Ground state energy in a wormhole space-time

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(Received 15 October 2001; published 8 April 2002)

The ground state energy of the massive scalar field with nonconformal coupling ξ on a short-throat flat-space wormhole background is calculated by using the zeta renormalization approach. We discuss the renormalization and relevant heat kernel coefficients in detail. We show that a stable configuration of wormholes can exist for $\xi > 0.123$. In the particular case of a massive conformal scalar field with $\xi = 1/6$, the radius of the throat of a stable wormhole $a \approx 0.16/m$. The self-consistent wormhole has the radius of throat $a \approx 0.0141 l_p$ and the mass of the scalar boson $m \approx 11.35 m_p$ (l_p and m_p are the Planck length and mass, respectively).

DOI: 10.1103/PhysRevD.65.084028 PACS number(s): 04.62.+v, 04.20.Gz, 04.70.Dy

I. INTRODUCTION

Wormholes are topological handles in space-time linking widely separated regions of a single universe, or "bridges" joining two different space-times. Interest in these configurations dates back at least as far as 1916 [1] with revivals of activity following both the classic work of Einstein and Rosen in 1935 [2] and the later series of works initiated by Wheeler in 1955 [3]. More recently, interest in the topic has been rekindled by the works of Morris and Thorne [4] and Morris, Thorne, and Yurtsever [5]. These authors constructed and investigated a class of objects they referred to as "traversable wormholes." Their work led to a flurry of activity in wormhole physics [6].

The central feature of wormhole physics is the fact that traversable wormholes are accompanied by unavoidable violations of the null energy condition, i.e., the matter threading the wormhole's throat has to be possessed of "exotic" properties. Classical matter does satisfy the usual energy conditions; hence, wormholes cannot arise as solutions of classical relativity and matter. If they exist, they must belong to the realm of semiclassical or perhaps quantum gravity. In the absence of a complete theory of quantum gravity, the semiclassical approach is beginning to play the most important role for examining wormholes. However, there are not many results concerning quantized fields on the wormhole background. Recently self-consistent wormholes in semiclassical gravity were studied numerically in our work [7]. Some arguments in favor of the possibility of existence of self-consistent wormhole solutions to the semiclassical Einstein equations have also been given by Khatsymovsky in Ref. [8].

Note that all the mentioned results were obtained within the framework of various approximations, whereas no one up to now has succeeded in exact calculations of vacuum expectation values on the wormhole background. The reason for this state of affairs consists in the considerable mathematical difficulties which one faces in trying to quantize a physical field on the wormhole background. To overcome these difficulties, in this work we will consider a simple model of the wormhole space-time: the short-throat flat-space wormhole. The model represents two identical copies of Minkowski space with spherical regions excised from each copy and with the boundaries of those regions to be identified. The space-time of this model is everywhere flat except for a two-dimensional singular spherical surface. Because of this fact it turns out to be possible to construct the complete set of wave modes of the massive scalar field and calculate the ground state energy.

The aim of our work is to calculate the ground state energy of the scalar field on the short-throat flat-space wormhole background using the zeta function regularization approach [9,10] which was developed in Refs. [11–13]. In the framework of this approach, the ground state energy of the scalar field ϕ is given by

$$E(s) = \frac{1}{2}\mu^{2s}\zeta_{\mathcal{L}}\left(s-\frac{1}{2}\right), \tag{1}$$

where

$$\zeta_{\mathcal{L}}(s) = \sum_{(n)} (\lambda_{(n)}^2 + m^2)^{-s} \tag{2}$$

is the zeta function of the corresponding Laplace operator. To make the eigenvalues $\lambda_{(n)}^2$ discrete we assume the field ϕ to be put into a large ball with the Dirichlet boundary condition. $\lambda_{(n)}^2$ are eigenvalues of the three dimensional Laplace operator $\mathcal{L}$

$$(\Delta - \xi R)\phi_{(n)} = \lambda_{(n)}^2 \phi_{(n)}, \tag{3}$$

where R is the curvature scalar.

The expression (1) is divergent in the limit $s \to 0$ which we are interested in. For renormalization we subtract from Eq. (1) its divergent part:

$$E^{\rm ren} = \lim_{s\to 0}[E(s) - E^{\rm div}(s)], \tag{4}$$

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where

$$E^{\mathrm{div}}(s)=\lim_{m\to\infty}E(s). \tag{5}$$

Because the heat kernel expansion of the zeta function is an asymptotic expansion for large mass, the divergent part has the following form:

$$E^{\mathrm{div}}(s)=\frac{1}{2}\left(\frac{\mu}{m}\right)^{2s}\frac{1}{(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\times\left\{B_0m^4\Gamma(s-2)+B_{1/2}m^3\Gamma\left(s-\frac{3}{2}\right)+B_1m^2\Gamma(s-1)+B_{3/2}m\Gamma\left(s-\frac{1}{2}\right)+B_2\Gamma(s)\right\}, \tag{6}$$

where B_α are the heat kernel coefficients. In this case the renormalized ground state energy (4) obeys the normalization condition

$$\lim_{m\to\infty}E^{\mathrm{ren}}=0. \tag{7}$$

The organization of the paper is as follows. In Sec. II we describe the space-time of a wormhole in the short-throat flat-space approximation and analyze the solution of the equation of motion for a massive scalar field. In Sec. III we obtain closed expressions for the zeta function and ground state energy and calculate the corresponding heat kernel coefficients. We also analyze the expression for the ground state energy for different radii of the throat. In Sec. IV we discuss our results. Appendixes A and B contain some technical details of the calculations.

We use units $\hbar=c=G=1$ (except in Sec. IV). The signature of the space-time, the sign of the Riemann and Ricci tensors, are the same as in the book by Hawking and Ellis [14].

II. A TRAVERSABLE WORMHOLE: THE SHORT-THROAT FLAT-SPACE APPROXIMATION

In this section we consider a simple model of a traversable wormhole. Assume that the throat of the wormhole is very short, and that the curvature in the regions outside the mouth of the wormhole is relatively weak. An idealized model of such a wormhole can be constructed in the following manner: Consider two copies of Minkowski space, $\mathcal{M}_+$ and $\mathcal{M}_-$, with the spherical coordinates $(t,r_\pm,\theta_\pm,\varphi_\pm)$. (Notice that $\mathcal{M}_+$ and $\mathcal{M}_-$ have a common time coordinate t. One may interpret this fact as the identification $t_+\leftrightarrow t_-$.) Excise from each copy the spherical region $r_\pm<a$, where a is the radius of the sphere, and then identify the boundaries of those regions: $(t,a,\theta_+,\varphi_+)\leftrightarrow(t,a,\theta_-,\varphi_-)$. The Riemann tensor for this model is identically zero everywhere except at the wormhole mouths where the identification procedure takes place. Generically, there will be an infinitesimally thin layer of exotic matter present at the mouth of the wormhole.

Such an idealized geometry can be described by the following metric:

$$ds^2=-dt^2+d\rho^2+r^2(\rho)(d\theta^2+\sin^2\theta d\varphi^2), \tag{8}$$

where ρ is the proper radial distance, $-\infty<\rho<\infty$, and the shape function $r(\rho)$ is

$$r(\rho)=|\rho|+a. \tag{9}$$

It is easily seen that in two regions $\mathcal{R}_+:\ \rho>0$ and $\mathcal{R}_-:\ \rho<0$ separately, one can introduce a new radial coordinate $r_\pm=\pm\rho+a$ and rewrite the metric (8) in the usual spherical coordinates:

$$ds^2=-dt^2+dr_\pm^2+r_\pm^2(d\theta^2+\sin^2\theta\, d\varphi^2).$$

This form of the metric explicitly indicates that the regions $\mathcal{R}_+:\ \rho>0$ and $\mathcal{R}_-:\ \rho<0$ are flat. However, note that this change of coordinates $r=|\rho|+a$ is not global, because it is ill defined at the throat $\rho=0$. Hence, as was expected, the space-time is curved at the wormhole throat. To illustrate this we calculate the scalar curvature $R(\rho)$ in the metric (8):

$$R(\rho)=-8a^{-1}\delta(\rho). \tag{10}$$

Let us now consider a scalar field ϕ in the space-time with the metric (8). The equation of motion of the scalar field is

$$(\Box-m^2-\xi R)\phi=0, \tag{11}$$

where m is the mass of the scalar field, and ξ is an arbitrary coupling with the scalar curvature R. In the metric (8), a general solution to Eq. (11) can be found in the following form:

$$\phi(t,\rho,\theta,\varphi)=e^{-i\omega t}u(\rho)Y_{ln}(\theta,\varphi), \tag{12}$$

where $Y_{ln}(\theta,\varphi)$ are spherical functions, $l=0,1,2,\ldots$, $n=0,\pm1,\pm2,\ldots,\pm l$, and the function $u(\rho)$ obeys the radial equation

$$u''+2\frac{r'}{r}u'+\left(\omega^2-\frac{l(l+1)}{r^2}-m^2-\xi R\right)u=0, \tag{13}$$

where a prime denotes the derivative $d/d\rho$. In the flat regions $\mathcal{R}_\pm$, where $r(\rho)=\pm\rho+a$, $r'(\rho)=\pm1$, and $R(\rho)=0$, Eq. (13) reads

$$u''+\frac{2}{\rho\pm a}u'+\left(\omega^2-m^2-\frac{l(l+1)}{(\rho\pm a)^2}\right)u=0. \tag{14}$$

A general solution of this equation can be written as

$$u_l^\pm[\lambda(\rho\pm a)]=A_l^\pm h_l^{(1)}[\lambda(\rho\pm a)]+B_l^\pm h_l^{(2)}[\lambda(\rho\pm a)], \tag{15}$$

where

$$\lambda=\sqrt{\omega^2-m^2},\quad |\omega|>m,$$

$h_l^{(i)}[z]$ are spherical Hankel functions, and $A_l^{\pm}, B_l^{\pm}$ are arbitrary constants.

