Supergravity and the Poincaré group

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(Received 25 May 2001; published 8 April 2002)

An action for $(3+1)$-dimensional supergravity genuinely invariant under the Poincaré supergroup is proposed. The construction of the action is carried out considering a bosonic Lagrangian invariant under both local Lorentz rotations and local Poincaré translations as well as under diffeomorphism, and therefore the Poincaré algebra closes off shell. Since the Lagrangian is invariant under the Poincaré supergroup, the supersymmetry algebra closes off shell without the need for auxiliary fields.

DOI: 10.1103/PhysRevD.65.084032 PACS number(s): 04.65.+e

I. INTRODUCTION

Recently it has been shown, by means of a first-order formalism, that the three- and five-dimensional supergravities studied by Achúcarro and Townsend [1] and by Chamseddine [2], respectively, as well as higher dimensional theories can be written as Chern-Simons theories [3,4]. The action for supergravity in $2+1$ dimensions $S=\int(\varepsilon_{abc}R^{ab}e^c+4\bar{\psi}D\psi)$, with ψ a two-component Majorana spinor, is invariant under Lorentz rotations, Poincaré translations, and supersymmetry transformations. The dreibein e^a_μ, the spin connection ω^{ab}_μ, and the gravitino ψ^a_μ transform as components of a connection for the super Poincaré group. This means that the supersymmetry algebra implied by the corresponding supersymmetry transformations is the super Poincaré algebra, i.e., the supersymmetry algebra closes off shell without the need for auxiliary fields.

The action for supergravity in $3+1$ dimensions $S=\int(\varepsilon_{abcd}R^{ab}e^ce^d+4\bar{\psi}e^a\gamma_a\gamma_5D\psi)$, where $\bar{\psi}$ is the Majorana conjugation of ψ, is not invariant under local Poincaré translations. The invariance of the action requires, in accordance with the 1.5 formalism, the vanishing of the supertorsion, which implies that the connection is no longer an independent variable. Rather, its variation is given in terms of δe^a and $\delta\psi$, and differs from that dictated by group theory. As a consequence the supersymmetry algebra, acting on the spinor field, closes off shell only with auxiliary fields.

The construction of a supergravity theory without auxiliary fields in $3+1$ dimensions has remained as an interesting open problem. This problem was studied a long time ago by Kaku, Townsend, and van Nieuwenhuizen [5,6]. They found that the action for conformal supergravity is invariant under both local supersymmetries (Q and S) corresponding to the square roots of the translations P_μ and conformal boosts K_μ ($Q=\sqrt{P}$ and $S=\sqrt{K}$). They also showed that the algebra of conformal supergravity closes off shell, unlike the gauge algebra of Poincaré supergravity. For a deeper analysis of this subject, see Refs. [5,6].

It is the purpose of this paper to show that it is also possible to construct four-dimensional supergravity without auxiliary fields provided one chooses the bosonic Lagrangian in an appropriate way. In fact, the correct Lagrangian for the bosonic sector is the Hilbert Lagrangian constructed with the help of the one-form vierbein defined by Stelle and West [7] and by Grignani and Nardelli [8]. This vierbein, also called the solder form [7,9] was considered as a smooth map between the tangent space to the space-time manifold M at a point P with coordinates x^μ and the tangent space to the internal AdS space at the point whose AdS coordinates are $\zeta^a(x)$, as the point P ranges over the whole manifold M. Figure 1 of Ref. [7] illustrates that such a vierbein $V^a_\mu(x)$ is the matrix of the map betweeen the tangent space $T_x(M)$ to the space-time manifold at x^μ and the tangent space $T_{\zeta(x)}(\{G/H\}_x)$ to the internal AdS space $\{G/H\}_x$ at the point $\zeta^n(x)$, whose explicit form is given by Eq. (3.19) of Ref. [7].

