

# Gauss-Bonnet black holes in AdS spaces

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We study the thermodynamic properties and phase structures of topological black holes in Einstein theory with a Gauss-Bonnet term and a negative cosmological constant. The event horizon of these topological black holes can be a hypersurface with positive, zero, or negative constant curvature. When the horizon is a zero curvature hypersurface, the thermodynamic properties of black holes are completely the same as those of black holes without the Gauss-Bonnet term, although the two black hole solutions are quite different. When the horizon is a negative constant curvature hypersurface, the thermodynamic properties of the Gauss-Bonnet black holes are qualitatively similar to those of black holes without the Gauss-Bonnet term. When the event horizon is a hypersurface with positive constant curvature, we find that the thermodynamic properties and phase structures of black holes drastically depend on the spacetime dimension  $d$  and the coefficient of the Gauss-Bonnet term: when  $d \geq 6$ , the properties of black holes are also qualitatively similar to the case without the Gauss-Bonnet term, but when  $d = 5$ , a new phase of locally stable small black holes occurs under a critical value of the Gauss-Bonnet coefficient, and beyond the critical value, the black holes are always thermodynamically stable. However, the locally stable small black hole is not globally preferred; instead a thermal anti-de Sitter space is globally preferred. We find that there is a minimal horizon radius, below which the Hawking-Page phase transition will not occur since for these black holes the thermal anti-de Sitter space is always globally preferred.

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## I. INTRODUCTION

In recent years black holes in anti-de Sitter (AdS) spaces have attracted a great deal of attention. There are at least two reasons responsible for this. First, in the spirit of AdS conformal field theory (CFT) correspondence [1–3], it has been convincingly argued by Witten [4] that the thermodynamics of black holes in AdS spaces (AdS black holes) can be identified with that of a certain dual CFT in the high temperature limit. With this correspondence, one can gain some insight into the thermodynamic properties and phase structures of strong 't Hooft coupling CFTs by studying the thermodynamics of AdS black holes.

Second, although the “topological censorship theorem” [5] still holds in asymptotically AdS spaces [6], it has been found that except for the spherically symmetric black holes whose event horizon is a sphere surface, black holes also exist with the event horizon being a zero or negative constant curvature hypersurface. These black holes are referred to as topological black holes in the literature. Because of the different horizon structures, these black holes behave in many aspects quite differently from the spherically symmetric black holes [7–19].

It is by now known that the AdS Schwarzschild black hole is thermodynamically unstable when the horizon radius is small, while it is stable for a large radius; there is a phase transition, named Hawking-Page phase transition [20], between the large stable black hole and a thermal AdS space. This phase transition is explained by Witten [4] as the confinement-deconfinement transition of the Yang-Mills

theory in the AdS-CFT correspondence. However, it is interesting to note that if the event horizon of AdS black holes is a hypersurface with zero or negative constant curvature, the black hole is always stable and the corresponding CFT is always dominated by the black hole. That is, there does not exist the Hawking-Page phase transition for AdS black holes with a Ricci flat or hyperbolic horizon [16].

Higher derivative curvature terms occur in many occasions, such as in the semiclassically quantum gravity and in the effective low-energy action of superstring theories. In the latter case, according to the AdS-CFT correspondence, these terms can be viewed as the corrections of large  $N$  expansion of boundary CFTs in the strong coupling limit. Due to the nonlinearity of Einstein equations, however, it is very difficult to find nontrivial exact analytical solutions of the Einstein equations with these higher derivative terms. In most cases, one has to adopt some approximation methods or find solutions numerically.

Among the gravity theories with higher derivative curvature terms, the so-called Lovelock gravity [21] has some special features in some sense. For example, the resulting field equations contain no more than second derivatives of the metric and have been proven to be free of ghosts when expanding about the flat space, evading any problems with unitarity. The Lagrangian of Lovelock theory is the sum of dimensionally extended Euler densities<sup>1</sup>

$$\mathcal{L} = \sum_i^n c_i \mathcal{L}_i, \quad (1.1)$$

<sup>1</sup>The gravity theory with a Gauss-Bonnet term was originally proposed by Lanczos [22] and independently rediscovered by Lovelock [21]. See also discussions in [23] and [24].

