Rotating charged mass shell: Dragging, antidragging, and the gyromagnetic ratio

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We calculate explicitly the system of a spherical shell of radius R, carrying (nearly) arbitrary mass M and charge q, and rotating slowly around an axis through its center. We discuss, mainly graphically in the plane of the model parameters M/R and q/R, the following properties of this system: The dragging of inertial frames which turns over to antidragging in part of the parameter space, the induced magnetic field, the angular momentum, the magnetic moment, and the gyromagnetic ratio. The latter is very near to the value 2 in an overwhelming part of the parameter space, and we argue that this signals a deep connection between general relativity and quantum theory which could be important in the search for quantum gravity.

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I. INTRODUCTION

It has been known since 1918 from the classic work of Thirring that in general relativity a rotating mass shell induces a dragging of inertial frames in its interior. In the work of Thirring this was a tiny effect since he confined himself to the weak field approximation. Later, however, it was shown by Brill and Cohen [1] that this dragging becomes complete (the dragging angular velocity coinciding with the angular velocity of the shell) in the collapse limit, and this result is considered as convincing evidence for the realization of (Machian type) relativity of rotation in general relativity, at least for appropriate model systems. Some authors considered also, in different approximations, the influence of a rotating mass shell on electromagnetic phenomena, especially on charges sitting inside the mass shell [2–4], one typical effect being the induction of a dipolar magnetic field. In [5] we treated in detail a shell of arbitrarily high charge sitting inside a shell of arbitrary mass in first order of the angular velocities of both shells. A comparison with the results of [3,4] could resolve an inconsistency concerning the Machian interpretation of the induced magnetic fields.

Aside from these models with charges sitting inside a rotating mass shell, it is of interest to treat rotating shells which carry at the same place mass and charge with identical angular velocities (isolator type material). In addition to the phenomena of dragging and the induction of a magnetic field, for such systems the question of their gyromagnetic ratio is of special interest. In [6] these questions were answered for a shell of arbitrary mass but small charge. In [7] and [8] the problem was in principle solved for arbitrary mass and charge but the physical discussion was confined to some special cases. In the present paper we calculate and discuss in all detail a spherical shell of radius R (in isotropic coordinates to be defined in Sec. II) with (nearly) arbitrary mass M and charge q, which rotates slowly around an axis through its center.

In Sec. II we calculate the static shell model. The only restrictions on the dimensionless parameters M/R and q/R are that the invariant radius of the shell is non-negative, that the model is really static (no collapse, no singularities), and that the total mass M is non-negative. We discuss the energy conditions for the shell material, but we allow for violation of these conditions. In Sec. III we treat (in analogy to [5]) the rotation of this charged shell in first order of the angular velocity ω. The (ordinary) differential equations for the dragging function and the magnetic field function can be solved explicitly and analytically (compare [7] and [8]), and the integration constants are completely and uniquely determined by the regularity at the shell center, the asymptotic flatness conditions, and by the continuity or discontinuity conditions at the shell. In Sec. IV we discuss the results, mainly in graphical form, since the analytic expressions are quite involved. The dragging is again complete in the collapse limit, is below completeness in all other cases, and changes sign (resulting in antidragging) in part of the parameter space where the weak energy condition is violated. The (constant) magnetic field inside the shell, measured in proper time, is zero in the collapse limit and in the limit of vanishing (invariant) radius of the shell, and is positive (in the direction of the rotation axis) in all other cases, i.e., it does not change sign due to a violation of the energy conditions. The most important and most surprising result may be that the gyromagnetic ratio G of the rotating shell is very near to the value $G=2$ in an overwhelming part of the parameter space. Only in a small strip around the parameters where, due to the violation of the energy conditions, the angular momentum of the system vanishes, G attains arbitrary other (positive and negative) values, and it attains of course the classical value $G=1$ in an appropriate weak field limit. We like to argue that the "robustness" of this value $G=2$, and its "coincidence" with the value $G\approx 2$ for the simplest rotating, charged quantum particles (electron and muon) hints to a deep connection between general relativity and quantum theory, which could serve as a guideline and control on the way to some future quantum gravity.

II. THE STATIC MODEL

According to a generalization of the Birkhoff theorem, a spherically symmetric, matter-free solution of the Einstein-Maxwell equations is automatically static and asymptotically

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flat, and can be represented by the Reissner-Nordström metric

$$ds^2=-F(\rho)d\tau^2+F(\rho)^{-1}d\rho^2+\rho^2 d\Omega^2, \tag{1}$$

with $F(\rho)=1-2M/\rho+q^2/\rho^2$, and $d\Omega^2=d\vartheta^2+\sin^2\vartheta d\varphi^2$. Therefore, our model of a spherically symmetric, massive, and charged shell is simply given by two pieces of this Reissner-Nordström metric: one for the region outside the shell, and one for the interior region. However, a matching of these Reissner-Nordström metrics with different mass and charge parameters obviously would not be continuous at the shell position. A global continuous metric is, however, desirable for the physical interpretation of the (nonlocal) dragging effects, and was also used in Refs. [1–7]. It can be reached by a transformation of the metric (1) to the isotropic form

$$ds^2=-e^{2U(r)}dt^2+e^{2V(r)}(dr^2+r^2 d\Omega^2). \tag{2}$$

Identification of Eqs. (2) and (1) results in

$$r(\rho)=\frac{1}{2D}(\sqrt{\rho^2-2M\rho+q^2}+\rho-M), \tag{3}$$

with an arbitrary constant D. For $F(\rho)>0$, i.e., outside of horizons, $r(\rho)$ is real. We simplify the following calculations by using dimensionless variables:

$$\alpha=\frac{M}{2R}, \quad \gamma=\frac{q}{2R}, \quad x=\frac{r}{R}. \tag{4}$$

In the exterior region $x\geqslant 1$ we set $D=1$ (then r and ρ coincide asymptotically), and identify t with τ. Then the exterior potentials read

$$V_1(x)=\log\left[\frac{(x+\alpha)^2-\gamma^2}{x^2}\right],$$

$$U_1(x)=\log\left[\frac{x^2-\alpha^2+\gamma^2}{(x+\alpha)^2-\gamma^2}\right]. \tag{5}$$

In the interior region $x<1$, we have to set $M=0$ and $q=0$ in order to guarantee regularity at the origin $\rho=0$. The interior metric is then automatically flat. The transformations $r=\rho/D$, $t=C\tau$ produce in the region $x<1$ the potentials

$$V_2(x)=\log D=\text{const}, \quad U_2(x)=-\log C=\text{const}, \tag{6}$$

and the continuity of $V(x)$ and $U(x)$ at $x=1$ leads to

$$D=1+2\alpha+\kappa, \quad C=\frac{D}{1-\kappa}, \tag{7}$$

with the useful abbreviation $\kappa=\alpha^2-\gamma^2$. We see that the continuity of the potentials $V(x)$ and $U(x)$ enforces in general nontrivial transformations between r and ρ, and t and τ. In order that the metric (2) with the potentials (5) and (6) represents a physically reasonable, static shell model, the parameters α and κ have to satisfy some restrictions: (i) In order that the invariant radius $\mathbf{R}=\rho(r=R)=DR$ of the shell (measuring the surface) is non-negative, we have to have $D\geqslant 0$, or $\kappa\geqslant-(1+2\alpha)$. (ii) In order that the shell is really static, and does not develop a horizon and a gravitational collapse, we have to have $C\geqslant 0$, or $\kappa\leqslant 1$. (iii) Obviously, the parameter κ has to fulfill $\kappa\leqslant\alpha^2$ (γ real), and the total mass of the shell should be non-negative: $\alpha\geqslant 0$.