Taking the limit $m\rightarrow 0$ in Eq. (3.19) of Ref. [7], we obtain $V^a_\mu(x)=D_\mu\xi^a+e^a_\mu$, which is the map between the tangent space $T_x(M)$ to the space-time manifold at x^μ and the tangent space $T_{\zeta(x)}(\{\mathrm{ISO}(3,1)/\mathrm{SO}(3,1)\}_x)$ to the internal Poincaré space $\{\mathrm{ISO}(3,1)/\mathrm{SO}(3,1)\}_x$ at the point $\zeta^n(x)$. The same result was obtained in Ref. [8] by gauging the action of a free particle defined in the internal Minkowski space.

II. GRAVITY AND THE POINCARÉ GROUP

In this section we shall review some aspects of the torsion-free condition in gravity. The main point of this section is to display the differences in the invariances of the Hilbert action when different definitions of the vierbein are used.

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A. The torsion-free condition in general relativity

The generators of the Poincaré group P_a and J_{ab} satisfy the Lie algebra

$$[P_a, P_b] = 0,$$
$$[J_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a,$$
$$[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc}. \quad (1)$$

Here the operators carry Lorentz indices not related to coordinate transformations. The Yang-Mills connection for this group is given by

$$A = A^A T_A = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab}. \quad (2)$$

Using the algebra (1) and the general form for the gauge transformations on A,

$$\delta A = \nabla \lambda = d\lambda + [A, \lambda] \quad (3)$$

with

$$\lambda = \rho^a P_a + \frac{1}{2} \kappa^{ab} J_{ab}, \quad (4)$$

we obtain that e^a and ω^{ab}, under Poincaré translations, transform as

$$\delta e^a = D\rho^a, \quad \delta \omega^{ab} = 0, \quad (5)$$

and under Lorentz rotations as

$$\delta e^a = \kappa^a_b e^b, \quad \delta \omega^{ab} = -D\kappa^{ab}, \quad (6)$$

where D is the covariant derivative in the spin connection ω^{ab}. The corresponding curvature is

$$F = F^A T_A = dA + AA$$
$$= T^a P_a + \frac{1}{2} R^{ab} J_{ab} \quad (7)$$

where

$$T^a = De^a = de^a + \omega^a_b e^b \quad (8)$$

is the torsion one-form, and

$$R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} \quad (9)$$

is the curvature two-form.

The Hilbert action

$$S_{EH} = \int \varepsilon_{abcd} R^{ab} e^c e^d \quad (10)$$

is invariant under diffeomorphism and under Lorentz rotations, but is not invariant under Poincaré translations. In fact,

$$\delta S_{EH} = 2 \int \varepsilon_{abcd} R^{ab} e^c \delta e^d$$
$$= 2 \int \varepsilon_{abcd} R^{ab} T^c \rho^d \quad (11)$$

where we see that the invariance of the action requires imposing the torsion-free condition

$$T^a = De^a = de^a + \omega^a_b e^b = 0, \quad (12)$$

which has effects on the algebra of local Poincaré transformations. If we impose this condition, then the local Poincaré translations take the form of a local change of coordinates, as we can see from the corresponding transformation laws

$$\delta_{tlp} e^a = D\rho^a + \kappa^a_b e^b, \quad (13)$$

$$\delta_{tgc} e^a = D\rho^a + \rho \cdot \omega^a_b e^b + \rho \cdot T^a. \quad (14)$$

The condition $T^a = 0$ permits replacing local Poincaré translations by a local change of coordinates which acts together with the local Lorentz transformations on the gauge fields as

$$\delta e^a = D\rho^a + \kappa^a_b e^b,$$
$$\delta \omega^{ab} = -D\kappa^{ab} + \varepsilon \cdot R^{ab}. \quad (15)$$

