Kink variety in systems of two coupled scalar fields in two space-time dimensions

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In this paper we describe the moduli space of kinks in a class of systems of two coupled real scalar fields in $1+1$ Minkowskian space-time. The main feature of the class is the spontaneous breaking of a discrete symmetry of (real) Ginzburg-Landau type that guarantees the existence of kink topological defects.

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I. INTRODUCTION

Research into the mathematical properties and physical meaning of topological defects in relativistic field theory has increased sharply since the mid 1970s. There has also been a parallel development in (nonrelativistic) condensed matter physics. Extended states and phase transitions—e.g., type II superconductivity—are related to the appearance of such exotic phenomena. Domain wall defects in the real world can be thought of as solitary waves propagating in a $(1+1)$-dimensional universe that self-repeats in the remaining two dimensions. Thus, investigations on the kink nature and behavior in $\lambda(\phi^4)_2$ or sine-Gordon models inform us about the properties of the simplest type of topological defect. Realistic theories, however, involve more than one scalar field and the study of $(1+1)$-dimensional N-scalar fields models in this respect is not only worthwhile but almost mandatory. Examples of theories with $N>1$, where one might be interested in looking at topological defects, include the linear sigma model, the Ginzburg-Landau theory of phase transitions, the supersymmetric Wess-Zumino model, supersymmetric (SUSY) QCD, etc.

Kinks are time-independent finite-energy solutions of the field equations that have been thoroughly investigated in the $N=1$ case (see, e.g., [1]). Much less is known about the kink variety in systems with two or more scalar fields (the reason for this is also clearly explained in [1]). To the best of our knowledge, however, there are these exceptions.

A deformation of the linear $O(2)$-sigma model, known in the literature as the Montonen-Sarker-Trullinger-Bishop (MSTB) model, exhibits a rich variety of kinks. The characteristics of any of these kink defects as well as the structure of the variety as a whole have been elucidated in a long series of papers (see Refs. [2–12]). The moduli space of kinks in an analogous deformation of the linear $O(3)$-sigma model has also been fully described in [13].

The search for kinks is tantamount to the solving of a mechanical problem, which is seldom solvable if $N\geqslant 2$. In [14] we described the kinks of two $N=2$ field-theoretical models associated with completely integrable mechanical systems; i.e., the same idea that works in the MSTB model and its $N=3$ generalization.

In [15], the kinks of the Wess-Zumino model are shown to be given by certain real algebraic curves in the complex plane.

Another favorable situation occurs when the field-theoretical model is the bosonic sector of a supersymmetric system. This is the case of the Wess-Zumino system and also happens in an $N=2$ model proposed in [21], which has been discussed and applied to describe several interesting physical contexts in a series of papers [16–25]. Throughout their work, Bazeia and co-workers identify only two kinds of kink: a topological one, with only the first component non-null, usually termed the TK1 kink, and a second topological kink that has both components non-null and is called the TK2 kink. In contrast with the MSTB model, where the TK1 kinks are unstable [10,11], and decay to the TK2 kinks [12], in the system of Bazeia and co-workers there is an interesting phenomenon of kink degeneracy: the TK1 and TK2 kinks have the same classical energy.

The main result to be shown in this paper is that the kink degeneracy is a continuous one rather than the discrete degeneracy implicit in [16–25]. We shall find a continuous family of kink solutions to the classical field equations, all of them degenerate in energy with the TK1 and TK2 kinks. The existence of this variety of kinks is possible because of the spontaneous breaking of a discrete internal symmetry group. The quotient of the kink variety by the symmetry group is the kink moduli space, a structure parallel to the moduli spaces of gauge theoretical topological defects such as vortices [26] or magnetic monopoles [27].

Identification of the kink variety is achieved through the solution of first-order, rather than second-order, field equations. In $(1+1)$-dimensional scalar field theories, first-order equations are available if, modulo a global sign, a superpotential is found. Note that the search for a superpotential is highly nontrivial if $N\geqslant 2$. Bazeia and co-workers, however, proposed a continuously differentiable superpotential in their model, which in turn guarantees the stability of any finite-energy solution of the associated first-order system of equations through the classical Bogomolńyi-Prasad-Sommerfield argument [28].

The existence of the superpotential tells us that we can understand the system as the bosonic sector of an $\mathcal{N}=1$ $(1+1)$-dimensional supersymmetric field theory, in which the kinks play a significant role as Bogomolńyi-Prasad-Sommerfield (BPS) states. We shall analyze the supersymmetric extension of this model in a future work, but we observe that the dimension of the kink moduli space in this system is such that the index introduced in [29] is zero, showing that the soliton supermultiplets are long or reducible.

All the foregoing statements are valid for any value of the single classically relevant coupling constant in the model. In this paper we shall show another new result: for certain values of the coupling constant there exists a second superpotential. Accordingly, a second system of first-order equations is available that also admits kink solutions, although the old and new solitons belong to different topological sectors of the configuration space. For the critical values where the second superpotential is found, there are two nonequivalent supersymmetric extensions of the same bosonic sector.