The solutions $u_l^{\pm}[\lambda(\rho\pm a)]$ were obtained in the flat regions $\mathcal{R}_{\pm}$ separately. To find a solution in the whole space-time we must impose matching conditions for $u_l^{\pm}[\lambda(\rho\pm a)]$ at the throat $\rho=0$. The first condition demands that the solution has to be continuous at $\rho=0$. This gives

$$u_l^-[-\lambda a]=u_l^+[\lambda a],$$

or

$$A_l^- h_l^{(1)}[-\lambda a]+B_l^- h_l^{(2)}[-\lambda a]-A_l^+ h_l^{(1)}[\lambda a]-B_l^+ h_l^{(2)}[\lambda a]=0. \tag{16}$$

To obtain the second condition we integrate Eq. (13) within the interval $(-\epsilon,\epsilon)$ and then go to the limit $\epsilon\to 0$. Taking into account the following relations:

$$r(\rho)=|\rho|+a,\quad r'(\rho)=\operatorname{sgn}\rho,\quad r''(\rho)=2\delta(\rho),$$

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} f(\rho)\delta(\rho)d\rho=f(0),$$

we find

$$\left.\frac{du_l^-[x]}{dx}\right|_{x=-\lambda a}=\left.\frac{du_l^+[x]}{dx}\right|_{x=\lambda a}+\frac{8\xi}{\lambda a}u_l^+[\lambda a]. \tag{17}$$

Substituting Eq. (15) into Eq. (17) gives

$$A_l^- h_l^{(1)\prime}[-\lambda a]+B_l^- h_l^{(2)\prime}[-\lambda a]-\left(h_l^{(1)\prime}[\lambda a]+\frac{8\xi}{\lambda a}h_l^{(1)}[\lambda a]\right)A_l^+-\left(h_l^{(2)\prime}[\lambda a]+\frac{8\xi}{\lambda a}h_l^{(2)}[\lambda a]\right)B_l^+=0, \tag{18}$$

where $h_l^{(i)\prime}[\pm\lambda a]=(dh_l^{(i)}[x]/dx)_{x=\pm\lambda a}$.

In addition to the two matching conditions (16) and (18) we must demand regular behavior of the scalar field at infinity. For this aim, we will consider a "box approximation," i.e., we will assume, in an intermediate stage of the calculations, that the wormhole space-time has a finite radius R, so that $|\rho|\leqslant R$, and we will go, in the end, to the limit $R\to\infty$. In the framework of the box approximation, we demand that the scalar field becomes equal to zero at the space-time bounds $\rho=\pm R$. Taking into account Eq. (15) gives

$$u_l^-[-\lambda(R+a)]=0,\quad u_l^+[\lambda(R+a)]=0,$$

or

$$A_l^- h_l^{(1)}[-\lambda(R+a)]+B_l^- h_l^{(2)}[-\lambda(R+a)]=0, \tag{19}$$

$$A_l^+ h_l^{(1)}[\lambda(R+a)]+B_l^+ h_l^{(2)}[\lambda(R+a)]=0. \tag{20}$$

The four conditions (16), (18), (19), and (20) obtained represent a homogeneous system of linear algebraic equations for four coefficients $A_l^{\pm}, B_l^{\pm}$. As is known, such a system has a nontrivial solution if and only if the matrix of coefficients is degenerate. Hence we get

$$\begin{vmatrix} h_l^{(1)}[-\lambda a] & h_l^{(2)}[-\lambda a] & -h_l^{(1)}[\lambda a] & -h_l^{(2)}[\lambda a] \\ h_l^{(1)\prime}[-\lambda a] & h_l^{(2)\prime}[-\lambda a] & -h_l^{(1)\prime}[\lambda a]-\frac{8\xi}{\lambda a}h_l^{(1)}[\lambda a] & -h_l^{(2)\prime}[\lambda a]-\frac{8\xi}{\lambda a}h_l^{(2)}[\lambda a] \\ h_l^{(1)}[-\lambda(R+a)] & h_l^{(2)}[-\lambda(R+a)] & 0 & 0 \\ 0 & 0 & h_l^{(1)}[\lambda(R+a)] & h_l^{(2)}[\lambda(R+a)] \end{vmatrix}=0. \tag{21}$$

After some algebra one can show that the determinant in the above formula is factorized, and so Eq. (21) can be reduced to the following two relations:

$$\Psi_l^1[\lambda]=0, \tag{22}$$

and

$$\Psi_l^2[\lambda]=0, \tag{23}$$

where the functions $\Psi_1[\lambda],\Psi_2[\lambda]$ are defined as follows:

$$\Psi_l^1[\lambda]\equiv\frac{i\lambda}{2}\sqrt{a(a+R)}\{h_l^{(1)}[\lambda(R+a)]h_l^{(2)}[\lambda a]-h_l^{(2)}[\lambda(R+a)]h_l^{(1)}[\lambda a]\}, \tag{24}$$

$$\Psi_l^2[\lambda]\equiv\frac{i\lambda^2 a}{8}\sqrt{a(a+R)}\left\{h_l^{(1)}[\lambda(R+a)]\times\left(\frac{4\xi}{\lambda a}h_l^{(2)}[\lambda a]+h_l^{(2)\prime}[\lambda a]\right)-h_l^{(2)}[\lambda(R+a)]\times\left(\frac{4\xi}{\lambda a}h_l^{(1)}[\lambda a]+h_l^{(1)\prime}[\lambda a]\right)\right\}. \tag{25}$$

We introduced additional factors in order to simplify the formulas that follow. These factors do not change the relations (22),(23). The significance of Eqs. (22) and (23) is that they determine the set of possible values of the wave number λ, i.e., the spectrum for the scalar field modes. Resolving Eq. (22) and Eq. (23) we can obtain two families, respectively,

$$\lambda^{(1)}_{lp_1}(a,R,\xi), \quad p_1=1,2,3,\dots, \tag{26a}$$

$$\lambda^{(2)}_{lp_2}(a,R,\xi), \quad p_2=1,2,3,\dots . \tag{26b}$$

III. GROUND STATE ENERGY AND HEAT KERNEL COEFFICIENTS

The ground state energy is given by

$$E=\frac{1}{2}\sum_{\alpha=1,2}\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}(2l+1)\sqrt{\lambda^{(\alpha)2}_{lp}+m^2}, \tag{27}$$

which is, in fact, the zero point energy of the massive scalar field. This expression is divergent. In the framework of the zeta function regularization method [9,10], the ground state energy is expressed in terms of the zeta function

$$E(s)=\frac{1}{2}\mu^{2s}\zeta_{\mathcal{L}}\left(s-\frac{1}{2}\right), \tag{28}$$

where

$$\zeta_{\mathcal{L}}\left(s-\frac{1}{2}\right)=\sum_{\alpha=1,2}\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}(2l+1)(\lambda^{(\alpha)^2}_{lp}+m^2)^{1/2-s} \tag{29}$$

is the zeta function associated with the Laplace operator $\hat{\mathcal{L}}=\Delta-m^2-\xi R$. The parameter μ, with dimension of mass, has been introduced in order to have the correct dimension for the energy. For simplicity we represent Eq. (28) in a slightly different form:

$$E(s)=\frac{1}{2}\left(\frac{\mu}{m}\right)^{2s}\zeta\left(s-\frac{1}{2}\right), \tag{30}$$

where we introduced the function with the dimension of energy

$$\zeta\left(s-\frac{1}{2}\right)=m^{2s}\zeta_{\mathcal{L}}\left(s-\frac{1}{2}\right), \tag{31}$$

which we shall also call the zeta function.

The solutions $\lambda^{(\alpha)}_{lp}(a,R,\xi)$ of Eqs. (22),(23) cannot be found in closed form. For this reason we use the method suggested in Ref. [11], which allows us to express the zeta function in terms of the eigenfunctions. The sum over p may be converted into a contour integral in the complex λ plane using the principal of argument: namely,

$$\zeta\left(s-\frac{1}{2}\right)=\frac{m^{2s}}{2\pi i}\sum_{\alpha=1,2}\sum_{l=0}^{\infty}(2l+1)\times\int_{\gamma}d\lambda(\lambda^2+m^2)^{1/2-s}\frac{\partial}{\partial\lambda}\ln\Psi^{\alpha}_{l}[\lambda], \tag{32}$$

where the contour γ runs counterclockwise and must enclose all solutions of Eqs. (22),(23). Shifting the contour to the imaginary axis, we obtain the following formula for the zeta function:

$$\zeta\left(s-\frac{1}{2}\right)=-m^{2s}\frac{\cos\pi s}{\pi}\sum_{\alpha=1,2}\sum_{l=0}^{\infty}(2l+1)\times\int_{m}^{\infty}dk(k^2-m^2)^{1/2-s}\frac{\partial}{\partial k}\ln\Psi^{\alpha}_{l}[ik], \tag{33}$$

where the functions (24),(25) on the imaginary axis $\lambda=ik$ read

$$\Psi^1_l[ik]=I_\nu[k(R+a)]K_\nu[ka]-K_\nu[k(R+a)]I_\nu[ka], \tag{34a}$$

$$\Psi^2_l[ik]=\left(\xi-\frac{1}{8}\right)\Psi^1_l[ik]+\frac{ka}{4}\{I_\nu[k(R+a)]K'_\nu[ka]-K_\nu[k(R+a)]I'_\nu[ka]\}, \tag{34b}$$

with

$$\nu=l+\frac{1}{2}.$$

The expression (33) may be simplified in the large box limit $R\gg a$, which we are interested in. Let us rewrite $\Psi^1_l[ik]$ in the following form:

$$\Psi^1_l[ik]=I_\nu[k(R+a)]K_\nu[ka]\left(1-\frac{K_\nu[k(R+a)]I_\nu[ka]}{I_\nu[k(R+a)]K_\nu[ka]}\right). \tag{35}$$

In the large box limit, the second term in the large parentheses obeys the inequality

$$\frac{K_\nu[k(R+a)]I_\nu[ka]}{I_\nu[k(R+a)]K_\nu[ka]}<e^{-2mR} \tag{36}$$

and gives an exponentially small contribution to the ground state energy.

