PHYSICAL REVIEW D, VOLUME 65, 084044

Coproduct and star product in field theories on Lie-algebra noncommutative space-times

Giovanni Amelino-Camelia and Michele Arzano
Dipartimento di Fisica, Università di Roma "La Sapienza," Place Aldo Moro 2, 00185 Roma, Italy
(Received 17 October 2001; published 9 April 2002)

We propose a new approach to field theory on κ-Minkowski noncommutative space-time, a popular example of Lie-algebra space-time. Our proposal is essentially based on the introduction of a star product, a technique which is proving to be very fruitful in analogous studies of canonical noncommutative space-times, such as the ones recently found to play a role in the description of certain string-theory backgrounds. We find to be incorrect the expectation, previously reported in the literature, that the lack of symmetry of the κ-Poincaré coproduct should lead to interaction vertices that are not symmetric under exchanges of the momenta of identical particles entering the relevant processes. We show that in κ-Minkowski the coproduct and the star product must indeed treat momenta in a nonsymmetric way, but the overall structure of interaction vertices is symmetric under exchange of identical particles. We also show that in κ-Minkowski field theories it is convenient to introduce the concepts of "planar" and "nonplanar" Feynman loop diagrams, again in close analogy with the corresponding concepts previously introduced in the study of field theories in canonical noncommutative space-times.

DOI: 10.1103/PhysRevD.65.084044 PACS number(s): 04.50.+h, 04.60.Ds

Noncommutative geometry is being used more and more extensively in attempts to unify general relativity and quantum mechanics. Some "quantum-gravity" approaches explore the possibility that noncommutative geometry might provide the correct fundamental description of space-time, while in other approaches noncommutative geometry turns out to play a role at the level of the effective theories that describe certain aspects of quantum gravity.

Two simple examples [1] are "canonical noncommutative space-times" ($\mu,\nu,\beta=0,1,2,3$)

$$[x_\mu,x_\nu]=i\theta_{\mu,\nu} \tag{1}$$

and "Lie-algebra noncommutative space-times"

$$[x_\mu,x_\nu]=iC^\beta_{\mu,\nu}x_\beta. \tag{2}$$

The canonical type (1) was originally proposed [2] in the context of attempts to develop a new fundamental picture of space-time. More recently, Eq. (1) is proving useful in the description of string theory in certain "B" backgrounds (see, e.g., Refs. [3–5]). String theory in these backgrounds admits a description (in the sense of effective theories) in terms of a field theory in the noncommutative space-times (1), with the tensor $\theta_{\mu,\nu}$ reflecting the properties of the specific background.

Among the Lie-algebra type (2) space-times, one of the most studied is κ-Minkowski [6,7] space-time ($l,m=1,2,3$),

$$[x_m,t]=\frac{i}{\kappa}x_m, \quad [x_m,x_l]=0. \tag{3}$$

One of us recently proposed [8] a new path toward quantum gravity which takes as the starting point κ-Minkowski space-time. It was shown in Ref. [8] that the Hopf algebra that characterizes the symmetries of κ-Minkowski space-time, one of the κ-Poincaré algebras [7,9], can be used to introduce the Planck length (here identified, up to a sign and perhaps a numerical factor of order 1, with $1/\kappa$) directly at the level of the relativity postulates. This could provide [8] the starting point for a path toward quantum gravity based on a, still being sought, nonflat (and dynamical) generalization of the κ-Minkowski noncommutative space-time.

The possible role in quantum gravity of noncommutative geometry, and particularly of the examples (1) and (2), has generated strong interest in the construction of field theories in these types of space-times. In this respect the canonical type confronts us with less severe technical challenges. Procedures based on the star product [1] can be applied straightforwardly to space-times of type (1), basically as a result of the fact that the product of two wave exponentials, $e^{ip_\mu x_\mu}$ and $e^{ik_\nu x_\nu}$, in these geometries has very simple properties:

$$e^{ip_\mu x_\mu}e^{ik_\nu x_\nu}=e^{-(i/2)p^\mu\theta_{\mu\nu}k^\nu}e^{i(p+k)_\mu x_\mu}, \tag{4}$$

i.e. the Fourier parameters p_μ and k_μ combine just as usual, $(p+k)_\mu$, with the only new ingredient of an overall phase factor that depends on $\theta_{\mu,\nu}$. This simplicity renders possible the development of field theories in which the tree-level propagator is undeformed and the only new ingredients are factors of the type $e^{-ip^\mu\theta_{\mu,\nu}k^\nu/2}$ in the interaction vertices. These properties follow from the analysis of products of fields, which are characterized by formulas of the type

$$\Phi(x)\Psi(x)=\frac{1}{(2\pi)^4}\int d^4p\,d^4k\,e^{ipx}\tilde{\Phi}(p)e^{ikx}\tilde{\Psi}(k)$$
$$=\frac{1}{(2\pi)^4}\int d^4p\,d^4k\,e^{-(i/2)p^\mu\theta_{\mu\nu}k^\nu}$$
$$\times e^{i(k+p)x}\tilde{\Phi}(p)\tilde{\Psi}(k). \tag{5}$$