For most of the critical values the second superpotential fails to be continuously differentiable at a finite number of points in the $\mathbb{R}^2$ internal space. In these cases, the second Bogomolńyi bound is not a topological quantity; it also depends on the values of the superpotential at the points where it is not differentiable. Kink orbits that cross those points are unstable and are solutions of the first-order equations only in one interval, not on the whole spatial line. Nevertheless, these kinks are solutions of the second-order equations.

A final comment: in concordance with the lifting of the kink translational degeneracy, we expect that the kink internal degeneracy will be removed in second order in the loop expansion of the energy in the quantum theory.

The paper is organized as follows. In Secs. II and III we introduce the Bazeia-Nascimento-Ribeiro-Toledo (BNRT) model discussed in [21] and identify a one-parametric family of kinks, which includes the TK1 and TK2 kinks, as BPS solutions. In Secs. IV and V we investigate the existence of a second decomposition in the manner of Bogomol'nyi. We find that this is possible for certain values of the coupling constant, for which we discover a second kink family.

II. THE BNRT MODEL

In the model introduced in [21] by Bazeia, Nascimento, Ribeiro, and Toledo, the scalar field is built from two components $\chi(y^\mu)=(\chi_1(y^\mu),\chi_2(y^\mu))$ and the dynamics is governed by the action

$$\bar{S}[\chi]=\int d^2y\left[\sum_{a=1}^{2}\partial_\mu\chi_a\partial^\mu\chi_a-\bar{U}(\chi_1,\chi_2)\right], \tag{1}$$

$$\bar{U}(\chi_1,\chi_2)=\frac{1}{2}\lambda^2(\chi_1^2-a^2)^2+\frac{1}{2}\lambda\mu(\chi_1^2-a^2)\chi_2^2+\frac{1}{8}\mu^2\chi_2^4+\frac{1}{2}\mu^2\chi_1^2\chi_2^2. \tag{2}$$

Here, λ and μ are coupling constants with dimensions of inverse length and a^2 is a non-dimensional parameter. We use a natural system of units, $\hbar=c=1$. The energy functional is

$$\bar{\mathcal{E}}[\chi]=\int dy\left[\frac{1}{2}\left(\frac{d\chi_1}{dy}\right)^2+\frac{1}{2}\left(\frac{d\chi_2}{dy}\right)^2+\bar{U}(\chi_1,\chi_2)\right] \tag{3}$$

where $\chi(y)=(\chi_1(y),\chi_2(y))\in\mathcal{C}=\{\text{Maps}(\mathbb{R},\mathbb{R}^2)/\bar{\mathcal{E}}[\chi(y)]<\infty\}$. Introducing nondimensional fields, variables, and parameters $\chi_b=2a\phi_b$, $y=(2\sqrt{2}/a\lambda)x$, and $\sigma=\mu/\lambda$, we obtain expressions that are simpler to handle. $\bar{\mathcal{E}}[\chi_1,\chi_2]=\sqrt{2}a^3\lambda\mathcal{E}[\phi_1,\phi_2]$ and the nondimensional energy functional—which depends on the single classically relevant coupling constant σ—is

$$\mathcal{E}[\phi]=\int dx\left[\frac{1}{2}\left(\frac{d\phi_1}{dx}\right)^2+\frac{1}{2}\left(\frac{d\phi_2}{dx}\right)^2+(4\phi_1^2+2\sigma\phi_2^2-1)^2+16\sigma^2\phi_1^2\phi_2^2\right]. \tag{4}$$

The Euler-Lagrange equations read

$$\frac{d^2\phi_1}{dx^2}=16\phi_1[4\phi_1^2+2\sigma(1+\sigma)\phi_2^2-1],$$

$$\frac{d^2\phi_2}{dx^2}=8\sigma\phi_2[4(\sigma+1)\phi_1^2+2\sigma\phi_2^2-1]. \tag{5}$$

In addition to the spatial parity and translational symmetries, there is a global or internal symmetry in this model: the reflection discrete group $\mathbb{G}=\mathbb{Z}_2\times\mathbb{Z}_2$ generated by the transformations $\pi_1:(\phi_1,\phi_2)\to(-\phi_1,\phi_2)$ and $\pi_2:(\phi_1,\phi_2)\to(\phi_1,-\phi_2)$ is also a symmetry subgroup of the system.

We shall focus our attention on the $\sigma>0$ regime, where the vacuum manifold is

$$\mathcal{M}=\left\{A_1=\left(\frac{1}{2},0\right);A_2=\left(-\frac{1}{2},0\right);B_1=\left(0,\frac{1}{\sqrt{2\sigma}}\right);B_2=\left(0,-\frac{1}{\sqrt{2\sigma}}\right)\right\}.$$

The action of $\mathbb{G}$ on $\mathcal{M}$ is summarized as follows: $\pi_1(A_1)=A_2$, $\pi_2(B_1)=B_2$. Therefore, $\mathcal{M}$ can be seen as the union of two disjoint vacuum orbits: $\mathcal{M}=A\sqcup B$, $A=\{A_1,A_2\}$, $B=\{B_1,B_2\}$. The vacuum moduli space $\bar{\mathcal{M}}=\mathcal{M}/\mathbb{G}$ is a set of two elements $\bar{\mathcal{M}}=\mathbf{A}\sqcup\mathbf{B}$, where $\mathbf{A}=A/(\mathbb{Z}_2\times\{e\})$ and $\mathbf{B}=B/(\{e\}\times\mathbb{Z}_2)$. The $\mathbb{G}=\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry of the action (1) is spontaneously broken to the $\{e\}\times\mathbb{Z}_2$ subgroup on the elements in the A orbit and to the $\mathbb{Z}_2\times\{e\}$ subgroup on the vacua of the B orbit.