The energy-momentum tensor of the shell has of course the form $T^\mu_\nu=\tau^\mu_\nu\delta(r-R)$, and can be calculated from Einstein's field equations

$$G^\mu_\nu=8\pi(T^\mu_\nu+S^\mu_\nu), \tag{8}$$

with $S^\mu_\nu=(1/4\pi)F^\mu_\lambda F^\lambda_\nu$ representing the electromagnetic contribution to the energy-momentum tensor, and F^μ_ν being the electromagnetic field tensor. Since in our isotropic metric form (2) the potentials $V(r)$ and $U(r)$ are by construction continuous at the shell position, the components τ^μ_ν are essentially determined by the discontinuities of the radial derivatives of V and U. Here and in the following, the indices 0,1,2,3 are used to denote, respectively, the variables t,r,ϑ,φ:

$$8\pi\tau^0_0=\frac{2}{RD^2}\frac{d}{dx}V_1(x=1)=-\frac{4(\alpha+\kappa)}{RD^3}, \tag{9}$$

$$8\pi\tau^2_2=8\pi\tau^3_3=\frac{1}{RD^2}\frac{d}{dx}[V_1(x=1)+U_1(x=1)]$$

$$=\frac{2\kappa}{RD^2(1-\kappa)}. \tag{10}$$

Obviously, the stresses in the shell vanish in the extreme Reissner-Nordström case $\kappa=0$, and they diverge in the collapse limit $\kappa=1$. In the sequel, it is useful to introduce the quantity

$$\Delta\tau=2\pi RCD^5(\tau^3_3-\tau^0_0)=\frac{C^2D}{2}(2\alpha+3\kappa-\kappa^2), \tag{11}$$

which appears also in the weak energy condition [9]: $T_{\mu\nu}u^\mu u^\nu\geqslant 0$ for all timelike vectors u^μ, being equivalent to $\Delta\tau\geqslant 0$, and $-\tau^0_0\geqslant 0$ for our shell. "Below" the parabola $(\kappa-\frac{3}{2})^2=2(\alpha+\frac{9}{8})$ in the (α,κ) plane, the energy condition $\Delta\tau\geqslant 0$ is violated. Below the line $\kappa=-\alpha$, also the other part $-\tau^0_0\geqslant 0$ of the weak energy condition is violated. (See Figs. 1–5.) For completeness let us also state the dominant energy condition: $|\tau^0_0|\geqslant|\tau^3_3|$ which is violated "above" the hyperbola $\kappa=\frac{1}{6}(1-4\alpha+\sqrt{1+16\alpha+16\alpha^2})$ approaching the asymptote $\kappa=\frac{1}{2}$ for $\alpha\to\infty$, i.e., in a strip below the collapse line $\kappa=1$. The strong energy condition reads $2\tau^3_3-\tau^0_0\geqslant 0$, and it is violated below the hyperbola $\kappa=-\alpha/(2+\alpha)$, approaching $\kappa=-1$ for $\alpha\to\infty$.

The electromagnetic field tensor belonging to the Reissner-Nordström metric (1) has only a radial component $-F_{\tau\rho}=E_\rho=q/\rho^2$. Transformation of this component to our isotropic coordinates, under the condition that the charge q be concentrated solely on the shell at $r=R$, leads to $-F_{tr}$

$=E_r=(q/r^2)e^{U-V}H(r-R)$, where $H(r-R)$ is the Heaviside function, and the potentials (5) have to be inserted for $r \geqslant R$. Herewith, and with the inhomogeneous Maxwell equation $(1/\sqrt{-g})(\sqrt{-g}F^{t\mu})_{,\mu}=4\pi j^t$, the charge density $\sigma=j^t$ at the shell at $r=R$ can be calculated: For the metric (2), we have $\sqrt{-g}=e^{(3V+U)}r^2\sin\vartheta$, and only $F^{tr}=e^{-2(U+V)}E_r$ is nonzero. Therefore, $\sqrt{-g}F^{tr}$ is equal to $q\sin\vartheta$ (independent of r) for $r\geqslant R$, and zero for $r<R$. The r derivative of this expression gives then a δ-function-type charge density:

$$\sigma(x)=\frac{qC}{4\pi R^3D^3}\delta(x-1). \tag{12}$$

III. FIRST ORDER ROTATION OF THE SHELL

In this section the notation and the details of the calculations are essentially taken from [5], Sec. III, but we repeat some of these calculations in order to make the present article self-contained. We write the rotational extension of the metric (2) for physical intuition in the form

$$ds^2=-e^{2U(r)}dt^2+e^{2V(r)}\{dr^2+r^2d\vartheta^2+r^2\sin^2\vartheta \times[d\varphi-\omega A(r)dt]^2\}, \tag{13}$$

neglecting, however, in the following all terms of second and higher order in ω. In this first order of ω, the potentials $U(r)$ and $V(r)$ can be taken over unchanged from Sec. II, and $A(r)$ is independent of ω. The radial dependence of the dragging function $A(r)$ is essentially given by the Einstein equation

$$G_3^0=-\frac{\omega\sin^2\vartheta}{2\rho^2}\frac{d}{d\rho}\left(\rho^4\frac{d}{d\rho}A(\rho)\right)=8\pi(T_3^0+S_3^0). \tag{14}$$

The electric components of the field tensor $F_{\mu\nu}$ reduce in our approximation to the field E_r from Sec. II. The magnetic components are of order ω, because a magnetic field results only from the induction due to the rotating shell. However, since there are no electric currents in the r and ϑ directions in our model, the component $B_\varphi=F_{r\vartheta}$ is identically zero, according to the inhomogeneous Maxwell equations. Therefore, in the first order of ω, there remains only one nontrivial homogeneous Maxwell equation

$$\frac{d}{dr}B_r+\frac{d}{d\vartheta}B_\vartheta=0, \tag{15}$$

and one nontrivial inhomogeneous Maxwell equation

$$4\pi j^\varphi=\frac{1}{\rho^2\sin^2\vartheta}\frac{d}{d\rho}[F(\rho)B_\vartheta]-\frac{1}{\rho^4\sin\vartheta}\frac{d}{d\vartheta}\left(\frac{B_\rho}{\sin\vartheta}\right)+\frac{\omega}{\rho^2}\frac{d}{d\rho}(\rho^2AE_\rho). \tag{16}$$

In the exterior region, Eq. (14) reads, with $E_\rho=q/\rho^2$:

$$-\frac{\omega\sin^2\vartheta}{4}\frac{d}{d\rho}\left(\rho^4\frac{d}{d\rho}A(\rho)\right)=qB_\vartheta. \tag{17}$$

This equation, together with the fact that in the limit $q\to 0$ also the magnetic field should vanish in our model, suggests the ansatz $B_\vartheta=\omega q f(r)\sin^2\vartheta$, with a dimensionless function $f(r)$. Equation (15) then enforces the form $B_r=\omega qRg(r)\sin\vartheta\cos\vartheta$, with $f(r)=-(R/2)dg(r)/dr$. Because of continuity across the shell, the forms for B_ϑ and B_r are also valid in the interior of this shell. Then Eqs. (14) and (16) constitute two coupled ordinary differential equations for the unknown functions $A(r)$ and $g(r)$.