It is easy to see that in space-times of Lie-algebra type, and in particular in κ-Minkowski space-time, the construction of field theories is less straightforward. A key point is that the type of simple deformation encoded in Eq. (4) is not sufficient. A more complicated rule for the product of two wave exponentials is clearly needed, basically to reflect the structure of the coproduct in the κ-Poincaré algebras [7]. The rule for the product of two wave exponentials treats in a profoundly[1] nonsymmetric way the two Fourier parameters. This has been a cause of concern with respect to the applicability of κ Minkowski in physics, since it appeared [10–12] inevitable to obtain cross sections which do not treat symmetrically pairs of identical incoming particles. The most significant result here being reported is that no such paradoxical predictions are found when field theory in κ-Minkowski space-time is constructed consistently.

Our first task is the one of giving a consistent description of the product of two fields in κ-Minkowski space-time in terms of deformed rules for the product of commuting fields. This is the basic task of the star-product procedure: the product of two operator-valued functions is described through the properties of an associated deformed product of ordinary functions. We proceed in analogy with the strategy (5) which is proving successful in the study of canonical noncommutative spacetimes, i.e. we want to introduce as auxiliary commuting variables some Fourier parameters. The proper formulation of the Fourier transform in κ-Minkowski space-time has been discussed in previous mathematical-physics studies [11–13]; it is based on the "ordered exponential" $:e^{ip_\mu x_\mu}:$ defined by[2]

$$:e^{ip_\mu x_\mu}: \equiv e^{ip_m x_m} e^{ip_0 x_0}. \tag{6}$$

While, as the reader can easily verify, wave exponentials of the type $e^{ip_\mu x_\mu}$ do not combine in a simple way in κ-Minkowski space-time, for wave exponentials of the type $:e^{ip_\mu x_\mu}:$ one finds [12] (using the Campbell-Baker-Hausdorff formula)

$$(:e^{ip_\mu x_\mu}:)(:e^{ik_\nu x_\nu}:) = :e^{i(p \dot{+} k)_\mu x_\mu}:, \tag{7}$$

where the notation "$\dot{+}$" has been here introduced to denote the deformed addition rule (no sum on repeated indices)

$$p_\mu \dot{+} k_\mu \equiv \delta_{\mu,0}(p_0+k_0) + (1-\delta_{\mu,0})(p_\mu + e^{p_0/\kappa}k_\mu), \tag{8}$$

i.e. the energy[3] addition is undeformed while 3-momenta are added according to $\vec{p}+e^{p_0/\kappa}\vec{k}$. This lack of symmetry under the $\vec{p},\vec{k}$ exchange is the one we mentioned in the opening remarks, and readers familiar with the κ-Poincaré research program will recognize that Eq. (8) is just the rule for the energy-momentum coproduct. As mentioned, the proper formulation [11–13] of the Fourier transform in κ-Minkowski space-time is based on the ordered exponentials $:e^{ip_\mu x_\mu}:$

$$\Phi(x) = \frac{1}{(2\pi)^2}\int d^4p :e^{ipx}: \tilde{\Phi}(p). \tag{9}$$

The canonical property of Fourier theory is then obtained [11] using the proper concept of a partial derivative in κ-Minkowski space-time:

$$\frac{\partial}{\partial x_m}:e^{ipx}: = :\frac{\partial}{\partial x_m}e^{ipx}:,$$

$$\frac{\partial}{\partial x_0}:e^{ipx}: = \kappa:(e^{ipx} - e^{ip(x+\Delta x_\kappa)}):, \tag{10}$$

where $(\Delta x_\kappa)_\mu \equiv -\delta_{\mu,0}/\kappa$.

Using Eq. (7) one easily finds that

$$\Phi(x)\Psi(x) = \frac{1}{(2\pi)^4}\int d^4p\, d^4k :e^{i(p\dot{+}k)x}: \tilde{\Phi}(p)\tilde{\Psi}(k). \tag{11}$$

Having established this simple property of the product of two fields in κ-Minkowski space-time we can now easily write a partition function (generating functional for Green functions) in energy-momentum space. This will allow us to explore whether the lack of symmetry of the coproduct leads to nonsensical results, as suspected in previous related studies [10–12], or instead an acceptable physical picture does emerge. We illustrate our line of analysis in the context of a scalar theory with quartic interaction ("$\lambda\Phi^4$ theory"). We start with a partition function in the noncommutative space-time

$$Z[J(x)] = \int \mathcal{D}[\phi] e^{i\int d^4x[(1/2)\partial^\mu\phi(x)\partial_\mu\phi(x) - (m^2/2)\phi^2(x) - (\lambda/24)\phi^4(x) + (1/2)J(x)\phi(x) + (1/2)\phi(x)J(x)]}. \tag{12}$$

[1] Here we are emphasizing that in κ-Minkowski space-time the $k\to p, p\to k$ symmetry of the product of wave exponentials is lost in a much more significant way than in Eq. (4). [In Eq. (4) the exchange of k and p only changes the sign of the constant phase factor.]