Because of the degeneracy and the discreteness of the vacuum manifold $\mathcal{M}$, the configuration space is the union of 16 topologically disconnected sectors. Keeping in mind the symmetries of the model, we identify the nontrivial topological sectors as the AA topological sector (formed by configurations of $\mathcal{C}$ that join the A_1 and A_2 vacua); the BB topological sector (configurations that connect the B_1 and B_2 vacua); and the AB sector (formed by configurations joining one vacuum in the A orbit with another vacuum in the B orbit).

We use the trial orbit method [1] to show the previously known kink solutions to Eqs. (5).

A. The TK1^{AA} kink

First, we try the curve

$$\gamma_{\mathrm{TK1}^{AA}}=\left\{\phi_2=0,-\frac{1}{2}\leqslant\phi_1\leqslant\frac{1}{2}\right\}.$$

This condition is compatible with Eqs. (5) and we find

$$\phi_1^{\mathrm{TK1}^{AA}}(x)=\pm\frac{1}{2}\tanh 2\sqrt{2}(x+a),\quad \phi_2^{\mathrm{TK1}^{AA}}(x)=0$$

as the one-component topological kinks in AA.

B. The TK2^{AA} kink

Second, we try the elliptic orbit

$$\gamma_{\mathrm{TK2}^{AA}}=\left\{\phi_1^2+\frac{\sigma}{2(1-\sigma)}\phi_2^2=\frac{1}{4},-\frac{1}{2}\leqslant\phi_1\leqslant\frac{1}{2}\right\}\tag{6}$$

in Eqs. (5) and find in the AA topological sector the two-component topological kinks

$$\phi_1^{\mathrm{TK2}^{AA}}(x)=\pm\frac{1}{2}\tanh 2\sqrt{2}\sigma(x+a),$$

$$\phi_2^{\mathrm{TK2}^{AA}}(x)=\pm\sqrt{\frac{1-\sigma}{2\sigma}}\mathrm{sech}\, 2\sqrt{2}\sigma(x+a),\tag{7}$$

henceforth referred to as TK2^{AA} kinks.

Note that the orbit (6) gives kink curves only in the $\sigma\in(0,1)$ range because if $\sigma\geqslant 1$ it becomes a hyperbola that does not connect the vacua. Moreover, Eq. (7) describes four different kinks according to the choices of the signs and one can obtain one from another by using the spatial parity and internal reflection symmetries.

The existence of one-component topological kinks—unnoticed in the literature about the model—in the BB topological sectors is obvious.

C. The TK1^{BB} kink

Third, we try the orbit

$$\gamma_{\mathrm{TK1}^{BB}}=\left\{\phi_1=0,-\frac{1}{\sqrt{2\sigma}}\leqslant\phi_2\leqslant\frac{1}{\sqrt{2\sigma}}\right\}$$

in the second-order field equations (5). We immediately find that the finite-energy solutions

$$\phi_1^{\mathrm{TK1}^{BB}}(x)=0,$$

$$\phi_2^{\mathrm{TK1}^{BB}}(x)=\pm\frac{1}{\sqrt{2\sigma}}\tanh 2\sqrt{\sigma}(x+a)$$

are the kinks that connect the B_1 and B_2 vacua.

III. THE MODULI SPACE OF KINKS IN THE *AA* TOPOLOGICAL SECTOR

In [16–25] the authors propose a superpotential for the model:

$$U(\phi_1,\phi_2)=\frac{1}{2}\left(\frac{\partial W}{\partial\phi_1}\right)^2+\frac{1}{2}\left(\frac{\partial W}{\partial\phi_2}\right)^2,$$

$$W(\phi)=4\sqrt{2}\left(\frac{1}{3}\phi_1^3-\frac{1}{4}\phi_1+\frac{\sigma}{2}\phi_1\phi_2^2\right).\tag{8}$$

The classical BPS states satisfy the system of first-order equations

$$\frac{d\phi_1}{dx}=\frac{\partial W}{\partial\phi_1}=\sqrt{2}(4\phi_1^2+2\sigma\phi_2^2-1),$$

$$\frac{d\phi_2}{dx}=\frac{\partial W}{\partial\phi_2}=4\sqrt{2}\sigma\phi_1\phi_2,\tag{9}$$

which are easier to solve than Eqs. (5). The superpotential $W(\phi_1,\phi_2)$ is a smooth function of the fields ϕ_1 and ϕ_2 at each point in $\mathbb{R}^2$. Therefore, according to the Bogomol'nyi arrangement

$$\mathcal{E}[\phi]=\int dx\sum_{a=1}^{2}\left(\frac{d\phi^a}{dx}-\frac{\partial W}{\partial\phi^a}\right)^2+\int\frac{\partial W}{\partial\phi^a}\frac{d\phi^a}{dx},$$

we have that

$$\mathcal{E}[\phi]=T[\phi]$$
$$=|W(\phi_1(\infty),\phi_2(\infty))-W(\phi_1(-\infty),\phi_2(-\infty))|$$

for all solutions of Eqs. (9) and the kink energy depends only on the topological sector of the solution.