Therefore, in the limit of a large box we have

$$\Psi^1_l[ik]\approx I_\nu[k(a+R)]K_\nu[ka], \tag{37a}$$

$$\Psi^2_l[ik]\approx I_\nu[k(a+R)]\left\{\left(\xi-\frac{1}{8}\right)K_\nu[ka]+\frac{ka}{4}K'_\nu[ka]\right\}. \tag{37b}$$

At this time we have to make a comment on the above formulas. Our approach is valid if the functions Ψ_l^m on the imaginary axis do not have zeros in the domain of integration in Eq. (33). This gives a restriction for ξ. The function $\Psi_l^1[ik]$ has no zeros on the imaginary axis, but the function $\Psi_l^2[ik]$ has simple zeros if $\xi>\frac{1}{4}$. Indeed, by using recurrence formulas for Bessel's function, let us represent the function $\Psi_l^2[ik]$ in the following form:

$$\Psi_l^2[ik]=I_\nu[k(a+R)]\left\{\left(\xi-\frac{1}{8}-\frac{\nu}{4}\right)K_\nu[ka]-\frac{ka}{4}K_{\nu-1}[ka]\right\}. \tag{38}$$

Since the Bessel's functions K_ν are positive, the expression in curly brackets may change sign and therefore the function $\Psi_l^2[ik]$ may have zeros, if

$$\xi-\frac{1}{8}-\frac{\nu}{4}>0. \tag{39}$$

The lowest boundary for ξ is 1/4 for $l=0$. More precisely, in this case we have

$$\Psi_0^2=\frac{1}{2k}\frac{1}{\sqrt{a(a+R)}}e^{kR}(1-e^{-2k(a+R)})\times\left(\xi-\frac{1}{4}-\frac{ka}{4}\right). \tag{40}$$

As long as $k>m$, the function Ψ_0^2 has a simple zero at the point $k=(4\xi-1)/a$ if

$$\xi>\frac{1}{4}+\frac{ma}{4}. \tag{41}$$

For this reason in this paper we will consider the ground state energy for $\xi<1/4$. In the opposite case we have to modify our approach.

Taking into account these formulas we may divide the zeta function, as well as the ground state energy (30), into two parts:

$$\zeta\left(s-\frac{1}{2}\right)=\zeta_R^{\text{ext}}\left(s-\frac{1}{2}\right)+\zeta_a^{\text{int}}\left(s-\frac{1}{2}\right), \tag{42}$$

where

$$\zeta_R^{\text{ext}}\left(s-\frac{1}{2}\right)=-\frac{2\beta_R^{2s}\cos\pi s}{\pi(a+R)}\sum_{l=0}^{\infty}\nu^{2-2s}\times\int_{\beta_R/\nu}^{\infty}dx\left(x^2-\frac{\beta_R^2}{\nu^2}\right)^{1/2-s}\times\frac{\partial}{\partial x}2\ln\{x^{-\nu}I_\nu[\nu x]\}, \tag{43}$$

$$\zeta_a^{\text{int}}\left(s-\frac{1}{2}\right)=-\frac{2\beta_a^{2s}\cos\pi s}{\pi a}\sum_{l=0}^{\infty}\nu^{2-2s}\times\int_{\beta_a/\nu}^{\infty}dx\left(x^2-\frac{\beta_a^2}{\nu^2}\right)^{1/2-s}\times\frac{\partial}{\partial x}\left\{\ln\{x^\nu K_\nu[\nu x]\}+\ln\left[x^\nu\left(\delta K_\nu[\nu x]+\frac{x\nu}{4}K_\nu'[\nu x]\right)\right]\right\}. \tag{44}$$

Here, $\beta_R=m(a+R)$, $\beta_a=ma$, $\delta=\xi-\frac{1}{8}$, and $\nu=l+\frac{1}{2}$.

The first part of the zeta function (43) depends only on the size of the box with throat $R'=R+a$ and the asymptotic structure of the space-time. It is exactly twice the expression in the flat Minkowski space time without a throat [11] calculated for a massive scalar field inside a ball of radius R' with the Dirichlet boundary condition. The factor of 2 is very easily explained: we consider a scalar field existing on a double-sided plane. The second part (44) does not depend on a boundary; it depends only on the radius of the throat a and the nonminimal coupling ξ. It contains information about the space-time under consideration. The same division of the zeta function into two parts has already been observed for the space-time of a thick cosmic string [12] and the space-time of a pointlike global monopole [13]. Because the first part of the zeta function (43) has already been analyzed in great detail, we proceed now to consideration of the second part (44).

Both expressions (43) and (44) and the ground state energy (30) are divergent in the limit $s\to 0$ which we are interested in. According to the renormalization procedure, we have to subtract from the regularized expression for the ground state energy (30) all terms that survive in the limit $m\to\infty$. This procedure corresponds to the subtraction of the five (three without the boundary) first terms of the DeWitt-Schwinger expansion [11–13].

Our goal now is to find in closed form the expansion of the zeta function (44) at the point $(-\frac{1}{2})$ as a power series over s (for arbitrary mass). For this reason we use the uniform asymptotic series over the inverse index for Bessel functions of large index and argument given in Ref. [15]. We subtract from and add to the integrand of Eq. (44) its uniform expansion up to terms proportional to ν^{-3}. After subtraction we may let $s\to 0$. The second part, which is the uniform expansion of the integrand, gives us the pole structure of the zeta function. In this way (see the details in Appendix A) we obtain the following series for the zeta function at the point $(-\frac{1}{2})$:

$$\zeta_a^{\text{int}}\left(s-\frac{1}{2}\right)_{s\to 0}=\frac{1}{(4\pi)^{3/2}a\Gamma\left(s-\frac{1}{2}\right)}\left\{b_0^a\beta_a^4\Gamma(s-2)+b_{1/2}^a\beta_a^3\Gamma\left(s-\frac{3}{2}\right)+b_1^a\beta_a^2\Gamma(s-1)\right.$$

$$+b^a_{3/2}\beta_a\Gamma\left(s-\frac{1}{2}\right)+b^a_2\Gamma(s)\Bigg\}$$

$$-\frac{1}{16\pi^2 a}\{b^a\ln\beta_a^2+\Omega^a[\beta_a]\}, \qquad (45)$$

$$\zeta_R^{\text{ext}}\left(s-\frac{1}{2}\right)_{s\to 0}=\frac{1}{(4\pi)^{3/2}(a+R)\Gamma\left(s-\frac{1}{2}\right)}\Bigg\{b^R_0\beta_R^4\Gamma(s-2)$$

$$+b^R_{1/2}\beta_R^3\Gamma\left(s-\frac{3}{2}\right)+b^R_1\beta_R^2\Gamma(s-1)$$

$$+b^R_{3/2}\beta_R\Gamma\left(s-\frac{1}{2}\right)+b^R_2\Gamma(s)\Bigg\}$$

$$-\frac{1}{16\pi^2(a+R)}\{b^R\ln\beta_R^2+\Omega^R[\beta_R]\}. \qquad (46)$$

Here

$$b^a_0=-\frac{8\pi}{3},\quad b^a_{1/2}=0,\quad b^a_1=32\pi\left[\xi-\frac{1}{6}\right],$$

$$b^a_{3/2}=64\pi^{3/2}\left[\xi-\frac{1}{8}\right]^2, \qquad (47)$$

$$b^a_2=\frac{8\pi}{3}\left[128\xi^3-64\xi^2+\frac{56}{5}\xi-\frac{68}{105}\right],$$

$$b^a_{5/2}=\frac{16}{3}\pi^{3/2}\left[96\xi^4-72\xi^3+21\xi^2-\frac{45}{16}\xi+\frac{93}{640}\right],$$

$$b^a=\frac{1}{2}b^a_0\beta_a^4-b^a_1\beta_a^2+b^a_2, \qquad (48)$$

and

$$b^R_0=\frac{8\pi}{3},\quad b^R_{1/2}=-4\pi^{3/2},\quad b^R_1=\frac{16}{3}\pi,\quad b^R_{3/2}=-\frac{1}{3}\pi^{3/2}, \qquad (49)$$

$$b^R_2=-\frac{32}{315}\pi,\quad b^a_{5/2}=-\frac{1}{60}\pi^{3/2},$$

$$b^R=\frac{1}{2}b^R_0\beta_R^4-b^R_1\beta_R^2+b^R_2. \qquad (50)$$

The above expressions (45) and (46) contain all terms that survive in the limit $s\to 0$. The details of calculation and a closed form for $\Omega^a[\beta_a]$ are outlined in Appendix A. The function $\Omega^\alpha[\beta_\alpha]$ tends to a constant for $\beta_\alpha\to 0$ and $\Omega^\alpha[\beta_\alpha]=-b^\alpha\ln\beta_\alpha^2+\sqrt{\pi}b^\alpha_{5/2}/\beta_\alpha+O(1/\beta_\alpha^2)$ for $\beta_\alpha\to\infty$ $(\alpha=a,R)$.

