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Curved space resolution of the singularity of fractional D3-branes on a conifold

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We construct a supergravity dual to the cascading $SU(N+M)\times SU(N)$ supersymmetric gauge theory (related to fractional D3-branes on a conifold according to Klebanov and co-workers) in the case when the three-space is compactified on S^3 and in the phase with unbroken chiral symmetry. The size of S^3 serves as an infrared cutoff on the gauge-theory dynamics. For a sufficiently large S^3 the dual supergravity background is expected to be nonsingular. We demonstrate that this is indeed the case: we find a smooth type IIB supergravity solution using a perturbation theory that is valid when the radius of S^3 is large. We consider also the case with the Euclidean world volume being S^4 instead of $R\times S^3$, where the supergravity solution is again found to be regular. This ''curved space'' resolution of the singularity of the fractional D3-branes on the conifold solution is analogous to the one in the nonextremal (finite temperature) case discussed in our previous work.

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I. INTRODUCTION

The gauge-theory–gravity duality[1] relates a gauge theory on the world volume of a large number of D-branes to supergravity backgrounds where the branes are replaced by the corresponding fluxes. In a particular realization of this duality, Klebanov and Witten (KW) [2] considered N regular D3-branes placed at a conical singularity in type IIB string theory. At a small 't Hooft coupling $g_sN\ll 1$, the system is best described by open strings and realizes $SU(N)\times SU(N)$ $\mathcal{N}=1$ supersymmetric gauge theory with two pairs of chiral multiplets A_i, B_j and a quartic superpotential at an infrared superconformal fixed point. In the limit of strong 't Hooft coupling this gauge theory is best described by type IIB supergravity compactified on $AdS_5\times T^{1,1}$, $T^{1,1}=[SU(2)\times SU(2)]/U(1)$, with N units of the Ramond-Ramond five-form flux through the $T^{1,1}$. If this is a genuine equivalence, then phenomena observed on the gauge theory side should have a dual description in string theory on $AdS_5\times T^{1,1}$. In particular, *any* deformation of the gauge theory visible in the large N limit should have a counterpart in the dual gravitational description, and vice versa.

Certain deformations, trivial on the gravity side, may have highly nontrivial analogues in gauge-theory dynamics. For example, the presence of the AdS_5 factor in the KW geometry is a reflection of the conformal symmetry of the dual gauge theory. In the Poincaré coordinates in AdS_5, its boundary, and thus the space-time where the gauge theory is formulated, is $R^{1,3}$. In the global parametrization of AdS_5 the boundary is $R\times S^3$. This gravitational background should correspond to the superconformal KW gauge theory defined on $R\times S^3$. From the supergravity perspective, going from the Poincaré to the global coordinates is a simple local coordinate transformation. However, on the gauge-theory side, this ''deformation'' drastically modifies the dynamics. Defined on a round three-sphere the gauge theory will have no zero modes:[2] it will have a mass gap in the spectrum of order of the inverse radius of S^3. The modification of the spectrum of the theory substantially modifies its thermodynamics. As in a similar system studied in [3], we expect a thermal phase transition in the S^3-compactified KW model, which, on the gravity side, should map into the Hawking-Page phase transition. We would like to emphasize that such a phase transition should occur only for the gauge theory defined on a curved space like S^3.

It is not known how to translate a generic gauge theory deformation into the dual supergravity description. For the particular deformations for which the dictionary is known, one typically encounters a naked singularity in the corresponding deformed geometry. Consider, for example, wrapping M D5-branes on the collapsed two-cycle of the conifold, in addition to N D3-branes put at its apex [4]. On the gauge-theory side this deformation corresponds to changing the gauge group to $SU(N+M)\times SU(N)$ with the same set of chiral multiplets and the superpotential as in the $M=0$ case. The dual supergravity background found in [5] was shown to have a naked singularity. Another example with a naked singularity in the bulk is provided by a large number on Neveu-Schwarz 5-branes (NS5-branes) wrapping a two-cycle of the resolved conifold in type IIB string theory [6]. The field-theory dual of this system can be interpreted as a compactification (in our language, a deformation) of the little string theory on S^2. Yet another, probably the simplest, example of generation of ir singularity is a mass deformation of the $\mathcal{N}=4\,SU(N)$ super Yang-Mills (SYM) theory dual to $AdS_5\times S^5$ compactification of type IIB string theory. Turning on a mass deformation on the gauge-theory side translates

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[1] For reviews and references see, e.g., [1].

[2] For the scalars, this follows from their coupling to the scalar curvature, required by the conformal invariance.

into turning on three-form fluxes on the gravity side [7–9]. At the linearized level, the fluxes diverge in the bulk, leading to a naked singularity.

A common feature of the discussed singularities is that they are produced by a well-defined deformation in the dual gauge-theory system. On the gravity side they occur in the bulk (as opposed to the boundary) of the geometry, which, according to the familiar uv-ir correspondence [10] expected in gauge-gravity duals, should reflect the IR physics of the gauge theory. If we can resolve the ir singularity induced by the deformation on the gauge-theory side, then the translation of the resolution mechanism to the gravity side should cure the naked singularity there as well.

This philosophy is rooted in the belief that there is a genuine equivalence between the two dual descriptions, and it was recently successfully applied, in particular, in Refs. [8, 11, 6] and in [12–16]. These two groups of papers differ in the type of mechanism used for the singularity resolution. In the former case, the singularity in the deformed gauge theory is resolved by nonperturbative phenomena, intrinsic to gauge theory, namely, the confinement and the chiral symmetry breaking. The resolution of the singularity proposed in the second group of papers is extrinsic to gauge theory: one puts the system at (sufficiently high) finite temperature.

In this paper we propose a more unified perspective on the issue of singularity resolution in gauge-gravity duals, and present a new specific example of the resolution mechanism. Although we shall concentrate on the case of the fractional D3-branes on conifold geometry [5] [Klebanov-Tseytlin (KT) background for short], we believe that our approach is generic and should be applicable to other cases as well.

An overview of the singularity resolution approaches given above underscores the similarity in all resolution mechanisms. As we have emphasized, in all cases the singularity is an ir phenomenon when viewed from the gauge-theory perspective. Then a natural way to resolve the singularity is to disallow the gauge theory to access low-energy states. This can be achieved as a result of a dynamical gauge-theory effect (generation of a mass gap in the spectrum due to confinement as in [8,11,6]) or by introducing an ir cutoff "by hand" (turning on a finite temperature[3] as in [12–16]). It is clear from this perspective that there should be many other ways to resolve the singularity: all one has to do is to introduce an ir cutoff on the field-theory side and to understand what that cutoff translates into on the gravity side of the duality. The corresponding supergravity background should contain an extra scale (the deformed conifold scale in [11], or the nonextremality parameter in [12–14], or the curvature of the "longitudinal" space in the examples considered below).

One possibility to introduce an ir cutoff is by "compactifying" the space on which the gauge theory is defined. As a specific realization of this proposal we shall consider the resolution of the singularity of the KT background by defining the dual gauge theory on $R\times S^3$ instead of 4D Minkowski space. The space compactification should provide an ir cutoff, and so for sufficiently large radius of the three-sphere we should expect the restoration of chiral symmetry in the dual field theory, and thus[4] a smooth dual supergravity background.

It should be emphasized that not all space compactifications (that provide an ir cutoff) may resolve the singularity of the supergravity dual. For example, compactifying the $SU(N+M)\times SU(N)$ gauge theory on a three-torus T^3 will not resolve the singularity.[5] We expect that a "good" (singularity-resolving) compactification is the one that lifts the zero modes of all of the gauge-theory fields, i.e., gauge bosons, fermions, and scalars. Let the space on which the gauge theory is defined be compactified on a d-dimensional manifold $\mathcal{M}_d$. There will not be massless gauge-boson modes, provided the first Betty number of $\mathcal{M}_d$ vanishes. The scalars will not have zero modes provided they are coupled to a nonzero scalar curvature of $\mathcal{M}_d$. Thus the second condition is a nonvanishing Ricci scalar of $\mathcal{M}_d$. One must also make sure that there are no fermionic zero modes. While the S^3 compactification satisfies these conditions, the T^3 one fails to do that.