The kink solutions of Eqs. (9) are the flow lines of grad W that start and end at elements of $\mathcal{M}$. It happens that A_1 and A_2 are, respectively, maxima and minima of W and that there are flow lines of grad W starting at A_1 and ending at A_2 (or vice versa). B_1 and B_2, however, are saddle points of W (see Fig. 1). Therefore, there are no flow lines of grad W between B_1 and B_2 (or vice versa). Nevertheless, flow lines of grad W between one point in the A orbit and another point in the B orbit (or vice versa) are possible. The flow lines of grad W thus provide kinks in the AA and the AB sectors with energies $E_{\mathrm{TK2}^{AA}}=\frac{4}{3}a^3\lambda$, $E_{\mathrm{TK2}^{AB}}=\frac{2}{3}a^3\lambda$.

To obtain the most general solution to the first-order system (9), we first integrate the first-order ordinary differential equation

$$\frac{d\phi_1}{d\phi_2}=\frac{4\phi_1^2+2\sigma\phi_2^2-1}{4\sigma\phi_1\phi_2}\tag{10}$$

which admits the integrating factor $|\phi_2|^{-2/\sigma}\phi_2^{-1}$ if $\sigma\neq 1$ and $\sigma\neq 0$, thereby allowing us to find all the flow lines as the family of curves

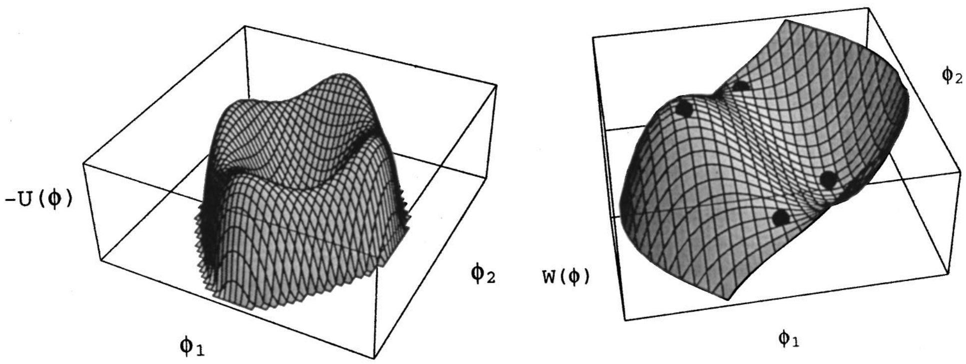


FIG. 1. The $U(\phi)$ potential (left) and the superpotential $W(\phi)$ (right).

$$\phi_1^2+\frac{\sigma}{2(1-\sigma)}\phi_2^2=\frac{1}{4}+\frac{c}{2\sigma}|\phi_2|^{2/\sigma} \tag{11}$$

parametrized by the real integration constant c. There is a critical value

$$c^S=\frac{1}{4}\frac{\sigma}{1-\sigma}(2\sigma)^{(\sigma+1)/\sigma}$$

and the behavior of a particular curve in the Eq. (11) family is described in the following items.

For $c\in(-\infty,c^S)$, formula (11) describes closed curves in the internal space $\mathbb{R}^2$ that connect the vacua A_1 and A_2 (see Fig. 2). Thus, they provide a kink family in the topological sector AA. Henceforth, we refer to these kinks as $\mathbf{TK2}^{AA}(c)$. A fixed value of c determines four members in the kink variety related among one another by spatial parity and internal reflections. The kink moduli space is defined as the quotient of the kink variety by the action of the symmetry group:

$$\mathcal{M}_{\rm K}=\frac{\mathcal{V}_{\rm K}}{\mathbb{P}\times\mathbb{G}}=(-\infty,c^S),$$

the real open half line parametrized by c. One sees that

$$\mathbf{TK2}^{AA}\equiv\mathbf{TK2}^{AA}(0),\mathbf{TK1}\equiv\lim_{c\to-\infty}\mathbf{TK2}^{AA}(c),$$

i.e., the $\mathrm{TK2}^{AA}$ kink is the $c=0$ member of the family (if $\sigma<1$) and the $\mathrm{TK1}^{AA}$ kink is not strictly included although it does appear at the boundary of $\mathcal{M}_{\rm K}$.

In the range $c\in(c^S,\infty)$, Eq. (11) describes open curves and no vacua are connected. These grad W flow lines are infinite-energy solutions that do not belong to the configuration space $\mathcal{C}$ (see Fig. 2).

At the other point of the boundary of $\mathcal{M}_{\rm K}$, $c=c^S$, we find the $\mathrm{TK2}^{AB}$ kinks, which are the separatrices between bounded and unbounded motion and the envelope of all kink orbits in the AA topological sector (see Fig. 2).