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where  $c_i$  is an arbitrary constant and  $\mathcal{L}_i$  is the Euler density of a  $2i$ -dimensional manifold,

$$\mathcal{L}_i = 2^{-i} \delta_{c_1 d_1 \dots c_i d_i}^{a_1 b_1 \dots a_i b_i} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_i b_i}^{c_i d_i}. \quad (1.2)$$

Here  $\mathcal{L}_0 = 1$  and hence  $c_0$  is just the cosmological constant.  $\mathcal{L}_1$  gives us the usual Einstein-Hilbert term and  $\mathcal{L}_2$  is the Gauss-Bonnet term. A spherically symmetric static solution of Eq. (1.1) has been found in the sense that the metric function is determined by solving for the real roots of a polynomial equation [25]. Since the Lagrangian (1.1) includes many arbitrary coefficients  $c_i$ , it is difficult to extract physical information from the solution. In Refs. [26,27], by restricting these coefficients to a special set so that the metric function can be readily determined by solving the polynomial equation, some exact, spherically symmetric black hole solutions have been found. Black hole solutions with nontrivial topology in this theory have been also studied in Refs. [17,19].

In this paper we will analyze black hole solutions in Einstein theory with a Gauss-Bonnet term and a negative cosmological constant, in which the Gauss-Bonnet coefficient is not fixed. In this theory a static, spherically symmetric black hole solution was first discovered by Boulware and Deser [28]. However, the thermodynamic properties of the solution were not discussed there. Here we will first generalize this solution to the case in which the horizon of black holes can be a positive, negative or zero constant curvature hypersurface, and then discuss thermodynamic properties and phase structures of black holes. Because of this Gauss-Bonnet term, some nontrivial and interesting features will occur.

## II. TOPOLOGICAL GAUSS-BONNET BLACK HOLES

The Einstein-Hilbert action with a Gauss-Bonnet term and a negative cosmological constant,  $\Lambda = -(d-1)(d-2)/2l^2$ , in  $d$  dimensions can be written down as [28]<sup>2</sup>

$$S = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left( R + \frac{(d-1)(d-2)}{l^2} + \alpha (R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2) \right), \quad (2.1)$$

where  $\alpha$  is the Gauss-Bonnet coefficient with dimension  $(length)^2$  and is positive in the heterotic string theory [28]. So we restrict ourselves to the case  $\alpha \geq 0$ .<sup>3</sup> Varying the action yields the equations of gravitational field

<sup>2</sup>The Gauss-Bonnet term is a topological invariant in four dimensions, so  $d \geq 5$  is assumed in this paper.

<sup>3</sup>We will make a simple discussion for the case  $\alpha < 0$  in Sec. III.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{(d-1)(d-2)}{2l^2} g_{\mu\nu} + \alpha \left( \frac{1}{2} g_{\mu\nu} (R_{\gamma\delta\lambda\sigma} R^{\gamma\delta\lambda\sigma} - 4R_{\gamma\delta} R^{\gamma\delta} + R^2) - 2R R_{\mu\nu} + 4R_{\mu\gamma} R^{\gamma}_{\nu} + 4R_{\gamma\delta} R^{\gamma}_{\mu}{}^{\delta}_{\nu} - 2R_{\mu\gamma\delta\lambda} R^{\gamma\delta\lambda}_{\nu} \right). \quad (2.2)$$

We assume the metric to be of the following form:

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2 h_{ij} dx^i dx^j, \quad (2.3)$$

where  $\nu$  and  $\lambda$  are functions of  $r$  only, and  $h_{ij} dx^i dx^j$  represents the line element of a  $(d-2)$ -dimensional hypersurface with constant curvature  $(d-2)(d-3)k$  and volume  $\Sigma_k$ . Without loss of the generality, one may take  $k = 1, -1$ , or  $0$ . Following Ref. [28] and substituting the ansatz (2.3) into the action (2.1), we obtain

$$S = \frac{(d-2)\Sigma_k}{16\pi G} \int dt dr e^{\nu+\lambda} \left[ r^{d-1} \varphi (1 + \tilde{\alpha} \varphi) + \frac{r^{d-1}}{l^2} \right]', \quad (2.4)$$

where a prime denotes derivative with respect to  $r$ ,  $\tilde{\alpha} = \alpha(d-3)(d-4)$ , and  $\varphi = r^{-2}(k - e^{-2\lambda})$ . From the action (2.4), one can find the solution

$$e^{\nu+\lambda} = 1,$$

$$\varphi (1 + \tilde{\alpha} \varphi) + \frac{1}{l^2} = \frac{16\pi G M}{(d-2)\Sigma_k r^{d-1}}, \quad (2.5)$$

from which we obtain the exact solution<sup>4</sup>

$$e^{2\nu} = e^{-2\lambda} = k + \frac{r^2}{2\tilde{\alpha}} \left( 1 \mp \sqrt{1 + \frac{64\pi G \tilde{\alpha} M}{(d-2)\Sigma_k r^{d-1}} - \frac{4\tilde{\alpha}}{l^2}} \right), \quad (2.6)$$

where  $M$  is the gravitational mass of the solution.<sup>5</sup> The solution with  $k=1$  and spherical symmetry was first found by Boulware and Deser [28]. Here we extend this solution to include the cases  $k=0$  and  $-1$ . Note that the solution (2.6) has two branches with a “ $-$ ” or “ $+$ ” sign. Moreover, there is a potential singularity at the place where the square root vanishes in Eq. (2.6), except for the singularity at  $r=0$ . Here

<sup>4</sup>It is not so obvious that the minisuperspace approach applies for nonspherically symmetric solutions in the gravity theory. However, it can be checked that the solution (2.6) indeed satisfies the equations (2.2) of motion. This is related to the fact that following [29], one can show that a Birkhoff-like theorem holds in the gravity theory (2.1).

<sup>5</sup>This gravitational mass can be obtained by substituting the solution (2.6) into the action (2.4) and then using the boundary term method. For this method, for example, see [27].

we mention that the theory (2.1) with  $\tilde{\alpha}=l^2/4$  discussed in Ref. [27] in five dimensions corresponds to the theory proposed in [30]; the solution with  $\tilde{\alpha}=l^2/4$  discussed in Refs. [26,27] in five dimensions was also included in Refs. [17,19]. If  $\tilde{\alpha}=0$ , namely, without the Gauss-Bonnet term, the solution (2.6) reduces to the one in [16], and the thermodynamics of the latter was discussed there.

When  $M=0$ , the vacuum solution in Eq. (2.6) is

$$e^{-2\lambda}=k+\frac{r^2}{2\tilde{\alpha}}\left(1\mp\sqrt{1-\frac{4\tilde{\alpha}}{l^2}}\right). \quad (2.7)$$

Since  $\tilde{\alpha}>0$ , one can see from the above that  $\tilde{\alpha}$  must obey  $4\tilde{\alpha}/l^2\leq 1$ , beyond which this theory is undefined. Thus, the action (2.1) has two AdS solutions with effective cosmological constants  $l_{\text{eff}}^2=(l^2/2)(1\pm\sqrt{1-4\tilde{\alpha}/l^2})$ . When  $4\tilde{\alpha}/l^2=1$ , these two solutions coincide with each other, resulting in  $e^{-2\lambda}=k+2r^2/l^2$  and that the theory has a unique AdS vacuum [26,27]. On the other hand, if  $\tilde{\alpha}<0$ , the solution (2.7) is still an AdS space if one takes the “−” sign, but becomes a de Sitter space if one takes the “+” sign and  $k=1$ . From the vacuum case, the solution (2.7) with both signs seems reasonable, from which we cannot determine which sign in Eq. (2.6) should be adopted. This problem can be solved by considering the propagation of gravitons on the background (2.7). It has been shown by Boulware and Deser [28] that the branch with the “+” sign is unstable and the graviton is a ghost, while the branch with the “−” sign is stable and is free of ghosts. This can also be seen from the case  $M\neq 0$ . When  $k=1$  and  $1/l^2=0$ , just as observed by Boulware and Deser [28], the solution is asymptotically a Schwarzschild solution if one takes the “−” sign, but is asymptotically an AdS Schwarzschild solution with a negative gravitational mass for the “+” sign, indicating the instability. Therefore the branch with the “+” sign in Eq. (2.6) is of less physical interest.<sup>6</sup> From now on, we will not consider the branch with the “+” sign.

