

Fluctuations of quantum fields via zeta function regularization

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(Received 28 January 2002; published 8 April 2002)

Explicit expressions for the expectation values and the variances of some observables, which are bilinear quantities in the quantum fields on a D -dimensional manifold, are derived making use of zeta function regularization. It is found that the variance, related to the second functional variation of the effective action, requires a further regularization and that the relative regularized variance turns out to be $2/N$, where N is the number of the fields, thus being independent of the dimension D . Some illustrating examples are worked through. The issue of the stress tensor is also briefly addressed.

DOI: 10.1103/PhysRevD.65.085031

PACS number(s): 11.10.Wx, 05.30.Jp, 11.15.Ex

I. INTRODUCTION

Vacuum fluctuations play an important role in many physical processes. The Casimir effect is one of its most interesting physical manifestations and it has been experimentally verified. It is also well known that the Casimir effect is related to the presence of a nonvanishing vacuum energy (see for example [1,2]). This fact mainly occurs when one is dealing with nontrivial space-times, where the spatial sections are topologically nontrivial spaces or manifolds with boundaries.

Another interesting issue where quantum vacuum effects are present is the quantum field theory in curved space-time [1,3,4]. Recall that within the semiclassical approach to quantum gravity, the basic equation reads

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G\langle T_{\mu\nu}\rangle, \quad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant and $T_{\mu\nu}$ is the vacuum expectation value of the matter stress tensor (we use $\hbar=c=1$). As a consequence, fluctuations of the stress tensor can induce fluctuations of the classical gravitational field and, in order to justify the semiclassical approximation, it appears very important to have reliable *a priori* estimates of these fluctuations.

Fluctuations of the stress tensor were studied in [5,6] making use of canonical methods. Fluctuations of Casimir forces were investigated in [7,8]. Alternatively, other authors [9] have investigated the same problem by making use of zeta function regularization.

With regard to this issue, it is well known that the notion of effective action (or effective potential) plays an important role as a powerful tool in relativistic quantum field theory. This quantity, however, is ill defined, since, within the Euclidean formulation and in the one-loop approximation, the one-loop effective action contains functional determinants of elliptic operators, which have to be regularized. Zeta-function regularization [10–12] (see also [13] for the gener-

alization to elliptic pseudo-differential operators and [14–16] for physical applications) was introduced by a number of authors as a convenient tool in order to deal with the evaluation of functional determinants. It permits one to give a meaning in the sense of analytic continuation—a mathematically very precise procedure—to quantities that are formally divergent.

In this paper, we would like to revisit the zeta function regularization approach for the evaluation of expectation values $\langle O \rangle$ and their quantum fluctuations. It is our opinion that zeta function regularization is a very powerful tool at our disposal, as compared with other methods and that this issue deserves a careful investigation.

We will mainly consider two quantities: $O = \phi^2$ and the stress tensor trace $O = T_{\mu}^{\mu}$ and their corresponding variances $\Delta O = \langle O^2 \rangle - \langle O \rangle^2$. Within our formalism, it is convenient to introduce the relative variance [9]

$$\Delta_r = \frac{\Delta O}{\langle O \rangle^2}. \quad (2)$$

A different relative variance, though directly related to the previous one, has been introduced by Kuo and Ford [6]. It reads

$$\Delta = \frac{\Delta O}{\langle O^2 \rangle} = \frac{\Delta_r}{1 + \Delta_r}. \quad (3)$$

Considering now the operator $O = T_{\mu}^{\mu}$, we observe that the trace of the Einstein equations reads

$$G_{\mu}^{\mu} + D\Lambda = 8\pi G\langle T_{\mu}^{\mu}\rangle. \quad (4)$$

Thus, fluctuations of the stress tensor trace induce classical fluctuations of the scalar curvature. Furthermore, this trace fluctuation may be used to have an estimate of the fluctuations of the whole stress tensor, since for conformally invariant quantum fields in homogeneous space-times one has

$$\langle T_{\mu\nu} \rangle = \frac{g_{\mu\nu}}{D} \langle T_{\alpha}^{\alpha} \rangle. \quad (5)$$

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The issue concerning the validity of semiclassical gravity has been discussed in [6,17].