Inside the shell, these equations decouple, and read

$$\frac{d}{dr}\left(r^4\frac{d}{dr}A\right)=0,\quad \frac{d^2}{dr^2}g-\frac{2}{r^2}g=0. \tag{18}$$

The solutions, which are regular at $r=0$, are given by

$$A_2(r)=\mu,\quad g_2(r)=\eta r^2/R^2, \tag{19}$$

with dimensionless constants μ and η, which have later to be fixed by continuity at the shell. Because of $A_2(r)=\text{const}$, the interior region stays flat in first order perturbation in ω, as is physically to be expected. The magnetic field components B_r and B_ϑ represent in Cartesian coordinates a constant field $B_z=(\omega q/R)\eta$ along the z axis, as is well known for the interior of a charged, rotating shell from classical electrodynamics.

In the exterior region, due to $B_\vartheta\sim f(r)\sim dg(r)/dr$, one integration of Eq. (17) is straightforward:

$$\frac{d}{d\rho}A(\rho)=\frac{1}{\rho^4}[2q^2Rg(\rho)-4MR^2\lambda], \tag{20}$$

with a dimensionless integration constant λ. Insertion of Eq. (20) into Eq. (16), together with $j^\varphi\equiv 0$ outside the shell, results in the differential equation for $g(\rho)$:

$$\frac{d}{d\rho}\left[F(\rho)\frac{d}{d\rho}g(\rho)\right]-\frac{2}{\rho^2}\left(1+\frac{2q^2}{\rho^2}\right)g(\rho)=-\frac{8MR\lambda}{\rho^4}. \tag{21}$$

We write the general solution of this equation in the form

$$g_1(r)=\lambda\hat{g}(\rho(r))+\xi\overline{g}(\rho(r))+\zeta\overline{\overline{g}}(\rho(r)), \tag{22}$$

with dimensionless integration constants ξ,ζ, where $\hat{g}(\rho)$ is a special solution of the inhomogeneous equation (21), and $\overline{g}(\rho)$, $\overline{\overline{g}}(\rho)$ are fundamental solutions of the corresponding homogeneous equation. As first observed in [7], and independently rediscovered in [5], these solutions can be given in explicit, analytic form:

$$\hat{g}(\rho)=\frac{4R}{3\rho},\quad \overline{g}(\rho)=\frac{1}{R^2}\left(\rho^2-3q^2+\frac{2q^4}{M\rho}\right), \tag{23}$$

$$\bar{\bar{g}}(\rho)=\frac{3M^2R}{4(M^2-q^2)^2}\left[\frac{2q^2}{3\rho}\left(1+\frac{2q^2}{M^2}\right)-\rho-M+R^2\bar{g}(\rho)S(\rho;M,q)\right], \tag{24}$$

with

$$S(\rho;M,q)=\begin{cases}\frac{1}{\sqrt{q^2-M^2}}\operatorname{arccot}\left(\frac{\rho-M}{\sqrt{q^2-M^2}}\right) & \text{for } q^2>M^2,\\ \frac{1}{2\sqrt{M^2-q^2}}\log\left(\frac{\rho-M+\sqrt{M^2-q^2}}{\rho-M-\sqrt{M^2-q^2}}\right) & \text{for } q^2<M^2,\end{cases}$$

where we define the branch of $y=\operatorname{arccot}(z)$ such that it goes from π to zero in the range $z\to-\infty$ to $z\to+\infty$.

Since $\bar{g}(\rho)$ diverges for $\rho\to\infty$, the integration constant ξ in Eq. (22) has to be set to zero. The normalization of $\bar{\bar{g}}(\rho)$ has been chosen such that it behaves asymptotically as R/ρ (independent of M and q). After transformation to Cartesian coordinates, all components of the magnetic field have the asymptotic behavior $B_i\sim\rho^{-3}\sim r^{-3}$, as is physically expected. With the general magnetic field function $g_1(r)$ available in the exterior region, we can also calculate the general dragging function $A_1(r)$ by integrating Eq. (20). If we write $A_1(r)$ in the suggestive form

$$A_1(r)=\frac{2q^2}{R^2}[\lambda\hat{A}(\rho(r))+\xi\bar{A}(\rho(r))+\zeta\bar{\bar{A}}(\rho(r))]+\frac{4MR^2\lambda}{3(\rho(r))^3}, \tag{25}$$

we get

$$\hat{A}(\rho)=-\frac{R^4}{3\rho^4},\quad \bar{A}(\rho)=R\left(-\frac{1}{\rho}+\frac{q^2}{\rho^3}-\frac{q^4}{2M\rho^4}\right), \tag{26}$$

$$\bar{\bar{A}}(\rho)=\frac{3MR^4}{8(M^2-q^2)^2}\left[-\frac{1}{\rho}+\frac{M}{\rho^2}+\frac{q^2+2M^2}{3\rho^3}-\frac{q^2(M^2+2q^2)}{3M\rho^4}+\left(1+\frac{2M}{R}\bar{A}(\rho)\right)S(\rho;M,q)\right]. \tag{27}$$

For $\rho\to\infty$, $\bar{\bar{A}}(\rho)$ behaves like $-R^4/4\rho^4$.