From Eq. (2.6), the mass of black holes can be expressed in terms of the horizon radius  $r_+$ ,

$$M=\frac{(d-2)\Sigma_k r_+^{d-3}}{16\pi G}\left(k+\frac{\tilde{\alpha}k^2}{r_+^2}+\frac{r_+^2}{l^2}\right). \quad (2.8)$$

The Hawking temperature of the black holes can be easily obtained by requiring the absence of conical singularity at the horizon in the Euclidean sector of the black hole solution. It is

<sup>6</sup>A detailed analysis of the solution (2.6) without the negative cosmological constant, namely,  $1/l^2=0$ , has been made in [32,33].

$$T=\frac{1}{4\pi}(e^{-2\lambda})'\Big|_{r=r_+} \\ =\frac{(d-1)r_+^4+(d-3)kl^2r_+^2+(d-5)\tilde{\alpha}k^2l^2}{4\pi l^2r_+(r_+^2+2\tilde{\alpha}k)}. \quad (2.9)$$

Usually entropy of black holes satisfies the so-called area formula. This is, the black hole entropy equals one-quarter of the horizon area. In gravity theories with higher derivative curvature terms, however, in general the entropy of black holes does not satisfy the area formula. To get the black hole entropy, in [17] we suggested a simple method according to the fact that as a thermodynamic system, the entropy of the black hole must obey the first law of black hole thermodynamics:  $dM=T dS$ . Integrating the first law, we have

$$S=\int T^{-1} dM=\int_0^{r_+} T^{-1}\left(\frac{\partial M}{\partial r_+}\right)dr_+, \quad (2.10)$$

where we have imposed the physical assumption that the entropy vanishes when the horizon of black holes shrinks to zero.<sup>7</sup> Thus once given the temperature and mass of black holes in terms of the horizon radius, one can readily get the entropy of black holes and need not know in which gravitational theory the black hole solutions are. Substituting Eqs. (2.8) and (2.9) into Eq. (2.10), we find the entropy of the Gauss-Bonnet black holes (2.6) is

$$S=\frac{\Sigma_k r_+^{d-2}}{4G}\left(1+\frac{(d-2)}{(d-4)}\frac{2\tilde{\alpha}k}{r_+^2}\right). \quad (2.11)$$

When  $k=1$ , it is in complete agreement with the one in [32], there the entropy of the Gauss-Bonnet black holes without the cosmological constant is obtained by calculating the Euclidean action of black holes. The heat capacity of black holes is

$$C=\left(\frac{\partial M}{\partial T}\right)=\left(\frac{\partial M}{\partial r_+}\right)\left(\frac{\partial r_+}{\partial T}\right), \quad (2.12)$$

where

<sup>7</sup>Note that for the  $k=-1$  black hole, there exists a minimal horizon radius. For these black holes, therefore the horizon cannot shrink to zero. However, it is known that the black hole entropy is a function of the horizon surface [31]. According to the second law of black hole mechanics, the black hole entropy can be expressed in terms of a polynomial of horizon radius  $r_+$  with positive exponents. As a result, although the black hole horizon cannot shrink to zero when  $k=-1$ , this method seems applicable as well. The results in [17] and in this paper show this point. For example, when  $\tilde{\alpha}=0$ , the formula (2.11) gives the entropy of AdS black holes in Einstein theory without the Gauss-Bonnet term. Obviously, in this case the resulting area formula (2.11) holds as well in the case of  $k=-1$ .