Comparing the above expression with that obtained by the Mellin transformation taking the trace of the heat kernel (in three dimensions),

$$\zeta\left(s-\frac{1}{2}\right)_{m\to\infty}=\frac{m^{2s}}{\Gamma\left(s-\frac{1}{2}\right)}\int_0^\infty dt\, t^{s-3/2}K[t]_{t\to 0}$$

$$=\frac{1}{(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\Bigg\{B_0m^4\Gamma(s-2)$$

$$+B_{1/2}m^3\Gamma\left(s-\frac{3}{2}\right)+B_1m^2\Gamma(s-1)$$

$$+B_{3/2}m\Gamma\left(s-\frac{1}{2}\right)+B_2\Gamma(s)+\cdots\Bigg\}, \qquad (51)$$

we obtain the heat kernel coefficients:

$$B_0=\frac{8\pi}{3}[(a+R)^3-a^3],$$

$$B_{1/2}=-4\pi^{3/2}(a+R)^2,$$

$$B_1=32\pi\left[\xi-\frac{1}{6}\right]a+\frac{16}{3}\pi(a+R),$$

$$B_{3/2}=64\pi^{3/2}\left[\xi-\frac{1}{8}\right]^2-\frac{1}{3}\pi^{3/2}, \qquad (52)$$

$$B_2=\frac{8\pi}{3a}\left[128\xi^3-64\xi^2+\frac{56}{5}\xi-\frac{68}{105}\right]-\frac{32}{315}\frac{\pi}{(a+R)},$$

$$B_{5/2}=\frac{16}{3}\frac{\pi^{3/2}}{a^2}\left[96\xi^4-72\xi^3+21\xi^2-\frac{45}{16}\xi+\frac{93}{640}\right]$$

$$-\frac{1}{60}\frac{\pi^{3/2}}{(a+R)^2}.$$

Using the above scheme we also calculated the coefficient $B_{5/2}$, which we will need later for the analysis. We should like to note the difference between Eqs. (45),(46) and Eq. (51). Equation (51) is an asymptotic expansion of the zeta function over the inverse mass $m\to\infty$ but the formulas (45),(46) are correct for arbitrary mass m and small $s\to 0$. In fact, we extracted the asymptotic (for $m\to\infty$) part of the zeta function in the form (51) and saved the finite part of it. In the limit $m\to\infty$ the finite part tends to zero and the two formulas are in agreement. This is the reason that the function $\Omega^\alpha[\beta_\alpha]=-b^\alpha\ln\beta_\alpha^2+\sqrt{\pi}b^\alpha_{5/2}/\beta_\alpha+O(1/\beta_\alpha^2)$ for $\beta_\alpha\to\infty$ $(\alpha=a,R)$.

As long as the space-time under consideration has a singular two-dimensional surface Σ with codimension 1, we cannot use the standard formulas obtained for a smooth background, and we have to utilize the formulas obtained by Gilkey, Kirsten, and Vassilevich in Ref. [16]. The heat kernel coefficients (52) coincide exactly with those obtained from general formulas in three dimensions given in Ref. [16]. We have to take into account that the extrinsic curvature tensor of the surface Σ is obtained as the covariant derivative of the *outward* unit normal vector N_α:

$$K_{\alpha\beta}=\nabla_\alpha N_\beta . \quad (53)$$

For this reason this vector has coordinates $N_\alpha=(0,\pm 1,0,0)$ on the spheres $\rho=\pm R$, and

$$\mathrm{tr}\, K=\frac{2}{R+a} \quad (54)$$

in both cases. In Appendix A we found general formulas for the arbitrary heat kernel coefficients and traced them in manifest form up to b_3.

To obtain the ground state energy we have to subtract from our expressions (30), (43), and (44) all terms that will survive in the limit $m\to\infty$. Then we set $s=0$ and the radius of the box $R\to\infty$. Therefore we arrive at the following expression:

$$E^{\mathrm{ren}}=-\frac{1}{32\pi^2 a}\{b^a \ln \beta_a^2+\Omega^a[\beta_a]\}. \quad (55)$$

A similar general structure for the ground state energy in the massless case was obtained first by Blau, Visser, and Wipf [9] using dimensional considerations only, and it was confirmed by detailed calculations in Refs. [12,13].

Using the above-mentioned behavior of $\Omega^a[\beta_a]$, the ground state energy tends to zero for large radius of the throat:

$$E^{\mathrm{ren}}\approx-\frac{b_{5/2}^a}{32\pi^{3/2} m a^2}, \quad a\to\infty, \quad (56)$$

and it is divergent for small radius of the throat:

$$E^{\mathrm{ren}}\approx-\frac{b_2^a}{16\pi^2 a}\ln(ma), \quad a\to 0. \quad (57)$$

The numerical calculations of the ground state energy E^{ren}/m [Eq. (55)] as a function of $\beta_a=ma$ is depicted in Figs. 1 and 2 for $\xi=\frac{1}{6}$ and $\xi=0$, respectively. The details of the numerical calculations are analyzed in Appendix B.

IV. DISCUSSION

We have calculated the ground state energy of the massive scalar field on a short-throat flat-space wormhole background [see Eq. (55)]. It can be written down in the form[1]

$$E^{\mathrm{ren}}=-\frac{\hbar c}{a} f(\beta_a), \quad (58)$$

where $\beta_a=mca/\hbar$, and $f(\beta_a)$ is a function of β_a which has the asymptotic

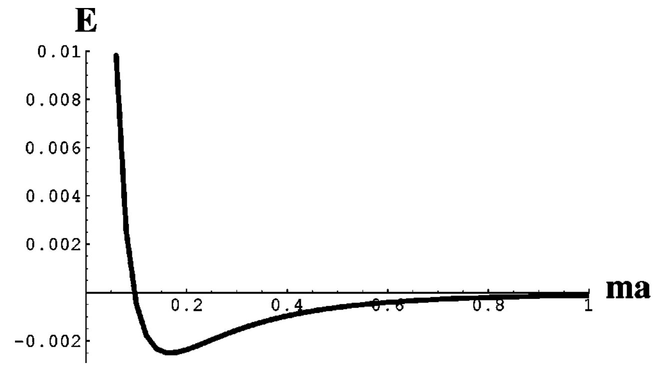

FIG. 1. The ground state energy $\mathbf{E}=E^{\mathrm{ren}}/m$ as a function of ma for fixed mass m and $\xi=1/6$. The energy has a minimum at the point $ma\approx 0.16$ with depth $E_{\min}/m\approx-0.0025$.

$$f(\beta_a)\approx\frac{b_2^a}{16\pi^2}\ln \beta_a, \quad \beta_a\to 0,$$

$$f(\beta_a)\approx\frac{b_{5/2}^a}{32\pi^{3/2}\beta_a}, \quad \beta_a\to\infty.$$

To characterize the behavior of the ground state energy as a function of ξ we note that the coefficient $b_{5/2}$ is positive for all values of ξ and hence the ground state energy tends to -0 as $\beta_a\to\infty$. In the limit $\beta_a\to 0$, the behavior of the ground state energy is determined by the sign of b_2 [see Eq. (57)] and depends on ξ. For $\xi<\xi_*\approx 0.123$, b_2 is negative and the ground state energy tends to minus infinity; otherwise it tends to plus infinity. This difference in asymptotic behavior at $\beta_a\to 0$ results in two qualitatively different pictures describing the behavior of the ground state energy. In the first case $\xi<\xi_*$, the ground state energy is monotonically increasing from $-\infty$ to 0 and has no extremum (see Fig. 2); while in the second case $\xi>\xi_*$, it has a global minimum. For example, in Fig. 1 the graph of E^{ren}/m versus β_a is shown for $\xi=\frac{1}{6}$. It is seen that the ground state energy has a minimum at $\beta_a\approx 0.16$ with depth $E_{\min}/m\approx-0.0025$.

Let us now speculate about the result obtained. Suppose that the quantum massive scalar field plays the role of the

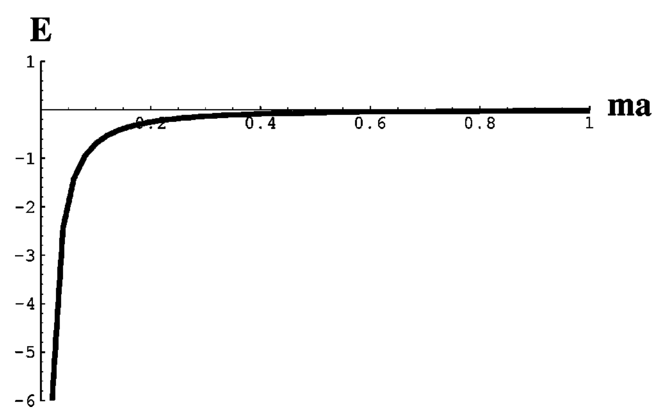

FIG. 2. The ground state energy $\mathbf{E}=E^{\mathrm{ren}}/m$ as a function of ma for fixed mass m and $\xi=0$. There is no minimum energy; it is always negative.

[1]In this section we use dimensional units G, c, and $\hbar$.

"exotic" matter maintaining the existence of the short-throat flat-space wormhole in a self-consistent manner. This means that the semiclassical Einstein equations have to be satisfied:

$$G_{\mu\nu}=\frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle^{\rm ren}, \tag{59}$$

where $G_{\mu\nu}$ is the Einstein tensor, and $\langle T_{\mu\nu}\rangle$ are the renormalized vacuum expectation values of the stress-energy tensor of the scalar field. The total energy in a static space-time is given by

$$E=\int_V \varepsilon\sqrt{g^{(3)}}d^3x, \tag{60}$$

where $\varepsilon=-\langle T_t^t\rangle^{\rm ren}=-G_t^t c^4/8\pi G$ is the energy density, and the integral is calculated over the whole space. In the spherically symmetric metric (8) we obtain

$$E=-\frac{c^4}{2G}\int_{-\infty}^{\infty}G_t^t r^2(\rho)d\rho. \tag{61}$$

Using the relations $G_t^t=2r''/r+(r'^2-1)/r^2$ and $r(\rho)=|\rho|+a$ we can calculate

$$E=-\frac{2c^4 a}{G}. \tag{62}$$

Note that the total energy is negative.