We also recall that the first variation of the effective action is related to the vacuum expectation value of physical quantities, while the second variation of the effective action is associated with the variance. Within the zeta function regularization procedure, one has to deal with traces of complex powers of elliptic operators. The first variation of the effective action is well defined by the use of zeta function regularization, while the second variation is intrinsically ill defined, unless one makes use of suitable variations with disjoint supports. In the coincidence limit, the physically interesting case, one has to make use of a further regularization [9].

Our main result is the following: modulo regularization problems, the relative variance turns out to be $\Delta_r = 2/N$ [thus $\Delta = 2/(N+2)$], where N is the number of scalar fields in some multiplet. This result seems to be general, that is independent on the quantity one is dealing with, for example the stress tensor trace. Our results are compatible and should be compared with the ones recently obtained, for $N=1$, regarding the vacuum energy density fluctuations via smeared fields and point separation [18], which give relative variances of the order of unity albeit dimensionally dependent.

On the other hand, coming back to the case of N neutral scalar non-interacting fields, we recover a well known criterion for neglecting the quantum gravity fluctuations in the large N limit [19].

The content of the paper is as follows. In Sec. II, zeta function regularization and heat-kernel techniques are briefly summarized. In Sec. III, the first variation of the effective action computed is shown to be related to the vacuum expectation values of observables. In Sec. IV, the second variation is shown to be associated with the variance and the final result is presented with the help of a further analytic and *ad hoc* regularization. In Sec. V, some examples are presented. The paper ends with some concluding remarks and an Appendix, where the first and second variations of the trace of the complex power of an elliptic operator are explicitly computed.

II. ZETA FUNCTION REGULARIZATION OF THE EFFECTIVE ACTION

In this section, we will summarize some basic aspects of the heat-kernel and zeta function regularization methods. For the sake of simplicity we shall here restrict ourselves to scalar fields, but the method is also valid in more general situations. In the case of a neutral scalar field, the one-loop Euclidean partition function reads [12]

$$\Gamma^{(1)} = -\ln Z = \frac{1}{2} \ln \det \frac{A}{\mu^2}, \quad (6)$$

where μ^2 is a renormalization parameter.

To begin with, recall the definition of the zeta function related to an elliptic operator A

$$\Gamma(s) \zeta(s|A) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} e^{-tA} dt, \quad (7)$$

which is valid for large values of $\Re s$.

For small t , and for second order elliptic differential operators, it will be assumed that the heat trace has the following asymptotics:

$$\text{Tr} e^{-tA} = \sum_{r=0}^{\infty} K_r t^{(r-D)/2}, \quad (8)$$

where K_r are the integrated heat-kernel coefficients. In principle they may be computed (see, for example [20]). We will also assume the validity of the local heat-kernel expansion, namely

$$e^{-tA}(x) = \sum_{r=0}^{\infty} K_r(x) t^{(r-D)/2}. \quad (9)$$

If the asymptotic expansions (8) and (9) hold, a standard argument leads to the following meromorphic extension of the zeta function and its local counterpart [$J(s)$ being an analytic function]:

$$\Gamma(s) \zeta(s|A) = \sum_{r=0}^{\infty} \frac{K_r}{s + \frac{r-D}{2}} + J(s), \quad (10)$$

$$\Gamma(s) \zeta(s|A)(x) = \sum_{r=0}^{\infty} \frac{K_r(x)}{s + \frac{r-D}{2}} + J(s,x). \quad (11)$$

As a result, at $s=0$, the global and local zeta functions are regular and their derivatives exist. We also have $\zeta(0|A)(x) = K_D(x)$. As is well known, $K_D(x) = 0$ in any odd dimensional manifold without boundary. Under the hypothesis above, zeta function regularization corresponds to doing the following:

$$\ln \det \frac{A}{\mu^2} = -\zeta' \left(0 \left| \frac{A}{\mu^2} \right. \right) = -\lim_{s \rightarrow 0} \frac{d}{ds} \text{Tr} (\mu^{2s} A^{-s}). \quad (12)$$

However we would like just to recall that, in some situations, it might be necessary to generalize the above definition in the form

$$\ln \det \frac{A}{\mu^2} = -\lim_{s \rightarrow 0} \frac{1}{2} \frac{d^2}{ds^2} \text{Tr} (s \mu^{2s} A^{-s}). \quad (13)$$

When $\text{Tr}(A^{-s})$ is regular at $s=0$, the two definitions (12) and (13) coincide, but, in some cases, $\zeta(s|A)$ has a simple pole at $s=0$ and so Eq. (12) is no longer valid. This fact may well be present at the level of the effective action (see for example [21–23]).