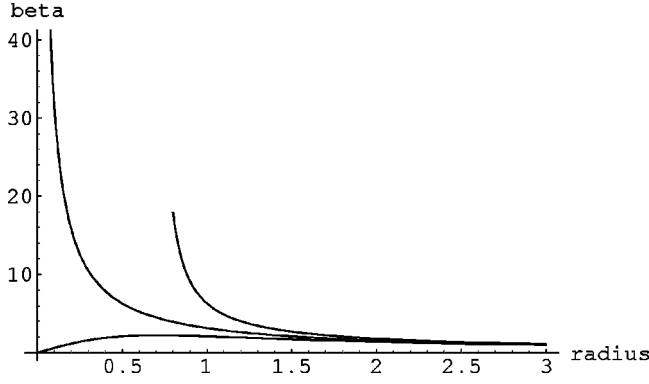


FIG. 1. The inverse temperature of topological black holes without the Gauss-Bonnet term. The three curves above from up to down correspond to the cases  $k = -1$ ,  $0$ , and  $1$ , respectively.

$$\begin{aligned} \frac{\partial M}{\partial r_+} &= \frac{(d-2)\Sigma_k}{4G} r_+^{d-5} (r_+^2 + 2\tilde{\alpha}k) T, \\ \frac{\partial T}{\partial r_+} &= \frac{1}{4\pi l^2 r_+^2 (r_+^2 + 2\tilde{\alpha}k)^2} [(d-1)r_+^6 \\ &\quad - (d-3)kl^2 r_+^4 \\ &\quad + 6(d-1)k\tilde{\alpha}r_+^4 \\ &\quad + 2(d-3)\tilde{\alpha}k^2 l^2 r_+^2 \\ &\quad - 3(d-5)\tilde{\alpha}k^2 l^2 r_+^2 \\ &\quad - 2(d-5)\tilde{\alpha}^2 k^2 l^2]. \end{aligned} \quad (2.13)$$

The free energy of black holes, defined as  $F = M - TS$ , is

$$\begin{aligned} F &= \frac{\Sigma_k r_+^{d-5}}{16\pi G(d-4)l^2(r_+^2 + 2\tilde{\alpha}k)} [-(d-4)r_+^6 + (d-4)kl^2 r_+^4 \\ &\quad - 6(d-2)k\tilde{\alpha}r_+^4 + (d-8)\tilde{\alpha}k^2 l^2 r_+^2 + 2(d-2)\tilde{\alpha}^2 kl^2]. \end{aligned} \quad (2.14)$$

Thus we give some thermodynamic quantities of Gauss-Bonnet black holes in AdS spaces. When  $\tilde{\alpha} = 0$ , these thermodynamic quantities reduce to corresponding ones in Ref. [16]. In Fig. 1 the inverse temperature  $\beta = 1/T$  of the black holes versus the horizon radius is plotted. We can see clearly different behaviors for the cases  $k = 1$ ,  $0$ , and  $-1$ : The inverse temperature always starts from infinity and monotonically decreases to zero in the cases  $k = 0$  and  $k = -1$ , while it starts from zero and reaches its maximum at a certain horizon radius and then goes to zero monotonically when  $k = 1$ . This indicates that for the cases  $k = -1$  and  $k = 0$ , the black holes are not only locally thermodynamically stable, but also globally preferred, while in the case of  $k = 1$ , the black hole is not locally thermodynamically stable for small radius, but it is for large radius. Therefore, for the  $k = 1$  case, there is a Hawking-Page phase transition. For details see [16].

When  $\tilde{\alpha} \neq 0$ , we see that those quantities drastically depend on the parameter  $\tilde{\alpha}$ , horizon structure  $k$  and the space-time dimension  $d$ . Below we will discuss each case according to the classification of horizon structures,  $k = 0$ ,  $k = -1$ , and  $k = 1$ , respectively.