We briefly discuss the $\sigma=1$ case. The $\sigma=0$ case is not interesting because the ϕ_2 dependence disappears in the potential: it is a "direct sum" of an $N=1$ ϕ^4 model and an $N=1$ free model. Integration of Eq. (10) when $\sigma=1$ gives

$$\phi_1^2-\phi_2^2\left(\frac{c}{2}+\log|\phi_2|\right)=\frac{1}{4} \tag{12}$$

where the kink trajectories now appear in the $c\in(-\infty,c^S]$ range, with $c^S=-1+\ln 2$. The description of the kink orbits is analogous to the description for $\sigma\neq1$ above.

A second step remains: the explicit dependence of the kinks with respect to the space coordinate can be obtained if we insert Eq. (11) into the second equation in Eq. (9),

$$h[\phi_2]=\int\frac{d\phi_2}{\phi_2\sqrt{\frac{1}{4}+(c/2\sigma)|\phi_2|^{2/\sigma}-[\sigma/2(1-\sigma)]\phi_2^2}}$$
$$=\int 4\sqrt{2}\sigma dx. \tag{13}$$

The kink solutions are

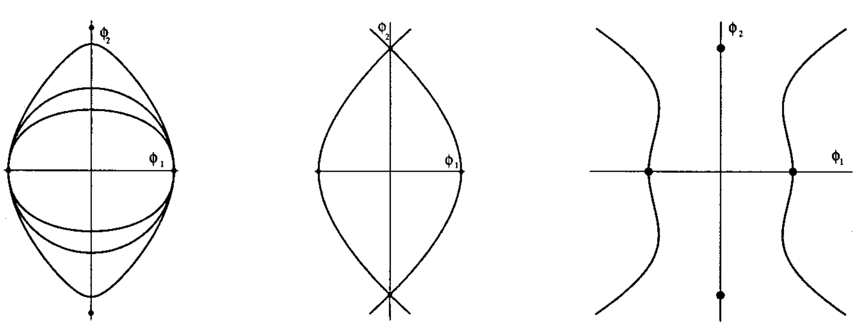


FIG. 2. Flow lines given by Eq. (11): for $c\in(-\infty,c^S)$ (left), $c=c^S$ (middle), and $c\in(c^S,\infty)$ (right).

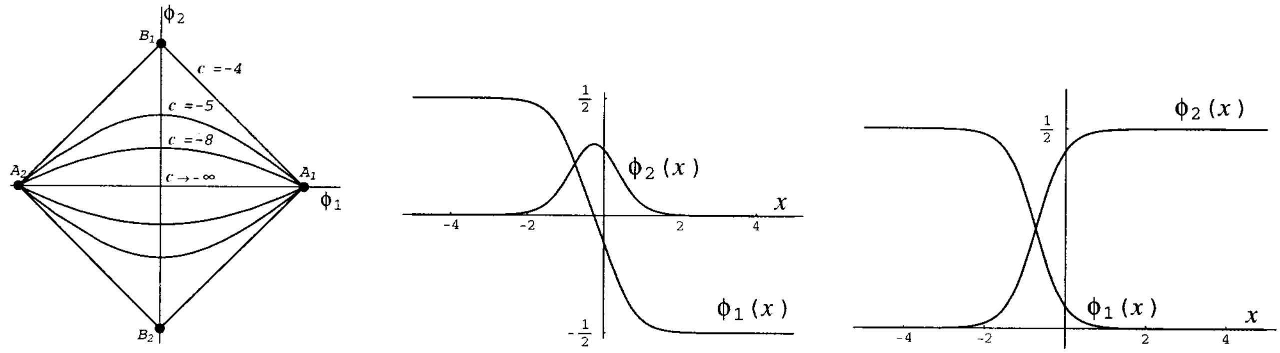


FIG. 3. Kink trajectories (left), a kink in the *AA* sector (middle), and a kink in the *AB* sector (right) in the case $\sigma=2$.

$$\phi_1^{\mathrm{K}}(x,c)=\pm\sqrt{\frac{1}{4}+\frac{c}{2\sigma}|h^{-1}(4\sqrt{2}\sigma x)|^{2/\sigma}-\frac{\sigma}{2(1-\sigma)}[h^{-1}(4\sqrt{2}\sigma x)]^2},\quad \phi_2^{\mathrm{K}}(x,c)=h^{-1}(4\sqrt{2}\sigma x).$$

In general, we cannot obtain the explicit dependence on x for the kink solutions because either we cannot integrate Eq. (13) or we cannot identify the inverse of $h(\phi)$. For certain values of the coupling constant, however, we can finish the task. We next show the family of $\mathrm{TK2}^{AA}$ kinks for $\sigma=2$ and $\sigma=\frac{1}{2}$.