One may also consider a Euclidean version and define the gauge theory on a curved 4D space-time, e.g., S^4 or K3. Then S^4 will lead to a resolution of the singularity (as we shall see below), but K3 will not, since it has $R_{mn}=0$ and thus does not lift the zero modes of the scalars.

Let us comment also on a peculiar relation between the space on which gauge theory is defined and its counterpart in the dual supergravity description. On the gauge-theory side we think of space-time being a manifold of fixed size. In the context of gauge-theory–gravity duality, the space-time where the gauge theory "lives" should be identified with a boundary submanifold of the dual 10D supergravity space-time. The size of this submanifold may obviously depend on other (transverse) directions. One example is $AdS_5\times S^5$ in global parametrization of the AdS_5, where the size of the spatial part of the boundary S^3 changes with the radial coordinate of AdS_5. Another example is provided by the duality discussed in [6], where the gauge theory arises from compactification of the little string theory on S^2 of fixed size. The size of the corresponding two-sphere in the dual supergravity background changes logarithmically with the radial coordinate [6].[6]

The rest of the paper is organized as follows. In Sec. II we

[3] The proposal to use a finite temperature as an ir cutoff to cloak naked singularities in five-dimensional gravity coupled to scalars was put forward in [17].

[4] The singularity of the KT background is related [11] to the chiral symmetry breaking in the dual field theory. This symmetry [reflected in the U(1) fiber symmetry of $T^{1,1}$] will be present in the generalized KT background we will construct.

[5] The gravity dual will be the original KT solution [5] with the spatial coordinates of the D3-brane world volume periodically identified.

[6] Related observations can be made in the case of other, more familiar, deformations of gauge theory. In [8] the authors studied the duality in the context of the mass-deformed $\mathcal{N}=4$ $SU(N)$ SYM theory. There, a constant mass deformation on the gauge-theory side is translated into a variable three-form flux in the gravity dual.

discuss the generalizations of the KT ansatz for the supergravity background dual to the cascading gauge theory compactified on $R\times S^3$ and S^4. Following the approach of [5,13,14], in Sec. III we derive the corresponding 1D effective action that generates the equations for the radial evolution of the functions parametrizing the background metric and matter fields. We then discuss the simplest supersymmetric solutions of these equations realizing the extremal fractional D3-brane KT background [5] and the $AdS_5\times T^{1,1}$ gravity dual to the KW gauge theory [2] compactified on $R\times S^3$ or S^4.

We then consider the deformations of $M_4=R\times S^3$ and $M_4=S^4$ compactifications of the KW model caused by switching on $P\neq 0$ units of fractional three-brane flux. As in the closely related work [14] on the nonextremal generalization of the KT background, being unable to solve the resulting equations exactly, we resort to a perturbation theory valid in the regime when the effective D3-brane charge (or the five-form flux) $K_\star$ is much larger than the fractional three-brane charge, $K_\star \gg P^2$. Physically, this approximation amounts to introducing an ir cutoff in the dual gauge theory at an energy scale high enough to mask the low-energy chiral symmetry breaking that is responsible for the generation of the KT singularity [11].[7]

In Sec. IV we construct a smooth supergravity solution interpolating between the S^4 compactification of the KW model in the ir and the KT model in the uv. In Sec. V we address the same problem in the technically more challenging case of the $R\times S^3$ compactification of the KT model. Both examples of regular compactifications of the KT model provide support to the general idea of resolving naked singularities in the bulk of gravitational duals to gauge theories by an ir cutoff produced by a ''boundary'' space compactification.

We conclude in Sec. VI with comments on constructing a gravitational dual to mass-deformed conformally compactified $\mathcal{N}=4$ supersymmetric Yang-Mills theories.

II. $R\times S^3$ AND S^4 GENERALIZATIONS OF THE KT BACKGROUND

Our aim will be to explore the generalization of the KT solution [5] for a fractional D3-brane on a conifold to the case when the constant radial distance slices of the ''parallel'' part of the metric have geometry $R\times S^3$ or S^4 (we shall consider the case of Euclidean signature). We shall argue that the corresponding solutions are regular (for large enough D3-brane charge compared to the fractional three-brane charge).

We shall start with the same ansatz as in [5,13] and simply replace $1+3$ ''longitudinal'' directions by $R\times S^3$ or by S^4. The treatment of the two cases will be very similar, and we will discuss them in parallel. There will be direct analogy with the nonextremal (finite temperature) case considered in [12–14].

[7]This is also the region of validity of the nonextremal deformation, i.e., of the finite temperature resolution of the KT singularity due to the chiral symmetry restoration studied in [14].

As in [5] we will impose the requirement that the background has Abelian symmetry associated with the U(1) fiber of $T^{1,1}$, as we will consider a phase where chiral symmetry is restored. In the case of $R\times S^3$ our general ansatz for a 10D (Euclidean signature) Einstein frame metric[8] will involve four functions x, y, z, and w of radial coordinate u

$$ds_{10E}^2=e^{2z}(dM_4)^2+e^{-2z}[e^{10y}du^2+e^{2y}(dM_5)^2], \tag{2.1}$$

$$(dM_4)^2=e^{-6x}\,dX_0^2+e^{2x}(dS^3)^2, \tag{2.2}$$

$$(dS^3)^2=d\alpha^2+\sin^2\alpha(d\beta^2+\sin^2\beta\, d\gamma^2). \tag{2.3}$$

Here the three-sphere replaces the three ''flat'' longitudinal directions of the three-brane and M_5 is a deformation of the $T^{1,1}$ metric:

$$(dM_5)^2=e^{-8w}e_\psi^2+e^{2w}(e_{\theta_1}^2+e_{\phi_1}^2+e_{\theta_2}^2+e_{\phi_2}^2), \tag{2.4}$$

$$e_\psi=\tfrac{1}{3}(d\psi+\cos\theta_1\ d\phi_1+\cos\theta_2\ d\phi_2),$$

$$e_{\theta_i}=\frac{1}{\sqrt{6}}d\theta_i\,,\qquad e_{\phi_i}=\frac{1}{\sqrt{6}}\sin\theta_i\ d\phi_i\,.$$

We choose the radius of S^3 to be 1 as it can be absorbed into a shift of x (and a rescaling of X_0).

In the case of S^3 replaced by R^3 (i.e., in the $x\to x+x_0$, $x_0\to\infty$ limit), this becomes the ansatz of [13], where the nonextremal D3-brane case was considered. The extremal D3-brane on the standard conifold and the more general fractional D3-brane KT solution have $x=w=0$. While in [13] a nonconstant function $x(u)$ $(=au)$ reflected the nonextremality of the background, in the present $R\times S^3$ case it will be nontrivial as a consequence of the curvature of S^3.

The ansatz in the S^4 case is the same as Eq. (2.1) but with $(dM_4)^2$ given by

$$(dM_4)^2=(dS^4)^2=d\alpha^2 +\sin^2\alpha[d\beta^2+\sin^2\beta(d\gamma^2+\sin^2\gamma\, d\delta^2)], \tag{2.5}$$

where the radius of S^4 is again chosen to be 1. Here there is no function x, i.e., the number of functions in the metric is the same as in the extremal case (however, in contrast to the standard KT case, here w will, in general, be nontrivial).

