Equilibrium of three collinear Kerr particles

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The equilibrium problem of three collinear Kerr particles is studied within the framework of an exact solution of Einstein's equations. The system of conditions defining the equilibrium of the particles is derived in an explicit form and particular equilibrium configurations involving black-hole and hyperextreme constituents are considered. It is demonstrated that in some equilibrium states a spinning particle can exhibit combined properties characteristic both of black holes and hyperextreme objects.

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I. INTRODUCTION

While static systems of collinear Schwarzschild and Curzon particles were investigated long ago [1–4], the study of collinear spinning particles became possible only after the development of modern solution generating techniques in the late 1970s and early 1980s [5–11]. Angular momentum permits us, in principle, to achieve gravitational equilibrium of the constituents comprising the system even when all the masses involved are positive, unlike in the static case where equilibrium is only possible for an odd number of particles (two Scwarzschild particles, for instance, can never be in equilibrium independently of the sign of their masses), provided that some of the constituents necessarily have negative mass. The stationary collinear systems are, therefore, of greater physical realism.

The first examples of gravitational equilibrium of two spinning collinear particles with positive individual masses were given by Dietz and Hoenselaers [12,13]. In Ref. [12] they derived a vacuum solution representing the exterior gravitational field of two rotating Curzon sources in equilibrium (a generalization of that result was given in Ref. [14]), and later on [13] they discovered the possibility of gravitational equilibrium of two identical hyperextreme constituents in the double-Kerr solution [15]. A numerical investigation of the analytical expressions for Komar masses [16] of the subextreme constituents carried out for the latter solution by Hoenselaers [17] permitted him to conjecture that two normal Kerr black holes possessing positive masses can never be in equilibrium (a rigorous proof of Hoenselaers' conjecture has been given recently by Manko and Ruiz [18]). In Ref. [19] equilibrium of the subextreme and hyperextreme Kerr constituents was claimed without a discussion of the sign of their individual masses. The explicit extended formulas determining equilibrium of any combination of the subextreme and hyperextreme constituents in the double-Kerr solution have been derived in Ref. [20], and there several particular examples of equilibrium of two nonidentical hyperextreme Kerr sources and of subextreme and hyperextreme constituents involving only positive masses have been given.

In our Letter [21] we attacked the balance problem of three collinear Kerr particles for the first time. Remarkably, the equilibrium of three Kerr black hole constituents with positive masses is possible, unlike in the aforementioned double-Kerr solution, although there still remains an open question of whether the appearance of a massless ring singularity which accompanies the known particular equilibrium states of black holes can be avoided for some specific values of the parameters. This advantage of three-body systems over two-body ones concerning the capacity to form the black hole equilibrium configurations might be a reflection of the intrinsic differences between systems composed of an odd and even number of particles if one takes into account that exact solutions describing the collinear Kerr particles are stationary generalizations of the Israel-Khan multi-Schwarzschild metric [4] for which such differences exist.

In the present paper we would like to consider various possibilities of the balance of three collinear Kerr particles. First of all, we would like to give new particular examples of three balancing black holes and of two black holes in equilibrium with a hyperextreme object. Then we shall supplement these two possible equilibrium configurations earlier discussed in Ref. [21] with two other ones, i.e., equilibrium of two hyperextreme and one subextreme constituents, and the case of three balancing hyperextreme Kerr sources. In Sec. II we shall present the extended metric describing the exterior gravitational field of three collinear arbitrary Kerr particles, and in Sec. III we shall derive the general balance conditions determining the equilibrium of the particles. Particular equilibrium states for all possible combinations of subextreme and hyperextreme constituents in the triple-Kerr solution possessing equatorial symmetry will be considered in Sec. IV. Concluding remarks are contained in Sec. V.

II. EXTENDED TRIPLE-KERR METRIC

Since the collinear Kerr particles can be either subextreme or hyperextreme objects, we need an extended solution of the

Einstein equations for their description. As is well known, the axisymmetric vacuum problem was reduced by Ernst to solving his famous equation [22]

$$(\mathcal{E}+\overline{\mathcal{E}})(\mathcal{E}_{,\rho,\rho}+\rho^{-1}\mathcal{E}_{,\rho}+\mathcal{E}_{,z,z})=2(\mathcal{E}_{,\rho}^2+\mathcal{E}_{,z}^2), \qquad (1)$$

where ρ and z are the Weyl-Papapetrou cylindrical coordinates, a bar means complex conjugation, and the subindices following a comma denote partial differentiation with respect to the indicated variables. If $\mathcal{E}$ is known, the metric coefficients f, γ, and ω entering the axisymmetric line element[1]

$$ds^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2 d\varphi^2]-f(dt-\omega d\varphi)^2, \quad (2)$$

can be obtained from the equations [22,23]

$$\mathcal{E}=f+i\Omega,$$

$$\omega_{,\rho}=-\rho f^{-2}\Omega_{,z}, \quad \omega_{,z}=\rho f^{-2}\Omega_{,\rho},$$

$$\gamma_{,\rho}=\frac{\rho}{(\mathcal{E}+\overline{\mathcal{E}})^2}(\mathcal{E}_{,\rho}\overline{\mathcal{E}}_{,\rho}-\mathcal{E}_{,z}\overline{\mathcal{E}}_{,z}),$$

$$\gamma_{,z}=\frac{2\rho}{(\mathcal{E}+\overline{\mathcal{E}})^2}\mathrm{Re}(\mathcal{E}_{,\rho}\overline{\mathcal{E}}_{,z}). \qquad (3)$$