In the self-consistent case the total energy must coincide with the ground state energy of the scalar field. Equating Eqs. (58) and (62) gives

$$\frac{2c^4a}{G}=\frac{\hbar c}{a}f(\beta_a),$$

or

$$a=l_P\sqrt{\frac{f(\beta_a)}{2}}, \tag{63}$$

where $l_P=\sqrt{\hbar G/c^3}$ is the Planck length. To make further estimations we take into account that in order to be stable a quantum system should be in the state with the minimum of ground state energy. This requirement can be satisfied in the case $\xi>\xi_*$. In particular, for $\xi=1/6$ the minimum $E_{\rm min}/mc^2\approx-0.0025$ is achieved at $\beta_a=mca/\hbar\approx0.16$. This gives $f(\beta_a)\approx4\times10^{-4}$, so that

$$a\approx0.0141 l_P \tag{64}$$

and

$$m\approx11.35 m_P, \tag{65}$$

where $m_P=(\hbar c/G)^{1/2}$ is the Planck mass.

Thus, our estimations have revealed that the self-consistent semiclassical wormhole, if it exists, should possess a throat of sub-Planckian radius, and the quantum scalar field maintaining the wormhole's existence should have super-Planckian mass. Of course, it should be noted that our consideration has been restricted to the toy model of a short-throat flat-space wormhole, and so one may expect that in more realistic models the results will be slightly changed.

Let us emphasize that the result obtained in this work for the wormhole configuration can be generalized. Really, the behavior of ground state energy for small (57) and large (56) values of the throat's radius a depends only on two dimensionless heat kernel coefficients b_2 and $b_{5/2}$, respectively. Instead of the radius a, we could use a typical system size λ (throat) and calculate the coefficients b_2 and $b_{5/2}$ on the corresponding background. Now let us consider the dimensionless ground state energy $E^{\rm ren}/m$. Obviously, it will depend only on the dimensionless combination $m\lambda$, and hence the limit of large (small) mass will correspond to the limit of large (small) size of the system. Since for renormalization we have to subtract the first five terms (up to b_2) of the expansion for large mass the ground state energy in this limit should be proportional to the next nonvanishing term of the expansion:

$$E^{\rm ren}\approx\frac{1}{2}\frac{1}{(4\pi)^{3/2}}\frac{b_{5/2}}{(m\lambda)^2}\left.\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(s-\frac{1}{2}\right)}\right|_{s\to0}=-\frac{b_{5/2}}{32\pi^{3/2}(m\lambda)^2}, \tag{66}$$

which coincides with Eq. (56). We would like to note that the coefficient $b_{5/2}$ is nonzero in the limit $R\to\infty$ for a background with singular scalar curvature, as was shown in Ref. [16]. For smooth, nonsingular geometrical characteristics of the background, it is zero and we have to take into account the next nonvanishing coefficient, which is b_3. In this case we have the following expression in the limit $m\lambda\to\infty$:

$$E^{\rm ren}\approx-\frac{b_3}{32\pi^2(m\lambda)^3}. \tag{67}$$

The origin of the logarithmic term, as well as the behavior for small size of the system is the following. The structure of poles of zeta functions does not depend on the parameters of the system m and λ. The subtraction of the asymptotics for large mass gives us the following contribution to the ground state energy:

$$\frac{(m\lambda)^{2s}-1}{2(\lambda m)(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\times\left\{b_0(\lambda m)^4\Gamma(s-2)+b_{1/2}(\lambda m)^3\Gamma\left(s-\frac{3}{2}\right)\right. \tag{68}$$

$$\left.+b_1(\lambda m)^2\Gamma(s-1)+b_{3/2}(\lambda m)\Gamma\left(s-\frac{1}{2}\right)+b_2\Gamma(s)\right\}_{s\to0}, \tag{69}$$

where b_α are dimensionless heat kernel coefficients. Taking the limit in this formula we observe that the heat kernel coefficients with integer indices will survive:

$$-\frac{1}{32\pi^2(\lambda m)}\left(\frac{1}{2}b_0(\lambda m)^4 - b_1(\lambda m)^2 + b_2\right)\ln(\lambda m)^2. \tag{70}$$

Therefore in the limit $\lambda \to 0$ one has

$$E^{\text{ren}} \approx -\frac{b_2 \ln(\lambda m)}{16\pi^2(\lambda m)}, \tag{71}$$

in agreement with Eq. (57).

Therefore the necessary condition that the ground state energy will possess a minimum is the following: the coefficients b_2 and the next nonvanishing coefficient ($b_{5/2}$ for singular curvature and b_3 for nonsingular) must be positive. If this is so, the discussion above is valid and the self-consistent semiclassical wormhole exists. The radius of the throat of a stable wormhole and the mass of the scalar boson in this case depend on the model of the wormhole and the value of the nonconformal coupling ξ.

ACKNOWLEDGMENTS

The authors would like to thank M. Bordag, J. S. Dowker, K. Kirsten, and D. Vassilevich for helpful comments on some items of the paper. This work was supported by the Russian Foundation for Basic Research Grant No. 99-02-17941.

APPENDIX A

The uniform asymptotic expansions of the modified Bessel's functions have the form below:

$$K_\nu[\nu x] = \sqrt{\frac{\pi t}{2\nu}} e^{-\nu\eta} \sum_{k=0}^{\infty} \frac{u_k[t]}{(-\nu)^k},$$

$$I_\nu[\nu x] = \sqrt{\frac{t}{2\pi\nu}} e^{\nu\eta} \sum_{k=0}^{\infty} \frac{u_k[t]}{\nu^k}, \tag{A1}$$

$$K'_\nu[\nu x] = -\sqrt{\frac{\pi}{2\nu x^2 t}} e^{-\nu\eta} \sum_{k=0}^{\infty} \frac{v_k[t]}{(-\nu)^k},$$

$$I'_\nu[\nu x] = \sqrt{\frac{1}{2\pi\nu x^2 t}} e^{\nu\eta} \sum_{k=0}^{\infty} \frac{v_k[t]}{\nu^k},$$

where

$$t = \frac{1}{\sqrt{1+x^2}}, \quad \eta = \sqrt{1+x^2} + \ln\frac{x}{1+\sqrt{1+x^2}},$$

$$u_{k+1}[t] = \frac{1}{2}t^2(1-t^2)u'_k[t] + \frac{1}{8}\int_0^t (1-5t^2)u_k[t]dt,$$

$$u_0[t] = 1, \tag{A2}$$

$$v_{k+1}[t] = u_{k+1}[t] + t(t^2-1)\left\{\frac{1}{2}u_k[t] + t u'_k[t]\right\},$$

$$v_0[t] = 1.$$

Taking into account these formulas in Eq. (44) we obtain a power series over s for the zeta function. The uniform asymptotic expansion (A1) up to ν^{-n} allows us to take into account terms up to m^{3-n}. Because we need all terms that survive in the limit $m \to \infty$ we use the uniform expansion up to $n=3$.

Therefore we have the following expression for the zeta function:

$$\begin{aligned}\zeta_a^{\text{int}}\left(s-\frac{1}{2}\right) = &-\frac{2\beta_a^{2s}\cos\pi s}{\pi a}\sum_{l=0}^{\infty}\nu^{2-2s} \\ &\times \int_{\beta_a/\nu}^{\infty} dx \left(x^2 - \frac{\beta_a^2}{\nu^2}\right)^{1/2-s} \\ &\times \frac{\partial}{\partial x}\left\{\ln\{x^\nu K_\nu[\nu x]\}\right. \\ &+ \ln\left[x^\nu\left(\delta K_\nu[\nu x] + \frac{x\nu}{4}K'_\nu[\nu x]\right)\right] \\ &\left.-\sum_{k=-1}^{3}(-\nu)^{-k}N_k\right\} - \frac{2\beta_a^{2s}\cos\pi s}{\pi a} \\ &\times \sum_{l=0}^{\infty}\nu^{2-2s}\int_{\beta_a/\nu}^{\infty}dx\left(x^2-\frac{\beta_a^2}{\nu^2}\right)^{1/2-s}\frac{\partial}{\partial x} \\ &\times \sum_{k=-1}^{3}(-\nu)^{-k}N_k, \end{aligned} \tag{A3}$$

where the functions N_p may be found in closed form for arbitrary index p using a simple program in the package MATHEMATICA. For $p \geqslant 0$ they are polynomial of degree $3p$ and have the following form:

$$N_p[t] = \sum_{k=0}^{p} a_{p,k} t^{p+2k}. \tag{A4}$$

The first five N_p are listed below:

$$N_0=0,\quad N_{-1}=2\eta,$$

$$N_1=\left[4\delta-\frac{1}{4}\right]t+\frac{1}{12}t^3, \tag{A5}$$

$$N_2=-8\left[\delta-\frac{1}{8}\right]^2 t^2-2\left[\delta-\frac{1}{8}\right]t^4-\frac{1}{8}t^6,$$

$$N_3=\frac{1}{3}\left[64\delta^3-24\delta^2+\frac{9}{2}\delta-\frac{19}{64}\right]t^3+\frac{1}{5}\left[40\delta^2-25\delta+\frac{169}{64}\right]t^5+\frac{1}{7}\left[\frac{49}{2}\delta-\frac{329}{64}\right]t^7+\frac{179}{576}t^9.$$

We should like to note that the expression (A3) is identical to the original one (44). The first term is finite in the limit $s\to 0$; all divergences are contained in the second part.