For the sake of simplicity, in the sequel we will assume the validity of Eqs. (10) and (11). The latter gives

$$\zeta(0|A)(x) = K_D(x). \quad (14)$$

For future use, we also observe that when $K_{D-2}(x) \neq 0$, the local zeta function has a simple pole at $s=1$. In fact we may write

$$\zeta(1+s|A)(x) = \frac{1}{\Gamma(1+s)} \frac{K_{D-2}(x)}{s} + G(1+s, x), \quad (15)$$

where $G(1, x)$ is a regular function. It is given by

$$G(1, x) = \text{PP}\zeta(1|A)(x) - \gamma K_{D-2}(x), \quad (16)$$

γ being the Euler's constant and

$$\text{PP}\zeta(1|A)(x) = \lim_{s \rightarrow 0} \left(\zeta(1+s|A)(x) - \frac{K_{D-2}(x)}{s} \right), \quad (17)$$

the finite part of $\zeta(s|A)$ at $s=1$. When we consider an odd dimensional manifold without a boundary, this singularity is absent.

III. THE FIRST VARIATION OF THE EFFECTIVE ACTION: VACUUM EXPECTATION VALUES

In this section, we will evaluate vacuum expectation values $\langle O \rangle$ of some specific quantities O , such as the stress tensor trace or conformal anomaly and the square of the field (fluctuation). These quantities involve the product of two quantum fields at the same point, x , and are therefore ill defined. They require a regularization. We shall consider a multiplet of N scalar fields denoted by ϕ in an external field, described through a classical action given by

$$S = \frac{1}{2} \int dx \phi L \phi, \quad (18)$$

where L is a suitable (matrix valued) differential operator defined on a D -dimensional smooth manifold.

To begin with, let us recall the formal trick that allows one to get the vacuum expectation value of the bilinear $O = \phi K \phi$ within a path integral approach [24]. We will consider two cases: $K=I$, the identity matrix, in the case of the field fluctuation $O = \phi^2$, and $K = c_1 + c_2 L$, in the case of stress tensor trace $O = T^\mu_\mu$. Here, c_1 and c_2 are constants and moreover, in the conformally invariant case, $c_1=0$. If we denote by $\alpha(x)$ a suitable classical source, we may consider the Euclidean partition function

$$Z(\alpha) = \int D\phi e^{- (1/2) \int dx \phi L(\alpha) \phi} = \left(\det \frac{L(\alpha)}{\mu^2} \right)^{-N/2}. \quad (19)$$

Here we are assuming that, in the massive case, the multiplet has the same common mass. In this way, there is no multiplicative anomaly (see, for example, [25]) and $L(\alpha) = L + \alpha K$ may be regarded as a simple differential operator.

A formal functional derivation leads to

$$\langle O(x) \rangle = -2 \frac{\delta \ln Z(\alpha)}{\delta \alpha} \Big|_{\alpha=0}. \quad (20)$$

Making use of Eq. (19) and zeta function regularization [see Eq. (A6) in the Appendix], we may give a meaning to the above formal expression, namely

$$\delta \ln Z(\alpha) = - \frac{N}{2} \lim_{s \rightarrow 0} \frac{d}{ds} [\mu^{2s} s \text{Tr}(L^{-s-1}(\alpha) \delta L(\alpha))]. \quad (21)$$

In the case $O = \phi^2$, $\delta L(\alpha) = \delta \alpha$. Thus

$$\langle \phi^2(x) \rangle = N \lim_{s \rightarrow 0} \frac{d}{ds} [s \mu^{2s} \zeta(s+1|L)(x)]. \quad (22)$$

In the equations above, $\zeta(s|L)(x)$ is the local zeta function. As a result, making use of the meromorphic expansion (15), one gets (see [26–28])

$$\langle \phi^2(x) \rangle = N \text{PP}\zeta(1|L)(x) + N K_{D-2}(x) \ln \mu^2. \quad (23)$$