$\sigma=2$. The vacuum points are the vertices of a square: $\mathcal{M}_{\sigma=2}=\{A_1=(\frac{1}{2},0),A_2=(-\frac{1}{2},0),B_1=(0,\frac{1}{2}),B_2=(0,-\frac{1}{2})\}$. The quadratures (13) can be solved explicitly and $h^{-1}[\phi_2]$ is a known analytical function. Thus,

$$\phi_1^{\mathrm{TK2}^{AA}}(x)=\pm\frac{1}{2}\,\frac{\sinh 4\sqrt{2}(x+a)}{\cosh 4\sqrt{2}(x+a)+b},$$

$$\phi_2^{\mathrm{TK2}^{AA}}(x)=\pm\frac{1}{2}\,\frac{\sqrt{b^2-1}}{\cosh 4\sqrt{2}(x+a)+b}$$

are the kink form factors. The integration constant b is related to c as $b=-c/\sqrt{c^2-16}$, and for $b\in(1,\infty)$ we find kinks in the *AA* topological sector.

If $c=c^S=-4$, $b=\infty$ we find the kinks in the *AB* sector

$$\phi_1^{\mathrm{TK2}^{AB}}(x)=\pm\tfrac{1}{4}[1-\tanh 2\sqrt{2}(x+a)],$$

$$\phi_2^{\mathrm{TK2}^{AB}}=\pm\tfrac{1}{4}[1+\tanh 2\sqrt{2}(x+a)],$$

and, replacing x by $-x$, its antikinks. The separatrices are placed on the edges of the above mentioned square $\phi_2=\pm\frac{1}{2}\pm\phi_1$. The kink trajectories in the *AA* topological sector form a dense family of curves enveloped by the kink orbits in the *AB* sector. See Fig. 3.

A rotation of 45° in $\mathbb{R}^2$, $\phi_1=(1/\sqrt{2})(\psi_1+\psi_2)$ and $\phi_2=(1/\sqrt{2})(\psi_1-\psi_2)$, shows that for this value of σ the system is noncoupled: $U_{\sigma=2}(\psi_1,\psi_2)=\frac{1}{32}(\psi_1^2-\frac{1}{8})^2+\frac{1}{2}(\psi_2^2-\frac{1}{8})^2$.

$\sigma=\frac{1}{2}$. The vacuum manifold is $\mathcal{M}_{\sigma=1/2}=\{A_1=(\frac{1}{2},0),A_2=(-\frac{1}{2},0),B_1=(0,1),B_2=(0,-1)\}$. By the same procedure as above, we obtain

$$\phi_1^{\mathrm{TK2}^{AA}}(x)=\pm\frac{1}{2}\,\frac{\sinh 2\sqrt{2}(x+a)}{\cosh 2\sqrt{2}(x+a)+b},$$

$$\phi_2^{\mathrm{TK2}^{AA}}(x)=\pm\frac{1}{\sqrt{1+b^{-1}\cosh 2\sqrt{2}(x+a)}},\qquad (14)$$

where we have introduced $b=1/\sqrt{1-4c}$. In the $b\in(0,\infty)$ range, the above solutions are kinks that connect the A_1 and

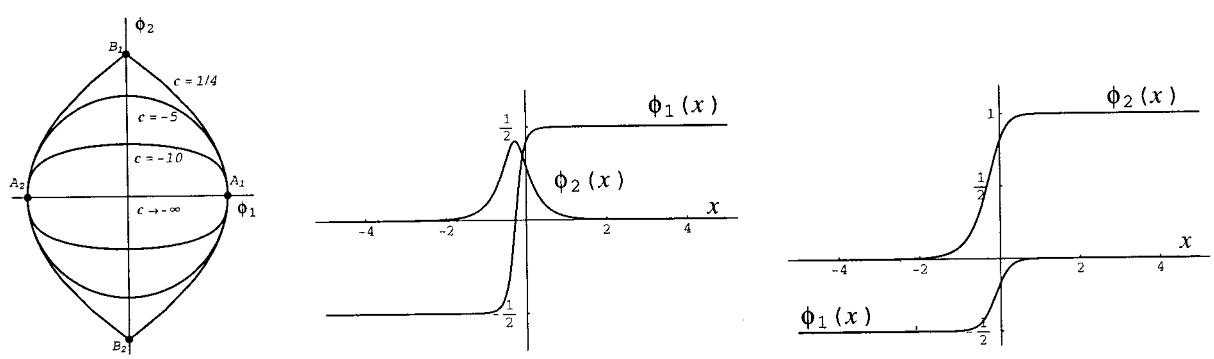


FIG. 4. Kink curves (left), a kink in the *AA* sector (middle), and a kink in the *AB* sector (right).

A_2 vacua (see Fig. 4). If $\sigma=\frac{1}{2}$, Eq. (11) becomes $\phi_1^2+\frac{1}{2}\phi_2^2=\frac{1}{4}+c\phi_2^4$, which can be written as $(1+2\phi_1-\phi_2^2)(1-2\phi_1-\phi_2^2)=0$ for $c=C^S=\frac{1}{4}$. There are kinks on parabolic trajectories joining points in the A and B vacuum orbits,

$$\phi_1^{\mathrm{TK2}^{AB}}(x)=\pm\tfrac{1}{4}[1-\tanh\sqrt{2}(x+a)],$$

$$\phi_2^{\mathrm{TK2}^{AB}}(x)=\pm\sqrt{\tfrac{1}{2}[1+\tanh\sqrt{2}(x+a)]},$$

and, replacing $x+a$ by $-x-a$, we obtain their antikinks.