[2] There is of course an equally valid alternative ordering prescription in which the time-dependent exponential is placed to the left [12] (while we are here choosing the convention with the time-dependent exponential to the right).

[3] We are here taking the liberty to denominate "energy" the Fourier parameter in the 0th direction (and similarly for the other three Fourier parameters we use "3-momentum"). This terminology may appear unjustified in the present context, but it is actually meaningful in light of the results on κ-Minkowski space-time reported in Ref. [11].

The observation (11) allows us to rewrite this partition function in energy-momentum space. We omit the tedious steps of this derivation, but we note here some useful formulas which reflect the type of care required by the lack of symmetry of the coproduct. Introducing

$$\delta^{(4)}(k)=\frac{1}{(2\pi)^4}\int d^4x{:}e^{ikx}{:}, \tag{13}$$

which holds in κ-Minkowski space-time [12], the required manipulations of the partition function will of course lead to the emergence of terms going like $\delta^{(4)}(p\dot{+}k)$. The lack of symmetry of $p\dot{+}k$ then requires some care; the relevant formulas are

$$\int d^4k\,\delta^{(4)}(k\dot{+}p)f(k)=\mu(p_0)f(\dot{-}p), \tag{14}$$

$$\int d^4k\,\delta^{(4)}(p\dot{+}k)f(k)=e^{-3p_0/\kappa}\mu(p_0)f(\dot{-}p), \tag{15}$$

where we introduced a function $\mu(p_0)$, possibly reflecting the properties of a nontrivial measure of integration over κ-energy-momentum space [which, in the sense reflected by Eq. (8), is not flat] and we also introduced the notation $\dot{-}p$

$$(\dot{-}p)_\mu\equiv\delta_{\mu,0}(-p_0)+(1-\delta_{\mu,0})(-e^{-p_0/\kappa}p_\mu). \tag{16}$$

Readers familiar with the κ-Poincaré research program will recognize Eq. (16) as the rule for the "antipode" [in fact $p\dot{+}(\dot{-}p)=0$]. On the function $\mu(p_0)$ we will only observe and use the fact that it can depend on energy momentum only through the 0th component (energy). As discussed in detail in Ref. [11], it is clear that the measure for integration over κ-energy-momentum space should depend only on energy. There are various candidates for this measure, but for our analysis this ambiguity could only affect the form of the function $\mu(p_0)$. Since we focus here on the relation between the nonsymmetric coproduct of κ-Poincaré and the structure of the conservation rules that characterize Green functions in field theory on κ-Minkowski space-time, we can postpone the investigation of the measure ambiguity to future studies. Concerns about the conservation rules have been the most serious obstacle for the construction of physical theories based on κ-Minkowski space-time, and we shall show that these concerns can be straightforwardly addressed within our approach, independently of the form of $\mu(p_0)$. We shall therefore keep track of factors of the type $\mu(p_0)$, but never assume anything about the form of the function $\mu(p_0)$. While the existence of alternative choices [11] of measure (leading to different forms of our $\mu(p_0)$) is an "aesthetic concern" for the κ-Minkowski research program (all the candidates identified in Ref. [11] are acceptable, but one would like to have a theory predicting the measure), the possibility that Green functions might be characterized by conservation rules that do not respect symmetry under exchange of the momenta of identical incoming (outgoing) particles represents a "quantitative concern" for the applicability of κ-Minkowski in physics. As anticipated, it is on this second, more alarming, problem which we focus here our attention.

Our first objective is to examine the structure of the tree-level propagator. For this first result we can of course switch off the coupling λ. Using Eqs. (10), (14) and (15), one obtains from Eq. (12)

$$Z^0[J(k)]\equiv\{Z[J(k)]\}_{\lambda=0}=\int \mathcal{D}[\phi]e^{i/2\int d^4k\mu(k_0)[\phi(\dot{-}k)[\mathcal{C}_\kappa(k)-m^2]\phi(k)+J(k)\phi(\dot{-}k)+\phi(k)J(\dot{-}k)]}, \tag{17}$$

where $\mathcal{C}_\kappa$ is the κ-Poincaré mass Casimir[4]

$$\mathcal{C}_\kappa(k)=\kappa^2(e^{k_0/\kappa}+e^{-k_0/\kappa}-2)-\vec{k}^2e^{-k_0/\kappa}. \tag{18}$$

It is convenient to introduce the normalized partition function

$$\overline{Z^0}[J(k)]\equiv\frac{Z^0[J(k)]}{Z^0[0]}, \tag{19}$$

and from Eq. (17) with simple manipulations one finds that

$$\overline{Z^0}[J(k)]=\exp\left(-\frac{i}{2}\int d^4k\mu(k_0)\frac{J(k)J(\dot{-}k)}{\mathcal{C}_\kappa(k)-m^2}\right). \tag{20}$$