As for the matter fields, we shall assume that the dilaton Φ may depend on u, and our ansatz for the p-form fields (the

[8]This metric can be brought into a more familiar form $ds_{10E}^2=h^{-1/2}(r)(dM_4)^2+h^{1/2}(r)[g^{-1}(r)dr^2+r^2\,ds_5^2]$, where $h=e^{-4z-4x}$, $r=e^{y+x+w}$, $g=e^{-8x}$, $e^{10y+2x}\,du^2=g^{-1}(r)dr^2$. When $w=0$ and $e^{4y}=r^4=1/4u$, the transverse 6D space is the standard conifold with $M_5=T^{1,1}$. Small u thus corresponds to large distances in 5D and vice versa. In the AdS_5 region large u is near the origin of AdS_5 space, while $u=0$ is its boundary.

same in the $R\times S^3$ and S^4 cases) will be exactly as in the extremal KT case [5] and in [13]:[9]

$$F_3 = P e_\psi \wedge (e_{\theta_1}\wedge e_{\phi_1} - e_{\theta_2}\wedge e_{\phi_2}),$$
$$B_2 = f(u)(e_{\theta_1}\wedge e_{\phi_1} - e_{\theta_2}\wedge e_{\phi_2}), \tag{2.6}$$
$$F_5 = \mathcal{F} + *\mathcal{F},$$
$$\mathcal{F} = K(u) e_\psi \wedge e_{\theta_1}\wedge e_{\phi_1}\wedge e_{\theta_2}\wedge e_{\phi_2},$$
$$K(u) = Q + 2Pf(u), \tag{2.7}$$

where, as in [5], the expression for K follows from the Bianchi identity for the five-form. The constants Q and P are proportional to the numbers N and M of standard and fractional D3-branes; their precise normalizations (see [18]) will not be important here.

In what follows, we shall first derive the corresponding system of type IIB supergravity equations of motion describing the radial evolution of the six unknown functions of u: x, y, z, w, K, Φ (five functions y, z, w, K, Φ in the S^4 case). We shall then discuss its solutions aiming to show that there exists a smooth interpolation (in radial coordinate only) between (i) a nonsingular short-distance region where the 10D background is approximately $AdS_5\times T^{1,1}$ written in the coordinates where the $u=$const slice is $R\times S^3$ or S^4, and (ii) a long-distance region where the 10D background approaches the KT solution. We shall start with the short-distance ($u=\infty$ or $\rho=0$) region, i.e., $AdS_5\times T^{1,1}$ space (with the radius determined by the effective charge K_*) and show that by doing perturbation theory in the small parameter $P^2/K_* \ll 1$ one can match it onto the KT asymptotics at large distances ($u\to 0$ or $\rho\to\infty$). The crucial point will be that $O(P^2/K_*)$ perturbations will be regular at small distances. This will be exactly parallel to the discussion of the nonextremal case in [14] where the starting point in the ir was a regular nonextremal D3-brane (black hole in AdS_5) solution with large (above critical) Hawking temperature.

We shall assume, for notational simplicity, that the value of the radius L of the short-distance limit space $AdS_5\times T^{1,1}$ is 1. That corresponds to the choice of the normalizations where the effective three-brane charge is ($g_s=1$)

$$K_* = 4, \quad \text{i.e.,} \quad L=1. \tag{2.8}$$

In discussing the $O(P^2)$ deformation it will be useful to compare the three possible regular starting points—the $AdS_5\times T^{1,1}$ space in the three different parametrizations, where the constant radial slice is R^4, $R\times S^3$, and S^4, respectively:

$$ds_{10}^2 = e^{2\rho}(dR^4)^2 + d\rho^2 + (dT^{1,1})^2,$$
$$-\infty < \rho < \infty, \tag{2.9}$$

$$ds_{10}^2 = \cosh^2\rho\, dX_0^2 + \sinh^2\rho (dS^3)^2 + d\rho^2 + (dT^{1,1})^2, \quad 0<\rho<\infty, \tag{2.10}$$

$$ds_{10}^2 = \sinh^2\rho (dS^4)^2 + d\rho^2 + (dT^{1,1})^2, \quad 0<\rho<\infty. \tag{2.11}$$

While these three spaces (with the Euclidean AdS_5 metric written in Poincaré, global, and "hyperboloid" parametrizations) are related locally by the coordinate transformations, these involve changing all five of the coordinates, i.e., the radial, but also the angular ones. It is the assumption that the 10D deformation (2.1) of the factorized metrics (2.9), (2.10), and (2.11) when the three-form flux (2.6) is switched on depends only on the corresponding radial coordinate ρ (which is different in the three cases) that makes the resulting solutions different. Since the 10D metric is no longer a direct product, different choices of the radial coordinate (or of the metric on the $\rho=$const slice) lead to inequivalent 10D equations and thus inequivalent D3-brane solutions no longer related by a local coordinate transformation *beyond* the short-distance $AdS_5\times T^{1,1}$ limit.

In particular, while the Poincaré patch metric (2.9) leads to the KT solution which is singular in the ir (for $\rho\to-\infty$), that singularity is resolved in $R\times S^3$ and S^4 where the $\rho\to 0$ limit is described by Eqs. (2.10) and (2.11), respectively.

III. ACTION FOR EQUATIONS OF RADIAL EVOLUTION AND SPECIAL CASES

As in [5,13,14] we shall first derive the 1D effective action that generates the equations for the radial evolution of unknown functions. Computing the scalar curvature for the metric (2.1) we find that in the $R\times S^3$ case (2.1),(2.2) $\int d^{10}x\sqrt{G}R$ is proportional to $I_{gr}=\int du\, L_{gr}$, where

$$L_{gr}(R\times S^3) = 5y'^2 - 3x'^2 - 2z'^2 - 5w'^2 + \tfrac{3}{2}e^{-2x+10y-4z} + e^{8y}(6e^{-2w} - e^{-12w}). \tag{3.1}$$

The corresponding expression in the S^4 case (2.1),(2.5) is

$$L_{gr}(S^4) = 5y'^2 - 2z'^2 - 5w'^2 + 3e^{10y-4z} + e^{8y}(6e^{-2w} - e^{-12w}). \tag{3.2}$$

Note that it can be formally obtained from Eq. (3.1) by setting[10]

$$x = \text{const}, \quad e^{-2x} = 2. \tag{3.3}$$

[9] The reason that the form of the ansatz is the same is that it is formulated in terms of the transverse space geometry only (the "parallel" or "electric" part of F_5 is then fixed by the self-duality condition).

[10] The coefficient 2 accounts for the ratio of the values of the Ricci scalars of S^3 and S^4.

The new term in L_{gr} [Eq. (3.1)] compared to the (non)extremal $R\times R^3$ case in [5,13] is the first potential term that reflects the curvature of $R\to S^3$ (or S^4).[11]

The matter part L_m of the effective type IIB Lagrangian [contributions of the dilaton, three-form fields, and the five-form following from Eqs. (2.6),(2.7)] is essentially the same as in the KT case [5] and [13] since L_m does not depend on the function x and the structure of M_4. As a result, $L=L_{gr}+L_m=T-V$, where

$$T=5y'^2-3x'^2-2z'^2-5w'^2-\frac{1}{8}\Phi'^2 -\frac{1}{4}e^{-\Phi+4z-4y-4w}\frac{K'^2}{4P^2}, \tag{3.4}$$

$$V=-\frac{3}{2}e^{-2x+10y-4z}-e^{8y}(6e^{-2w}-e^{-12w}) +\frac{1}{4}e^{\Phi+4z+4y+4w}P^2+\frac{1}{8}e^{8z}K^2. \tag{3.5}$$

The equations of motion that follow from L should be supplemented by the "zero-energy" constraint $T+V=0$. As in [14], we will use the five-form flux function $K(u)=Q+2Pf(u)$ instead of $f(u)$ in Eq. (2.6).