Integrating over x with the help of the integral

$$\int_{\beta/\nu}^{\infty}dx\,x\left(x^2-\frac{\beta^2}{\nu^2}\right)^{1/2-s}(1+x^2)^{-p/2}=\frac{\Gamma\left(\frac{3}{2}-s\right)\Gamma(s+(p-3)/2)}{2\Gamma(p/2)}\left(\frac{\nu}{\beta}\right)^{p-3+2s}\times\left(1+\frac{\nu^2}{\beta^2}\right)^{-s-(p-3)/2} \tag{A6}$$

and taking the limit $s\to 0$ in the first term, we get

$$\zeta_a^{\rm int}\left(s-\frac{1}{2}\right)_{s\to 0}=-\frac{1}{16\pi^2 a}A_f[\beta_a]+\frac{1}{(4\pi)^{3/2}a\Gamma\left(s-\frac{1}{2}\right)}\sum_{k=-1}^{3}(-1)^k A_k[\beta_a], \tag{A7}$$

where

$$A_f[\beta_a]=32\pi\sum_{l=0}^{\infty}\nu^2\int_{\beta/\nu}^{\infty}dx\sqrt{x^2-\frac{\beta^2}{\nu^2}}\frac{\partial}{\partial x}\times\left(\ln K_\nu(\nu x)+\ln\left[\delta K_\nu(\nu x)+\frac{x\nu}{4}K'_\nu(\nu x)\right]+2\nu\eta(x)+\frac{1}{\nu}N_1-\frac{1}{\nu^2}N_2+\frac{1}{\nu^3}N_3\right), \tag{A8}$$

$$A_{-1}=4\pi\beta^2\Gamma\left(s-\frac{1}{2}\right)\sum_{l=0}^{\infty}\frac{\mathcal{Z}(0,l+s-1)}{\Gamma(l+s+1/2)}, \tag{A9}$$

$$A_p=-8\pi^{3/2}\beta^{1-p}\sum_{k=0}^{p}\frac{a_{p,k}}{\Gamma(l+s+1/2)}\times\mathcal{Z}\left(2k,s+k+\frac{p-1}{2}\right), \tag{A10}$$

$$\mathcal{Z}(p,s)=\Gamma(s)\sum_{l=0}^{\infty}\frac{2\nu}{(1+\nu^2/\beta_a^2)^s}\left(\frac{\nu}{\beta_a}\right)^p. \tag{A11}$$

The first four A_p are listed below:

$$A_0[\beta_a]=0,$$

$$A_1[\beta_a]=-8\pi\left[\left(4\delta-\frac{1}{4}\right)\mathcal{Z}(0,s)+\frac{1}{6}\mathcal{Z}(2,s+1)\right],$$

$$A_2[\beta_a]=\frac{4\pi^{3/2}}{\beta_a}\left[16\left(\delta-\frac{1}{8}\right)^2\mathcal{Z}\left(0,s+\frac{1}{2}\right)+4\left(\delta-\frac{1}{8}\right)\mathcal{Z}\left(2,s+\frac{3}{2}\right)+\frac{1}{8}\mathcal{Z}\left(4,s+\frac{5}{2}\right)\right], \tag{A12}$$

$$A_3[\beta_a]=-\frac{16\pi}{3\beta_a^2}\left[\left(64\delta^3-24\delta^2+\frac{9}{2}\delta-\frac{19}{64}\right)\mathcal{Z}(0,s+1)+\frac{2}{5}\left(40\delta^2-25\delta+\frac{169}{64}\right)\mathcal{Z}(2,s+2)+\frac{4}{35}\left(\frac{49}{2}\delta-\frac{329}{64}\right)\mathcal{Z}(4,s+3)+\frac{179}{2520}\mathcal{Z}(6,s+4)\right].$$

To find the heat kernel coefficients we have to take the limit $m\to\infty$ in Eq. (A7). The asymptotic expansion of $\mathcal{Z}(0,q)$ over inverse powers of β_a^2 was found in Ref. [13]:

$$\mathcal{Z}(0,s)=\beta_a^2\Gamma(s-1)+2\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\Gamma(l+s)\beta_a^{-2l}\zeta_H\left(-1-2l,\frac{1}{2}\right), \tag{A13}$$

where $\zeta_H(s,a)$ is the Hurwitz zeta function

$$\zeta_H(s,a)=\sum_{l=0}^{\infty}(l+a)^{-s},\quad s>1. \tag{A14}$$

The other functions $\mathcal{Z}(2k,s+k+(p-1)/2)$ in Eq. (A10) are expressed in terms of $\mathcal{Z}(0,q)$ by the relation

$$\mathcal{Z}\left(2k,s+k+\frac{p-1}{2}\right)=\sum_{n=0}^{k}\frac{k!}{n!(k-n)!}\frac{\Gamma(k+(p-1)/2+s)}{\Gamma(n+(p-1)/2+s)}\times\mathcal{Z}\left(0,n+\frac{p-1}{2}+s\right). \tag{A15}$$

Taking into account the above formulas we obtain the following formulas for the heat kernel coefficients:

$$b_n=-\frac{1}{\Gamma(s-2+n)}\sum_{p=0}^{n}\alpha_{n-p-1}(2p-1), \tag{A16}$$

$$b_{n+1/2}=\frac{1}{\Gamma\left(s-\frac{3}{2}+n\right)}\sum_{p=0}^{n}\alpha_{n-p-1}(2p), \tag{A17}$$

where $(l,p\geqslant 0)$

$$\alpha_{-1}(-1)=\frac{8\pi}{3}\Gamma(s-2),$$

$$\alpha_l(-1)=16\pi(-1)^l\frac{\zeta_H\left(-1-2l,\frac{1}{2}\right)}{l!(1-2l)}\Gamma(s-1+l),$$

$$\alpha_l(p)=-8\pi^{3/2}\sum_{k=0}^{p}\frac{a_{p,k}}{\Gamma\left(k+\frac{p}{2}\right)}z_l(p,k), \tag{A18}$$

$$z_{-1}(p,k)=\sum_{n=0}^{k}(-1)^n\frac{k!}{n!(k-n)!}\frac{\Gamma(k+(p-1)/2+s)}{n+(p-3)/2+s},$$

$$z_l(p,k)=2\frac{(-1)^l}{l!}\zeta_H\left(-1-2l,\frac{1}{2}\right)\sum_{n=0}^{k}(-1)^n\frac{k!}{n!(k-n)!}\times\frac{\Gamma(k+(p-1)/2+s)}{\Gamma(n+(p-1)/2+s)}\Gamma\left(l+n+\frac{p-1}{2}+s\right).$$

Using these formulas, one obtains the following expressions for the heat kernel coefficients:

$$b_0^a=-\frac{8\pi}{3},\quad b_{1/2}^a=0,\quad b_1^a=32\pi\left[\xi-\frac{1}{6}\right],$$

$$b_{3/2}^a=64\pi^{3/2}\left[\xi-\frac{1}{8}\right]^2,$$

$$b_2^a=\frac{8\pi}{3}\left[128\xi^3-64\xi^2+\frac{56}{5}\xi-\frac{68}{105}\right],$$

$$b_{5/2}^a=\frac{16}{3}\pi^{3/2}\left[96\xi^4-72\xi^3+21\xi^2-\frac{45}{16}\xi+\frac{93}{640}\right], \tag{A19}$$

$$b_3^a=\frac{8\pi}{3}\left[\frac{4096}{5}\xi^5-\frac{4096}{5}\xi^4+\frac{35584}{105}\xi^3-\frac{1088}{15}\xi^2+\frac{848}{105}\xi-\frac{144}{385}\right].$$

The coefficient $b_{k/2}$ is a polynomial of $(k-1)$th order over ξ. The coefficient b_2 changes its sign at the point $\xi_*\approx 0.123$ and $b_{5/2}$ is positive for arbitrary ξ.

Our problem now is to take the limit $s\to 0$ in the second part of Eq. (A3). Because the function $\mathcal{Z}(p,s)$ with $p=2,4,\ldots$ may be expressed in terms of $\mathcal{Z}(0,s)$ only, let us analyze it in detail. Let us suppose for a moment that $\beta_a<1$ and represent $\mathcal{Z}(0,s)$ as a power series over β_a:

$$\mathcal{Z}(0,s)=2\beta_a^{2s}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma(n+s)\beta_a^{2n}\times\zeta_H\left(2n+2s-1,\frac{1}{2}\right). \tag{A20}$$

The gamma function $\Gamma(s)$ has simple poles at the points $s=0,-1,-2,\ldots$ and the Hurwitz zeta function $\zeta_H(s,p)$ has a simple pole only at one point $s=1$. They have the following expansion near their poles:

$$\Gamma(s-n)_{s\to 0}=\frac{(-1)^n}{n!}\left(\frac{1}{s}+\Psi[n+1]\right)+O(s),$$

$$\zeta_H(s+1,p)_{s\to 0}=\frac{1}{s}-\Psi[p]+O(s), \tag{A21}$$

where $\Psi[x]$ is the digamma function.