In the process of integration of the magnetic field function g, and of the dragging factor A in the two regions, we had to introduce a total of four nontrivial integration constants: μ, η, λ, and ζ. These constants have now to be fixed by the continuity conditions for these functions at the shell position $r=R$, or $\rho=RD$, and by the appropriate discontinuity conditions for their radial derivatives. In this connection it should be remarked quite generally that the function $S(\rho;M,q)$, introduced in Eq. (24), attains the relatively simple values

$$RS(RD;M,q)=\begin{cases}\frac{1}{2\sqrt{-\kappa}}\operatorname{arccot}\left(\frac{1+\kappa}{2\sqrt{-\kappa}}\right) & \text{for } \kappa<0,\\ 1 & \text{for } \kappa=0,\\ \frac{1}{2\sqrt{\kappa}}\log\left(\frac{1+\sqrt{\kappa}}{1-\sqrt{\kappa}}\right) & \text{for } \kappa>0,\end{cases} \tag{28}$$

which makes the introduction of the parameter $\kappa=\alpha^2-\gamma^2$ especially profitable. Furthermore, the r and ρ derivatives of all functions at the outer and inner edge of the shell are connected by the factors

$$\left.\frac{d\rho}{dr}\right|_{R_+}=\frac{D}{C}=1-\kappa,\quad \left.\frac{d\rho}{dr}\right|_{R_-}=D. \tag{29}$$

The continuity conditions $g_1(r=R)=g_2(r=R)$ and $A_1(r=R)=A_2(r=R)$ lead to homogeneous, linear equations between the integration constants

$$-\eta+\frac{4}{3D}\lambda+\bar{\bar{g}}(RD)\zeta=0, \tag{30}$$

$$-\frac{1}{8}\mu+\frac{\alpha D-\gamma^2}{3D^4}\lambda+\gamma^2\bar{\bar{A}}(RD)\zeta=0. \tag{31}$$

The discontinuities of the radial derivatives of $A(r)$ and $g(r)$ are determined, respectively, by the mass and charge currents of the shell. The energy-momentum tensor $T^\mu_\nu=\tau^\mu_\nu\delta(r-R)$ of the shell has of course to satisfy the eigenvalue equations $T^\mu_\nu u^\nu=-\varrho_0 u^\mu$, where $u^\mu=u^0(1,0,0,\omega)$ is the purely axial, and, due to rigid rotation, ϑ-independent four-velocity vector of the shell matter, and ϱ_0 is the rest-energy density. Comparison of the components $\mu=0$ and $\mu=3$ of the eigenvalue equations, together with the metric form (13) and the definition (11), gives, in first order of ω,

$$8\pi\tau^0_3=\frac{4\omega RC\sin^2\vartheta}{D^3}(1-\mu)\Delta\tau. \tag{32}$$

On the other hand, integration of the Einstein equation (14), together with Eq. (20), gives

$$8\pi\tau^0_3=-\frac{1}{2}\omega R^2C^2\sin^2\vartheta\frac{d}{dr}A_1(r=R)=\frac{4\omega RC\sin^2\vartheta}{D^3}\left[\left(\alpha-\frac{4\gamma^2}{3D}\right)\lambda-\gamma^2\bar{\bar{g}}(RD)\zeta\right]. \tag{33}$$

Comparison of Eqs. (32) and (33) leads to the inhomogeneous equation

$$\Delta\tau\,\mu+\left(\alpha-\frac{4\gamma^2}{3D}\right)\lambda-\gamma^2\overline{\overline{g}}(RD)\zeta=\Delta\tau. \tag{34}$$

In Eq. (16), obviously only the ρ derivatives of B_ϑ and E_ρ contribute localized currents proportional to $\delta(x-1)$, with the result

$$4\pi j^{\varphi}=\frac{\omega qC}{R^3D^3}\left[-\frac{R^2}{2CD}\frac{d^2}{dr^2}g(r=R)+\mu\ \delta(x-1)\right]. \tag{35}$$

If we assume that the charge elements have the same four-velocity $u^{\mu}=u^0(1,0,0,\omega)$ as the mass elements (as is, e.g., the case, if the shell consists of isolator material), j^{φ} can also be expressed by the charge density from Eq. (12):

$$4\pi j^{\varphi}=4\pi\omega\sigma(x)=\frac{\omega qC}{R^3D^3}\delta(x-1). \tag{36}$$

Comparison of Eqs. (35) and (36) gives, after integration from $x=1-\epsilon$ to $x=1+\epsilon$, the last inhomogeneous equation between the integration constants

$$\mu+\frac{1}{CD}\eta+\frac{1}{2C^2}\left[\frac{4}{3D^2}\lambda-R\frac{d}{d\rho}\overline{\overline{g}}(RD)\zeta\right]=1. \tag{37}$$

In total, Eqs. (30), (31), (34), and (37) comprise the inhomogeneous, linear system

$$\begin{pmatrix} 0 & -1 & \frac{4}{3D} & \overline{\overline{g}}(RD)\\ -\frac{1}{8} & 0 & \frac{\alpha D-\gamma^2}{3D^4} & \gamma^2\overline{\overline{A}}(RD)\\ \Delta\tau & 0 & \alpha-\frac{4\gamma^2}{3D} & -\gamma^2\overline{\overline{g}}(RD)\\ 1 & \frac{1}{CD} & \frac{2}{3C^2D^2} & \frac{-R}{2C^2}\frac{d}{d\rho}\overline{\overline{g}}(RD)\end{pmatrix}\begin{pmatrix}\mu\\ \eta\\ \lambda\\ \zeta\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \Delta\tau\\ 1\end{pmatrix}, \tag{38}$$

from which all four integration constants μ, η, λ, and ζ can be calculated in a straightforward manner. We have numerically checked that the determinant of this system is nonzero (in fact negative) in the whole physical region of the model parameters α and κ, as described in Sec. II.

IV. RESULTS AND DISCUSSION

We see essentially five properties of our massive and charged rotating shell which deserve a detailed analysis and physical discussion: This is first the dragging constant μ in its dependence on the model parameters $\alpha=M/2R$ and $\kappa=(M^2-q^2)/4R^2$, and the question whether the usual dragging of the inertial frames turns over to antidragging (negative μ) for such model parameters for which the energy conditions in the shell material are violated. Secondly, it is of interest how the strength η of the (constant) magnetic field, induced inside the shell, depends on α and κ, and whether it also can become zero or change the sign. In detail, it is more natural to analyze not η by itself but the constant $\tilde{\eta}=\eta/C$, defining the interior magnetic field, measured by a coordinate-stationary observer in his proper time τ. Thirdly, the total angular momentum of our system can be read off from $\lim_{r\to\infty}A(r)=4MR^2\lambda/3r^3$ as $J=\frac{2}{3}\omega MR^2\lambda$, i.e., it is essentially given by the constant λ. A sometimes useful alternative definition of J can be given, according to Tolmans theorem, by the Komar-like integral

$$J=\int_0^{\infty}dr\int_0^{\pi}d\vartheta\int_0^{2\pi}d\varphi\sqrt{-g}[T^0_3+S^0_3]$$
$$=\frac{4\omega R^3}{3}[(1-\mu)\Delta\tau+\gamma^2\eta]. \tag{39}$$