The new potential term $e^{-2x+10y-4z}$ in Eq. (3.5) associated with the scalar curvature of the four-space, in general, leads to breaking of supersymmetry and thus to a nontrivial modification of the extremal KT solution. In the nonextremal case discussed in [13] this term was absent and the equation for x simply gave $x=au$, with a being the nonextremality parameter. In the $R\times S^3$ case where the function x is no longer a "modulus" it cannot be easily decoupled. In the S^4 case the new potential term in Eq. (3.2) provides a nontrivial mixing between the y,z.

Let us first consider some special solutions of the equations following from this action.

A. Flat four-space case: Extremal KT solution

Let us first recall the solution in the $M_4=R\times R^3$ case [corresponding formally to the "infinite radius" limit $x=\infty$ of Eq. (3.5)]. The crucial observation made in [5] is that in the absence of the $e^{-2x+10y-4z}$ term the Lagrangian (3.4), (3.5) admits a superpotential, i.e.,

$$L=g_{ij}(\phi'^i+g^{ik}\partial_k W)(\phi'^j+g^{jl}\partial_l W)-2W'.$$

As a result, there exists a special Bogomol'nyi-Prasad-Sommerfield solution satisfying $\phi'^i+g^{ik}\partial_k W=0$ and thus also the zero-energy constraint. In the present case [5,19]

$$W=\tfrac{1}{4}e^{4y}(3e^{4w}+2e^{-6w})-\tfrac{1}{8}e^{4z}K, \tag{3.6}$$

and the corresponding system of first order equations is

$$x'=0,\quad y'+\tfrac{1}{5}e^{4y}(3e^{4w}+2e^{-6w})=0,$$

$$w'-\tfrac{3}{5}e^{4y}(e^{4w}-e^{-6w})=0, \tag{3.7}$$

$$\Phi'=0,\quad K'+2P^2e^{\Phi+4y+4w}=0,$$

$$z'+\tfrac{1}{4}e^{4z}K=0. \tag{3.8}$$

Choosing the special solution $w=0$ we then find[12] [5]

$$x=w=\Phi=0,\quad e^{-4y}=4u,\quad K=K_0-\frac{P^2}{2}\ln u, \tag{3.9}$$

$$e^{-4z}=h=h_0+\left(K_0+\frac{P^2}{2}\right)u-\frac{P^2}{2}u\ln u, \tag{3.10}$$

where $h_0=0$ if we omit the standard asymptotically flat region (as we shall assume below).

B. $K=$const ($P=0$) case: $AdS_5\times T^{1,1}$ with $M_4=R\times S^3$ or $M_4=S^4$

Setting first the fractional three-brane flux to zero $P=0$ (i.e., $K=K_*=$const and also $\Phi=f=0$), we get from Eqs. (3.4),(3.5)

$$L=5y'^2-3x'^2-2z'^2-5w'^2+\tfrac{3}{2}e^{-2x+10y-4z} +e^{8y}(6e^{-2w}-e^{-12w})-\tfrac{1}{8}K_*^2e^{8z}. \tag{3.11}$$

Here the first term in the potential is the contribution of the curvature of S^3, the second is the curvature of the (w-deformed) $T^{1,1}$ space, and the last one is the negative 5D cosmological constant originating from the five-form flux contribution. Shifting z and x we may set the D3-brane charge parameter K_* to some fixed value, e.g., $K_*=4$ as in Eq. (2.8).

Since $w=0$ is an obvious special solution, in this case we get

$$L=5y'^2-3x'^2-2z'^2+\tfrac{3}{2}e^{-2x+10y-4z} +5e^{8y}-2e^{8z}. \tag{3.12}$$

In the standard "flat" D3-brane case, i.e., in the absence of the $e^{-2x+10y-4z}$ term, this system is easily integrated giving

[11] Its scaling under shifts of x,y,z follows directly from the structure of the metric (2.1). Shifting x or z to restore explicitly the inverse radius parameter of S^3 or S^4 as its coefficient, one may then recover the $R\times R^3$ case in the limit when this parameter goes to zero. As in [13], in the absence of matter terms $w=0$ is a consistent fixed point of the equations of motion, corresponding to M_5 in Eq. (2.4) replaced by the standard $T^{1,1}$. Note also that a special solution of the equations $R_{mn}=0$ that follow from this gravitational action is R times a cone over $S^3\times T^{1,1}$ or a cone over $S^4\times T^{1,1}$.

[12] $u=1/4r^4$ where r is the standard radial coordinate in the D3-brane solution.

us the extremal ($x=0$) or nonextremal ($x=au$) solution for the D3-brane on a conifold. The case of

$$y=z \tag{3.13}$$

then corresponds to the $AdS_5\times T^{1,1}$ limit (2.9) where the M_5 part of the metric (2.1) factorizes.

In general, while it is not clear how to solve the system that follows from Eq. (3.12) analytically, it is easy to see that the $5+5$ factorized case (3.13) is still a special solution. Here we end up with

$$L=3\left(y'^2-x'^2+\tfrac{1}{2}e^{-2x+6y}+e^{8y}\right). \tag{3.14}$$

The corresponding equations have the following solution:[13]

$$e^{4x}=\tanh\rho,\quad e^{4y}=\sinh^3\rho\cosh\rho,$$

$$d\rho=-e^{4y}\,du, \tag{3.15}$$

where we have set the only integration constant (the origin of ρ) to zero.[14] The metric is thus given by Eq. (2.10), i.e., is the product of AdS_5 in the global parametrization and $T^{1,1}$ (both with scale $L=1$). Large ρ corresponds to the boundary and small ρ to the origin of AdS_5.

In the S^4 case (3.2) setting $K=K_*=\text{const}$ gives [e.g., using Eq. (3.3) in Eq. (3.11)]

$$L=5y'^2-2z'^2-5w'^2+3e^{10y-4z}+e^{8y}(6e^{-2w}-e^{-12w})-\tfrac{1}{8}K_*^2e^{8z}, \tag{3.16}$$

or, for $w=0$ and $K_*=4$,

$$L=5y'^2-2z'^2+3e^{10y-4z}+5e^{8y}-2e^{8z}. \tag{3.17}$$

The meaning of the three terms in the potential is again the curvature of S^4, the curvature of $T^{1,1}$, and the negative cosmological term produced by the F_5 flux. Equivalently,

$$L=3n'^2-30m'^2+3e^{6n}+e^{8n}(5e^{-16m}-2e^{-40m}),$$

$$z=n-5m,\quad y=n-2m. \tag{3.18}$$

In general, this system does not admit a superpotential (wrapping the Euclidean three-brane world volume over S^4 breaks supersymmetry). The special easily solvable case is the fixed point $m=0$, i.e., $y=z$ [Eq. (3.13)] or the case of factorization $M_{10}\to M_5\times T^{1,1}$. Here we are left with just with one function y satisfying the zero-energy constraint [there is thus an obvious superpotential; cf. Eq. (3.14)]

$$y'^2=e^{6y}+e^{8y}, \tag{3.19}$$

so that

$$z=y=\ln\sinh\rho,\quad d\rho=-e^{4y}\,du, \tag{3.20}$$

where we again set $\rho_0=0$.[15] Then the metric becomes equal to Eq. (2.11), with the AdS_5 part written in the parametrization where the topology of the radial slices is S^4.

It is useful to stress again that the three $AdS_5\times T^{1,1}$ metrics (2.9), (2.10), and (2.11), although related locally by the coordinate transformations, are obtained from *inequivalent* 1D actions. This reflects the inequivalence of the corresponding radial coordinates, and leads also to very different properties of the corresponding fractional brane ($P\neq 0$) deformations of these backgrounds discussed below.