All divergences of the function $\mathcal{Z}(0,s)$ [Eq. (A20)] in the limit $s\to 0$ are contained in the first two terms with $n=0,1$. The rest of the series is finite and we set $s=0$ in it. Therefore we obtain the following expression:

$$\mathcal{Z}(0,s)_{s\to 0}=2\beta_a^{2s}\left\{\frac{1}{2}\beta_a^2\Gamma(s-1)+\frac{1}{24}\Gamma(s)+\frac{1}{2}\beta_a^2[1-2\gamma-4\ln 2]-\frac{1}{12}[12\zeta_R'(-1)+\ln 2]\right\}+2\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\beta_a^{2n}\zeta_H\left(2n-1,\frac{1}{2}\right)+O(s), \tag{A22}$$

where $\zeta_R(s)$ is the Riemann zeta function and γ is the Euler constant.

The series in the above formula may be analytically continued for arbitrary values of β_a. First of all, using the series representation of the Hurwitz zeta function (A14) we represent this series in the following form:

$$j_2(\beta_a)=\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\beta_a^{2n}\zeta_H\left(2n-1,\frac{1}{2}\right)$$
$$=\sum_{l=0}^{\infty}\nu\left[-\ln\left(1+\frac{\beta_a^2}{\nu^2}\right)+\frac{\beta_a^2}{\nu^2}\right]. \qquad \text{(A23)}$$

Then, taking into account the integral representation for the logarithm,

$$\ln x=\int_0^x\frac{dt}{1+t},$$

and the closed expression for the series below

$$j_0(x^2)=\sum_{l=0}^{\infty}\frac{1}{\nu(\nu^2+x^2)}$$
$$=\frac{1}{2x^2}\left\{\Psi\left(\frac{1}{2}-ix\right)+\Psi\left(\frac{1}{2}+ix\right)-2\Psi\left(\frac{1}{2}\right)\right\}, \qquad \text{(A24)}$$

one has

$$j_2(\beta)=\beta^2\int_0^1 dx\,x\left\{\Psi\left(\frac{1}{2}-ix\beta\right)+\Psi\left(\frac{1}{2}+ix\beta\right)-2\Psi\left(\frac{1}{2}\right)\right\}. \qquad \text{(A25)}$$

The function on the right-hand side is analytical in the whole complex plane and therefore it gives the analytical continuation of the series $j_2(\beta_a)$ for arbitrary values of β_a. This representation of $j_2(\beta)$ is preferable for numerical calculations.

Using the same approach for the other $\mathcal{Z}(p,q)$ we arrive at the following formulas for $A_k[\beta_a]$:

$$A_{-1}[\beta_a]=-\frac{8\pi}{3}\beta_a^{2s}\left\{\frac{7}{160}\Gamma(s)-\frac{1}{4}\beta_a^2\Gamma(s-1)\right.$$
$$\left.-\beta_a^4\Gamma(s-2)\right\}+\omega_{-1}(\beta_a),$$

$$A_1[\beta_a]=-\frac{8\pi}{3}\beta_a^{2s}\left\{\left(\delta-\frac{1}{16}\right)\Gamma(s)\right.$$
$$\left.+12\left(\delta-\frac{1}{48}\right)\beta_a^2\Gamma(s-1)\right\}+\omega_1(\beta_a),$$

$$A_2[\beta_a]=4\pi^{3/2}\beta_a^{2s}\left\{16\delta^2\beta_a\Gamma\left(s-\frac{1}{2}\right)\right.$$
$$\left.+\frac{4}{3\beta_a}\left(\delta-\frac{1}{8}\right)^2\Gamma\left(s+\frac{1}{2}\right)\right\}+\omega_2(\beta_a),$$

$$A_3[\beta_a]=-\frac{8\pi}{3}\beta_a^{2s}\left\{128\delta^3-16\delta^2+\frac{1}{5}\delta+\frac{71}{3360}\right\}\Gamma(s)$$
$$+\omega_3(\beta_a), \qquad \text{(A26)}$$

where

$$\omega_{-1}(\beta_a)=8\pi\left\{\left[-\frac{7}{160}-\frac{7}{2}\zeta'(-3)+\frac{1}{240}\ln 2\right]\right.$$
$$+\beta_a^2\left[2\zeta'(-1)+\frac{1}{6}\ln 2+\frac{1}{4}\right]$$
$$\left.+\beta_a^4\left[\frac{1}{3}\gamma-\frac{13}{36}+\frac{2}{3}\ln 2\right]+j_3(\beta_a)\right\},$$

$$\omega_1(\beta_a)=-16\pi\left\{\left[\frac{1}{4}\zeta'(-1)+\frac{1}{48}\ln 2+\frac{1}{144}\right]\right.$$
$$+\delta\left[-4\zeta'(-1)-\frac{1}{3}\ln 2\right]+\beta_a^2\left[\left(\frac{1}{12}\gamma-\frac{1}{8}\right.\right.$$
$$\left.\left.+\frac{1}{6}\ln 2\right)+\delta(-4\gamma-8\ln 2+2)\right]$$
$$\left.+4\left(\delta-\frac{1}{16}\right)j_2(\beta_a)+\frac{1}{6}\beta_a^4 j_0(\beta_a^2)\right\},$$

$$\omega_2(\beta_a)=-8\pi^2\left\{-16\delta^2\beta_a-16\left[\delta-\frac{1}{8}\right]^2 j_{1/2,0}(\beta_a)\right.$$
$$\left.-2\left[\delta-\frac{1}{8}\right]j_{3/2,0}(\beta_a)-\frac{3}{32}j_{5/2,0}(\beta_a)\right\}, \qquad \text{(A27)}$$

$$\omega_3(\beta_a)=-\frac{32}{3}\pi\left\{\left[\frac{71}{3360}\ln 2+\frac{757}{20160}+\frac{71}{6720}\gamma\right]\right.$$
$$+\delta\left[\frac{1}{10}\gamma-\frac{4}{5}+\frac{1}{5}\ln 2\right]+\delta^2[-8\gamma+8-16\ln 2]$$
$$+\delta^3[128\ln 2+64\gamma]+\left[64\delta^3-24\delta^2+\frac{9}{2}\delta\right.$$
$$\left.-\frac{19}{64}\right]j_{1,1}(\beta_a)+\frac{2}{5}\left[40\delta^2-25\delta+\frac{169}{64}\right]j_{2,1}(\beta_a)$$
$$\left.+\frac{8}{35}\left[\frac{49}{2}\delta-\frac{329}{64}\right]j_{3,1}(\beta_a)+\frac{179}{420}j_{4,1}(\beta_a)\right\}.$$

Here we introduced the following notation:

$$j_3(\beta)=\sum_{n=3}^{\infty}\frac{(-1)^n}{n\left(n-\frac{1}{2}\right)(n-1)}\beta^{2n}\zeta_H\left(2n-3,\frac{1}{2}\right), \qquad \text{(A28)}$$

$$j_{p,q}(\beta)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(n+p)}{\Gamma(p)}\beta^{2n}\zeta_H\left(2n+q,\frac{1}{2}\right). \qquad \text{(A29)}$$

The functions $j_3(\beta)$ and $j_{p,q}(\beta)$ with integer p and q are expressed in terms of the function $j_0(\beta)$ only by the relations

$$j_3(\beta)=-2\beta^4\int_0^1 dx x(1-x)^2\left\{\Psi\left(\frac{1}{2}-i\beta x\right)+\Psi\left(\frac{1}{2}+i\beta x\right)-2\Psi\left(\frac{1}{2}\right)\right\},$$

$$j_{1,1}(\beta)=-xj_0(x)|_{x=\beta^2},$$

$$j_{2,1}(\beta)=-2xj_0(x)-x^2j_0'(x)|_{x=\beta^2}, \quad (A30)$$

$$j_{3,1}(\beta)=-3xj_0(x)-3x^2j_0'(x)-\frac{1}{2}x^3j_0''(x)|_{x=\beta^2},$$

$$j_{4,1}(\beta)=-4xj_0(x)-6x^2j_0'(x)-2x^3j_0''(x)-\frac{1}{6}x^4j_0'''(x)|_{x=\beta^2}.$$

For half-integer indexes p and q in Eq. (A29) we use the integral representation for the Hurwitz zeta function from Ref. [17]:

$$\zeta_H(s,v)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-(v-1)x}x^{s-1}\,dx}{e^x-1}, \quad (A31)$$

and obtain the following formulas:

$$j_{1/2,0}(\beta)=-\beta\int_0^\infty\frac{dx}{\sinh(x)}J_1(2\beta x),$$

$$j_{3/2,0}(\beta)=-\beta\int_0^\infty\frac{dx}{\sinh(x)}\{3J_1(2\beta x)-2\beta xJ_2(2\beta x)\}, \quad (A32)$$

$$j_{5/2,0}(\beta)=-\beta\int_0^\infty\frac{dx}{\sinh(x)}\left\{5J_1(2\beta x)-\frac{20}{3}\beta xJ_2(2\beta x)+\frac{4}{3}\beta xJ_3(2\beta x)\right\}.$$

Substituting the formulas obtained in Eq. (A7) one has

$$\zeta_a^{\rm int}\left(s-\frac{1}{2}\right)_{s\to 0}=\frac{\beta_a^{2s}}{(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)}\left\{b_0^am^4\Gamma(s-2)+b_{1/2}^am^3\Gamma\left(s-\frac{3}{2}\right)+b_1^am^2\Gamma(s-1)+b_{3/2}^am\Gamma\left(s-\frac{1}{2}\right)+b_2^a\Gamma(s)\right\}-\frac{1}{16\pi^2a}\Omega^a(\beta_a), \quad (A33)$$

where

$$\Omega^a(\beta_a)=A_f^a(\beta_a)+\sum_{k=-1}^{3}(-1)^k\omega_k(\beta_a). \quad (A34)$$

In the limit $s\to 0$ an additional finite contribution appears due to multiplying $s\ln\beta_a^2$ and the gamma functions in the curly brackets in Eq. (A33). Because the gamma function has simple poles at the points $0,-1,-2,\ldots,$ the heat kernel coefficients with integer indices will give finite contributions and we arrive at the following expression:

$$\zeta_a^{\rm int}\left(s-\frac{1}{2}\right)_{s\to 0}=\frac{1}{(4\pi)^{3/2}a\Gamma\left(s-\frac{1}{2}\right)}\left\{b_0^a\beta_a^4\Gamma(s-2)+b_{1/2}^a\beta_a^3\Gamma\left(s-\frac{3}{2}\right)+b_1^a\beta_a^2\Gamma(s-1)+b_{3/2}^a\beta_a\Gamma\left(s-\frac{1}{2}\right)+b_2^a\Gamma(s)\right\}-\frac{1}{16\pi^2a}\{b^a\ln\beta_a^2+\Omega^a[\beta_a]\}. \quad (A35)$$

Therefore we have divided the zeta function into two parts: the asymptotic singular part of the zeta function in standard form and a finite contribution.