The commutator of two local Poincaré translations can now be computed and gives

$$[\delta(\rho_2), \delta(\rho_1)] = \delta(\kappa) \quad (16)$$

with $\kappa^{ab} = \rho_1^\lambda \rho_2^\nu R^{ab}_{\lambda\nu}$. Furthermore, one finds

$$[\delta(\kappa^{ab}), \delta(\rho^c)] = \delta(\rho'^d) \quad \text{with} \quad \rho'^a = \rho_b \kappa^{ba} \quad (17)$$

and

$$[\delta(\kappa_2), \delta(\kappa_1)] = \delta(\kappa_3) \quad \text{with} \quad \kappa_3 = [\kappa_1, \kappa_2]. \quad (18)$$

This means that, for nonvanishing R^{ab}, the local Poincaré translations no longer commute, but their commutator is a local Lorentz transformation proportional to the Riemann curvature. The rest of the algebra is unchanged. Thus an effect of the torsion-free condition is that the Poincaré algebra closes only on shell, but does not close off shell.

Another consequence of the torsion-free condition [4] is that it is an equation of motion of the action, which implies that the invariance of the action under diffeomorphisms does not result from the transformation properties of the fields alone, but that it is a property of their dynamics as well. The problem stems from the identification between diffeomorphism, which is a genuine invariance of the action, and local Poincaré translation, which is not a genuine invariance.

The torsion-free condition breaks local translation invariance in Lorentz space, and uniquely identifies the origin of the local Lorentz frame with the space-time point at which it is constructed.

B. Gravity invariant under the Poincaré group

Now we show that the formalism proposed by Stelle and West [7] and by Grignani and Nardelli [8] (SWGN) leads to a formulation of general relativity where Hilbert's action is invariant both under local Poincaré translations and under local Lorentz transformations as well as under diffeomorphism, and therefore the Poincaré algebra closes off shell.

The key ingredients of the SWGN formalism are the so called Poincaré coordinates $\xi^a(x)$ which behave as vectors under ISO(3,1) and are involved in the definition of the one-form vierbein V^a, which is not identified with the component e^a of the gauge potential, but is given by

$$V^a = D\xi^a + e^a = d\xi^a + \omega^a_b \xi^b + e^a. \tag{19}$$

Since $\zeta^a, e^a, \omega^{ab}$ under local Poincaré translations change as

$$\delta\zeta^a = -\rho^a, \quad \delta e^a = D\rho^a, \quad \delta\omega^{ab} = 0, \tag{20}$$

and under local Lorentz rotations change as

$$\delta\zeta^a = \kappa^a_b \zeta^b, \quad \delta e^a = \kappa^a_b e^b, \quad \delta\omega^{ab} = -D\kappa^{ab}, \tag{21}$$

we have that the vierbein V^a is invariant under local Poincaré translations,

$$\delta V^a = 0, \tag{22}$$

and, under local Lorentz rotations, transforms as

$$\delta V^a = \kappa^a_b V^b. \tag{23}$$

The space-time metric is postulated to be

$$g_{\mu\nu} = \eta_{ab} V^a_\mu V^b_\nu \tag{24}$$

with $\eta_{ab} = (-1,1,1,1)$. Thus the corresponding curvature is given by Eq. (7), but now Eq. (8) does not correspond to the space-time torsion because the vierbein is not given by e^a. The space-time torsion $\mathcal{T}^a$ is given by

$$\mathcal{T}^a = DV^a = T^a + R^{ab}\xi_b. \tag{25}$$

The Hilbert action can be rewritten as

$$S_{EH} = \int \varepsilon_{abcd} V^a V^b R^{cd} \tag{26}$$

which is invariant under general coordinate transformations and under local Lorentz rotations, as well as under local Poincaré translations. In fact,

$$\delta S_{EH} = \int \varepsilon_{abcd}\, \delta(R^{ab} V^c V^d), \tag{27}$$

$$\delta S_{EH} = 2\int \varepsilon_{abcd} R^{ab} V^c \delta V^d = 0. \tag{28}$$

Thus the action is genuinely invariant under the Poincaré group *without imposing a torsion-free condition*.