When D is odd and the manifold is without boundary, we simply have

$$\langle \phi^2(x) \rangle = N \zeta(1|L)(x). \quad (24)$$

In the other case, namely when one is dealing with the stress tensor trace, $\delta L(\alpha) = \delta \alpha (c_1 + c_2 L)$. As a consequence,

$$\langle T^\mu_\mu(x) \rangle = N \lim_{s \rightarrow 0} \frac{d}{ds} [s \mu^{2s} (c_1 \zeta(s+1|L)(x)) + c_2 \zeta(s|L)(x)], \quad (25)$$

and, as a result,

$$\langle T^\mu_\mu(x) \rangle = c_1 \langle \phi^2(x) \rangle + c_2 N K_D(x). \quad (26)$$

In the conformally coupled case $c_1=0$ and one has the usual conformal anomaly, due only to quantum effects.

IV. THE SECOND VARIATION OF THE EFFECTIVE ACTION: THE VARIANCE

We have seen that the first variation of the effective action is associated with the vacuum expectation value $\langle O \rangle$ of bilinear quantities in quantum fields. Let us show now that the second variation of the effective action is related to the variance $\Delta O = \langle O^2 \rangle - \langle O \rangle^2$.

To begin with, the second variation of the partition function (19) gives

$$\langle O(x) O(y) \rangle = 4 \frac{1}{Z(\alpha)} \frac{\delta^2 Z(\alpha)}{\delta \alpha(x) \delta \alpha(y)} \Big|_{\alpha=0}. \quad (27)$$

On the other hand, we have the identity

$$\frac{\delta^2 \ln Z(\alpha)}{\delta\alpha(x)\delta\alpha(y)} = \frac{1}{Z(\alpha)} \frac{\delta^2 Z(\alpha)}{\delta\alpha(x)\delta\alpha(y)} - \frac{1}{Z^2(\alpha)} \frac{\delta Z(\alpha)}{\delta\alpha(x)} \frac{\delta Z(\alpha)}{\delta\alpha(y)}. \quad (28)$$

As a consequence, the variance is given by

$$\begin{aligned} \Delta(x,y) &= \langle O(x)O(y) \rangle - \langle O(x) \rangle \langle O(y) \rangle \\ &= 4 \frac{\delta^2 \ln Z(\alpha)}{\delta\alpha(x)\delta\alpha(y)} \Big|_{\alpha=0}. \end{aligned} \quad (29)$$

Now, it appears convenient to introduce the relative variance

$$\Delta_r(x,y) = \frac{\Delta^2(x,y)}{\langle O(x) \rangle \langle O(y) \rangle}. \quad (30)$$

The coincidence case, $x=y$, is particularly interesting from the physical point of view. It is formally given by

$$\Delta_r(x) = \Delta_r(x,y)|_{x=y} = \frac{\delta^2 \ln Z(\alpha)}{\delta^2 \alpha(x)} \left(\frac{\delta \ln Z(\alpha)}{\delta \alpha(x)} \right)^{-2}. \quad (31)$$

Let us evaluate the second variation of $\ln Z(\alpha)$. In general, in the coincidence limit one gets an ill defined quantity and a further regularization is required, as explained in the Appendix. Making use of Eqs. (21), (13) and (A10) in the Appendix, one has

$$\begin{aligned} \delta_2 \delta_1 \ln Z(\alpha)(\varepsilon) &= \frac{N\mu^{2\varepsilon}}{2\Gamma^2(1+\varepsilon)} \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty du u^\varepsilon \int_0^\infty dv v^\varepsilon (u+v)^s \right. \\ &\quad \left. \times \text{Tr}(e^{-uA} \delta_2 A e^{-vA} \delta_1 A) \right]. \end{aligned} \quad (32)$$

For $\varepsilon > 0$ and sufficiently large, the integrand is regular at $s=0$, and we have

$$\begin{aligned} \delta_2 \delta_1 \ln Z(\alpha)(\varepsilon) &= \frac{N\mu^{2\varepsilon}}{2\Gamma^2(1+\varepsilon)} \left[\int_0^\infty du u^\varepsilon \int_0^\infty dv v^\varepsilon \right. \\ &\quad \left. \times \text{Tr}(e^{-uA} \delta_2 A e^{-vA} \delta_1 A) \right] \\ &= N \frac{\mu^{2\varepsilon}}{2} \text{Tr}(A^{-1-\varepsilon} \delta_2 A A^{-1-\varepsilon} \delta_1 A). \end{aligned} \quad (33)$$

This is our general formula for the second variation of the functional determinant. It is in agreement with a result obtained in [13].