Indeed, it can be easily checked [e.g., by comparing the terms proportional to $(\Delta\tau)^k$] that both expressions for J coincide. As a fourth property we consider the magnetic moment of the shell. From classical electrodynamics it is well known that a magnetic moment $\mathbf{m}$ produces a magnetic field $\mathbf{B}(\mathbf{r})=[3\mathbf{e}_r(\mathbf{e}_r\cdot\mathbf{m})-\mathbf{m}]r^{-3}$, resulting in a component $B_r=(2m/r)\sin\vartheta\cos\vartheta$ in isotropic coordinates. Comparing this with the asymptotic behavior $\lim_{r\to\infty}g(r)=(\zeta+\frac{4}{3}\lambda)R/r$ of the magnetic field of our rotating shell, allows us to read off its magnetic moment as

$$m=\frac{1}{3}\omega qR^2\left(\frac{3}{2}\zeta+2\lambda\right), \tag{40}$$

where $\frac{1}{3}\omega qR^2$ is the magnetic moment of a rotating charged shell in classical electrodynamics. (We do not know of any formula for the magnetic moment as a volume integral in a curved manifold.) Therefore, for calculating and discussing the magnetic moment m, we need, besides λ, in an essential way the integration constant ζ. Fifthly, a quantity of high physical interest is the gyromagnetic ratio of our shell system. Quite generally, the gyromagnetic ratio of a charged, rotating system is defined by $G=2Mm/qJ$. Inserting the results for J and m, gives

$$G=2+\frac{3\zeta}{2\lambda}. \tag{41}$$

The physical constants μ, $\tilde{\eta}$, λ, ζ, and G, in their dependence on the model parameters α and κ, are in principle given explicitly and analytically by solving the linear system (38), and inserting the values $\overline{\overline{g}}(RD)$, $d\overline{\overline{g}}(RD)/d\rho$, and $\overline{\overline{A}}(RD)$ from Eqs. (24), (27), and (28). Since these expressions are, however, algebraically quite involved, we find that the simplest and most informative presentation of our results can be given graphically by drawing the level lines of μ, $\tilde{\eta}$, λ, ζ, and G for representative values in the (α,κ) plane.

For the dragging constant μ, this is done in Fig. 1. We see that in the collapse limit $\kappa \to 1$ we have total dragging $\mu = 1$ in the interior of the charged mass shell. This result was already derived in [7], and it generalizes the important (Machian) result of Brill and Cohen [1] that a collapsing rotating mass shell leads to total dragging, to a strongly charged shell. Also for arbitrary $\kappa < 1$ and $\alpha \to \infty$, the constant μ reaches the value $\mu = 1$. Figure 1 shows also increasingly oblique curves for decreasing values of μ until we arrive at a curve (with slope $d\kappa/d\alpha = -\frac{3}{4}$ at the origin, and the asymptotic slope $d\kappa/d\alpha \approx -0.44$) the value $\mu = 0$: no dragging. This curve lies everywhere in the energy condition violating region $\Delta\tau < 0$, depicted in gray shadow, but "above" the region $-\tau_0^0 < 0$, depicted in dark gray shadow. Below the curve $\mu = 0$ we reach negative values of the dragging constant μ. This interesting antidragging phenomenon was already observed in [7], Sec. VI, case D, but only for the weak field region $\alpha \ll 1$, $\gamma \ll 1$, where μ behaves like $\mu = \frac{8}{9}(3\alpha + 4\kappa)$. Physically, the antidragging phenomenon is of course caused by the violation of the weak energy condition of the shell material. (In a trivial manner, we could have antidragging already for the uncharged, weakly massive Thirring model in the case $M < 0$. In our model class, "playing" with the two parameters M/R and q/R allows us to have antidragging also for the positive total mass of the system.) In the limit of a shell with vanishing invariant radius [i.e., $\kappa = -(1+2\alpha)$] the dragging constant μ reaches even the value $\mu \to -\infty$. For vanishing total mass, respectively energy, we have antidragging in the whole admissible κ interval $-1 \leqslant \kappa \leqslant 0$, starting from the value $\mu = -8q^2/9R^2$ for $\kappa \ll 1$, and growing monotonically to the value $-\infty$ for $\kappa \to -1$. This antidragging behavior is again a consequence of a violation of the energy conditions: Because of $\alpha = 0$ and $\kappa < 0$ obviously all components $T^{\mu\nu}$ of the energy-momentum tensor of the shell are negative, according to Eqs. (9) and (10). But this is to be expected according to the positive energy theorem: For total energy $M = 0$ "nontrivial phenomena" occur only if the energy conditions are violated, otherwise necessarily spacetime is globally flat (which in our model class results only for $\alpha = \kappa = 0$).

Figure 2 depicts in a similar way the constant $\tilde{\eta}$, representing the (constant) magnetic field inside the shell, measured by a coordinate-stationary observer in his proper time τ. This constant is zero in the collapse limit $\kappa = 1$, as already observed and discussed in [6] and [7], but also in the limit of a shell with vanishing invariant radius. Otherwise, $\tilde{\eta}$ is positive in the whole physical range of the parameters α, κ, and reaches arbitrarily high values, e.g., on the line $\kappa = -\alpha$ (i.e., $\tau_0^0 = 0$) for $\alpha \to \infty$ it grows like α. For fixed values of κ, and $\alpha \to \infty$, $\tilde{\eta}$ attains finite values. In the weak field limit $\alpha \ll 1$, $\gamma \ll 1$, $\tilde{\eta}$ attains of course the limit $\tilde{\eta} = \frac{2}{3}$, known from classical electrodynamics. In the collapse limit $\kappa = 1$, the gravitational and electromagnetic fields in the exterior region $r > R$ constitute the Kerr-Newman field (in first order of ω), according to the no hair theorem (see also [5]).

In Fig. 3 we have plotted the level lines of the angular momentum J (measured in units of $R^3\omega$). In the (α,κ) region with $\Delta\tau \geqslant 0$ the function J is monotonically increasing for fixed values of κ, and increasing values of α. In the part of the parameter space with $\Delta\tau < 0$, there exists a curve with $J = 0$. There the positive contribution of the electromagnetic field to the total angular momentum compensates the negative mechanical ones, coming from the surfaces stresses and from the rest mass density. Below this curve J is negative, e.g., for $\alpha = 0$ and $-1 < \kappa < 0$. The expression (39) for J shows that this is due to the violation of the energy condition $\Delta\tau \geqslant 0$, since the electromagnetic contribution $\gamma^2\eta$ to $3J/4\omega R^3$ is, as already said, everywhere positive, according to Fig. 2, and since $\mu \leqslant 1$, according to Fig. 1. The angular momentum is, in addition to the case described above, zero in the trivial case $\alpha = \kappa = 0$, and for an infinitely small shell [i.e., $\kappa = -(1+2\alpha)$]. In the weak field limit J reaches the classical value $\frac{2}{3}MR^2\omega$ of Newtonian physics.