A powerful method for the construction of extended solutions of Eq. (1) equally applicable for treating the subextreme and hyperextreme sources was developed by Sibgatullin [11] on the basis of a brilliant employment of the opportunities contained in the Hauser-Ernst approach [9]. A starting point in Sibgatullin's integral equation method is an arbitrarily prescribed axis expression $e(z)\equiv\mathcal{E}(\rho=0,z)$ of the Ernst complex potential which is used, through the Riemann-Hilbert procedure of the continuation of analytic functions, for obtaining the corresponding $\mathcal{E}(\rho,z)$ satisfying Eq. (1) and valid in the whole (ρ,z) plane.

In Ref. [24] an extended metric for N collinear Kerr particles was considered and explicit formulas relating parameters of the metric to the axis data and multipole moments were obtained. The three-body case which is of interest to us corresponds to $e(z)$ of the form

$$e(z)=1+\sum_{l=1}^{3}\frac{e_l}{z-\beta_l}, \qquad (4)$$

where e_l and β_l are arbitrary complex constants. For $e(z)$ defined by Eq. (4), Ref. [24] gives the following expression for $\mathcal{E}(\rho,z)$:

$$\mathcal{E}=E_+/E_-, \quad E_\pm=\Lambda\pm\Gamma,$$

$$\Lambda=\sum_{1\leqslant i<j<k\leqslant 6}\lambda_{ijk}r_ir_jr_k, \quad \Gamma=\sum_{1\leqslant i<j\leqslant 6}\nu_{ij}r_ir_j,$$

$$\lambda_{ijk}=(-1)^{i+j+k}A_{ij}A_{ik}A_{jk}A_{i'j'}A_{i'k'}A_{j'k'}\widetilde{R}_i\widetilde{R}_j\widetilde{R}_k \times R_{i'}R_{j'}R_{k'} \quad (i',j',k'\neq i,j,k;i'<j'<k'),$$

$$\nu_{ij}=(-1)^{i+j}A_{ij}A_{i'j'}A_{i'k'}A_{i'l'}A_{j'k'}A_{j'l'}A_{k'l'}\widetilde{R}_i\widetilde{R}_j \times R_{i'}R_{j'}R_{k'}R_{l'} \quad (i',j',k',l'\neq i,j;i'<j'<k'<l'),$$

$$A_{mn}:=\alpha_m-\alpha_n, \quad r_n:=\sqrt{\rho^2+(z-\alpha_n)^2},$$

$$R_n:=(\alpha_n-\beta_1)(\alpha_n-\beta_2)(\alpha_n-\beta_3),$$

$$\widetilde{R}_n:=(\alpha_n-\overline{\beta}_1)(\alpha_n-\overline{\beta}_2)(\alpha_n-\overline{\beta}_3), \qquad (5)$$

where α_n are six parameters which can assume arbitrary real values or occur in complex conjugate pairs. Note that α_n enter Eq. (5) instead of three complex parameters e_l from Eq. (4), but the parametrizations $\{e_l,\beta_l\}$ and $\{\alpha_n,\beta_l\}$ are equivalent, each counting twelve arbitrary real constants. The relation of e_l to α_n is given by the formula

$$e_l=\frac{2\prod_{n=1}^{6}(\beta_l-\alpha_n)}{\prod_{\substack{k=1\\k\neq l}}^{3}(\beta_l-\beta_k)\prod_{k=1}^{3}(\beta_l-\overline{\beta}_k)}. \qquad (6)$$

The metric functions f, γ, and ω corresponding to the potential (5) have the form

$$f=\frac{E_+\overline{E}_-+\overline{E}_+E_-}{2E_-\overline{E}_-}, \quad e^{2\gamma}=\frac{E_+\overline{E}_-+\overline{E}_+E_-}{2\lambda\overline{\lambda}\prod_{n=1}^{6}r_n},$$

$$\omega=2\,\mathrm{Im}\,\sigma-\frac{4\,\mathrm{Im}\{(\overline{\Lambda}-\overline{\Gamma})G\}}{E_+\overline{E}_-+\overline{E}_+E_-},$$

$$G=-\sigma\Lambda+(z+\overline{\sigma})\Gamma+\sum_{1\leqslant i<j<k\leqslant 6}\lambda_{ijk}(\alpha_i+B_{jk})r_ir_jr_k - \sum_{1\leqslant i<j\leqslant 6}\nu_{ij}(B_{i'j'}+B_{k'l'})r_ir_j \quad (i',j',k',l' \neq i,j;i'<j'<k'<l'),$$

$$\sigma:=\beta_1+\beta_2+\beta_3 =\lambda^{-1}\left[\sum_{1\leqslant i<j\leqslant 6}\nu_{ij}+\sum_{1\leqslant i<j<k\leqslant 6}\lambda_{ijk}(\alpha_i+B_{jk})\right],$$

$$\lambda:=\sum_{1\leqslant i<j<k\leqslant 6}\lambda_{ijk}, \quad B_{mn}:=\alpha_m+\alpha_n. \qquad (7)$$