Let us consider the specific variation related to our observables. First, for $O = \phi^2$, $\delta L(\alpha) = \delta\alpha$, and in the coincidence limit we have

$$\langle \phi^2 \rangle(\varepsilon) = \frac{N\mu^{2\varepsilon}}{2} \zeta^2(1+\varepsilon|L)(x). \quad (34)$$

For $x \neq y$, $\zeta(1+\varepsilon|L)(x,y)$ is regular at $\varepsilon=0$, but in the coincidence limit $y \rightarrow x$, $\zeta(1|L)(x)$ is well defined only for odd D and boundary free manifolds.

In the conformally coupled case and for $O = T_\mu^\mu$, one has $\delta L(\alpha) = c_2 \delta\alpha L$. As a result

$$\langle T_\mu^\mu \rangle(\varepsilon) = \frac{c_2^2 N}{2} \zeta^2(\varepsilon|L)(x) = \frac{c_2^2 N}{2} \zeta^2(0|L)(x) + O(\varepsilon) \quad (35)$$

and we can remove the regularization parameter because $\zeta(s|L)(x)$ is regular at the origin.

Some remarks are in order. If we limit ourselves to the odd dimensional case, in a manifold without boundary, the analytic continuation works in a simple manner for both the quantity $O = T_\mu^\mu$ and $O = \phi^2$ and no scale dependence appears in the final expressions. As a consequence, it turns out that the relative variance,

$$\Delta_r = \frac{\langle O^2 \rangle - \langle O \rangle^2}{\langle O \rangle^2}, \quad (36)$$

is always equal to

$$\Delta_r = \frac{2}{N}, \quad (37)$$

and

$$\Delta = \frac{2}{N+2}. \quad (38)$$

In the even dimensional case, or/and in the presence of boundary, the situation is more complicated and a further renormalization seems unavoidable.

V. EXAMPLES

A. D -dimensional torus

In this section we will consider, as a first example, the D -dimensional torus. This is a symmetric flat manifold with finite volume and the local zeta function is simply the ratio of the global zeta function and the torus volume. Thus, we may limit ourselves to the discussion involving the zeta function. This is a general conclusion valid for every symmetric space.

The zeta function for the case of a massive ($N=1$) scalar field is given by

$$\zeta_D(s|L) = \left(\frac{4\pi^2}{R^2} \right)^{-s} Z_D(s; (mR)^2) \quad (39)$$

with [29–32]

$$\begin{aligned}
 Z_D(s; (mR)^2) &= (2\pi)^{D/2} (mR)^{D-2s} \frac{\Gamma(s-D/2)}{\Gamma(s)} \\
 &+ \frac{2^{s/2+D/4+1} \pi^s (mR)^{-s+D/2}}{\Gamma(s)} \\
 &\times \sum'_{\vec{n} \in \mathbf{Z}^D} (\vec{n}^2)^{s/2-D/4} \\
 &\times K_{D/2-s}(2^{3/2} \pi m R \sqrt{\vec{n}^2}). \quad (40)
 \end{aligned}$$

Here R is the radius of the torus and m the mass of the field. This expansion is exponentially convergent. It is to be seen not just as a big mass expansion (the convergence is then extremely fast), for it is valid in a very wide range of values of mR : $1 \simeq mR \langle \infty$.

In the case of a massless field, the convenient expression to be used is quite different (this is explained in detail in [29]):

$$\begin{aligned}
 Z_D(s) &= \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{D-1} \left[\pi^{j/2} \Gamma(s-j/2) \zeta_R(2s-j) \right. \\
 &\left. + 4\pi^s \sum_{n=1}^{\infty} \sum'_{\vec{n}_j \in \mathbf{Z}^j} n^{j/2-s} (\vec{n}_j^2)^{s/2-j/4} K_{j/2-s}(2\pi n \sqrt{\vec{n}_j^2}) \right]. \quad (41)
 \end{aligned}$$

On the other hand, for the sake of completeness, it is interesting to have a perturbative expression for the case when the mass of the field is very small but different from zero. This is obtained by means of binomial expansion in the equation defining the zeta function. The result is

$$Z_D(s; (mR)^2) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{k! \Gamma(s)} (mR)^{2k} Z_D(s+k). \quad (42)$$

This expression, combined with the preceding one, Eq. (41), yields the desired low mass expansion. Such expansions for the zeta function are not to be found in the literature. In fact, explicit expressions of the type (41) have appeared in the seminal paper [29] for the first time. They are convenient, on the one hand, because they exhibit the pole structure of the zeta function explicitly. On the other, they are useful from the computational point of view, because they consist of a term including the main contribution, together with a series that converges extremely fast: only a few first terms need to be computed in order to obtain results as accurate as desired.