Figure 4 shows a behavior of the magnetic moment m (measured in units of $\frac{1}{3}\omega q R^2$) rather similar to the angular

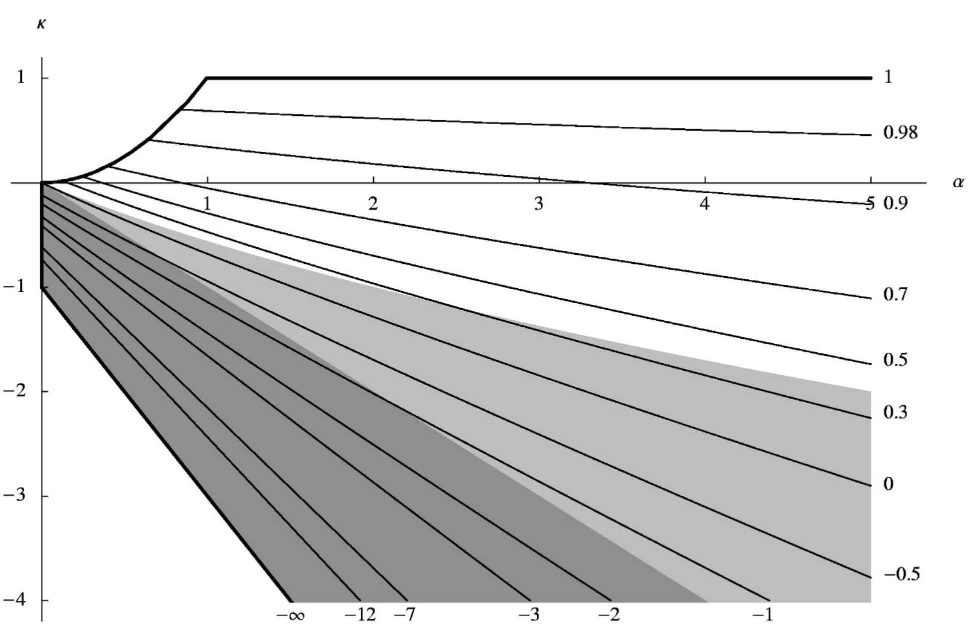

FIG. 1. Level lines for representative values of the interior, constant dragging function A in the physical region of the model parameters $\alpha = M/2R$ and $\kappa = (M^2 - q^2)/4R^2$. The energy condition violating region $\Delta\tau < 0$ is shown in light gray shadow; the region where the condition $-\tau_0^0 \geqslant 0$ also is violated is shown in dark gray shadow. The boundaries of the presented region in the (α,κ) plane are given (from above to below) by the conditions that the shell lies outside the horizon, the charge is real, the mass is non-negative, and the invariant radius of the shell is non-negative.

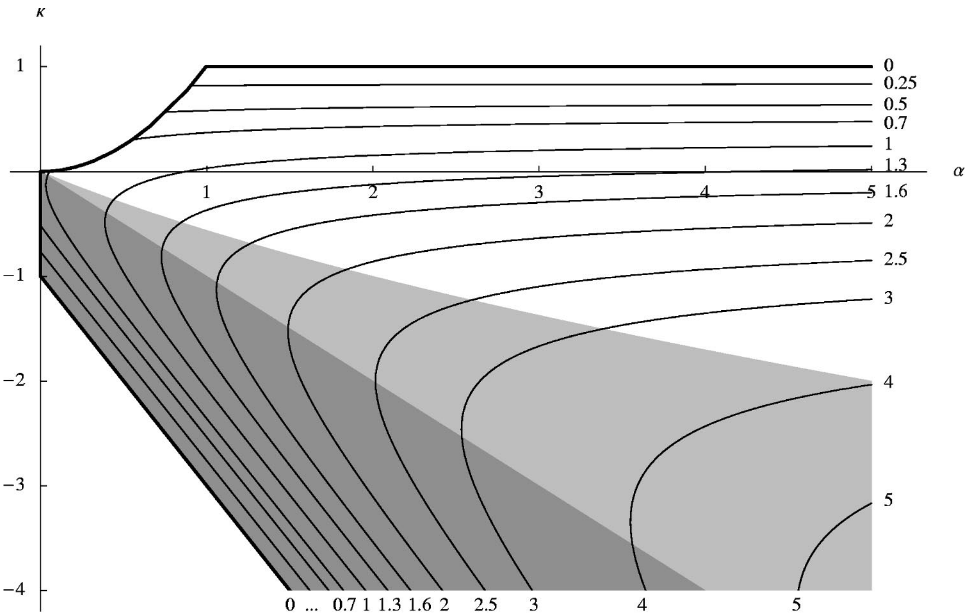

FIG. 2. Level lines for representative values of the interior, constant magnetic field B_z [in units of $\omega qC/R$, with C from Eq. (7)] in the physical region of the model parameters $\alpha = M/2R$ and $\kappa = (M^2 - q^2)/4R^2$. Concerning boundaries of the region, and gray shadows, see the caption of Fig. 1.

momentum J in its dependence on the model parameters α and κ. Only the zero of the function m does not start at $\alpha = \kappa = 0$, but at the value $\alpha = 0, \kappa \approx -0.15$. The dependence of the angular momentum J and the magnetic moment m as functions of the charge-to-mass and radius-to-mass ratios has been illustrated for special values of q/M in [8].

Finally, we come to the especially interesting and important discussion of the gyromagnetic ratio $G = 2 + (3\zeta/2\lambda)$ of our rotating shell. In the ratio $3\zeta/2\lambda$, the algebraically complicated determinant of the linear system (38) cancels, with the result

$$G - 2 = \frac{3\zeta}{2\lambda} = \frac{8\kappa^2(1-\kappa)[-\kappa^2 + \kappa(5+\alpha) + 3\alpha]}{P(5;4) + 3\alpha(1-\kappa)P(5;3)RS(\kappa)},$$

with $RS(\kappa)$ from Eq. (28), and where $P(i;j)$ denote polynomials of order i in κ, and order j in α. Notwithstanding these algebraic simplifications, the gyromagnetic ratio G has a very delicate and interesting behavior in the (α, κ) plane, as is shown in Fig. 5, and which comes about essentially through the interaction between strong gravitational and electric fields: G approaches the value 2 in the collapse limit $\kappa \to 1$, with a correction term $G - 2 = -4\epsilon/3\alpha(1+\alpha)^2$ for $\kappa = 1 - \epsilon$, which, although being linear in ϵ, becomes very small for large α. For fixed κ, and $\alpha \to \infty$, G approaches 2 with a correction term proportional to α^{-3} (and proportional to α^{-4} for $\kappa = 1$ and $\kappa = -3$). $G - 2$ is identically zero on a branch of the hyperbola $\kappa = \frac{1}{2}(5 + \alpha - \sqrt{25 + 22\alpha + \alpha^2})$, which approaches $\kappa = -3$ for $\alpha \to \infty$. Also in the lower part of the figure, G approaches the value $G = 2$ very rapidly, e.g., in all asymptotic directions with slope $-2 \leqslant d\kappa/d\alpha < -0.698$ the correction term is proportional to $\alpha^{-5/2}$. On the other hand, G diverges, due to $J = 0$, on a curve which starts with slope $d\kappa/d\alpha = -\frac{6}{5}$ at the origin, and reaches an