Since we are interested in the case of three separated particles, we assign the following order to the parameters α_n which determine the location of the particles on the symmetry axis:

[1]Throughout the paper units are used in which the speed of light and Newton's gravitational constant are equal to unity.

$$\mathrm{Re}\,\alpha_1 \geqslant \mathrm{Re}\,\alpha_2 > \mathrm{Re}\,\alpha_3 \geqslant \mathrm{Re}\,\alpha_4 > \mathrm{Re}\,\alpha_5 \geqslant \mathrm{Re}\,\alpha_6. \tag{8}$$

Then, as usual, a pair of real-valued α's, say α_1 and α_2, will define a subextreme Kerr black hole, the segment $\alpha_2 \leqslant z \leqslant \alpha_1$ being its Killing horizon, and a pair of complex conjugate α's, say α_1 and $\alpha_2 = \overline{\alpha}_1$, will define a hyperextreme constituent. In the latter case the whole mass of the particle will be concentrated along the cut joining the points α_1 and $\overline{\alpha}_1$, which suggests a possible interpretation of the hyperextreme Kerr constituents as relativistic disks.

The total mass M and total angular momentum J of our three-body system are defined by the expressions

$$M = -\sum_{1\leqslant i<j\leqslant 6} \nu_{ij}/\lambda,$$

$$J = \mathrm{Im}\left\{\left[\sum_{1\leqslant i<j\leqslant 6} \nu_{ij} B_{ij} + M \sum_{1\leqslant i<j<k\leqslant 6} \lambda_{ijk}(\alpha_i + B_{jk})\right] \Big/ \lambda\right\} \tag{9}$$

which are readily obtainable using the results of Ref. [24].

It is advantageous for the search of equilibrium configurations of the Kerr particles to introduce the constant objects X_n via the formula

$$X_n := \frac{\tilde{R}_n}{R_n} = \frac{(\alpha_n - \overline{\beta}_1)(\alpha_n - \overline{\beta}_2)(\alpha_n - \overline{\beta}_3)}{(\alpha_n - \beta_1)(\alpha_n - \beta_2)(\alpha_n - \beta_3)},$$

$$n = 1,2,3,4,5,6 \tag{10}$$

verifying the relation

$$X_n \tilde{X}_n = 1, \tag{11}$$

where the "tilde operator" again means the conjugation of the constants β_l exclusively.

The set $\{\alpha_n, X_n\}$ is equivalent to the set $\{\alpha_n, \beta_l\}$, and we shall use it in the next sections for the definition of the equatorially symmetric case of three collinear particles and for finding particular equilibrium configurations since the numerical values of X_n are more easily detected by computer programs than the respective β_l. In terms of X_n the formulas (5), (7), (9) remain the same except for the form of λ_{ijk} and ν_{ij} which slightly changes (after cancelling the common factor $\Pi_{n=1}^{6} R_n$) to become

$$\lambda_{ijk} = (-1)^{i+j+k} A_{ij} A_{ik} A_{jk} A_{i'j'} A_{i'k'} A_{j'k'} X_i X_j X_k,$$

$$\nu_{ij} = (-1)^{i+j} A_{ij} A_{i'j'} A_{i'k'} A_{i'l'} A_{j'k'} A_{j'l'} A_{k'l'} X_i X_j. \tag{12}$$

Mention that the quantities X_n defined by Eq. (10) relate the results obtained in Ref. [24] with the aid of Sibgatullin's method to the "complexified Kinnersley-Chitre transformation" approach employed by Ernst [25]. It should also be remarked that the derivation of many mathematical characteristic features of the metric (7), the balance conditions among them, is simpler using the quantities R_n and not X_n.

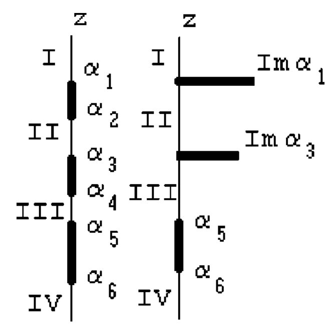

FIG. 1. Two examples of the systems of three Kerr particles. Sections I–IV of the symmetry axis should be regular for any particular equilibrium configuration.

To conclude this section, it might be worthy to point out that in view of the evidence on the uniqueness of the double-Kerr solution for the description of two Kerr particles [26], the triple-Kerr metric defined by Eqs. (5) and (7) can be also considered a unique one for the description of stationary systems consisting of three collinear Kerr particles.

III. BALANCE CONDITIONS

Conditions determining the equilibrium of three Kerr particles due to the balance of the gravitational attraction and spin-spin repulsion forces are obtainable by requiring that the elementary flatness of the parts of the symmetry axis be outside the location of the material sources. Mathematically this is equivalent to the vanishing of the metric functions γ and ω on the parts of the z-axis exterior to and between the particles [27,12]:

$$\gamma^{(\text{I-IV})} = \omega^{(\text{I-IV})} = 0, \tag{13}$$

where the segments I and IV are the upper and lower parts of the symmetry axis, i.e., $\mathrm{Re}\alpha_1 < z < \infty$ and $-\infty < z < \mathrm{Re}\alpha_6$, respectively, and segments II and III are the parts of the symmetry axis between the particles, i.e., $\mathrm{Re}\alpha_3 < z < \mathrm{Re}\alpha_2$ and $\mathrm{Re}\alpha_5 < z < \mathrm{Re}\alpha_4$, respectively (see Fig. 1).