The variations of the action with respect to ζ^a, e^a, and ω^{ab} lead to the following equations:

$$\varepsilon_{abcd} \mathcal{T}^b R^{cd} = 0, \tag{29}$$

$$\varepsilon_{abcd} V^b R^{cd} = 0, \tag{30}$$

$$\varepsilon_{[aecd}\zeta_{b]} V^e R^{cd} + \varepsilon_{abcd} V^c \mathcal{T}^d = 0, \tag{31}$$

which reproduce the correct Einstein equations [8]:

$$\mathcal{T}^a = DV^a = 0, \tag{32}$$

$$\varepsilon_{abcd} V^b R^{cd} = 0. \tag{33}$$

The commutator of two local Poincaré translations is given by

$$[\delta(\rho_2), \delta(\rho_1)] = 0, \tag{34}$$

i.e., the local Poincaré translations now commute. The rest of the algebra is unchanged. Thus the Poincaré algebra closes off shell. This fact has deep consequences in supergravity.

III. SUPERGRAVITY IN 3+1 DIMENSIONS WITHOUT AUXILIARY FIELDS

In this section we shall review some aspects of the torsion-free condition in supergravity. The main point of this section is to show that the SWGN formalism permits constructing a supergravity invariant under local Lorentz rotations and under local Poincaré translations as well as under local supersymmetry transformations. This means that the super Poincaré algebra closes off shell without the need for any auxiliary fields.

A. The torsion-free condition in $N=1$ supergravity

$D=3+1$, $N=1$ supergravity is based on the super poincaré algebra

$$[P_a, P_b] = 0, \tag{35}$$

$$[J_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a, \tag{36}$$

$$[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc}, \tag{37}$$

$$[J_{ab}, Q^\alpha] = -\frac{1}{2}(\gamma_{ab})^\alpha_\beta Q^\beta, \tag{38}$$

$$[P_a, Q_\beta] = 0, \tag{39}$$

$$[Q^\alpha, Q_\beta] = \frac{1}{2}(\gamma^a)^\alpha_\beta P_a. \tag{40}$$

The connection for this group is given by

$$A = A^A T_A = e^a P_a + \frac{1}{2}\omega^{ab} J_{ab} + \bar{Q}\psi. \tag{41}$$

Using the algebra (40) and the general form for gauge transformations on A,

$$\delta A = D\lambda = d\lambda + [A,\lambda], \tag{42}$$

with

$$\lambda = \rho^a P_a + \frac{1}{2}\kappa^{ab}J_{ab} + \bar{Q}\varepsilon, \tag{43}$$

we obtain that e^a, ω^{ab}, and ψ under Poincaré translations transform as

$$\delta e^a = D\rho^a, \quad \delta\omega^{ab} = 0, \quad \delta\psi = 0; \tag{44}$$

under Lorentz rotations as

$$\delta e^a = \kappa^a_b e^b, \quad \delta\omega^{ab} = -D\kappa^{ab}, \quad \delta\psi = \frac{1}{4}\kappa^{ab}\gamma_{ab}\psi; \tag{45}$$

and under supersymmetry transformations as

$$\delta e^a = \frac{1}{2}\bar{\varepsilon}\gamma^a\psi, \quad \delta\omega^{ab} = 0, \quad \delta\psi = D\varepsilon. \tag{46}$$

The consistency of the propagation of the massless Rarita-Schwinger field in a classical gravitational background field is proved by contracting its field equation

$$\gamma_5 e^a \gamma_a D\psi = 0 \tag{47}$$

by the covariant derivative D,

$$D(\gamma_5 e^a \gamma_a D\psi) = 0,$$

$$\gamma_5\gamma_a T^a D\psi + \gamma_5 e^a \gamma_a DD\psi = 0. \tag{48}$$

The Einstein equation and the Bianchi identity reduce Eq. (48) to an identity.