### A. The case of $k = 0$

In this case we have

$$\begin{aligned} T &= \frac{(d-1)r_+}{4\pi l^2}, \\ S &= \frac{\Sigma_k}{4G} r_+^{d-2}, \\ C &= \frac{(d-2)\Sigma_k}{4G} r_+^{d-2}, \\ F &= -\frac{\Sigma_k}{16\pi G} \frac{r_+^{d-1}}{l^2}, \end{aligned} \quad (2.15)$$

where  $r_+^{d-1} = 16\pi G l^2 M / (d-2)\Sigma_k$ . It is interesting to note that these thermodynamic quantities are independent of the parameter  $\tilde{\alpha}$ . That is, these quantities have the completely same expressions as those [16] for black holes without the Gauss-Bonnet term. We therefore conclude that in the case  $k = 0$ , the black holes with and without the Gauss-Bonnet term have completely the same thermodynamic properties, although the two solutions are quite different, which can be seen from Eq. (2.6). In particular, we note here that the entropy of the Gauss-Bonnet black holes still satisfies the area formula in the case  $k = 0$ .

### B. The case of $k = -1$

As the case [16] without the Gauss-Bonnet term, there is also the so-called ‘‘massless’’ black hole and ‘‘negative’’ mass black hole in the Gauss-Bonnet black hole (2.6). When  $M = 0$ , the black hole has the horizon radius

$$r_+^2 = \frac{l^2}{2} \left( 1 \pm \sqrt{1 - \frac{4\tilde{\alpha}}{l^2}} \right), \quad (2.16)$$

with Hawking temperature  $T = 1/2\pi r_+$ . Here there are two ‘‘massless’’ black hole solutions, corresponding to two branches in the solution (2.6). But the black hole with smaller horizon radius belongs to the unstable branch.

Given a fixed  $\tilde{\alpha}$ , the smallest black hole has the horizon radius

$$r_{\min}^2 = \frac{(d-3)l^2}{2(d-1)} \left( 1 + \sqrt{1 - \frac{(d-1)(d-5)}{(d-3)^2} \frac{4\tilde{\alpha}}{l^2}} \right). \quad (2.17)$$

The black hole is an extremal one; it has vanishing Hawking temperature and the most ‘‘negative’’ mass



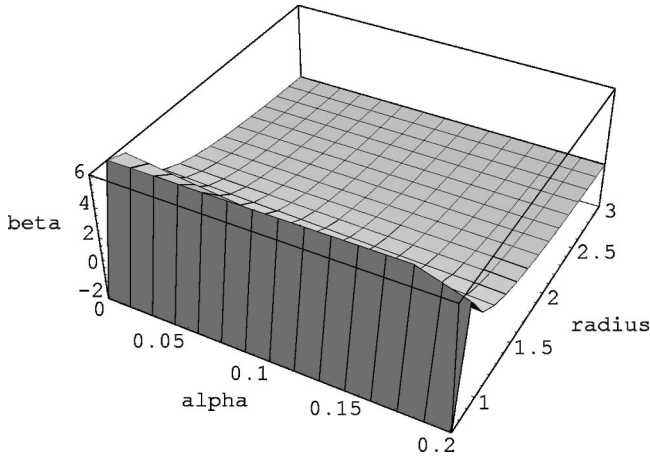


FIG. 2. The inverse temperature of the  $k=-1$  Gauss-Bonnet black holes in  $d=6$  dimensions.

$$M_{\text{ext}} = -\frac{(d-2)(d-3)\Sigma_k l^2 r_{\text{min}}^{d-5}}{16\pi G(d-1)^2} \left( 1 - \frac{d-1}{d-3} \frac{4\tilde{\alpha}}{l^2} + \sqrt{1 - \frac{(d-1)(d-5)}{(d-3)^2} \frac{4\tilde{\alpha}}{l^2}} \right). \quad (2.18)$$

When  $4\tilde{\alpha}/l^2=1$ , the smallest radius is  $r_{\text{min}}^2=l^2/2$  and  $M_{\text{ext}}=0$ , independent of the spacetime dimension  $d$ . But in this case, the Hawking temperature does not vanish. It is  $T=1/\sqrt{2}\pi l$ . This is an exceptional case.