By construction, the metric (5), (7) verifies

$$\gamma^{(\text{I,IV})} = \omega^{(\text{I})} = 0, \tag{14}$$

i.e., the function γ is zero on both upper and lower parts of the symmetry axis, whereas ω, in general, vanishes only on the interval $\mathrm{Re}\alpha_1 < z < \infty$. The condition for vanishing ω on the lower part of the z axis, $\omega^{(\text{IV})} = 0$, coincides with the condition of the asymptotic flatness of the triple-Kerr metric

$$\mathrm{Im}\left\{\sum_{l=1}^{3} \frac{2\prod_{n=1}^{6}(\beta_l - \alpha_n)}{\prod_{\substack{k=1\\k\neq l}}^{3}(\beta_l - \beta_k)\prod_{k=1}^{3}(\beta_l - \tilde{\beta}_k)}\right\} = 0. \tag{15}$$

In what follows, we shall refer to the Kerr particles defined by the pairs (α_1, α_2), (α_3, α_4), and (α_5, α_6) as to the upper, medium, and lower constituents, respectively. The remaining four balance conditions involve two segments (II and III) between the particles, and their explicit form can be obtained by considering the coefficients at leading powers of z in the numerators and denominators of γ and ω since both these functions assume constant values on the symmetry axis.

Taking into account that at region II we have $r_1=\alpha_1-z$, $r_2=\alpha_2-z$, and $r_i=z-\alpha_i$, $i=3,4,5,6$, the condition $\gamma^{(\mathrm{II})}=0$ can be written as

$$\sum_{i=3}^{6}\lambda_{12i}+\sum_{3\leqslant i<j<k\leqslant 6}\lambda_{ijk}=0, \tag{16}$$

while the condition $\omega^{(\mathrm{II})}=0$ assumes the form

$$\gamma_{12}+\gamma'_{12}+\sum_{3\leqslant i<j<k\leqslant 6}(\gamma_{ij}+\gamma'_{ij})+\sum_{i=3}^{6}(\alpha_1+B_{2i})\times(\lambda_{12i}+\lambda'_{12i})+\sum_{3\leqslant i<j<k\leqslant 6}(\alpha_i+B_{jk})\times(\lambda_{ijk}+\lambda'_{ijk})=0, \tag{17}$$

where we have introduced the quantities

$$\lambda'_{ijk}=(-1)^{i+j+k}A_{ij}A_{ik}A_{jk}A_{i'j'}A_{i'k'}A_{j'k'}X_{i'}X_{j'}X_{k'},$$

$$\nu'_{ij}=(-1)^{i+j}A_{ij}A_{i'j'}A_{i'k'}A_{i'l'}A_{j'k'}A_{j'l'}A_{k'l'}\times X_{i'}X_{j'}X_{k'}X_{l'}. \tag{18}$$

Turning now to region III, we have $r_i=\alpha_i-z$, $i=1,2,3,4$, $r_5=z-\alpha_5$, $r_6=z-\alpha_6$. Then the condition $\gamma^{(\mathrm{III})}=0$ takes the form

$$\sum_{i=1}^{4}\lambda_{i56}+\sum_{1\leqslant i<j<k\leqslant 4}\lambda_{ijk}=0, \tag{19}$$

and the condition $\omega^{(\mathrm{III})}=0$ leads to the equation

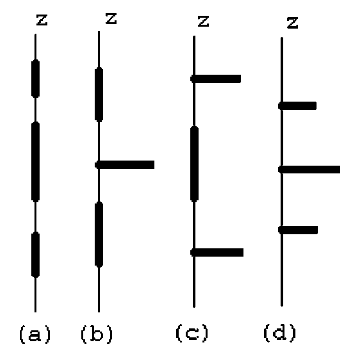

FIG. 2. Four possible types of the triple-Kerr systems possessing reflectional symmetry.

$$\sum_{\substack{1\leqslant i\leqslant 4\\5\leqslant j\leqslant 6}}(\gamma_{ij}+\gamma'_{ij})+\sum_{i=1}^{4}(\alpha_i+B_{56})(\lambda_{i56}+\lambda'_{i56})+\sum_{1\leqslant i<j<k\leqslant 4}(\alpha_i+B_{jk})(\lambda_{ijk}+\lambda'_{ijk})=0. \tag{20}$$

Equations (15)–(17), (19), (20) constitute a complete set of conditions determining the equilibrium of three collinear Kerr particles. It should be underlined that these equations are valid for the whole range of the parameters α_n, i.e., for any combination of the subextreme and hyperextreme constituents. This can be easily seen in the case of Eqs. (16) and (19) for γ, but the derivation of Eqs. (17) and (20) for ω involves multiple complex conjugate quantities and, therefore, originally has been carried out by us separately for all possible combinations of real- and complex-valued α_n. Remarkably, all the cases have finally permitted a unified representation.

