. [24]):

$$\sum_{k=1}^{N}\Lambda_k^4 B_1\ldots\hat{B}_{k-1}\hat{B}_k\ldots B_N, \tag{3.57}$$

where hats indicate that the corresponding fields are omitted and $0\simeq N$. Let us, for the moment, assume that all the Λ_k are equal complex numbers. Then Eq. (3.57) vanishes identically:

$$\left(\sum_{k=1}^{N}\alpha^{-4k}\right)\alpha^2\Lambda^4v^{2N-4}=0. \tag{3.58}$$

Hence, in the vacuum (3.56), the instantons in the partially broken $SU(2)$ groups do not contribute to the curve and τ parameter of the low-energy $U(1)$.

Now we need to justify our assumption of equal phases of the Λ_k^4 factors (the assumption of equal couplings is inherent to the idea of deconstruction). To this end, note that the $SU(2)^N$ field theory has N anomalous global $U(1)$ symmetries with parameters ω_k, acting as follows:

$$Q_k\to e^{i\omega_k}Q_k, \tag{3.59}$$

$$\Lambda_k^4=\to e^{2i\omega_k+2i\omega_{k-1}}\Lambda_k^4,$$

[7]Note that $\int d^2\mu\,\mu_{\dot\alpha}\mu^{\dot\beta}=\delta_{\dot\alpha}^{\dot\beta}/2$, $\int d^2\mu\,\mu_{\dot\alpha}\mu_{\dot\beta}=-\varepsilon_{\dot\alpha\dot\beta}/2$, etc.

[8]The relation (3.56) holds more generally, i.e., the VEVS of the $SU(2)_D$-breaking adjoint also obey $a_k=\alpha^k a$, as is evident from the brane picture.

where $k=1,\ldots,N$ and $k=0$ is identified with $k=N$. The transformations of Λ_k^4 reflect the $U(1)$ anomalies. From the last line in Eq. (3.59) it follows that the θ parameters transform as follows:

$$\theta_k \to \theta_k + 2\omega_k + 2\omega_{k-1}. \tag{3.60}$$

It is easy to see, by writing Eq. (3.60) as an $N\times N$ matrix equation, that for odd N all θ parameters can be put to zero by field redefinitions. Thus, the Λ_k can be assumed real from the very beginning, justifying our assumption of equal phases. In the case of even N, the rank of the matrix in Eq. (3.60) is $N-1$ and there is one physical θ parameter—the combination:

$$\theta_{phys} = \frac{1}{N}\sum_{k=1}^{N} (-1)^{k+1}\theta_k. \tag{3.61}$$

It is easy to verify that θ_{phys} is invariant under Eq. (3.60) only for even N. By appropriate field redefinitions, any choice of θ_k can be brought to the form $\theta_k=(-1)^{k+1}\theta_{phys}$ for some θ_{phys}. It follows that for N even, plugging $\Lambda_k^4 = e^{i(-1)^{k+1}\theta_{phys}}\Lambda^4$ and the VEVs (3.56) into (3.57), the contribution of partially broken instantons is proportional to $\sum_{k=1}^{N/2} e^{i4\pi k/N/2}=0$ (for $N>4$). Thus, along the flat direction (3.56), the contributions of instantons in the partially broken gauge groups cancel.

The brane picture suggests that the world volume theory becomes $\mathcal{N}=2$ in the infrared (i.e. large v, at least for fixed N); at large v the branes are far away from the orbifold fixed point and thus do not "feel" the reduced supersymmetry. This leads to the hope that more nonperturbative quantities could be matched between the deconstructed and continuum theory than just the agreement of τ parameters considered in this paper. We leave this for future study.

It is also worth commenting that it may be more natural to relate the continuum theory to the deconstructed theory along this special flat direction, despite the fact that the modulus that appears in the Seiberg-Witten curve does not receive broken instanton corrections in either case. Other operators might still receive such corrections, and the nonperturbative matching of those operators between the deconstructed and continuum theories may be nontrivial. For example, along generic flat directions in the deconstructed theory operators like $\cosh(\pi AR)$, which are related to the operator T in the large N limit, are expected to receive nonperturbative correction due to the dynamics in partially broken gauge groups. On the other hand, it is natural to conjecture that along the special flat direction considered in this section all such corrections vanish.

F. Large radius limit

The exact result for the curve should reproduce correctly the infrared behavior in the large-R limit. The 5D $SU(2)$ theory has been studied in [5]; for analysis of general 5D theories see [6]. In the 5D uncompactified case, the nonrenormalization theorem restricts the prepotential to contain at most cubic terms. The coefficient of the cubic term is related to the coefficient of the Chern-Simons term. In the $SU(2)$ theory that we are considering, a tree-level Chern-Simons term is not allowed; the only contribution to the CS coefficient occurs at one loop along the Coulomb branch and is computed in [7]. We will check, in what follows, that the curve (2.23) reproduces these results in the large-R limit.