From these expressions, the first and second variations of the effective action are immediate to compute. Essentially, what we get are expressions of the following kind: (i) in the massless case (p is a natural number, a small one for any of the operators here considered)

$$\frac{Z_D(p)}{[Z_D(p-1)]^2} = \frac{\Gamma(p-1)^2}{2^{p-2} \Gamma(p)} \frac{\sum_{j=0}^{D-1} [\pi^{j/2} \Gamma(p-j/2) \zeta_R(2p-j) + 4\pi^2 S_{D,j}(p)]}{\left\{ \sum_{j=0}^{D-1} [\pi^{j/2} \Gamma(p-1-j/2) \zeta_R(2p-2-j) + 4\pi^2 S_{D,j}(p-1)] \right\}^2}, \quad (43)$$

where the $S_{D,j}(p)$ are fast convergent series providing only corrections to the main terms, (ii)

$$\frac{Z_D(p; (mR)^2)}{[Z_D(p-1; (mR)^2)]^2} = \frac{Z_D(p)}{[Z_D(p-1)]^2} \left[1 + 2p \left(\frac{Z_D(p)}{[Z_D(p-1)]^2} - \frac{Z_D(p+1)}{[Z_D(p)]^2} \right) (mR)^2 + \dots \right], \quad (44)$$

in the case of a field of very small mass (after doing a small-mass expansion as described above), and (iii)

$$\frac{Z_D(p; (mR)^2)}{[Z_D(p-1; (mR)^2)]^2} = \frac{(mR)^{2p-4} \Gamma(p-1)^2}{(2\pi)^{D/2} \Gamma(p)} \frac{[\Gamma(p-D/2) + 2^{p/2-D/4+1} \pi^{p-D/2} (mR)^{p-D/2} + \Sigma_D(p)]}{[\Gamma(p-1-D/2) + 2^{p/2-D/4+1/2} \pi^{p-1-D/2} (mR)^{p-1-D/2} + \Sigma_D(p-1)]^2}, \quad (45)$$

in the massive case, where again $\Sigma_D(p)$ is a very fast convergent series. These expressions simplify very much when poles of the gamma function appear (for even dimension D). They are quite easy to deal with. As advanced before, no singularity appears for D odd, and the whole expression remains, in that case (the series being well approximated by a couple of terms).

B. The Casimir slab

As a second example, we will revisit the computation of the local zeta function related to the Casimir slab, namely a massless scalar quantum field confined in the x direction between infinite parallel planar Dirichlet boundaries, located at $x=0$ and $x=a$. In this case, the local zeta function is not

trivial, due to the presence of the planar boundaries (see, for example [2,3,14,15]).

To start with, recall the form of the local diagonal heat kernel, which depends only on the confining coordinate x ; i.e.,

$$\langle x|e^{-tL}|x\rangle = \frac{2}{a(4\pi t)^{(D-1)/2}} \sum_{n=1}^{\infty} \sin^2 \frac{n\pi x}{a} \times \exp\left(-\frac{n^2\pi^2 t}{a^2}\right). \quad (46)$$

Here $L = -\nabla^2$ is the Dirichlet Laplacian in the slab. Making use of the trigonometric duplication formula for the sine and Poisson-Jacobi re-summation formula, we may rewrite the above expression in the form

$$\langle x|e^{-tL}|x\rangle = \frac{1}{(4\pi t)^{D/2}} \left(\sum_{n=-\infty}^{\infty} \exp\left(-\frac{a^2 n^2}{t}\right) - \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(na+x)^2}{t}\right) \right). \quad (47)$$