IV. PARTICULAR EQUILIBRIUM STATES

It does not look possible to investigate the algebraic system (15)–(17), (19), (20) analytically, so one is forced to resort to numerical calculations in order to obtain particular equilibrium states. But even the task of finding numerical roots of this system proves to be very complicated in the general case, thus suggesting some further simplifications. It is reasonable to single out the systems possessing a reflectional symmetry with respect to the $z=0$ plane which permit

TABLE I. Purely black-hole equilibrium states.

α_1	5	5	3	3
α_2	2	2	0.5	0.5
α_3	0.8	1	0.25	0.25
X_1	$0.9+0.436i$	$0.8-0.6i$	$0.9-0.436i$	$0.8-0.6i$
X_2	$0.707+0.707i$	$0.694-0.72i$	$0.564-0.826i$	$0.53-0.848i$
X_3	$0.575+0.818i$	$0.642-0.767i$	$0.486-0.874i$	$0.46-0.888i$
$M_u=M_l$	3.537	1.869	1.432	1.001
M_m	1.823	5.553	1.33	2.501
$J_u=J_l$	-67.156	67.255	13.547	13.807
J_m	121.312	-166.255	-28.804	-35.828

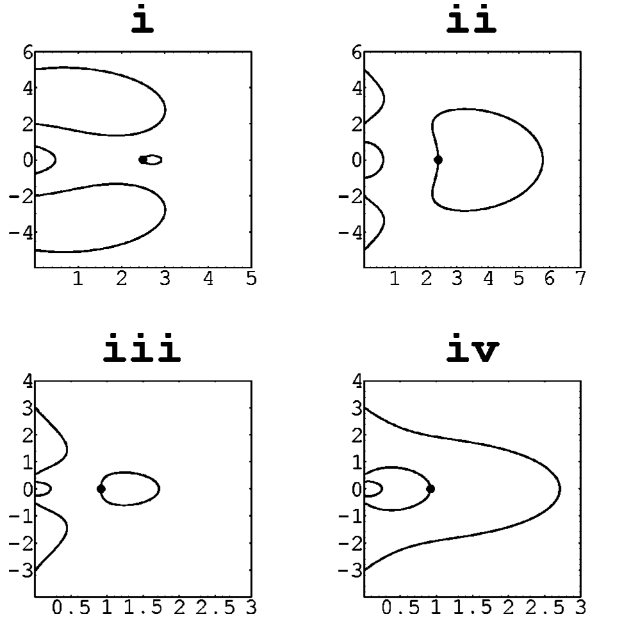

FIG. 3. Stationary limit surfaces corresponding to the equilibrium configurations from Table I.

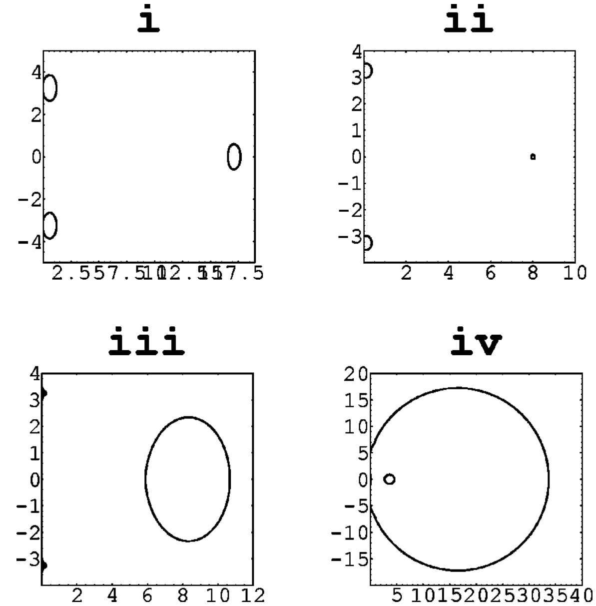

FIG. 4. Stationary limit surfaces corresponding to the equilibrium configurations from Table II.

the following nontrivial interesting combinations of three collinear Kerr particles: (a) a superposition of three subextreme (black hole) constituents, (b) two black holes superposed with a hyperextreme object, (c) two hyperextreme objects superposed with a black hole, and (d) a superposition of three hyperextreme constituents (see Fig. 2).

In the equatorially symmetric case the balance problem considerably simplifies since segment IV of the symmetry axis becomes identical to segment I, and segment III becomes identical to segment II, so that Eq. (15) is satisfied automatically, and Eqs. (19) and (20) coincide with Eqs. (16) and (17), respectively. Therefore, only the latter two equations should be solved to get the equilibrium states. In addition, they take a much simpler form since, making use of the general relations obtained in Ref. [24], it can be shown that the metric (7) is equatorially symmetric if α_n and X_n are subjected to the following restrictions:

$$\alpha_6=-\alpha_1, \quad \alpha_5=-\alpha_2, \quad \alpha_4=-\alpha_3,$$
$$X_1X_6=X_2X_5=X_3X_4=-1. \tag{21}$$