IV. STRATEGY OF FINDING $P\neq 0$ SOLUTION AND S^4 CASE

Being unable to solve the system of equations that follows from (3.4), (3.5) in general, we need to resort to perturbation theory similar to the one used in [14]. Our aim will be to show that starting from the asymptotic KT geometry at large ρ one may smoothly interpolate to a *regular* $AdS_5\times T^{1,1}$ geometry (with large enough effective charge $K_*\gg P^2$) at small ρ with the metric having a nontrivial scalar curvature of $\rho=\text{const}$ slices, i.e., Eq. (2.10) or Eq. (2.11).

Following the same strategy as used in [14] in the finite temperature case we shall start with the $AdS_5\times T^{*1,1}$ background (2.11) expected to be a good approximation in the small ρ region if $K_*=K(\rho\to 0)$ is sufficiently large, and solve the supergravity equations perturbatively to leading order in $P^2/K_*\ll 1$. We shall see that the leading deformation of the $AdS_5\times T^{1,1}$ background is *regular* at small ρ.

If one starts instead with the "flat" $AdS_5\times T^{1,1}$ metric (2.9), this perturbative expansion reproduces the exact form of the KT solution already at the first order of perturbation theory in P^2/K_* [note that the correction terms in Eqs. (3.9), (3.10) are linear in P^2]. Here, however, the perturbation (and the exact solution) is *singular* in the short-distance region [which in the case of Eq. (2.9) corresponds to $\rho\to-\infty$]. As was explained in [14], introducing nonextremality (i.e., replacing AdS_5 by the black hole background with sufficiently high temperature) resolves the singularity, making the perturbative solution regular. We shall see that a similar resolution is provided by the curvature of the "parallel" three-brane directions.

As was already mentioned above, to simplify the presentation we shall assume that the value of the five-form flux at $\rho\to 0$ is fixed as in Eq. (2.8), so that the radius of AdS_5 is 1 as in Eqs. (2.9)–(2.11). The expansion parameter is then simply P^2.

The full system of second order equations following from Eqs. (3.4),(3.5) in the $R\times S^3$ case is similar to the one in [14]:

$$x''-\tfrac{1}{2}e^{-2x-4z+10y}=0, \tag{4.1}$$

[13] Note that while for $q\neq 0$ or $y\neq z$ Eq. (3.12) does not admit a superpotential, it exists for Eq. (3.14) [cf. Eq. (3.6)]: $W=(3/4)[(1/2)e^{-2x+2y}+e^{4y}]$.

[14] Here $u=\ln\tanh\rho+\sinh^{-2}\rho$, so that $u(\rho\to\infty)\to 2e^{-2\rho}$ and $u(\to 0)\to 1/2\rho^2$.

[15] Here $u=\cosh\rho(1-2\sinh^2\rho)/3\sinh^3\rho+2/3$, so that $u(\rho\to 0)\to 1/3\rho^3$, $u(\rho\to\infty)\to 4e^{-4\rho}$.

$$10y'' - 8e^{8y}(6e^{-2w} - e^{-12w}) - 30x'' + \Phi'' = 0, \qquad (4.2)$$

$$10w'' - 12e^{8y}(e^{-2w} - e^{-12w}) - \Phi'' = 0, \qquad (4.3)$$

$$\Phi'' + e^{-\Phi+4z-4y-4w}\left(\frac{K'^2}{4P^2} - e^{2\Phi+8y+8w}P^2\right) = 0, \qquad (4.4)$$

$$4z'' - K^2 e^{8z} - e^{-\Phi+4z-4y-4w}\left(\frac{K'^2}{4P^2} + e^{2\Phi+8y+8w}P^2\right) - 12x'' = 0, \qquad (4.5)$$

$$(e^{(-\Phi+4z-4y-4w}K')' - 2P^2 K e^{8z} = 0. \qquad (4.6)$$

The integration constants are subject to the zero-energy constraint $T+V=0$. It is easy to see that because of the extra S^3-curvature term $e^{-2x-4z+10y}$ in the potential this system does not (in contrast to the nonextremal case [12]) admit a special solution with constant dilaton and self-dual three-forms.[16] In [13] we needed to relax this first order system to get a nonsingular nonextremal solution. Here we do not have a choice—all functions (in particular, w) are to be nontrivial in general.

In the S^4 case we get instead[17]

$$10y'' - 8e^{8y}(6e^{-2w} - e^{-12w}) - 30e^{10y-4z} + \Phi'' = 0, \qquad (4.7)$$

$$10w'' - 12e^{8y}(e^{-2w} - e^{-12w}) - \Phi'' = 0, \qquad (4.8)$$

$$\Phi'' + e^{-\Phi+4z-4y-4w}\left(\frac{K'^2}{4P^2} - e^{2\Phi+8y+8w}P^2\right) = 0, \qquad (4.9)$$

$$4z'' - K^2 e^{8z} - e^{-\Phi+4z-4y-4w}\left(\frac{K'^2}{4P^2} + e^{2\Phi+8y+8w}P^2\right) - 12e^{10y-4z} = 0, \qquad (4.10)$$

$$(e^{-\Phi+4z-4y-4w}K')' - 2P^2 K e^{8z} = 0, \qquad (4.11)$$

with the zero-energy constraint

$$5y'^2 - 2z'^2 - 5w'^2 - \frac{1}{8}\Phi'^2 - \frac{1}{4}e^{-\Phi+4z-4y-4w}\frac{K'^2}{4P^2} - 3e^{10y-4z} - e^{8y}(6e^{-2w} - e^{-12w}) + \frac{1}{4}e^{\Phi+4z+4y+4w}P^2 + \frac{1}{8}e^{8z}K^2 = 0. \qquad (4.12)$$

This system is simpler than in the $R\times S^3$ case, and in the remainder of this section we shall concentrate on its solution for the first $O(P^2)$ deformation away from the $AdS_5\times T^{1,1}$ metric (2.11).

A. Asymptotics of regular S^4 solution

Let us first discuss the expected behavior of the solution in the two asymptotic regions $\rho\to 0$ $(u\to\infty)$ and $\rho\to\infty$ $(u\to 0)$, i.e., in the short-distance and long-distance limits in 10D space. We would like the solution to have a regular short-distance limit which has the form (2.11) (up to possible constant rescalings)

$$\rho\to 0: \quad y\to \ln\rho + y_*, \quad z\to \ln\rho + z_*, \quad w\to w_*, \quad \Phi\to\Phi_*, \quad K\to K_*. \qquad (4.13)$$

At large distances $(\rho\to\infty)$ the solution is expected to approach the extremal KT solution (3.9),(3.10), i.e. [note that according to Eq. (3.20) $u(\rho\to\infty)\to 4e^{-4\rho}$],

$$\rho\to\infty: \quad w\to 0, \quad \Phi\to 0, \quad e^y\to \tfrac{1}{2}e^{\rho}, \quad K\to 2P^2\rho, \quad e^{-4z}\to 8P^2\rho e^{-4\rho}. \qquad (4.14)$$

To demonstrate the existence of a regular solution that interpolates between these two asymptotics we shall start with Eq. (2.11) which is valid for $P=0$, and find its deformation order by order in P^2. We shall see that (under a proper choice of integration constants) the leading $O(P^2)$ perturbations are *regular* at $\rho\to 0$, so that we indeed match onto the short-distance asymptotics (4.13). It turns out that the leading $O(P^2)$ correction is already enough to match onto the expected KT long-distance asymptotics (4.14).