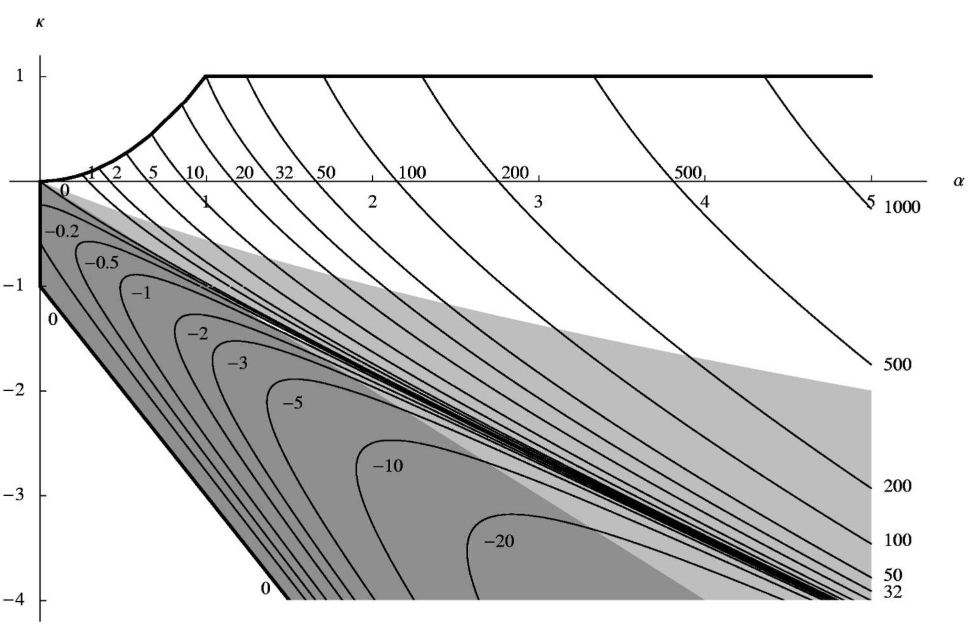

FIG. 3. Level lines for representative values of the angular momentum J (measured in units of ωR^3) in the physical region of the model parameters $\alpha = M/2R$ and $\kappa = (M^2 - q^2)/4R^2$. Concerning boundaries of the region, and gray shadows, see the caption of Fig. 1.

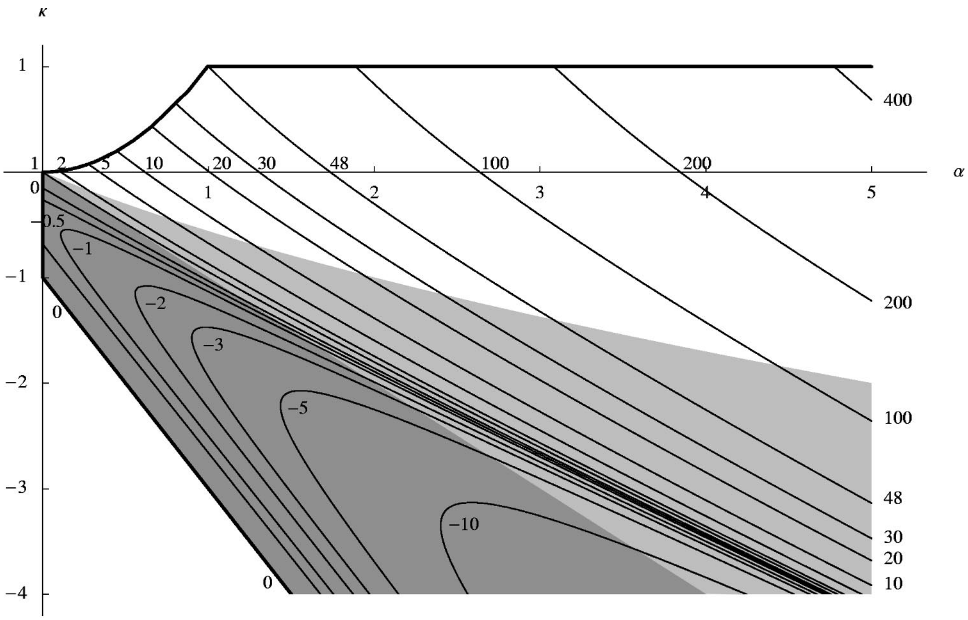

FIG. 4. Level lines for representative values of the magnetic moment m (measured in units of $\frac{1}{3}\omega q R^2$) in the physical region of the model parameters $\alpha = M/2R$ and $\kappa = (M^2 - q^2)/4R^2$. Concerning boundaries of the region, and gray shadows, see the caption of Fig. 1.

asymptotic slope $d\kappa/d\alpha \approx -0.697$. Below this curve there exists a region with negative values of G, due to $J<0$, and $m>0$ in this part of parameter space (see Figs. 3 and 4). However, the region with $G<0$ is extremely small: it starts with a width $\Delta\kappa \approx 0.15$ at $\alpha = 0$, which diminishes very rapidly with growing α. The reason for this is that the magnetic moment $m = \frac{1}{3}\omega q R^2(\frac{3}{2}\zeta + 2\lambda)$ changes from a positive to negative sign at the lower end of the region $G<0$, because there the contribution of $\frac{3}{2}\zeta$ (which is positive below the curve $G=2$) can no longer compensate the negative contribution from 2λ. At the left edge of the figure ($\alpha = 0, -1 \leqslant \kappa \leqslant 0$) we have $G \equiv 0$. Remarkable is also that the weak field limit of G (for $\alpha \ll 1$, $\kappa \ll 1$) is ''direction dependent'': $G \to [1 + (5\kappa/6\alpha)]^{-1}$. Only approaching the origin of the (α,κ) plane from the upper-right quadrant (in the under-extreme or extreme Reissner-Nordström case), leads to the classical value $G=1$.

Some special cases of this behavior of the gyromagnetic ratio G have already been discussed in [6–8] (see also [10] for the behavior of G in a non-shell-like model system). We find it very interesting, however, how G behaves in the whole physical range of the parameters α and κ, especially the ''extreme'' behavior of G around the line $J=0$, and the fact that G is very near to the value $G=2$ in an overwhelming part of the (α,κ) plane. The latter fact may be seen more impressively in the three-dimensional plot $G(\alpha,\kappa)$ of Fig. 6 than from the level lines of Fig. 5. (This fact, for the special value $q/M = 1.01$, may also be read off from Fig. 5a of [8], but it was not commented and discussed there in a more general way.)