From Eq. (20) one can obtain the tree-level propagator using a straightforward generalization of the usual formulation adopted for field theory in commutative space-times:

$$G_0^{(2)}(p,\dot{-}p')=-\left.\frac{\delta^2\overline{Z^0}[J(k)]}{\delta J(\dot{-}p)\,\delta J(p')}\right|_{J=0}. \tag{21}$$

For the functional derivatives required by Eq. (21) one also needs appropriately generalized definitions:

[4]We remind the reader that in the κ-Poincaré mass-casimir relation $\mathcal{C}_\kappa(k)=m^2$ the mass parameter m, which here appears also in the Lagrangian, is not to be identified with the rest energy $E(\vec{k}=0)$. The physical mass M, which is the rest energy, is obtained from m through the relation [7] $m^2=\kappa^2\sinh^2[M/(2\kappa)]$. (Note that, however, m and M differ only at order $1/\kappa^2$.)

$$\frac{\delta F(f(p))}{\delta f(k)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{F[f(p)+\varepsilon\,\delta^{(4)}(p\dot{+}(\dot{-}k))] - F[f(p)]\}, \tag{22}$$

$$\frac{\delta F[f(p)]}{\delta f(\dot{-}k)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{F[f(p)+\varepsilon\,\delta^{(4)}(p\dot{+}k)] - F[f(p)]\}. \tag{23}$$

Using Eqs. (14), (15), (22), (23) and the property $\mathcal{C}_\kappa(\dot{-}p)=\mathcal{C}_\kappa(p)$, from Eq. (21) one easily obtains the tree-level propagator:

$$G_0^{(2)}(p,\dot{-}p') = \frac{i}{2}\mu(-p_0)\mu(p_0) \times \frac{\delta^{(4)}(p\dot{+}(\dot{-}p'))+\delta^{(4)}((\dot{-}p')\dot{+}p)}{\mathcal{C}_\kappa(p)-m^2}. \tag{24}$$

One first point that deserves to be emphasized is the central role played by the mass Casimir $\mathcal{C}_\kappa$ in the structure of this tree-level propagator.[5] This result on the role of $\mathcal{C}_\kappa$ is consistent with the expectations on the tree-level propagator formulated in previous preliminary attempts [10,11] to develop field theory in κ-Minkowski space-time. However, in addition to the role of $\mathcal{C}_\kappa$, it is perhaps even more important to observe that the two $\delta^{(4)}$ in Eq. (24) enforce the same trivial conservation condition; in fact, $\delta^{(4)}((\dot{-}p')\dot{+}p) = e^{3p_0/\kappa}\delta^{(4)}(p\dot{+}(\dot{-}p')) = e^{3p_0/\kappa}\delta^{(4)}(p-p')$. This is a first reassuring sign that our approach to field theory in κ-Minkowski space-time can provide the correct way to handle the nontrivial coproduct structure of κ-Poincaré: in spite of the nonsymmetric and nonlinear coproduct structure, our result preserves the usual property that energy momentum is conserved along the tree-level propagator. In order to be able to investigate the properties of the propagator beyond tree level and in order to establish the form of the tree-level vertex we now push our analysis one step further, by analyzing the $O(\lambda)$ contributions to the Green functions. For this we must of course reinstate $\lambda\neq 0$, i.e. we need to analyze $\overline{Z}[J(k)]$ rather than $\overline{Z}^0[J(k)]$. It turns out to be useful to rely on the following relation between $\overline{Z}[J(k)]$ and $\overline{Z}^0[J(k)]$, which one easily obtains with manipulations analogous to the ones described above:

[5]This is a first characteristic result of our Lie-algebra noncommutative space-time; in fact, in canonical noncommutative space-times the tree-level propagator is governed by the undeformed casimir E^2-p^2 (but of course eventually, through loop effects, even in canonical noncommutative space-times the propagator does reflect the space-time deformation [4]).

$$\overline{Z}[J(k)] = \exp\left[i\frac{\lambda}{24}\int \delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right) \times \prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\frac{\delta}{\delta J(\dot{-}k_j)}\right] \times \overline{Z}^0[J(k)], \tag{25}$$

where

$$\xi(k_{j,0}) \equiv 2[\mu(k_{j,0})+\mu(-k_{j,0})e^{3k_{j,0}/\kappa}]^{-1}, \tag{26}$$

and we introduced a compact ordered-sum notation

$$\dot{\sum_{k_1,k_2,k_3,k_4}} \equiv k_1\dot{+}k_2\dot{+}k_3\dot{+}k_4. \tag{27}$$

Of course, for the $O(\lambda)$ contributions to the propagator and the vertex we only need the $O(\lambda)$ approximation of $\overline{Z}[J(k)]$

$$\overline{Z^{(1)}}[J(k)] = i\frac{\lambda}{24}\int \delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right) \times \prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\frac{\delta}{\delta J(\dot{-}k_j)}\overline{Z}^0[J(k)]. \tag{28}$$