With relations (21) taken into account, Eq. (16) can be rewritten as

$$\begin{aligned}A_{12}[A_{13}^2A_{23}^2(X_1^2X_2^2X_3^2+1)+B_{13}^2B_{23}^2(X_1^2X_2^2+X_3^2)]\\+4\alpha_3B_{12}[\alpha_1A_{23}B_{23}X_1X_3(X_2^2+1)\\-\alpha_2A_{13}B_{13}X_2X_3(X_1^2+1)]=0,\end{aligned} \tag{22}$$

and Eq. (17) takes the form

$$\begin{aligned}B_{12}\{\alpha_1A_{13}B_{13}X_1[B_{23}^2(X_2^2-X_3^2)+A_{23}^2(X_2^2X_3^2-1)]\\-\alpha_2A_{23}B_{23}X_2[B_{13}^2(X_1^2-X_3^2)+A_{13}^2(X_1^2X_3^2-1)]\\+4\alpha_1\alpha_2\alpha_3X_3[A_{23}B_{23}X_1(X_2^2-1)\\-A_{13}B_{13}X_2(X_1^2-1)]\}\\+A_{12}[A_{13}^2A_{23}^2(B_{12}+\alpha_3)(X_1^2X_2^2X_3^2-1)\\+B_{13}^2B_{23}^2(B_{12}-\alpha_3)(X_1^2X_2^2-X_3^2)]=0.\end{aligned} \tag{23}$$

TABLE II. Equilibrium configurations of two black-hole and one hyperextreme constituents.

α_1	3.5	3.5	3.5	5
α_2	3	3	3	1
α_3	$-17i$	$-8i$	$-8i$	$-4i$
X_1	$0.65+0.76i$	$0.5+0.866i$	$0.5+0.866i$	$0.5-0.866i$
X_2	$0.984+0.448i$	$1-0.011i$	$0.453-0.892i$	$0.82-0.573i$
X_3	$0.15i$	$0.081i$	$-2.706i$	$-0.657i$
$M_u=M_l$	1.176	0.457	-0.297	5.429
M_m	4.1	1.105	7.465	22.959
$J_u=J_l$	-2.012	-0.228	0.129	-90.678
J_m	80.279	10.037	-74.504	-967.622

TABLE III. Equilibrium states of a black hole and two hyperextreme objects.

$\alpha_1=\overline{\alpha_2}$	$2-2.8i$	$2-4.558i$	$2-3.163i$	$2-1.94i$
α_3	1.5	1.5	1.8	1.5
$X_1=\overline{X}_2^{-1}$	$1.2-0.8i$	$1.2-1.1i$	$1.2-0.8i$	$1.2-0.5i$
X_3	$0.787-0.617i$	$0.534-0.845i$	$0.822-0.57i$	$0.92-0.392i$
$M_u=M_l$	2.204	1.386	2.179	2.997
M_m	2.079	4.285	2.419	0.749
$J_u=J_l$	34.075	40.074	35.866	37.791
J_m	-55.619	-81.051	-58.787	-51.767

In Ref. [21] we have already presented two numerical solutions of the system (22), (23), but both of them were of the (a) and (b) types since in that paper we were primarily interested in the question of how equilibrium of two Kerr black holes can be achieved by placing a third Kerr constituent between them. Below we shall give more examples of the equilibrium states of three subextreme Kerr black holes, two black holes, and one hyperextreme constituent; we shall also complement our consideration of the balance problem with the remaining two possibilities of the equilibrium of three Kerr particles in items (c) and (d) above.

The equilibrium states which will be considered below are all characterized by the particular values of the parameters $\alpha_1,\alpha_2,\alpha_3,X_1,X_2,X_3$, and by the corresponding individual Komar masses M_u,M_m,M_l (u, m, and l denote, respectively, the upper, medium, and lower constituents) and angular momenta J_u,J_m,J_l. In view of the reflectional symmetry, we have $M_u=M_l$, $J_u=J_l$, and it is advantageous to calculate the masses and angular momenta of the equilibrium configurations using Tomimatsu's formulas [28]

$$M=-\frac{1}{4}\int_H \omega\Omega_{,z}\,dz,$$

$$J=-\frac{1}{4}\int_H \omega\left(1+\frac{1}{2}\omega\Omega_{,z}\right)dz, \tag{24}$$

where H stands for the horizon of a black-hole constituent and ω is calculated at the horizon. Then, after evaluating M and J for any chosen black-hole constituent, the mass and angular momentum of the remaining black-hole or hyperextreme constituent(s) can be found using M, J and M_t, J_t. In the case of three hyperextreme Kerr particles the Komar masses and angular momenta should be calculated via formulas (23) of Ref. [20].

(A) *Equilibrium of three Kerr black holes.* In Table I four different purely black-hole equilibrium states are given (the numerical values are presented up to three decimal places). Searching for the roots of the system (22), (23) we were fixing the values of $\alpha_1,\alpha_2,\alpha_3$, X_1, and finding the values of X_2 and X_3 taking into account that the latter two quantities in the black-hole case have the property $|X_2|^2=|X_3|^2=1$. The corresponding stationary limit surfaces, i.e., the surfaces on which $f=0$, are plotted in Fig. 3.[2] The dots denote massless ring singularities lying in the equatorial ($z=0$) plane, whose origin is due to the rupture of the stationary limit surface of the medium black-hole constituent because of a highly large value of the respective angular momentum. The possibility for angular momenta per unit mass to exceed the corresponding masses in a system of several Kerr black holes was reported in Ref. [29], and the three-black-hole case illustrates this effect even more visually than the two-black-hole one.