Our ansatz for the leading perturbative solution that differs from Eq. (2.11) by the $O(P^2)$ terms will be

$$K = 4 + 2P^2 k(\rho), \quad \Phi = P^2\phi(\rho), \quad w = P^2 w(\rho), \qquad (4.15)$$

$$y = y_0(\rho) + P^2\xi(\rho), \quad z = y_0(\rho) + P^2\eta(\rho), \quad y_0(\rho)\equiv \ln\sinh\rho. \qquad (4.16)$$

We shall look for solutions for the perturbations k,ϕ,w,ξ,η which are regular at $\rho\to 0$,

$$\rho\to 0: \quad k,\phi,w,\xi,\eta\to \text{const}, \qquad (4.17)$$

in agreement with Eq. (4.13). We will find then that at large ρ the solution matches onto the KT asymptotics (4.14):

$$\rho\to\infty: \quad w,\phi,\xi\to 0, \quad k\to\rho, \quad \eta\to -\tfrac{1}{8}\rho. \qquad (4.18)$$

[16] If we set $K'^2 - 4P^3 e^{2\Phi+8y+8w} = 0$, i.e., $K' = -2P^2 e^{\Phi+4y+4w}$, then Eq. (4.6) implies that z should be subject to the first order equation in (3.8), but this is not consistent with Eq. (4.5) unless $x''=0$.

[17] This system is related to the $R\times S^3$ one by setting $e^{-2x}=2$ in Eqs. (4.1)–(4.6) after using Eq. (4.1) in Eqs. (4.2),(4.5).

B. Solution for $O(P^2)$ perturbations

Substituting Eq. (4.15) into the system (4.7)–(4.12) we get

$$10\xi''-320e^{8y_0}\xi-60e^{6y_0}(5\xi-2\eta)+\phi''+O(P^2)=0, \quad (4.19)$$

$$10w''-120e^{8y_0}w-\phi''+O(P^2)=0, \quad (4.20)$$

$$\phi''+k'^2-e^{8y_0}+O(P^2)=0, \quad (4.21)$$

$$4\eta''-128e^{8y_0}\eta-24e^{6y_0}(5\xi-2\eta)-(16k+1)e^{8y_0}-k'^2 +O(P^2)=0, \quad (4.22)$$

$$k''-4e^{8y_0}+O(P^2)=0, \quad (4.23)$$

$$2y_0'(5\xi'-2\eta')-\tfrac{1}{4}k'^2+e^{8y_0}[\tfrac{1}{4}+2k-8(5\xi-2\eta)] -6e^{6y_0}(5\xi-2\eta)+O(P^2)=0. \quad (4.24)$$

Here the prime stands for the derivative over u, with $du=-e^{-4y0}d\rho$ [see Eq. (3.20)]. Changing to the derivatives over ρ we finish with

$$k''+4y_0'k'-4=0,$$

$$k'\equiv\frac{dk}{d\rho}=-e^{-4y_0}\frac{dk}{du}, \quad y_0'=\coth\rho, \quad (4.25)$$

$$\phi''+4y_0'\phi'+k'^2-1=0, \quad (4.26)$$

$$w''+4y_0'w'-12w+\tfrac{1}{10}(k'^2-1)=0, \quad (4.27)$$

$$\xi''+4y_0'\xi'-32\xi-6e^{-2y_0}(5\xi-2\eta)-\tfrac{1}{10}(k'^2-1)=0, \quad (4.28)$$

$$\eta''+4y_0'\eta'-32\eta-6e^{-2y_0}(5\xi-2\eta)-\tfrac{1}{4}(k'^2+1+16k)=0, \quad (4.29)$$

$$y_0'(5\xi'-2\eta')-(3e^{-2y_0}+4)(5\xi-2\eta) -\tfrac{1}{8}(k'^2-1-8k)=0. \quad (4.30)$$

The deformation of the background is thus driven by the perturbation $k(\rho)$ of the effective three-brane charge K; solving for $k(\rho)$ first we then determine the source terms in the linear equations for the remaining perturbations. Equation (4.25) is readily solved:

$$k=-\frac{5}{6}+\rho\coth\rho\left(1-\frac{1}{2\sinh^2\rho}\right)+\frac{1}{2\sinh^2\rho}, \quad (4.31)$$

where we have fixed the two integration constants so as to satisfy the condition (4.17) of regularity at small ρ: $k(0)=0$. Indeed, $k(\rho\to 0)\to\frac{2}{5}\rho^2+O(\rho^4)$. We also get the expected matching onto the KT asymptotics (4.18): $k(\rho\to\infty)\to\rho$.

The solution for the dilaton perturbation (4.26) is then

$$\phi=\frac{13}{72}-\frac{1}{2\sinh^2\rho}+\frac{3\rho^2+2\rho\coth\rho-1}{8\sinh^4\rho} -\frac{\rho^2}{8\sinh^6\rho}, \quad (4.32)$$

where again we have fixed the integration constants so as to have the regularity at small ρ, $\phi(\rho\to 0)=\rho^2/10+O(\rho^4)$. At large ρ the dilaton perturbation exponentially approaches zero, in agreement with Eq. (4.18).

The three equations for the gravitational perturbations w, ξ, η have a similar structure (as was also the case in [14]). For w we get

$$w''+4\coth\rho\, w'-12w+q(\rho)=0, \quad (4.33)$$

$$q\equiv\frac{1}{10}(k'^2-1) =\frac{1}{10}\left[\frac{(12\rho-8\sinh 2\rho+\sinh 4\rho)^2}{640\sinh^8\rho}-1\right].$$

Note that the source term is regular at small ρ: $q(\rho\to 0)\to -\frac{1}{10}+\frac{8}{125}\rho^2+\cdots$, and $q(\rho\to\infty)\to\frac{12}{5}e^{-2\rho}+O(\rho e^{-4\rho})$. As a result, this equation has a regular solution near $\rho=0$: $w=w_*+(\frac{6}{5}w_*+\frac{1}{100})\rho^2+\cdots$.[18] It is easy to see (following the analysis in [14] or by numerical integration) that this regular short-distance asymptotics is smoothly connected to the long-distance asymptotics $w\to\frac{3}{20}e^{-2\rho}\to 0$.

The equations for ξ (4.28) and η (4.29) are coupled through the $5\xi-2\eta$ term, but the equation for this combination can be easily integrated. In fact, its solution can be found from the constraint (4.30):

$$\nu'+p_1(\rho)\nu+p_2(\rho)=0, \quad \nu\equiv 5\xi-2\eta, \quad (4.34)$$

$$p_1\equiv-\left(\frac{3}{\sinh^2\rho}+4\right)\tanh\rho,$$

$$p_2\equiv-\frac{1}{8}\tanh\rho(k'^2-1-8k).$$

This gives

$$\nu=-S(\rho)\int d\rho\, S^{-1}(\rho)p_2(\rho),$$

$$S\equiv e^{-\int d\rho\, p_1(\rho)}=\sinh^3\rho\cosh\rho. \quad (4.35)$$

The resulting expression for ν (given in terms of the dilogarithm function) is regular at small ρ, $\nu(\rho\to 0)=\frac{1}{8}\rho^2+ +O(\rho^3)$, while for large ρ we get $\nu\to\frac{1}{4}\rho$, in agreement with Eq. (4.18).

[18] Note that Eq. (4.33) can be put also in the following form (which is of the same type that appeared in [14]): $\tilde{w}''-2(6+\cosh 2\rho/\sinh^2\rho)\tilde{w}+\sinh^2\rho\, q(\rho)=0$, $w=e^{-2y_0}\tilde{w}=\sinh^{-2}\rho\,\tilde{w}$.

We are left with only one equation to solve − for ξ (or for η) [Eq. (4.33)],

$$\xi'' + 4\coth\rho\,\xi' - 32\xi + v(\rho) = 0, \tag{4.36}$$

$$v \equiv -\tanh\rho\left[\frac{6}{\sinh^2\rho}\nu + \frac{1}{10}(k'^2-1)\right].$$

Its analysis is the same as for Eq. (4.33). Since the source v here is again regular at $\rho\to 0$, $v = v_0 + O(\rho^2)$, $v_0 = -\frac{13}{20}$, the solution for ξ is also regular, $\xi = \xi_* + (\frac{16}{5}\xi_* - v_0/10)\rho^2 + O(\rho^4)$. As in the case of Eq. (4.33), we are also able to connect this to the required large ρ asymptotics (4.18), i.e., $\xi \sim e^{-2\rho} \to 0$.