From this formula it is straightforward to obtain the $O(\lambda)$ contribution to the propagator:

$$\begin{aligned} G_\lambda^{(2)}(p,\dot{-}p') &= \left(-\left.\frac{\delta^2\overline{Z}^1[J(k)]}{\delta J(\dot{-}\mathrm{p})\,\delta J(p')}\right|_{J=0}\right)_{connected} \\ &= i\frac{\lambda}{24}\int \prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\,\delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right) \\ &\times\left[\left.\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(\dot{-}p)\,\delta J(\dot{-}k_2)}\right|_{J=0} \left.\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(p')\,\delta J(\dot{-}k_3)}\right|_{J=0} \right. \\ &\left.\times \left.\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(\dot{-}k_1)\,\delta J(\dot{-}k_4)}\right|_{J=0} + \mathcal{P}_{k_1,k_2,k_3,k_4}\right], \end{aligned} \tag{29}$$

where $\mathcal{P}_{k_1,k_2,k_3,k_4}$ denotes permutation of the momenta k_1,k_2,k_3,k_4 (the term explicitly written out in the square brackets is only one of 24 terms obtained by permutations of k_1,k_2,k_3,k_4).

Let us focus on the contribution to $G_\lambda^{(2)}$ coming from the first term in the square brackets of Eq. (29). Using again the same observation that took us from Eqs. (21) to (24), we can rewrite this contribution as

$$i\frac{\lambda}{24}\int\prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right)\left(-\frac{i}{2}\right)^3$$
$$\times\left(\mu(p_0)\mu(-p_0)\frac{1}{2}\frac{\delta^{(4)}(p\dot{+}k_2)+\delta^{(4)}(k_2\dot{+}p)}{\mathcal{C}_\kappa(p)-m^2}\right)$$
$$\times\left(\mu(p_0')\mu(-p_0')\frac{1}{2}\frac{\delta^{(4)}(\dot{-}p'\dot{+}k_3)+\delta^{(4)}(k_3\dot{+}(\dot{-}p'))}{\mathcal{C}_\kappa(p')-m^2}\right)$$
$$\times\left(\mu(k_{4,0})\mu(-k_{4,0})\frac{1}{2}\frac{\delta^{(4)}(k_1\dot{+}k_4)+\delta^{(4)}(k_4\dot{+}k_1)}{\mathcal{C}_\kappa(k_1)-m^2}\right),\quad(30)$$

and integrating over k_1,k_2,k_3 we find that Eq. (30) is proportional to

$$\delta^{(4)}[\dot{-}k_4\dot{+}(\dot{-}p)\dot{+}p'\dot{+}k_4]\sim\delta^{(4)}(p-p').\qquad(31)$$

The same observation applies to the other contributions to $G_\lambda^{(2)}$ that correspond to "planar diagrams": the 16 permutations, out of the 24 permutations encoded in $\mathcal{P}_{k_1,k_2,k_3,k_4}$, that are such that the external momenta are attached to consecutive internal lines (i.e. either to k_1,k_2 or k_2,k_3 or k_3,k_4 or k_4,k_1). The difference between these "planar" diagrams and the diagrams, which can be described as "nonplanar," that correspond to the remaining eight permutations, in which instead the external lines are attached to nonconsecutive lines, is meaningful in our κ-Minkowski field theory, since the coproduct sum $\dot{\Sigma}_{k_1,k_2,k_3,k_4}$ is not invariant under permutations.[6] While planar diagrams, as shown in Eqs. (30) and (31), provide contributions that are proportional to the *ordinary* $\delta^{(4)}(p-p')$, i.e. correspond to ordinary conservation of energy momentum, nonplanar diagrams introduce a new structure in the propagator. Let us consider, for example, the contribution

$$i\frac{\lambda}{24}\int\prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right)$$
$$\times\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(\dot{-}p)\,\delta J(\dot{-}k_2)}\Bigg|_{J=0}\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(p')\,\delta J(\dot{-}k_4)}\Bigg|_{J=0}$$
$$\times\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(\dot{-}k_1)\,\delta J(\dot{-}k_3)}\Bigg|_{J=0},\qquad(32)$$

that corresponds to

$$i\frac{\lambda}{24}\int\prod_{j=1}^{4}\frac{d^4k_j}{2\pi}\xi(k_{j,0})\delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right)\left(-\frac{i}{2}\right)^3$$
$$\times\left(\mu(p_0)\mu(-p_0)\frac{1}{2}\frac{\delta^{(4)}(p\dot{+}k_2)+\delta^{(4)}(k_2\dot{+}p)}{\mathcal{C}_\kappa(p)-m^2}\right)$$
$$\times\left(\mu(p_0')\mu(-p_0')\frac{1}{2}\frac{\delta^{(4)}(\dot{-}p'\dot{+}k_4)+\delta^{(4)}(k_4\dot{+}(\dot{-}p'))}{\mathcal{C}_\kappa(p')-m^2}\right)$$
$$\times\left(\mu(k_{3,0})\mu(-k_{3,0})\frac{1}{2}\frac{\delta^{(4)}(k_1\dot{+}k_3)+\delta^{(4)}(k_3\dot{+}k_1)}{\mathcal{C}_\kappa(k_1)-m^2}\right).\quad(33)$$