To begin, consider the perturbative part of the τ-parameter in the deconstructed theory. It is clear from the expression in Eq. (2.45) that the instanton contributions vanish in the $R\to\infty$ limit (the instantons, which are Euclidean particles in 5D, have infinite action in this limit and so can not contribute to the path integral), hence the perturbative part of τ (in the $\overline{DR}$ scheme) is

$$\frac{\tau_{pert}}{4\pi i} = \frac{1}{4\pi^2}\log\frac{4v^{2N}\sinh^2\pi AR}{\Lambda^{2N}}. \tag{3.62}$$

Let us make some comments on the meaning of $\tau_{pert}(A)$. Using the product formula $\sinh x = x\Pi_{n>0}(1+x^2/(n^2\pi^2))$, we can rearrange equation (3.62) as follows:

$$\frac{\tau_{pert}}{4\pi i} = \frac{1}{4\pi^2}\log\frac{A^2}{\Lambda_D^2} + \frac{1}{4\pi^2}\sum_{n\neq 0}\log\left(A^2+\frac{n^2}{R^2}\right) - \log\frac{n^2}{R^2}. \tag{3.63}$$

The formula (3.63) has a simple physical interpretation. It gives the perturbative running of the diagonal $SU(2)$ gauge coupling as a function of the scale A; recall that Im $\tau_{pert}(A) \sim 1/g_D^2(A)$. The leading $\sim\log A$ term accounts for the running of the 4D coupling at small scales A, obeying $\Lambda_D \ll A \ll 1/R$. The sum over $n\neq 0$ correctly (i.e., consistent with the symmetries) takes into account the contributions of the KK modes to the running. To see this, note that for fixed A, the main contribution to the sum in Eq. (3.63) comes from modes $n\leq AR$, while the contribution of KK modes with $n \geq AR$ cancels between the two terms in the sum. Hence, modes of mass greater than A decouple from the running of the Wilsonian coupling, consistent with our interpretation of $\tau_{pert}(A)$.

Next, we can also consider the limit of large R and fixed A. In this limit, as discussed in the beginning of this section, only the linear term in A (corresponding to a trilinear prepotential) survives in τ:

$$\left.\frac{\tau_{pert}}{4\pi i}\right|_{large\text{-}R} \to \frac{1}{4\pi^2}\left(2\pi RA - \log\left(\frac{\Lambda}{v}\right)^{2N}\right). \tag{3.64}$$

Using the definition of Λ from Eq. (3.24):

$$\Lambda^4 = 16v^4\exp\left(-\frac{8\pi^2}{g^2(2v)}\right) = v^4\exp\left(-\frac{8\pi^2}{g^2(v)}\right), \tag{3.65}$$

we then obtain, at large N:

$$\frac{\tau_{pert}}{4\pi i} = 2\pi R\left(\frac{N}{2\pi R g^2(v)} + \frac{A}{4\pi^2}\right) = 2\pi R\left(\frac{1}{g_5^2} + \frac{A}{4\pi^2}\right). \tag{3.66}$$

The interpretation of the two terms in Eq. (3.66) is as follows. The overall $2\pi R$ factor can be interpreted as an integration over the extra dimension and the (dimensionful) combination $(2\pi R/N)g^2(v)=v^{-1}g^2(v)=g_5^2$ as the 5D gauge coupling. The real part of the term linear in A gives the power-law running of the coupling [38] (recall that in the "Weyl wedge" of the 5D theory $\mathrm{Re}\,A>0$ [5]). The imaginary part of the second term originates in the one-loop 5D Chern-Simons term mentioned above. The imaginary part of R in Eq. (3.66) can be made to vanish by choosing v real or, as already mentioned in Sec. III B, be interpreted as an expectation value of a field in the background supergravity multiplet.

IV. CONCLUSIONS

We have considered nonperturbative effects in theories with extra dimensions from several different perspectives: exact results, explicit instanton calculations and deconstructed extra dimensions. For definiteness we have focused on the 5D $SU(2)$ theory with eight super-charges. We have shown how to perform an explicit one-instanton calculation in this theory by using two towers of instanton solutions obtained from large gauge transformations acting on the ordinary 4D instanton. Our results are in agreement with an improved version of the exact results obtained for this model in [4]. In the second part of the paper, we have considered the deconstructed version of the same theory. We have shown that the Seiberg-Witten curve for the deconstructed model is in agreement with exact results and an explicit instanton calculation for the continuum theory, thus providing the first nonperturbative evidence in favor of deconstruction.

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