Equation (47) does not take into account the back reaction of the spin-3/2 field on the gravitational field. It turns out that this back reaction of the spin-3/2 field on the gravitational field and on itself can be taken into account by a generalizing of Weyl's lemma [10]:

$$\mathcal{D}e^a_\mu = \partial_\nu e^a_\mu - \omega^a_{b\mu}e^b_\nu - \frac{1}{4}\bar{\psi}_\mu\gamma^a\psi_\nu - \Gamma^\lambda_{\mu\nu}e^a_\lambda = 0, \tag{49}$$

which implies that the corresponding torsion is given by

$$\hat{T}^a = T^a - \frac{1}{2}\bar{\psi}\gamma^a\psi. \tag{50}$$

Supergravity is the theory of the gravitational field interacting with a spin-3/2 Rarita-Schwinger field [11,12]. In the simplest case there is just one spin-3/2 Majorana fermion, usually called the gravitino. The theory is described by the action

$$S = \int \varepsilon_{abcd}e^a e^b R^{cd} + 4\bar{\psi}\gamma_5 e^a\gamma_a D\psi. \tag{51}$$

The variation of Eq. (51) with respect to $e^a, \omega^{ab}, \bar{\psi}$ leads to the equations for supergravity

$$\varepsilon_{abcd}e^b R^{cd} + 2\bar{\psi}\gamma_5\gamma_a D\psi = 0, \tag{52}$$

$$\varepsilon_{abcd}e^c\hat{T}^d = 0 \quad \text{where} \quad \hat{T}^a = T^a - \frac{1}{2}\bar{\psi}\gamma^a\psi, \tag{53}$$

$$\gamma_5 e^a\gamma_a D\psi = 0. \tag{54}$$

Since $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and $\gamma_\mu = e^a_\mu\gamma_a$, we can write Eqs. (51), (52), (53), and (54) as

$$L = -2\sqrt{-g}R + 4\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma, \tag{55}$$

$$R^{\tau\mu} - \frac{1}{2}g^{\tau\mu}R + 2\varepsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma^\tau D_\nu\psi_\rho = 0, \tag{56}$$

$$\hat{T}^\alpha_{\mu\nu} = T^\alpha_{\mu\nu} - \frac{1}{2}\bar{\psi}_\mu\gamma^\alpha\psi_\nu = 0, \tag{57}$$

where $T^\alpha_{\mu\nu} = \Gamma^\alpha_{[\mu\nu]}$, and

$$\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma = 0. \tag{58}$$

The action (51) is invariant under diffeomorphism, under local Lorentz rotations, and under local supersymmetry transformations, but it is not invariant under Poincaré translations. In fact, under local Poincaré translations,

$$\delta S = 2\int \varepsilon_{abcd}R^{ab}\left(T^c - \frac{1}{2}\bar{\psi}\gamma^c\psi\right)\rho^d + \text{surf. term}, \tag{59}$$

$$\delta S = 2\int \varepsilon_{abcd}R^{ab}\hat{T}^c\rho^d + \text{surf. term}. \tag{60}$$

The invariance of the action requires the vanishing of the torsion

$$\hat{T}^a = 0, \tag{61}$$

which implies that the connection is no longer an independent variable. Rather, its variation is given in terms of δe^a and $\delta\psi$, and differs from the one dictated by group theory.

An effect of the supertorsion-free condition on the local Poincaré superalgebra is that all commutators on e^a, ψ close except the commutator of two local supersymmetry transformations on the gravitino. For this commutator on the vierbein one finds

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]e^a = \frac{1}{2}\bar{\varepsilon}_2\gamma^a D\varepsilon_1 - \frac{1}{2}\bar{\varepsilon}_1\gamma^a D\varepsilon_2 = \frac{1}{2}D(\bar{\varepsilon}_2\gamma^a\varepsilon_1). \tag{62}$$

With $\rho^a = \frac{1}{2}\bar{\varepsilon}_2\gamma^a\varepsilon_1$, we can write