We conclude that both the matter and the gravitational perturbations are *regular* at small ρ, and match onto the KT solution at large ρ.

It is instructive to see explicitly why replacing S^4 by R^4, i.e., going back to the original KT ansatz, gives a singular solution, i.e., why repeating the above perturbative analysis in the R^4 case leads to singular $O(P^2)$ corrections, even though the starting point—$AdS_5\times T^{1,1}$ space in Poincaré coordinates (2.9)—is nonsingular (see also [14]). Omitting the potential term associated with the curvature of S^4 in Eqs. (4.7),(4.29),(4.12) and using the ansatz (4.15),(4.16) with $y_0 = -\frac{1}{4}\ln(4u) = \rho$ [cf. Eqs. (3.9),(2.9)] we finish with the following system of equations that replaces Eqs. (4.25)–(4.30) ($y_0' = 1$):[19]

$$k'' + 4k' - 4 = 0, \quad \phi'' + 4\phi' + k'^2 - 1 = 0, \tag{4.37}$$

$$w'' + 4w' - 12w + \tfrac{1}{10}(k'^2-1) = 0,$$

$$\xi'' + 4\xi' - 32\xi - \tfrac{1}{10}(k'^2-1) = 0, \tag{4.38}$$

$$\eta'' + 4\eta' - 32\eta - \tfrac{1}{4}(k'^2 + 1 + 16k) = 0, \tag{4.39}$$

$$5\xi' - 2\eta' - 4(5\xi - 2\eta) - \tfrac{1}{8}(k'^2 - 1 - 8k) = 0. \tag{4.40}$$

Fixing the integration constants so as to achieve the maximal possible regularity of functions at $\rho = -\infty$ we get

$$k = \rho, \quad \phi = 0, \quad w = 0, \quad \eta = -\frac{1}{32} - \frac{1}{8}\rho, \quad \xi = 0. \tag{4.41}$$

This reproduces Eqs. (3.9),(3.10) [note that $e^{-4z} = e^{-4y_0}(1 + P^2\eta + \cdots)$,] and thus leads to a short-distance singularity at $\rho \to -\infty$. It is singular behavior of the "source function" k that translates into the singularity of the gravitational perturbation η. At the same time, in the nonextremal case in [14] and in the present S^4 case (4.31) (and in the $R\times S^3$ case discussed below) the function k has a *regular* short-distance limit.

V. $P\neq 0$ SOLUTION IN $R\times S^3$ CASE

The case of compactification on S^3, although technically more complicated, can be analyzed analogously to the S^4 case. We will construct a smooth supergravity renormalization group (RG) flow interpolating between a conformal compactification of the KW geometry at the origin, and the asymptotically KT geometry to the leading order in P^2. The full second order system is given by[20] Eqs. (4.1)–(4.6). The starting point for the deformation by the three-form fluxes is the $AdS_5\times T^{1,1}$ space in the global parametrization (2.10). In what follows we will use the radial coordinate t related to ρ in Eq. (2.10) as

$$t = \tanh^2\rho, \tag{5.1}$$

and to u in Eqs. (4.1)–(4.6) as

$$\frac{du}{dt} = \frac{e^{z-5y}}{2\sqrt{t}(1-t)}. \tag{5.2}$$

Here $t\to 0_+$ and $t\to 1_-$ are the short-distance and the long-distance limits of the 10D space, respectively.[21] Let us also introduce the functions

$$f_1 = e^{12x-4z}, \quad f_2 = t^2 e^{-4z-4x},$$
$$f_3 = e^{4y-16w-4z}, \quad f_4 = e^{4y+4w-4z}, \tag{5.3}$$

so that the deformed 10D metric (2.1) takes the form

$$ds^2_{10E} = f_1^{-1/2} dX_0^2 + t f_2^{-1/2}(dS^3)^2 + \frac{dt^2}{4t(1-t)^2} + f_3^{1/2}e_\psi^2 + f_4^{1/2}(e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2). \tag{5.4}$$

The reason for the redefinitions (5.2),(5.3) is that by using $f_i(t)$ it is easier to construct a solution perturbative in P^2 to Eqs. (4.1)–(4.6). For $P=0$ we recover the $AdS_5\times T^{1,1}$ space in the global parametrization (2.10):

$$f_1 = f_2 = (1-t^2), \quad f_3 = f_4 = 1, \tag{5.5}$$

with unit radius corresponding to the choice of $K=4$.

Our anzatz for a perturbative solution that differs from Eq. (5.5) by $O(P^2)$ terms will be similar to Eqs. (4.15), (4.16):

$$f_1(t) = (1-t)^2 + P^2\varphi_1(t),$$

[19] The derivative here is over ρ, which here takes values $-\infty < \rho < \infty$, with $\rho \to -\infty$ being the short-distance limit.

[20] The integration constants are subject to the zero-energy constraint as explained above.

[21] These are correspondingly the ir and the uv regimes of the holographically dual gauge theory.

$$f_2(t)=(1-t)^2+P^2\varphi_2(t), \tag{5.6}$$

$$f_3(t)=1+P^2\varphi_3(t),$$

$$f_4(t)=1+P^2\varphi_4(t),$$

$$K(t)=4+2P^2k(t), \quad \Phi(t)=P^2\phi(t).$$

From Eq. (5.4) it is clear that to avoid a singularity in the metric at $t\to 0_+$ we should have

$$\varphi_2(t)\to 0, \quad \varphi_{1,3,4}(t)\to \text{const}. \tag{5.7}$$

Also, to reproduce the $P=0$ values of the dilaton and of the regular D3-brane charge K at $t=0$ we shall assume that

$$\phi(t)\to 0, \quad k(t)\to 0. \tag{5.8}$$

At large distances ($t\to 1_-$) the solution is expected to approach the extremal KT solution (3.9),(3.10):

$$\varphi_1(t)\to 2k(t)e^{-4k(t)}, \quad \varphi_2(t)\to 2k(t)e^{-4k(t)}, \tag{5.9}$$

$$\varphi_3(t)\to \tfrac{1}{2}k(t), \quad \varphi_4(t)\to \tfrac{1}{2}k(t),$$

$$\phi(t)\to 0, \quad k(t)\to +\infty.$$

Notice that because $k(t\to 1_-)\to +\infty$, the perturbative expansion (5.6) necessarily breaks down there, so that, strictly speaking, we should not expect to reproduce the precise form of the KT asymptotics (5.9). This is indeed what we will find. We will recover asymptotically the warped product of the two factors $R\times S^3$ (with a finite S^3) and $T^{1,1}$, with the warp factors differing from the corresponding ones in the KT geometry by subleading logarithmic corrections. The same phenomenon was also observed in [14].