Integrating over k_1,k_2,k_3 we can observe that this contribution is proportional to

$$\delta^{(4)}(\dot{-}k_3\dot{+}(\dot{-}p)\dot{+}k_3\dot{+}p')\sim\delta(p_0-p_0')\delta^{(3)}(e^{p_0/\kappa}\vec{k}_3-\vec{p}+\vec{k}_3+e^{k_{3,0}/\kappa}\vec{p}').\qquad(34)$$

While energy conservation is still ordinary also for nonplanar diagrams, momentum conservation is modified and it is modified in a way that cannot even be described as a modified conservation law: the terms involving the loop integration momentum $\vec{k}_3$ do not cancel each other out in the $\delta^{(3)}$ of Eq. (34). It is easy to verify that these nonplanar contributions, while not implementing exactly the ordinary energy-momentum conservation, are still mainly centered around ordinary energy-momentum conservation (assuming reasonably good behavior at infinity of the expression under integration). There is therefore some deviation from the ordinary conservation of energy momentum, a sort of fuzzy conservation of momentum, but it is plausible that the full theory (whose construction will also require the measure that we are here treating as an unknown) would only predict a very small (κ^{-1} suppressed) deviation from the ordinary conservation of energy momentum, possibly consistent with observational limits. This dynamical issue is beyond the scopes of the analysis here being reported, which focuses on kinematics, and we postpone it to future studies.

Our next, and final, task is the study of the tree-level vertex. The $O(\lambda)$ contribution to the four-point Green function can be expressed in terms of $\overline{Z}^{(1)}$ through

$$G_\lambda^{(4)}(p_1,p_2,\dot{-}p_3,\dot{-}p_4)=\frac{\delta^4\overline{Z}^{(1)}[J(k)]}{\delta J(\dot{-}p_1)\,\delta J(\dot{-}p_2)\,\delta J(p_3)\,\delta J(p_4)}\Bigg|_{J=0},\qquad(35)$$

and the tree-level vertex, sum of connected graphs contributing to $G_\lambda^{(4)}(p_1,p_2,\dot{-}p_3,\dot{-}p_4)$, turns out to have the form

[6] Analogous comments apply to field theories in canonical noncommutative space-time, where indeed a completely analogous concept of planar and nonplanar diagrams has been introduced [3–5].

$$G_\lambda^{(4)}(p_1,p_2,\dot{-}p_3,\dot{-}p_4)_{connected}$$
$$=\frac{i\lambda}{24}\int\left(\prod_{l=1}^{4}\frac{d^4k_l}{2\pi}\xi(k_{l,0})\right)\delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right)$$
$$\times\left(\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(\dot{-}p_1)\,\delta J(\dot{-}k_1)}\bigg|_{J=0}\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(\dot{-}p_2)\,\delta J(\dot{-}k_2)}\bigg|_{J=0}\right.$$
$$\times\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(p_3)\,\delta J(\dot{-}k_3)}\bigg|_{J=0}\frac{\delta^2\overline{Z}^0(J(k))}{\delta J(p_4)\,\delta J(\dot{-}k_4)}\bigg|_{J=0}$$
$$\left.+\mathcal{P}_{k_1,k_2,k_3,k_4}\right). \tag{36}$$

From Eq. (36) one can easily obtain more explicit formulas for the tree-level vertex $G_{tree}^{(4)}(p_1,p_2,\dot{-}p_3,\dot{-}p_4)_{connected}$. As mentioned we are primarily interested in establishing what are the conservation rules implemented at the vertex and how they are related to the nonsymmetric structure of the coproduct. In this respect it is important to observe that each of the 24 terms generated by the permutations of k_1,k_2,k_3,k_4 is characterized by a different conservation rule. This is completely different from the behavior of the $\kappa\rightarrow\infty$ limit (the limit in which our noncommutative space-time turns into the ordinary commutative Minkowski space-time), in which all permutations $\mathcal{P}_{k_1,k_2,k_3,k_4}$ lead to the same conservation rule $\delta^{(4)}(-p_1-p_2+p_3+p_4)$. In order to render more explicit these comments on the faith of energy-momentum conservation in the vertices of field theories on κ-Minkowski space-time, it is sufficient to observe that the term written out explicitly in Eq. (36) (the other 23 contributions are implicitly introduced through the permutations $\mathcal{P}_{k_1,k_2,k_3,k_4}$) can be rewritten as

$$\frac{i\lambda}{24}\int\left(\prod_{l=1}^{2}\frac{d^4k_l}{2\pi}\xi(k_{l,0})\mu(p_{l,0})\mu\right.$$
$$\left.\times(-p_{l,0})\frac{1}{2}\frac{\delta^{(4)}(p_l\dot{+}k_l)+\delta^{(4)}(k_l\dot{+}p_l)}{\mathcal{C}_\kappa(p_l)-m^2}\right)$$
$$\times\left(\prod_{m=3}^{4}\frac{d^4k_m}{2\pi}\xi(k_{m,0})\mu(p_{m,0})\mu(-p_{m,0})\right.$$
$$\left.\times\frac{1}{2}\frac{\delta^{(4)}(\dot{-}p_m\dot{+}k_m)+\delta^{(4)}(k_m\dot{+}(\dot{-}p_m))}{\mathcal{C}_\kappa(p_m)-m^2}\right)$$
$$\times\delta^{(4)}\left(\dot{\sum_{k_1,k_2,k_3,k_4}}\right). \tag{37}$$