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]e^a = D\rho^a. \tag{63}$$

This means that, in the absence of the torsion-free condition, the commutator of two local supersymmetry transformations on the vierbein is a local Poincaré translation. However, the action is invariant by construction under general coordinate transformations and not under local Poincaré translations. The general coordinate transformation and the local Poincaré translation can be identified if we impose the torsion-free condition: since $\rho^a = \rho^\nu e^a_\nu$ we can write

$$D_\mu \rho^a = (\partial_\mu \rho^\nu) e^a_\nu + \rho^\nu (\partial_\nu e^a_\mu) + \frac{1}{2}\rho^\nu (\overline{\psi}_\mu \gamma^a \psi_\nu) + \rho^\nu \omega^{ab}_\nu e_{\mu b} + \rho^\nu T^a_{\mu\nu}. \tag{64}$$

This means that, if $T^a_{\mu\nu}=0$, then the following commutator is valid:

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_{GCT}(\rho^\mu) + \delta_{LLT}(\rho^\mu \omega^{ab}_\mu) + \delta_Q(\rho^\nu \overline{\psi}_\nu) \tag{65}$$

where we can see that P in $\{Q,Q\}=P$, i.e., local Poincaré translations, is replaced by general coordinate transformations in addition to two other gauge symmetries. The structure constants defined by this result are field dependent [6], which is a property of supergravity not present in Yang-Mills theory.

The commutator of two local supersymmetry transformations on the gravitino is given by

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]\psi = \frac{1}{2}(\sigma_{ab}\varepsilon_2)[\delta(\varepsilon_1)\omega^{ab}] - \frac{1}{2}(\sigma_{ab}\varepsilon_1) \times [\delta(\varepsilon_2)\omega^{ab}]. \tag{66}$$

The condition $\hat{T}^a = 0$ leads to $\omega^{ab} = \omega^{ab}(e,\psi)$ which implies that the connection is no longer an independent variable and its variation $\delta(\varepsilon)\omega^{ab}$ is given in terms of $\delta(\varepsilon)e^a$ and $\delta(\varepsilon)\psi$. Introducing $\delta(\varepsilon)\omega^{ab}(e,\psi)$ into Eq. (66) we see that, without the auxiliary fields, the gauge algebra does not close, as is shown by Eq. (10) of Ref. [6]. Therefore the condition $\hat{T}^a=0$ breaks not only local Poincaré invariance, but also the supersymmetry transformations. We show that, if we use the SWGN formalism, the gauge algebra closes without the auxiliary fields because it is not necessary to impose the torsion-free condition.

B. Supergravity invariant under the Poincaré group

Analogous to the pure gravity case, the action for supergravity in $3+1$ dimensions is not invariant under local Poincaré translations. The invariance of the action requires, in accord with 1.5 formalism, the vanishing of the torsion $\hat{T}^a$, which implies that the connection is no longer an independent variable. Rather, its variation is given in terms of δe^a and $\delta\psi$, and differs from the one dictated by group theory. As a consequence the supersymmetry algebra, acting on the spinor field, closes off shell only with auxiliary fields. How to construct supergravity in four dimensions without auxiliary fields is an interesting open problem. Now we will show that this construction is possible.

The massless Rarita-Schwinger field is a spin-3/2 field that can be described by a Majorana vector-spinor ψ_μ that satisfies the equation

$$\gamma_5 V^a \gamma_a d\psi = 0 \tag{67}$$

where now

$$V^a = D\xi^a + e^a. \tag{68}$$

The coupling of this field to a gravitational field satisfying the free Einstein equations is achieved by minimal coupling. According to this prescription, it is possible to generalize the free spin-3/2 equation consistently to include interaction with a gravitational background field, as

$$\gamma_5 V^a \gamma_a D\psi = 0. \tag{69}$$

Consistency of the propagation of a spin-3/2 particle in a classical gravitational background field is proved by contracting Eq. (69) with another derivative D:

$$D(\gamma_5 V^a \gamma_a D\psi) = 0,$$

$$\gamma_5 \gamma_a \mathcal{T}^a D\psi + \gamma_5 V^a \gamma_a DD\psi = 0. \tag{70}$$

The Einstein equation and the Bianchi identity reduce Eq. (70) to an identity. Of course, using the free Einstein equation implies that we have not taken into account the back reaction of the spin-3/2 field on the gravitational field. In that sense the gravitational field here is just a fixed classical background field. The more general situation, in which both the gravitational field and the Rarita-Schwinger field are dynamical, is the situation encountered in supergravity.