IV. THE SECOND SUPERPOTENTIAL: $\sigma=2$

For $\sigma=2$, $U(\phi)=(4\phi_1^2+4\phi_2^2-1)^2+64\phi_1^2\phi_2^2$ does not change if we exchange the field components. There is a second superpotential in the model for $\sigma=2$: $W'(\phi_1,\phi_2)=W(\phi_2,\phi_1)$. A second arrangement in the manner of Bogomolńyi using $W'(\phi_1,\phi_2)$ provides another system of first-order differential equations:

$$\frac{d\phi_1}{dx}=\frac{\partial W'}{\partial\phi_1}=8\sqrt{2}\phi_1\phi_2,$$

$$\frac{d\phi_2}{dx}=\frac{\partial W'}{\partial\phi_2}=\sqrt{2}(4\phi_1^2+4\phi_2^2-1).$$

The flow lines of $\mathrm{grad}\,W'$ connect B_1 and B_2, which are, respectively, the maximum and the minimum of W', whereas A_1 and A_2 are W' saddle points. We thus obtain a new family of topological kinks, now in the BB sector, with the roles of ϕ_1 and ϕ_2 interchanged: if $b\in(1,\infty)$,

$$\phi_1^{\mathrm{TK2}^{BB}}(x)=\pm\frac{1}{2}\frac{\sqrt{b^2-1}}{\cosh 4\sqrt{2}(x+a)+b},$$

$$\phi_2^{\mathrm{TK2}^{BB}}(x)=\pm\frac{1}{2}\frac{\sinh 4\sqrt{2}(x+a)}{\cosh 4\sqrt{2}(x+a)+b}$$

are the two-component topological kinks in the BB sector. If $c\to-\infty(b\to1)$ we find the $\mathrm{TK1}^{BB}$ kink, and if $c=4(b\to\infty)$ the separatrix kinks in the AB sector are reached at the boundary of the component of the moduli space of kinks that belong to the BB sector. The kink energy sum rules are $E_{\mathrm{TK2}^{AA}}=E_{\mathrm{TK2}^{BB}}=2E_{\mathrm{TK2}^{AB}}=\frac{4}{3}a^3\lambda$.

V. THE MODULI SPACE OF NON-BPS KINKS IN THE BB TOPOLOGICAL SECTOR: $\sigma=1/2$

If $\sigma=\frac{1}{2}$, there is also a second superpotential

$$W'(\phi_1,\phi_2)=\frac{\sqrt{2}}{3}\sqrt{\phi_1^2+\phi_2^2}(4\phi_1^2+\phi_2^2-3) \tag{15}$$

that also solves the first equation in Eq. (8). The second system of first-order equations

$$\frac{d\phi_1}{dx}=\pm\frac{\partial W'}{\partial\phi_1}=\pm\frac{\sqrt{2}\phi_1(4\phi_1^2+3\phi_2^2-1)}{\sqrt{\phi_1^2+\phi_2^2}},$$

$$\frac{d\phi_2}{dx}=\pm\frac{\partial W'}{\partial\phi_2}=\pm\frac{\sqrt{2}\phi_2(2\phi_1^2+\phi_2^2-1)}{\sqrt{\phi_1^2+\phi_2^2}} \tag{16}$$

rules the flows generated by $\pm\mathrm{grad}\,W'$ in the system. W' is not differentiable at the origin and the flows of $\pm\mathrm{grad}\,W'$

$$\frac{d\phi_2}{d\phi_1}=\frac{\phi_2(2\phi_1^2+\phi_2^2-1)}{\phi_1(4\phi_1^2+3\phi_2^2+1)} \tag{17}$$

are undefined at $O\equiv(0,0)\in\mathbb{R}^2$. Note that B_1 and B_2 are both minima of W', whereas A_1 and A_2 are W' saddle points. The origin is the maximum of W' and thus the flow lines of $\mathrm{grad}\,W'$ run from O to either B_1 or B_2. To obtain a kink orbit, we must glue at O a γ_- flow line of $\mathrm{grad}\,W'$ smoothly with a γ_+ flow-line of $-\mathrm{grad}\,W'$. Because the flows are undefined at O, we expect that an infinite number of lines will meet at the origin.

The Bogomolńyi splitting must take this into account and the energy of the kink solutions of Eqs. (16),

$$\mathcal{E}[\phi]=\int_{-a}^{\infty}dx\frac{1}{2}\left\|\frac{d\phi}{dx}-\frac{\partial W'}{\partial\phi}\right\|^2+\int_{-\infty}^{-a}dx\frac{1}{2}\left\|\frac{d\phi}{dx}-\frac{\partial W'}{\partial\phi}\right\|^2+T(\gamma_+)+T(\gamma_-),$$

$$T=T(\gamma_+)+T(\gamma_-)$$

$$=|W'(B_1)-W'(O)|+|W'(B_2)-W'(O)|.$$

$\mathcal{E}[\phi_{\mathrm{TK2}^{BB}}]=T(\gamma_+)+T(\gamma_-)$ is not topological; it depends on the value of the superpotential at the origin, a sign of instability [10,11]. The kink energy sum rules are $E_{\mathrm{TK2}^{BB}}=2E_{\mathrm{TK2}^{AA}}=4E_{\mathrm{TK2}^{AB}}=\frac{8}{3}a^3\lambda$ and the $\mathrm{TK2}^{BB}$ kinks decay to two $\mathrm{TK2}^{AB}$ plus one $\mathrm{TK2}^{AA}$ kinks.