Then doing the k_1,k_2,k_3,k_4 integrations over the internal four-momenta it is easy to establish that this contribution to the tree-level vertex enforces the conservation rule

$$\delta^{(4)}(\dot{-}p_1\dot{+}(\dot{-}p_2)\dot{+}p_3\dot{+}p_4), \tag{38}$$

which corresponds to ordinary energy conservation, $-p_{1,0}-p_{2,0}+p_{3,0}+p_{4,0}=0$, but enforces a nontrivial and nonsymmetric rule of conservation of 3-momenta: $-e^{(-p_{1,0}/\kappa)}\vec{p}_1 \quad -e^{-(1/\kappa)(p_{1,0}+p_{2,0})}\vec{p}_2 \quad +e^{-(1/\kappa)(p_{1,0}+p_{2,0})}\vec{p}_3 + e^{(1/\kappa)(-p_{1,0}-p_{2,0}+p_{3,0})}\vec{p}_4=0$.

The lack of symmetry of Eq. (38) is just of the type feared in previous preliminary analyses [10,11] of field theory in κ-Minkowski space-time, and was responsible for some skepticism toward the physical applicability of such field theories (since, of course, a physically acceptable theory should describe collision processes in a way that treats symmetrically, e.g., two identical incoming particles). However, in our approach to field theory in κ-Minkowski space-time (37) is just one of 24 contributions to the vertex (36). The other 23 terms are characterized by permutations of the conservation rule (38), which span over all possible ways to order the (deformed) sum of p_3, p_4, $(\dot{-}p_1)$, and $(\dot{-}p_2)$. The overall structure of our interaction vertex is fully symmetric[7] under exchanges of the momenta that enter it, thereby fulfilling the condition for physical applicability of our field theory. Our analysis shows that the new ingredient introduced by the κ deformation is not the loss of particle-exchange symmetry, feared in previous studies, but rather a revision of the concept of energy-momentum conservation for scattering processes: since our vertex is not characterized by an overall δ-function, but instead it is split up into 24 pieces each with its own different δ-function, in a given scattering process, with incoming particles characterized by four-momenta p_1 and p_2, it becomes impossible to predict the sum of the 3-momenta of the outgoing particles. The theory only predicts that one of our 24 energy-momentum-conservation rules must be satisfied and assigns (equal) probabilities to each of these 24 channels.

These properties of vertices in κ-Minkowski space-time represent a rather significant departure from conventional physics, but they do make sense physically (identical particles are treated symmetrically), and we are therefore providing a key tool for testing whether Nature makes use of κ-Minkowski space-time. Making the reasonable assumption [8] that κ should be of the order of the Planck scale one easily checks that our prediction for new (non)conservation rules at the vertex is consistent with all available low-energy data. (In the limit $p_0/\kappa\ll 1$ the 24 different conservation rules that characterize our κ-deformed vertex collapse into a single, and ordinary, conservation rule.) There is, however, one experimental context in which our κ-deformed vertex

[7]The fact that some sort of symmetrization of the coproduct should emerge in the formalism had been conjectured in Ref. [11], where the puzzles implied by a naive implementation of the nonsymmetric coproduct were analyzed in detail. Our result provides the correct realization of this conjectured symmetrization (and the structure of our result is significantly different from the simpler ansatzae considered in Ref. [11]).

could have observably large consequences: astrophysics observations sensitive to the value of the kinematic threshold for certain particle-production processes. It is of encouragement for our proposal that observations of ultra-high-energy cosmic rays [14] and of Markarian501 photons [15] have recently obtained data that appear to be in conflict with conventional theories and appear to require [16,17,19,18,20,21] a deformation of the kinematic conservation rules applied to collision processes. κ-Minkowski space-time has been considered [19,20] as a possible source of such a deformation of kinematics, but the previously conjectured lack of symmetry of the vertex did not allow direct comparison with data (and a simple-minded ansatz for the symmetrization of the coproduct was shown not to lead to interesting results [20]). Our analysis now renders possible this comparison with data, and the fact that the data appear to be in conflict with the conventional structure of vertices encourages us to hope that the κ-deformed vertex might play a role in the solution of the experimental paradox, but we postpone this delicate phenomenological analysis to a future study.[8]