Equation (69) does not take into account the back reaction of the spin-3/2 field on the gravitational field; namely, the spin-3/2 field itself can act as a source for the gravitational field. Since the spin-3/2 field is coupled to the gravitational field through the covariant derivative D, this also induces a coupling of the gravitino field with itself. It turns out that this back reaction of the spin-3/2 field on the gravitational field and on itself can be taken into account by generalizing Weyl's lemma [10]:

$$\mathcal{D}_\nu V^a_\mu = \partial_\nu V^a_\mu - \omega^a_{b\mu} V^b_\nu - \frac{1}{4}\overline{\psi}_\mu \gamma^a \psi_\nu - \Gamma^\lambda_{\mu\nu} V^a_\lambda = 0. \tag{71}$$

This implies that the corresponding torsion is given by

$$\hat{\mathcal{T}}^a = \mathcal{T}^a - \frac{1}{2}\overline{\psi}\gamma^a\psi. \tag{72}$$

Within the (SWGN) formalism the action for supergravity can be rewritten as

$$S = \int \varepsilon_{abcd} V^a V^b R^{cd} + 4\overline{\psi}\gamma_5 V^a \gamma_a D\psi \tag{73}$$

which is invariant under local Lorentz rotations:

$$\delta V^a = \kappa^a_b V^b, \quad \delta\psi = \frac{1}{4}\kappa^{ab}\gamma_{ab}\psi, \tag{74}$$

i.e., under $\delta\omega^{ab}=-D\kappa^{ab}$, $\delta e^a=\kappa^a_b e^b$, $\delta\psi=\frac{1}{4}\kappa^{ab}\gamma_{ab}\psi$, and $\delta\xi^a=\kappa^a_b\xi^b$; under local Poincaré translations

$$\delta V^a=0, \quad \delta\psi=0, \tag{75}$$

i.e., under $\delta\omega^{ab}=0$, $\delta e^a=D\rho^a$, $\delta\psi=0$, and $\delta\xi^a=-\rho^a$; and under local supersymmetry transformations

$$\delta V^a=\tfrac{1}{2}\bar{\varepsilon}\gamma^a\psi, \quad \delta\psi=D\varepsilon, \tag{76}$$

i.e., under $\delta\omega^{ab}=0$, $\delta e^a=\frac{1}{2}\bar{\varepsilon}\gamma^a\psi$, $\delta\psi=D\varepsilon$, and $\delta\xi^a=0$.

This means that the action (73) is invariant without the need to impose a torsion-free condition. We can see that, in the local Poincaré superalgebra, all commutators on e^a,ψ close including the commutators of two local supersymmetry transformations on the vierbein and on the gravitino. In fact, for this commutator on the vierbein one finds

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]e^a=D\rho^a \tag{77}$$

where $\rho^a=\bar{\varepsilon}_2\gamma^a\varepsilon_1$, i.e., the commutator of two local supersymmetry transformations on the vierbein is a local Poincaré translation.

The commutator of two local supersymmetry transformations on the gravitino is given by

$$[\delta(\varepsilon_1),\delta(\varepsilon_2)]\psi=\frac{1}{2}(\sigma_{ab}\varepsilon_2)[\delta(\varepsilon_1)\omega^{ab}] - \frac{1}{2}(\sigma_{ab}\varepsilon_1)[\delta(\varepsilon_2)\omega^{ab}]=0 \tag{78}$$

because now the connection is an independent variable, i.e., $\delta(\varepsilon)\omega^{ab}=0$ in accordance with group theory.