Using the same approach for $\zeta_R^{\rm ext}(s-\frac{1}{2})$ one has

$$\zeta_R^{\rm ext}\left(s-\frac{1}{2}\right)_{s\to 0}=\frac{1}{(4\pi)^{3/2}(a+R)\Gamma\left(s-\frac{1}{2}\right)}\left\{b_0^R\beta_R^4\Gamma(s-2)+b_{1/2}^R\beta_R^3\Gamma\left(s-\frac{3}{2}\right)+b_1^R\beta_R^2\Gamma(s-1)+b_{3/2}^R\beta_R\Gamma\left(s-\frac{1}{2}\right)+b_2^R\Gamma(s)\right\}-\frac{1}{16\pi^2(a+R)}\{b^R\ln\beta_R^2+\Omega^R[\beta_R]\}. \quad (A36)$$

By virtue of the fact that in the limit $m\to\infty$ the above formulas must give us the asymptotic expansion (51), the function $\Omega[\beta]$ has the following behavior $[\beta_\alpha\to\infty,(\alpha=a,R)]$:

$$\Omega^{\alpha}[\beta_\alpha]=-b^\alpha \ln \beta_\alpha^2-\frac{2\sqrt{\pi}}{\Gamma\left(s-\frac{1}{2}\right)}\sum_{k=5}^{\infty} b_{k/2}^{\alpha}\beta_\alpha^{4-k}\times\Gamma\left(s+\frac{k}{2}-2\right)\Bigg|_{s\to 0}$$
$$=-b^\alpha \ln \beta_\alpha^2+\sum_{k=5}^{\infty} b_{k/2}^{\alpha}\beta_\alpha^{4-k}\Gamma\left(\frac{k}{2}-2\right)$$
$$=-b^\alpha \ln \beta_\alpha^2+\sqrt{\pi}b_{5/2}^{\alpha}\beta_\alpha^{-1}+O(\beta_\alpha^{-2}). \quad \text{(A37)}$$

APPENDIX B

The main problem for numerical calculation of the ground state energy is the term $A_f[\beta]$ given by Eq. (A8). The series in Eq. (A8) is poorly convergent. To calculate this expression let us, first of all, represent it in the following form:

$$\overline{A_f}\equiv\frac{A_f}{32\pi}=\sum_{l=0}^{\infty}\sigma_\nu, \quad \text{(B1)}$$

where

$$\sigma_\nu=\nu^2\int_{\beta/\nu}^{\infty}dx\sqrt{x^2-\frac{\beta^2}{\nu^2}}\frac{\partial}{\partial x}\left(\ln K_\nu(\nu x)+\ln\left[\delta K_\nu(\nu x)+\frac{x\nu}{4}K_\nu'(\nu x)\right]+2\nu\eta(x)+\frac{1}{\nu}N_1-\frac{1}{\nu^2}N_2+\frac{1}{\nu^3}N_3\right), \quad \text{(B2)}$$

and divide the series into two parts:

$$\overline{A_f}=\sum_{l=0}^{N}\sigma_\nu+\sum_{l=N+1}^{\infty}\sigma_\nu. \quad \text{(B3)}$$

The first sum we calculate numerically. The calculations become lighter because the Bessel functions of the second kind with half-integer indexes are polynomial with a simple exponent factor [15]. In the second sum we use the uniform expansion of the integrand over inverse powers of the index ν. Since we have already subtracted the first three terms N_1, N_2, and N_3, the uniform expansion of the integrand will start from ν^{-4}, and we obtain the following expression:

$$\sum_{l=N+1}^{\infty}\sigma_l=\sum_{l=N+1}^{\infty}\nu^2\int_{\beta/\nu}^{\infty}dx\sqrt{x^2-\frac{\beta^2}{\nu^2}}\frac{\partial}{\partial x}\times\sum_{p=4}^{\infty}(-\nu)^{-p}N_p[t], \quad \text{(B4)}$$

where $N_p[t]$ is the polynomial of degree $3p$:

$$N_p[t]=\sum_{k=0}^{p}a_{p,k}t^{p+2k}. \quad \text{(B5)}$$

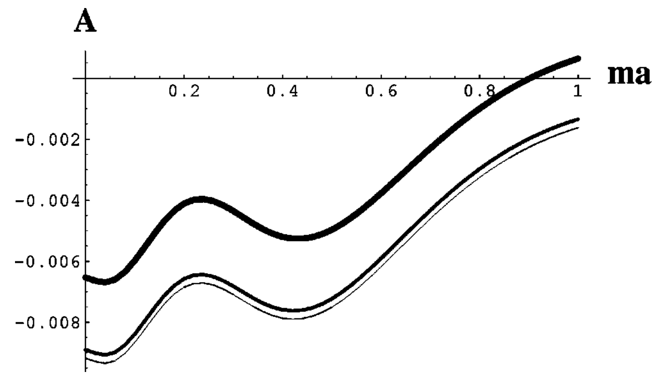

FIG. 3. The function $A=A_f/32\pi$ as a function of ma for $\xi=1/6$. Thick curve is zero term ($l=0$) contribution. The curve of medium thickness is the contribution of the first 11 terms up to $l=10$. The thin curve reproduces the calculations with high precision (up to $p=8$ of the uniform expansion).

The coefficients $a_{p,k}$ for $p=1,2,3$ may be singled out from Eq. (A5).

Then one takes derivatives and integrals in Eq. (B4) and changes the sums over p and l. After this we arrive at the following formula:

$$\sum_{l=N+1}^{\infty}\sigma_l=\sum_{p=4}^{\infty}\overline{N_p}, \quad \text{(B6)}$$

where

$$\overline{N_p}=\frac{\sqrt{\pi}}{2}(-1)^{3-p}\sum_{k=0}^{p}a_{p,k}\frac{\Gamma(p/2-1/2+k)}{\Gamma(p/2+k)}\times h[p,p+2k,\beta,N], \quad \text{(B7)}$$

$$h[p,q,\beta,N]=\sum_{l=N+1}^{\infty}\nu^{2-p}\left(1+\frac{\beta^2}{\nu^2}\right)^{-(q-1)/2}. \quad \text{(B8)}$$

The function h may be found in closed form for integer p and q.

The above function h may be estimated by

$$h[p,q,\beta,N]\approx\left(N+\frac{3}{2}\right)^{2-p}, \quad \text{(B9)}$$

and the series (B6) is quickly convergent for large N. We use $N=10$ and in order to work with precision 10^{-10} it is enough to take the expansion up to $p=8$ (five terms). In fact, this procedure converts the poorly convergent series to a quickly convergent series over N^{2-p}.

To illustrate the above approach we reproduce in Fig. 3 the three steps of calculation of $\overline{A_f}$: (i) the zero term (thick curve), (ii) the contribution of the first 11 terms up to $l=10$ (medium thickness), and (iii) the exact curve (up to $p=8$ in the uniform expansion, thin curve). Ten terms $l=1$–10 give us a the correction of 36% for the zero term, and the series from $l=11$ to ∞ gives us a 4% correction.

[1] L. Flamm, Phys. Z. **17**, 48 (1916).
[2] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).
[3] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[4] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[5] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
[6] The reader can find some references on wormhole physics in the excellent book by M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (AIP, Woodbury, NY, 1995).
[7] S. V. Sushkov, Phys. Lett. A **164**, 33 (1992); A. Popov and S. V. Sushkov, in *Quantum Field Theory Under the Influence of External Conditions*, edited by M. Bordag (Teubner, Leipzig, 1996); D. Hochberg, A. Popov, and S. V. Sushkov, Phys. Rev. Lett. **78**, 2050 (1997).
[8] V. Khatsymovsky, Phys. Lett. B **429**, 254 (1998).
[9] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977); S. K. Blau, M. Visser, and A. Wipf, Nucl. Phys. **B310**, 163 (1988).
[10] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
[11] M. Bordag, J. Phys. A **28**, 755 (1995); M. Bordag and K. Kirsten, Phys. Rev. D **53**, 5753 (1996); M. Bordag, K. Kirsten, and E. Elizalde, J. Math. Phys. **37**, 895 (1996); M. Bordag, E. Elizalde, K. Kirsten, and S. Leseduarte, Phys. Rev. D **56**, 4896 (1997); M. Bordag, K. Kirsten, and D. Vassilevich, *ibid.* **59**, 085011 (1999).
[12] N. R. Khusnutdinov and M. Bordag, Phys. Rev. D **59**, 064017 (1999).
[13] E. R. Bezerra de Mello, V. B. Bezerra, and N. R. Khusnutdinov, J. Math. Phys. **42**, 562 (2001).
[14] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[15] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, DC, 1964).
[16] P. B. Gilkey, K. Kirsten, and D. V. Vassilevich, Nucl. Phys. **B601**, 125 (2001).
[17] H. Bateman and A. Erdélyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.