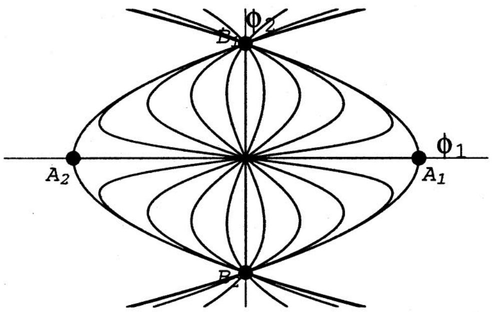
Using parabolic variables, we have shown that the integration of Eq. (16) reduces to quadratures in Ref. [14]. The translation of our results to Cartesian coordinates is as follows.

The kink orbits that solve Eq. (17) satisfy the equation

$$16e^{4\sqrt{2}c}\phi_1^2(\phi_1^2+\phi_2^2)+(1-e^{4\sqrt{2}c})^2\phi_2^4(2\phi_1-\phi_2^2+1)\times(2\phi_1+\phi_2^2-1)=0 \tag{18}$$

and are plotted in Fig. 5. Here c is a real integration constant.

Analytically, the variety of $\mathrm{TK2}^{BB}(c)$ kinks is given by



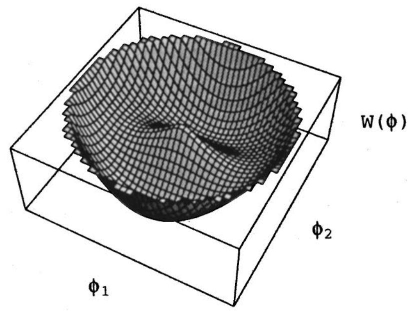


FIG. 5. $\mathrm{TK2}^{BB}(c)$ Kink family (left) and the superpotential W' (right).

$$\phi_1^{\mathrm{TK2}^{BB}}(x)=\frac{\sinh 2\sqrt{2}c\,\sinh 2\sqrt{2}(x+a)}{\cosh^2 2\sqrt{2}(x+a)+2\cosh 2\sqrt{2}c\,\cosh 2\sqrt{2}(x+a)+1},\tag{19}$$

$$\phi_2^{\mathrm{TK2}^{BB}}(x)=\frac{-\sinh 2\sqrt{2}(x+a)}{\sqrt{\cosh^2 2\sqrt{2}(x+a)+2\cosh 2\sqrt{2}c\,\cosh 2\sqrt{2}(x+a)+1}}.\tag{20}$$

In addition to the soliton center $x=-a$, the kink family is parametrized by c.

Because the spatial translation $T_a:x\to x+a$ leads from one solution to another and

$$\pi_1(\phi_1^{\mathrm{TK2}^{BB}}(x;c),\phi_2^{\mathrm{TK2}^{BB}}(x;c))=(\phi_1^{\mathrm{TK2}^{BB}}(x;-c),\phi_2^{\mathrm{TK2}^{BB}}(x;-c)),$$

the moduli space of $\mathrm{TK2}^{BB}$ kinks—the quotient of the (19), (20) kink variety by the action of T_a and π_1—is the open half line $c\in(0,\infty)$. If, moreover, we take quotient by $P:x+a\to -x-a$, the antikinks are also included in the moduli space.

The asymptotic behavior

$$\lim_{x\to\pm\infty}\phi_1^{\mathrm{TK2}^{BB}}(x;c)=0,\quad \lim_{x\to\pm\infty}\phi_2^{\mathrm{TK2}^{BB}}(x;c)=\mp 1$$

fits in with the boundary behavior, guaranteeing finite energy to the $\mathrm{TK2}^{BB}(c)$ kinks. They are not stable because all of them cross the origin:

$$\phi_1^{\mathrm{TK2}^{BB}}(-a;c)=0,\quad \phi_2^{\mathrm{TK2}^{BB}}(-a;c)=0.$$

Thus, only if $x\in(-\infty,-a)$ are Eqs. (19), (20) solutions of the first-order equations (16) with the $+$ sign, whereas they solve Eq. (16) with the $-$ sign in the $x\in(-a,\infty)$ range, or vice versa. It can easily be proved, however, that these solutions satisfy the second-order differential equations (5).

Things are different at the boundary of the moduli space, the union of the $c=0$ and $c=\infty$ points. Looking at the formula (18) we find the $\mathrm{TK1}^{BB}$ kink as the $c=0$ limit of the kink variety, whereas the $\mathrm{TK1}^{AA}$ kink and two TK^{AB} kinks—that exist on different parabolic branches—meet at $c=\infty$.

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