Now, it is well known that a value $G=2$ results in quantum mechanics if the simplest rotating object (a spin-$\frac{1}{2}$ particle without inner structure) is minimally coupled (according to the gauge principle) to an electromagnetic field, and

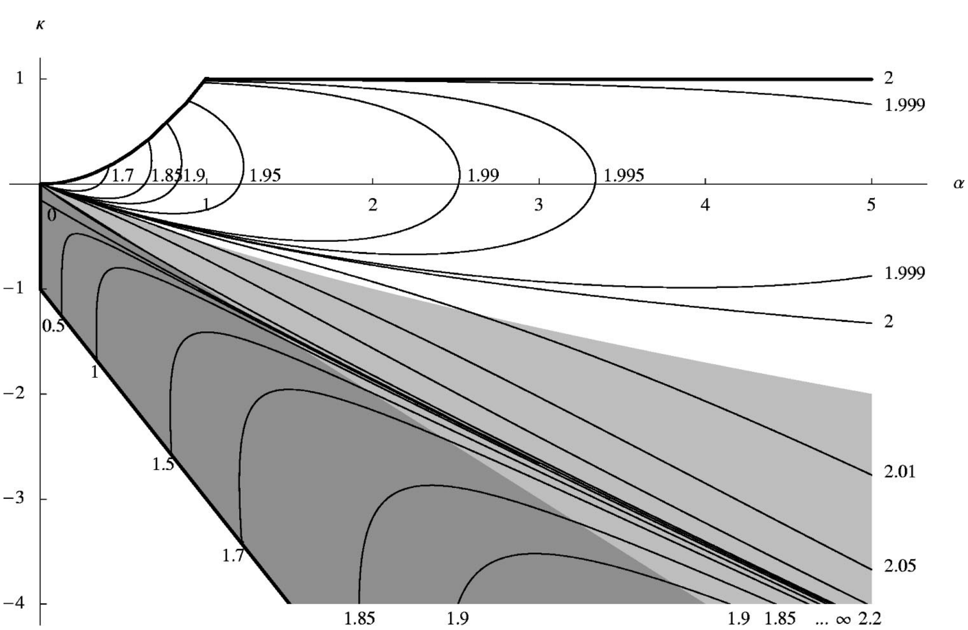

FIG. 5. Level lines for representative values of the gyromagnetic ratio G in the physical region of the model parameters $\alpha = M/2R$ and $\kappa = (M^2 - q^2)/4R^2$. Concerning boundaries of the region, and gray shadows, see the caption of Fig. 1.

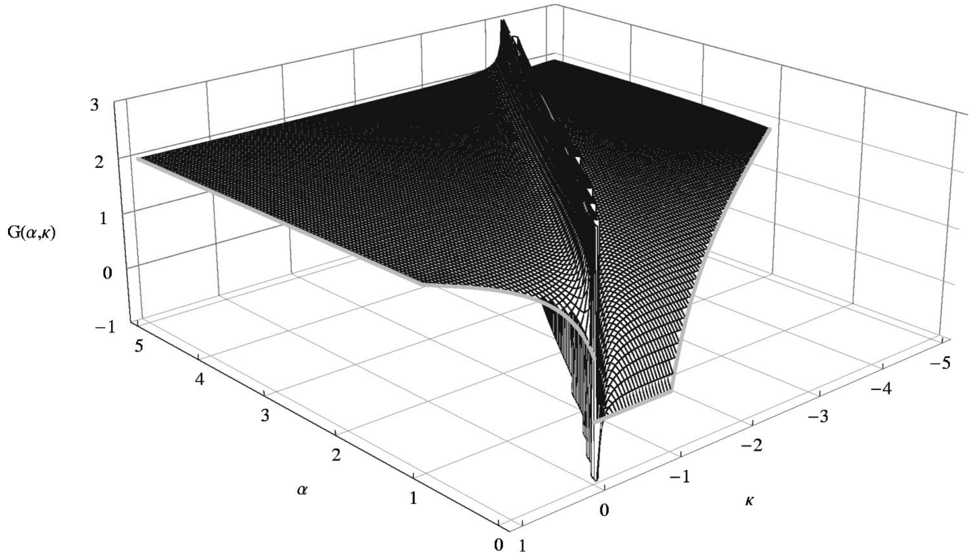

FIG. 6. A three-dimensional plot of the gyromagnetic ratio $G(\alpha,\kappa)$, showing especially drastic that $G\approx 2$ in an overwhelming part of the physical (α,κ) plane.

this equally in a non-relativistic (Galilei-covariant) and in a relativistic (Poincaré-covariant) scheme. Furthermore, the value $G=2$ is nearly perfectly realized in nature for some of the most elementary particles: the electron and the muon. And there are even strong arguments for $G=2$ also for other elementary particles of the standard model, e.g., for the W boson [11]. A value $G=2$ is, however, hardly reachable in any classical model (without strong gravity), and if it is reachable in special models (see, e.g., [12]), $G=2$ is by no means a natural or preferred value. Insofar as it is very remarkable that general relativity predicts for some of its simplest and most unique solutions, the Kerr-Newman class of back hole solutions, also the value $G=2$, as was first observed by Carter [13]. (This result extends also to the charged Tomimatsu-Sato solutions [14,15], and to a large class of other solutions.) That the same value results for rotating, charged shell models in the collapse limit [6,7], is to be expected from the no hair theorems. The new and surprising result of our analysis is that this value $G=2$ is ''extremely robust'' in the sense that in a big part of the parameter space (α,κ) in Figs. 5 and 6, and not only in the collapse limit, G deviates from the value $G=2$ only by a very small amount. And this is true not only in regions of (α,κ) where, due to the shell structure of our model, the stresses $\tau_2^2=\tau_3^3$ are unrealistically high (as compared to the energy density $-\tau_0^0$) but also for values $|\kappa|\ll 1$ where the stresses are arbitrary small.

We like to argue that this ''naturalness'' of $G=2$ for a large class of rotating shell systems with strong gravitational and electric fields, and its ''coincidence'' with the value $G\approx 2$ for the most elementary quantum particles, signals a deep connection between general relativity and quantum theory, which should serve as a guideline and control in any attempt to build some ''quantum gravity.'' (Remember the importance of the coincidence between inertial and gravitational mass as a cornerstone of Einstein's general relativity theory, and the importance of the coincidence between the classical and quantum Rutherford formula for the birth of quantum mechanics.) To make this argument even more convincing, it would be helpful to extend our analysis of the gyromagnetic ratio to rapidly rotating charged mass shells, or even to full (non-shell-like) charged bodies. It is, however, to be expected that such an analysis can be sucessfully performed, if any, only numerically. And it is to be expected that the results depend (severely?) on the detailed material properties of the rotating bodies, as they show up, e.g., in the centrifugal deformation, in the nonspherical distribution of the mass and charge densities, and in possible differential rotations (compare [16] and [17]). Indeed, such a numerical study of the G factor for more realistic charged, rotating bodies, and also for high values of ω, will now be started by a group in Meudon (E. Gourgoulhon and J. Novak). In contrast to [7] we resist, however, from describing quantum particles like the electron and the proton literally by our models. Besides the unrealistic shell structure, such an identification would have the irritating consequences that one would be in the range of extremely weak electric and gravitational fields (values $\alpha\approx 10^{-43}$ for the electron, and $\alpha\approx 10^{-36}$ for the proton), that the velocity ωR (calculated from $J=\frac{1}{2}\hbar$) would be in the range of 250 c for the electron, and bigger than 1000 c for the proton (what would also be inconsistent with our first order approximation in ω), and that the ''radius'' would be much smaller for the proton than for the electron. In conclusion, it may be remarked that the value $G=2$ was also derived for a special (Majumdar-Papapetrou-like) class of supergravity solitons [18].

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