Now, changing the radial coordinate according to Eq. (5.2), performing the redefinitions (5.3) in Eqs. (4.1)–(4.6), and substituting the expansion (5.6) into the resulting system of equations, we obtain a coupled system of second order equations for $\varphi_{1,2,3,4}(t),\phi(t),k(t)$. In particular, for $k(t)$ we find

$$t(1-t)^2k''+(1-t)(2-t)k'-1=0. \tag{5.10}$$

The solution of Eq. (5.10) with the correct boundary conditions is

$$k(t)=-\tfrac{1}{2}\ln(1-t). \tag{5.11}$$

For the dilaton perturbation we find

$$t(t-1)^2\phi''+(1-t)(2-t)\phi'+\tfrac{1}{4}(t-1)=0, \tag{5.12}$$

and its appropriate solution is

$$\phi(t)=-\frac{1}{4t}[t\,\mathrm{Li}_2(t)+\ln(1-t)(1-t+\ln t)]. \tag{5.13}$$

Next, let us consider the equations for φ_3 and φ_4. Introducing

$$\varphi_{34}(t)\equiv\varphi_4-\varphi_3, \tag{5.14}$$

we obtain (using the already determined functions)

$$2t(1-t)^2\varphi_{34}''+2(1-t)(2-t)\varphi_{34}' -\tfrac{2}{3}\varphi_{34}+(t-1)=0. \tag{5.15}$$

The solution of Eq. (5.15) with the correct asymptotics is

$$\varphi_{34}(t)=\frac{t+2}{2(1-t)}[\mathrm{Li}_2(t)+\ln(1-t)\ln t] -\frac{5t+1}{4t}\ln(1-t)-\frac{3}{2}. \tag{5.16}$$

Substituting the already determined functions into the equation for φ_3 we find

$$(1-t)^4\left(\frac{t^2}{1-t}\varphi_3'\right)'-8t(1-t)\varphi_3 +\frac{1}{4}t(t^2-28t+27)-2t(t+2)\ln(1-t)\ln t +2(7t^2-6t-1)\ln(1-t)-4(t^2+2)\mathrm{Li}_2 t=0. \tag{5.17}$$

Although Eq. (5.17) looks complicated, the general solution can still be found:

$$\varphi_3(t)=\frac{1}{12t(1-t)^2}[I_1(t)+I_2(t)] +\frac{t^2+6t+3}{(1-t)^2}(\alpha_1+3\alpha_2\ln t) +\alpha_2\frac{51t^2+48t+1}{t(1-t)^2}, \tag{5.18}$$

where

$$I_1(t)=-t(t^2+6t+3)\int_0^t\frac{dx}{x(1-x)^6} \times[51x^2+48x+1+3x(x^2+6x+3)\ln x] \times[x^3-3x^2+27x+4(7x^2-6x-1)\ln(1-x) -8x(x+2)\ln(x)\ln(1-x)-24x^2\,\mathrm{Li}_2(x)], \tag{5.19}$$

and

$$I_2(t)=[51t^2+48t+1+3t(t^2+6t+3)\ln t] \times\int_0^t\frac{dx}{(1-x)^6}(x^2+6x+3) \times[x^3-28x^2+27x+4(7x^2+6x-1)\ln(1-x) -8x(x+2)\ln(x)\ln(1-x)-24x^2\,\mathrm{Li}_2(x)]. \tag{5.20}$$

Both integration constants α_1 and α_2 are uniquely fixed by the boundary conditions. For $t\to 0$ we find

$$\varphi_3(t)=\frac{\alpha_2}{t}+\alpha_2(50+9\ln t)+3\alpha_1+\alpha_2 O(t\ln t)+O(t). \tag{5.21}$$

From Eq. (5.21) we see that the analyticity of φ_3 at the origin requires $\alpha_2=0$. In the limit $t\to 1_-$ we get

$$\varphi_3(t)=\frac{10s}{(1-t)^2}-\frac{8s}{(1-t)}+s+\frac{1}{8}-\frac{1}{4}\ln(1-t)+O\big((1-t)\ln(1-t)\big), \tag{5.22}$$

where

$$s\equiv\alpha_1+\tfrac{1}{120}\,[I_1(t\to 1_-)+I_2(t\to 1_-)]. \tag{5.23}$$

It is straightforward to verify that the sum $I_1(t\to 1_-)+I_2(t\to 1_-)$ is actually finite.[22] From Eq. (5.22) we conclude that to get the KT asymptotic for φ_3 as given by Eq. (5.9) we have to tune $s=0$. Then Eq. (5.23) uniquely fixes the coefficient α_1.

We did not find the exact analytical solutions for φ_1,φ_2, but it is possible to show that the regularity at $t\to 0$ fixes all the integration constants but one. In general, one finds

$$\varphi_1(t)=\gamma(1-t)^2+\sum_{i=1}^{\infty} d_{1i}t^i,$$

$$\varphi_2(t)=\sum_{i=1}^{\infty} d_{2i}t^i, \tag{5.24}$$

where d_{1i} and d_{3i} are some (uniquely) determined coefficients and γ is an arbitrary integration constant. The presence of γ reflects the freedom of rescaling of the time coordinate X_0 in Eq. (5.4). This arbitrary constant has no effect on the uv $(t\to 1_-)$ asymptotic, where we find

$$\varphi_1(t)\to\tfrac{1}{16}(1-t)^2[\ln(1-t)]^2,$$

$$\varphi_2(t)\to\varphi_1(t). \tag{5.25}$$

Unlike the solution for the φ_3 perturbation,[23] which precisely reproduces the corresponding KT asymptotic, the precise form of the KT asymptotics for φ_1,φ_2 would be [see Eqs. (5.9),(5.11)]

$$\varphi_1(t)\to-(1-t)^2\ln(1-t),$$

$$\varphi_2(t)\to\varphi_1(t). \tag{5.26}$$

[22] Numerically, we find that $I_1(t\to 1_-)+I_2(t\to 1_-)\approx -7.753297$.

[23] Recall that this function determines the asymptotic warp factor of the $T^{1,1}$ space in the $R\times S^3$ compactification of the KT geometry.

The (subleading) difference between Eqs. (5.25) and (5.26) should not be surprising. Much like what happens in the nonextremal deformation of the KT solution [14], our perturbative expansion breaks down at $t\to 1_-$.

VI. CONCLUDING REMARKS

In this paper we have argued that naked bulk singularities of gravitational backgrounds dual to gauge theories can be resolved by introducing an analogue of an ir cutoff in gauge theories into the supergravity background. As a new explicit realization of this proposal we demonstrated the resolution of the singularity of fractional D3-branes on a conifold background by the compactification of the gauge-theory spacetime on $R\times S^3$ or S^4 with sufficiently large radius.

Unlike the original KT solution [5], the resulting supergravity backgrounds discussed here are nonsupersymmetric. This should not be too surprising, as our solutions are certain deformations of the KT background which had only $\mathcal{N}=1$ supersymmetry in four dimensions. An interesting question is whether one can preserve supersymmetry in the process compactification of gauge theories with reduced supersymmetry, and what would be their gravity duals.

A promising starting point to address this question is the so called $\mathcal{N}=2^*$ RG flow describing a mass-deformed $\mathcal{N}=4$ gauge theory. The corresponding supergravity solution was found by Pilch and Warner (PW) [20], and the realization of the gauge-gravity duality in this case was explained in detail [21,22]. It is straightforward to construct a linearized solution for the gravitational background dual to the mass-deformed $R\times S^3$ compactified $\mathcal{N}=4$ SYM theory. In fact, the solution (and its supersymmetries) are precisely the same as in the original PW construction. The physical explanation for this is that the linearized solution effectively probes the uv dynamics of the gauge theory, where the compactification is actually irrelevant.

A highly nontrivial question is whether one can find the full nonlinear (supersymmetric?) solution in this case. An intuitive reason for why this solution may exist is the following. As explained in [21], the PW flow is dual in the ir to a special vacuum point in the $\mathcal{N}=2^*$ moduli space. Neither the gauge theory nor gravity is able to explain what picks out this particular vacuum. The problem can be resolved if we assume the existence of an analogous RG flow for the compactified $\mathcal{N}=4$ SYM theory. Indeed, the adjoint scalar coupling to the curvature of S^3 would lift all of the moduli space, apart from an isolated point; the conjecture is that the $\mathcal{N}=2^*$ vacuum of the PW flow is precisely the one surviving under the S^3 compactification. Finally, there is an interesting enhancon phenomenon in the PW geometry. The size of the S^3 in the "compactified" flow produces a new mass scale in the geometry. One could imagine a phase transition originating from an interplay between the mass scale in the $\mathcal{N}=4$ deformation and the scale introduced by the S^3.

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