All the Seeley-DeWitt $K_r(x)$ coefficients vanish, but the first one $K_0(x) = 1$. As a consequence, we may anticipate that $\langle T_\mu^\mu(x) \rangle$ is vanishing. Making use of the Mellin transform and the above expression for the heat kernel, and analytically regularizing the integral [33]

$$\int_0^\infty dt t^z = 0, \quad (48)$$

the analytic continuation for the local zeta function may be obtained. We present the result in a simple and symmetric form (for another equivalent form corresponding to $D=4$, see [26])

$$\zeta(s|L)(x) = \frac{\Gamma\left(\frac{D}{2} - s\right) a^{2s-D}}{(4\pi)^{D/2} \Gamma(s)} [2\zeta_R(D-2s) - \zeta_H(D-2s, x/a) - \zeta_H(D-2s, 1-x/a)]. \quad (49)$$

In the above expression, $\zeta_R(z)$ and $\zeta_H(z, q)$ are the Riemann and Hurwitz zeta functions respectively. This result can also be obtained by making use of the re-summation techniques explained in [34,14].

Some comments are in order. The local expression we have obtained is already in the renormalized form. Furthermore, the related trace involves a (infinite, but trivial) volume V_T in the transverse directions and an integration over x . Since

$$\int_0^1 dq \zeta_H(z, q) = 0, \quad (50)$$

the resulting zeta function reads

$$\zeta(s|L) = \frac{2V_T \Gamma\left(\frac{D}{2} - s\right) a^{2s-D+1}}{(4\pi)^{D/2} \Gamma(s)} \zeta_R(D-2s), \quad (51)$$

and this is exactly the zeta function associated with the Casimir slab configuration.

As far as the application to the evaluation of the vacuum expectation value is concerned, first, let us consider $D > 2$. The zeta function is regular at $s=1$ and the result is

$$\langle \phi^2(x) \rangle = \frac{\Gamma\left(\frac{D}{2} - 1\right) a^{2-D}}{(4\pi)^{D/2}} [2\zeta_R(D-2) - \zeta_H(D-2, x/a) - \zeta_H(D-2, 1-x/a)]. \quad (52)$$

The expression is finite everywhere and gives a vanishing result at $x=0$ and $x=a$. Due to the simple geometry of the planar boundaries, the boundary divergences are not present.

Things are different in $D=2$. In this case, the zeta function has a pole at $s=1$ and the dependence on the scale μ appears. For the sake of completeness, we give the result. It reads

$$\langle \phi^2(x) \rangle = \frac{1}{2\pi} \left[\gamma + \ln \frac{a\mu}{\pi} + \ln \sin \frac{\pi x}{a} \right]. \quad (53)$$

In this case, the boundary divergences are present.

A few other exact analytic results obtained via zeta function regularization can be found in [26].

VI. CONCLUDING REMARKS

In this paper, we have revisited the use of the zeta function regularization approach to the evaluation of expectation values of physical quantities and their related quantum variances. The former are associated with the evaluation of the first variation of the effective action, while the latter are related to the second variation. We have shown that the first variation can be regularized by use of zeta function techniques, and explicit expressions for the vacuum expectation values have been exhibited. For issues concerning the second variation, in general, zeta function regularization works well only when one is dealing with off diagonal terms, since the coincident limit is highly singular. A further analytic regularization has to be introduced to treat the coincidence limit.

We may summarize our results as follows. For a multiplet of N scalar fields and $O = T_\mu^\mu$, in the conformally coupled case, analytic regularization gives a relative variance exactly equal to $2/N$, independently on the dimension of the boundary free manifold. For $O = \phi^2$, we have obtained again a relative variance exactly equal to $2/N$, but now limited to the odd dimensional case without boundary. These restrictions can be removed in some particular situations, like the case of the Casimir slab, where one is dealing with a flat manifold with flat boundaries. The even dimensional case seems to require however further renormalization.

The formalism can be directly applied to the expectation value of the stress tensor (see [9]), but only after much more

effort and work. Again, the first variation of the effective action with respect to the metric tensor gives $\langle T_{\mu\nu} \rangle$. With regard to this issue, for an evaluation using local zeta function regularization see [27,35].

The second variation is related to the variance, and for the off diagonal case, there is again factorization. In the special case of homogeneous space-times and for a conformally coupled scalar field, one recovers the $O=T_\mu^\mu$ case.