From the transformation laws of the fields ω^{ab},ξ^a one can see that the commutators of local supersymmetry transformations on ω^{ab},ξ^a close.

This proves that, if we use the SWGN formalism, the gauge algebra closes without auxiliary fields, because it is not necessary to impose the torsion-free condition.

In the context of a genuinely first-order formalism, i.e., where the spin connection ω^{ab} transforms independently of the graviton field e^a, of the gravitino field ψ, and of the Poincaré field ξ, the field equations can be obtained by varying Eq. (73) with respect to $\bar{\psi}$, e^a, and ω^{ab} and with respect to ξ^a:

$$\gamma_5 V^a\gamma_a D\psi=0, \tag{79}$$

$$\varepsilon_{abcd}V^bR^{cd}+2\bar{\psi}\gamma_5\gamma_a D\psi=0, \tag{80}$$

$$\varepsilon_{abcd}V^c\hat{\mathcal{T}}^d+\varepsilon_{[aecd}\xi_{b]}V^eR^{cd}+2\xi_{[b}\bar{\psi}\gamma_5\gamma_{a]}D\psi=0. \tag{81}$$

The field equation corresponding to the variation of Eq. (73) with respect to ξ^a is not an independent equation. In fact, taking the covariant derivative D of Eq. (80) we obtain the same equation that one obtains by varying the action (73) with respect to ξ^a.

From the action (73) and Eqs. (79), (52), and (53) we can see that, once the gauge $\xi^a=0$ is chosen, from the equations

$$V^a=D\xi^a+e^a,$$

$$\mathcal{T}^a=DV^a,$$

it follows that $V^a=e^a$, $\mathcal{T}^a=T^a=De^a$, $\hat{\mathcal{T}}^a=\hat{T}^a$, and that the Lagrangian of the action (73) takes the form (51) and Eqs. (79),(80),(81) take the forms (52),(53),(54), which are the equations for supergravity theory as developed in Refs. [6,11,12]. The action reduces to the usual action for supergravity if we choose $\xi^a=0$, i.e., if we uniquely identify the origin of the local Lorentz frame with the space-time point at which it is constructed.

IV. COMMENTS

We have shown in this work that the successful formalism used by Stelle and West [7] and by Grignani and Nardelli [8] to construct an action for $(3+1)$ dimensional gravity invariant under the Poincaré group can be generalized to supergravity in $3+1$ dimensions. The extension to other even dimensions remains an open problem. The main result of this paper is that we have shown that, in order to construct a supergravity theory invariant under local Lorentz rotations and under local Poincaré translations as well as under local supersymmetry transformations, it is necessary to use the SWGN formalism. This means using the vierbein V^a which involves in its definition the so called "Poincaré coordinates" $\xi^a(x)$. This field can be interpreted, at the supergravity level, as some kind of auxiliary field which permits the supersymmetry algebra to close off shell without the need for any auxiliary fields.

It is perhaps interesting to note that, if one considers, following Ref. [8], $g_{\mu\nu}=V^a_\mu V^b_\nu\eta_{ab}$ and $\gamma_\mu=V^a_\mu\gamma_a$, one can write the Lagrangian of the action (73) in the form (55). This means that, if one considers the theory constructed in terms of the space-time metric $g_{\mu\nu}$ and the gravitino field ψ_μ, ignoring the underlying formulation, the theory described in our paper is completely equivalent to the theory developed in Refs. [11,12,6]. No trace of the new structure of the vierbein existing in the underlying formulation of the theory can be found at the metric level. This could permit one to interpret the local Poincaré translation as some kind of hidden symmetry of the theory when it is constructed in terms of the space-time metric $g_{\mu\nu}$.

ACKNOWLEDGMENTS

This work was supported in part by FONDECYT through Grants No. 1010485 and No. 1000305, in part by Dirección de Investigación, Universidad de Concepción through Grant No. 98.011.023-1.0, and in part by UCV through Grant UCV-DGIP No. 123.752/00.

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