Three black holes in equilibrium can have disconnected individual stationary limit surfaces, as follows from Figs. 3(i)–(iii), or can form a joint stationary limit surface, as in Fig. 3(iv). Note also that equilibrium states of three black holes are characterized by counter-rotation of the middle constituent to the two other constituents, unlike in the two-body case. It is interesting to point out that the systems of balancing black holes as a whole satisfy the inequality $M_t^4>J_t^2$ valid for a single subextreme Kerr black hole [30,22]. Each balancing black hole verifies Smarr's mass formula [31]

$$M=\frac{1}{4\pi}\kappa S+2\frac{J}{\omega^H}, \tag{25}$$

where κ, S, and ω^H are the surface gravity, area of the horizon, and value of the metric coefficient ω on the horizon, respectively. To see this, one has only to use Tomimatsu's formula [32]

$$\frac{1}{4\pi}\kappa_i S_i=\frac{1}{2}(\alpha_{2i-1}-\alpha_{2i}). \tag{26}$$

(B) *Equilibrium of two black holes and a hyperextreme object*. Qualitatively, the equilibrium states of two black-hole and one hyperextreme constituents are similar to the black-hole-hyperextreme equilibrium configurations in the extended double-Kerr solution considered in Ref. [20]. In Table II we give some typical examples of such a balance with the corresponding stationary limit surfaces plotted in Fig. 4. The first two of them are well behaved in the sense that they involve only positive individual Komar masses and their stationary limit surfaces consist of three disconnected regions

[2] In Figs. 3–5 the horizontal and vertical axes define, respectively, ρ and z coordinates.

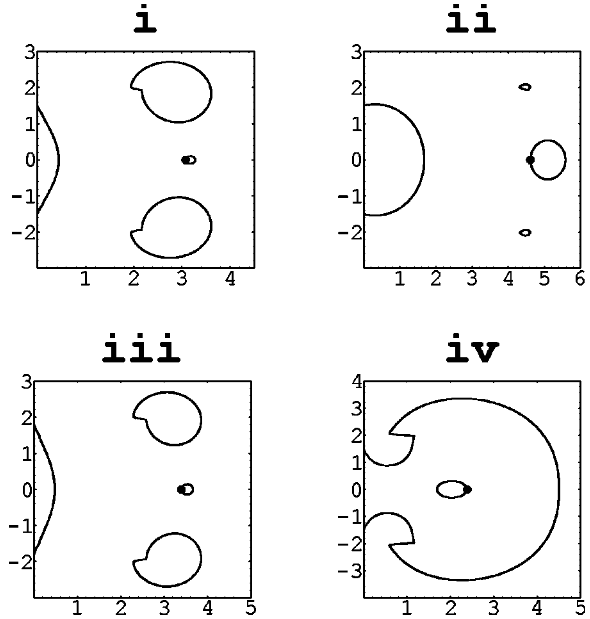

FIG. 5. Stationary limit surfaces corresponding to the equilibrium configurations from Table III.

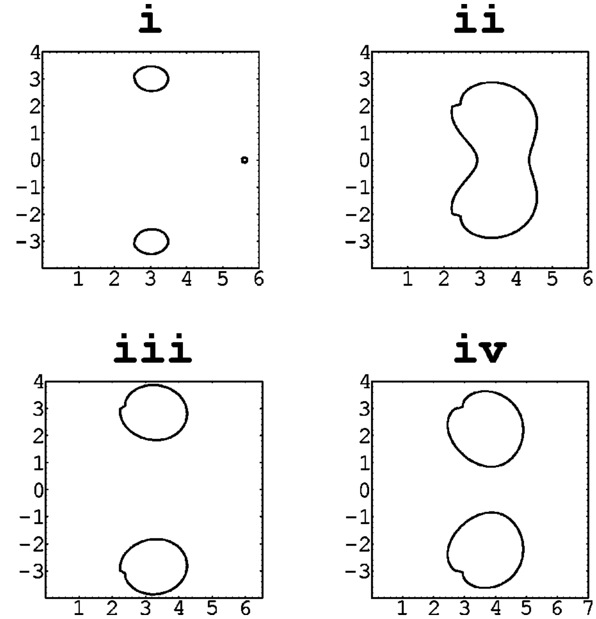

FIG. 6. Stationary limit surfaces corresponding to the equilibrium configurations from Table IV.

without any anormalies; however, the black-hole constituents satisfy the inequality $M^4_{u,l}<J^2_{u,l}$ valid for a single Kerr hyperextreme particle. The third example shows that for a given set of the constants α_1, α_2, α_3, X_1 there exist different numerical solutions of the system (22), (23) for X_2 and X_3, and unlike in the third example, the values X_2 and X_3 of the third example lead to the negative masses of the black-hole constituents and, consequently, to two ring singularities at $\rho\simeq 0.143$, $z\simeq\pm 3.25$ denoted by dots in Fig. 4 (iii). The last example illustrates that the type (b) equilibrium configurations can be achieved by three corotating constituents; however, in this case the Kerr particles form a kind of a stationary limit surface preventing a test particle to reach the parts of the symmetry axis separating the constituents without crossing it.