We close with some remarks summarizing the results here obtained. We took off from some preliminary attempts [10,11] to construct field theories in κ-Minkowski space-time. Those previous studies had identified some intruiging features, which might render such theories attractive as possible tools in quantum-gravity research, but they had also led to the alarming suspect that the highly nontrivial structure of the κ-Poincaré coproduct might render these theories unacceptable for physical applications. Our approach was mostly inspired by the previous study reported in Ref. [11], but we proposed that the correct starting point for deriving κ-Minkowski Green functions must be the corresponding generating functional, just like in commutative space-times. In the familiar commutative space-times one can easily guess the Green functions from the structure of the action, and the previous attempts of construction of field theory in κ-Minkowski space-time relied on the assumption that very similar guesswork could be adopted in Lie-algebra noncommutative spacetimes; however, this is clearly not the case. In particular, previous studies found that the interaction vertices would not enjoy symmetry under the exchange of identical incoming particles [10,11], while from our more fundamental generating-functional starting point no such loss of symmetry was encountered. Actually, at tree level the whole theory appears to be very intuitive, without any dramatic departures from the structure of field theory in commutative spacetimes. Our tree-level results are sufficient (at least in principle, but the required phenomenological analysis appears to be rather challenging) for testing κ-Minkowski space-time through comparison with the recent exciting results of observations of ultra-high-energy cosmic rays and of Markarian501 photons.

Beyond tree level the nature of the departures from the conventional classical space-time picture appears to become rather dramatic in κ-Minkowski space-time at least from the field-theory perspective here adopted. Evidence of such departures is found in the non-planar-diagrams sector of our one-loop analysis of the self-energy. Whereas the theory at tree level enforces a standard concept of energy-momentum conservation (although in an appropriately adapted sense, reflecting the quantum, rather than classical, symmetries of κ-Minkowski space-time), beyond tree-level energymomentum conservation is enforced only in a ''fuzzy'' way: energy momentum is conserved on average (no preferred direction of the violations) but violations of energy-momentum conservation are to be expected in any given particle-propagation process. Within our purely kinematical analysis (which allowed us to set aside a delicate issue concerning the choice of measure for integration over energy-momentum variables) we are of course not in the position to estimate the quantitative significance of this effect, but clearly there is strong motivation for future studies to focus on this issue since it might lead to significant effects (it is even possible that the anomalous effects would be so large to be in conflict with available data, thereby ruling out applications of κ-Minkowski space-time in space-time physics). The issue of possible departures from energy-momentum conservation for particle-propagation processes had already emerged in a previous attempt [10] to construct field theories in κ-Minkowski space-time. However, within the approach adopted in Ref. [10] this effect could not be studied consistently since the limited starting point there adopted (guessing Feynman rules directly from an action, rather than deriving them from a generating functional of Green functions) did not give rise to the full structure of interaction vertices.[9]

While our study has removed one key reason (pertaining to kinematics) of skepticism concerning the applicability of

[8]Some readers might feel that developing a κ-Minkowski phenomenology could be premature at this stage, since dynamics in this spacetime is still very poorly understood. However, especially through the results reported in the present study, a clear picture of kinematics in κ-Minkowski space-time is emerging, and significant progress can be achieved by comparing to data this kinematical predictions of κ-Minkowski space-time. If these experimental tests give negative results we will be able to conclude that κ-Minkowski space-time can only be of interest to mathematicians, whereas positive results of these tests would clearly provide additional motivation to the physics community for the understanding of dynamics in κ-Minkowski space-time.

[9]For example, the four-point vertex constructed in Ref. [10] corresponds to only one of the 24 nonequivalent permutations of external momenta that characterize the vertex we constructed from the generating functional (and actually, even if one wanted to choose one among the 24, it is difficult to imagine an argument that could single out that particular one). In addition, the approach developed in Ref. [10] fails to uncover the key role played by the difference between planar and nonplanar diagrams, which instead was imposed on us by our constructive procedure based on the generating functional of Green functions. These issues concerning the role played by the differences between planar and nonplanar diagrams appear to be a general feature [3–5] of field theory in noncommutative spacetimes, and it is encouraging that our approach based on the generating functional of Green functions proved to be consistent with this general expectation.

κ-Minkowski space-time the fact that dynamics in κ-Minkowski space-time is still very poorly understood (partly because of the above-mentioned issues concerning the loop-integration measure) remains of significant concern. In particular, κ-Minkowski space-time is an example of ''Lie-algebra space-time noncommutativity'' (the space coordinates commute among themselves) and it is well known (see, e.g., Ref. [22]) that ''canonical space-time noncommutativity'' naturally leads to severe nonunitarity problems. Lie-algebra noncommutative space-times are profoundly different from canonical ones, and therefore these unitarity concerns do not automatically apply to κ-Minkowski space-times,[10] but clearly this issue deserves high priority in future theory studies.

Although we focused here on a specific example of Lie-algebra noncommutative space-time, κ-Minkowski, it appears likely that the techniques we developed will be applicable also to field theories in other Lie-algebra noncommutative space-times, which are all affected by similar problems associated with the highly nontrivial structure of the coproduct.

[10] In particular, we expect that the peculiarities of the light-cone in κ-Minkowski [8] would play a non-trivial role in the analysis of unitarity.

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