ACKNOWLEDGMENTS

We are grateful to Valter Moretti for a useful discussion. This investigation has been supported by the cooperation agreement INFN (Italy)–DGICYT (Spain). E.E. has been financed also by DGI/SGPI (Spain), project BFM2000-0810, and by CIRIT (Generalitat de Catalunya), contract 1999SGR-00257.

APPENDIX

In order to compute the variation of the trace of an elliptic invertible operator A , one has to take into account the fact that the variation (deformation) of A does not commute with A . From $A^{-1}A=I$, we have

$$\delta A^{-1} = -A^{-1} \delta A A^{-1}. \quad (\text{A1})$$

For the calculation of complex powers of A we can use the Cauchy-Dunford representation

$$A^{-s} = -\frac{1}{2\pi i} \int_C dz z^{-s} (A-z)^{-1}, \quad (\text{A2})$$

in which C is a suitable contour on the complex z plane. As a consequence, we have

$$\delta A^{-s} = \frac{1}{2\pi i} \int_C dz z^{-s} (A-z)^{-1} \delta A (A-z)^{-1}. \quad (\text{A3})$$

Making use of the two representations (for $\text{Re } z > 0$ and $\text{Re } s > 0$)

$$\Gamma(s)(A-z)^{-s} = \int_0^\infty dt t^{s-1} e^{tz} e^{-tA} \quad (\text{A4})$$

and

$$\frac{1}{\Gamma(z)} = -\frac{1}{2\pi i} \int_C dw (-w)^{-z} e^{-w}, \quad (\text{A5})$$

one gets

$$\delta \text{Tr} A^{-s} = -s \text{Tr}(A^{-s-1} \delta A). \quad (\text{A6})$$

If the operator A is self-adjoint, then there exist eigenvalues and eigenvectors, λ_n and Ψ_n , such that $A \Psi_n = \lambda_n \Psi_n$, and we have

$$\begin{aligned} \delta \text{Tr} A^{-s} &= \frac{1}{2\pi i} \int_C dz z^{-s} \text{Tr}((A-z)^{-2} \delta A) \\ &= \frac{1}{2\pi i} \int_C dz z^{-s} \sum_n \frac{1}{(z-\lambda_n)^2} \langle \Psi_n, \delta A \Psi_n \rangle \\ &= -s \sum_n \lambda_n^{-s-1} \langle \Psi_n, \delta A \Psi_n \rangle \\ &= -s \text{Tr}(A^{-s-1} \delta A). \end{aligned} \quad (\text{A7})$$

For the second variation, making use again of Eqs. (A4) and (A5), one gets

$$\begin{aligned} \delta_2 \delta_1 \text{Tr} A^{-s} &= -s \text{Tr}(\delta_2 A^{-s-1} \delta_1 A) \\ &= -s \frac{1}{2\pi i} \int_C dz z^{-s-1} \text{Tr}((A-z)^{-1} \delta_2 A \\ &\quad \times (A-z)^{-1} \delta_1 A). \end{aligned} \quad (\text{A8})$$

This expression is valid only if $\text{Supp } \delta_1$ has void intersection with $\text{Supp } \delta_2$. Since we are interested in the coincidence limit $y=x$, we have to introduce an additional analytic regularization. The simplest one is to replace $(A-z)^{-1}$ with $(A-z)^{-1-\varepsilon}$ and employ the representation in terms of the Mellin transform:

$$\Gamma(1+\varepsilon)(A-z)^{-1-\varepsilon} = \int_0^\infty du u^\varepsilon e^{uz} e^{-uA}, \quad (\text{A9})$$

in Eq. (A10) and then use again Eq. (A5). As a result, we arrive at the formula

$$\begin{aligned} \delta_2 \delta_1 \text{Tr} A^{-s}(\varepsilon) &= -\frac{1}{\Gamma^2(1+\varepsilon)} \frac{1}{\Gamma(s)} \\ &\quad \times \int_0^\infty du u^\varepsilon \int_0^\infty dv v^\varepsilon (u+v)^s \\ &\quad \times \text{Tr}(e^{-uA} \delta_2 A e^{-vA} \delta_1 A). \end{aligned} \quad (\text{A10})$$

This representation for the second variation of the trace of a complex power of an elliptic operator A is in agreement with the one obtained in [9], making use of a different method based on Schwinger's perturbative expansion.

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