(C) *Two hyperextreme objects in equilibrium with a black hole*. While searching for the equilibrium states of this type, it is convenient to use X_3 and the imaginary part of α_1 as unknowns in Eqs. (22), (23), fixing all other quantities. A peculiar feature of the particular equilibrium configurations given in Table III is that they possess a massless ring singularity in the equatorial plane (see Fig. 5) which corresponds to the black-hole constituent. The origin of the singularity is the same as in the case (A), i.e., the rupture of the stationary limit surface due to the "dual" properties of the middle constituent: it is a black hole with a horizon defined by the rod $[-\alpha_3,\alpha_3]$, and at the same time its mass and angular momentum verify the inequality $M^4_m<J^2_m$ characteristic of a single hyperextreme Kerr particle. As a consequence, the corresponding stationary limit surface consists of two topologically different parts, the S^2 topology representing the basic black-hole nature of the constituent, and the $S^1\times S^1$ topology originating from the mass-angular-momentum inequality which is inherent to the hyperextreme Kerr sources. As in the purely black-hole equilibrium configurations, the counter-rotation of the constituents comprising type (c) systems seems necessary to achieve the balance.

TABLE IV. Equilibrium states of three hyperextreme objects.

$\alpha_1=\overline{\alpha_2}$	$3-3i$	$2-3i$	$3-3i$	$3-3i$
α_3	$-5.553i$	$-4.663i$	$-6.059i$	$-6.304i$
$X_1=\overline{X_2}^{-1}$	$2.1-1.2i$	$1.8-0.6i$	$1.9-0.4i$	$1.9-0.4i$
X_3	$0.542i$	$0.054i$	$0.147i$	$-1.81i$
$M_u=M_l$	1.89	2.309	2.474	1.166
M_m	0.64	0.018	0.231	−1.664
$J_u=J_l$	−11.844	−13.262	−15.228	−1.164
J_m	16.661	1.072	5.611	−33.744

(D) *Equilibrium states of three hyperextreme Kerr particles.* The equatorial symmetry implies that in this case α_3 is pure imaginary, hence it is most convenient to solve the system (22), (23) for α_3 and X_3. The equilibrium states given in Table IV provide some interesting information about the balance of three hyperextreme Kerr particles. In Fig. 6 we have plotted the stationary limit surfaces corresponding to the equilibrium configurations of Table IV, and one can see that these surfaces may consist of three disconnected regions, each region being related to the respective hyperextreme constituent; of two disconnected regions, when the middle particle has no stationary limit surface; and of a single region which is formed as the result of merging of the individual stationary limit surfaces in one. The first three equilibrium states are characterized by the positive Komar masses of the particles and by the counter-rotation of the middle constituent with respect to the other two particles. The last example is of special interest even though the middle particle has a negative mass. The two remarkable feature of the latter equilibrium state are, first, that the intermediate hyperextreme constituent verifies the inequality $M^4_m>J^2_m$ characteristic of a single subextreme Kerr black hole, and, secondly, that in spite of its negative mass, this constituent is not accompanied by a ring singularity outside the symmetry axis. The former property illustrates well that hyperextreme objects, such as black holes, can exhibit ''dual'' nature; the latter feature clearly shows that a massless ring singularity has a topological origin related to the formation of the stationary limit surface since such a singularity does not appear if a hyperextreme object possessing negative mass has no stationary limit surface.[3]

V. CONCLUSIONS

The main conclusion which can be drawn from the analysis of the extended triple-Kerr metric carried out in our paper is that the collinear Kerr particles described by the solution [24], such as the collinear Schwarzschild particles described by the Israel-Khan solution [4], display a different aptitude with regard to forming the equilibrium configurations, depending on whether the number of particles is even or odd. Since the principle difference between the stationary vacuum systems composed of an even or odd number of particles seems to be the capacity to form the physically meaningful black-hole equilibrium states, it is tempting to conjecture that no one system having an even number of collinear Kerr particles possesses purely black-hole equilibrium configurations with positive Komar masses of all of its constituents. To support this conjecture, it would be, of course, interesting to consider the case of four Kerr particles, but apparently it would contain by far more technical difficulties than the case of the triple-Kerr solution. We leave the quadruple-Kerr equilibrium problem as an interesting and important future task.

The remarkable phenomena revealed by our study of three Kerr particles are the ''duality'' properties which the balancing particles, both the black holes and the hyperextreme objects, are capable to develop, and the absence of ring singularities outside the symmetry axis when a hyperextreme object with negative mass has no stationary limit surface. Further work is needed, however, to discover the mechanisms standing behind these phenomena which, in our opinion, should manifest themselves even stronger in the systems with a larger number of particles.

Although the numerical solution of the balance equations (22), (23) exhibits no difficulty, providing one with particular equilibrium states, we have been unable, in spite of our effort, to solve them analytically, which would be likely to establish general relations between the parameters defining the balance of the Kerr particles. In this respect we wonder if any approximate method similar to that recently employed by Bonnor [34] for treating the two-body problem might be useful for obtaining such general physical relations in the three-body case.

[3]As was shown by Hoenselaers and Perjés [33], the singularities of an axisymmetric vacuum metric are defined by zeros of the denominator of Ernst's potential $\mathcal{E}$, hence they all lie on the stationary limit surface $f=0$.

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