From the solution (2.6), one can find that in order for the solution to have a black hole horizon, the horizon radius must obey

$$r_+^2 \geq 2\tilde{\alpha}. \quad (2.19)$$

Thus the smallest radius (2.17) gives a constraint on the allowed value of the parameter  $\tilde{\alpha}$ :

$$r_{\text{min}}^2 \geq 2\tilde{\alpha}, \quad (2.20)$$

which leads to  $4\tilde{\alpha}/l^2 \leq 1$ . Since the theory is defined in the region  $4\tilde{\alpha}/l^2 \leq 1$ , the condition (2.20) is always satisfied. Due to the existence of the smallest black holes (2.17), we see from Eq. (2.9) that except for the case  $4\tilde{\alpha}/l^2=1$ , the temperature of the black hole always starts from zero at the smallest radii, corresponding to the extremal black holes and monotonically goes to infinity as  $r_+ \rightarrow \infty$ . In the case  $4\tilde{\alpha}/l^2=1$ , the temperature starts from  $1/\sqrt{2}\pi l$  at  $r_+^2=l^2/2$ . This can also be verified by looking at the behavior of the heat capacity (2.12). After considering the fact that  $r_+^2 \geq 2\tilde{\alpha}$  and  $4\tilde{\alpha}/l^2 \leq 1$ , it is easy to show that the heat capacity is always positive. In Fig. 2 we plot the inverse temperature of black holes in six dimensions versus the parameter  $\tilde{\alpha}/l^2$  and the horizon radius  $r_+/l$ .

Among the smaller black holes (2.17), the smallest one is  $r_+^2=l^2/2$  when  $4\tilde{\alpha}/l^2=1$ ; its free energy is zero. Therefore the free energy is always negative for other black holes since

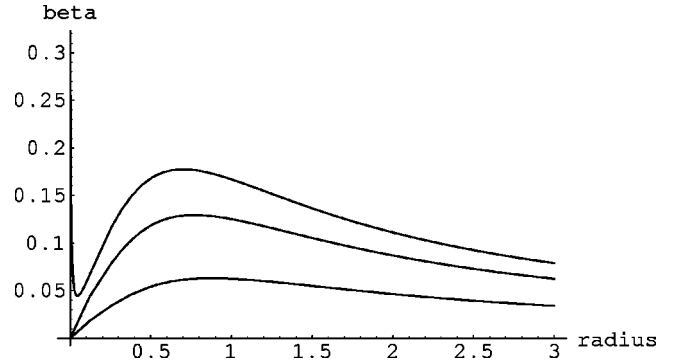


FIG. 3. The inverse temperature of the  $k=1$  Gauss-Bonnet black holes with  $\tilde{\alpha}/l^2=0.001$ . The three curves from up to down correspond to  $d=5$ ,  $6$ , and  $d=10$ , respectively.

the heat capacity is always positive. As a result, the thermodynamic properties of the black holes with the Gauss-Bonnet term are qualitatively similar to those of black holes without the Gauss-Bonnet term: These black holes are always stable not only locally, but also globally.

In addition, let us note that except for the singularity at  $r=0$ , the black hole solution (2.6) has another singularity at

$$r_s^{d-1} = \frac{4\tilde{\alpha}r_+^{d-3}}{1-4\tilde{\alpha}/l^2} \left( 1 - \frac{\tilde{\alpha}}{r_+^2} - \frac{r_+^2}{l^2} \right), \quad (2.21)$$

when  $M_{\text{ext}} < M < 0$ . But both singularities are shielded by the event horizon  $r_+$ .

### C. The case of $k=1$

This case is very interesting. From the temperature (2.9) one can see that the case  $d=5$  is quite different from the other cases  $d \geq 6$ . When  $d=5$ , the temperature starts from zero at  $r_+=0$  and goes to infinity as  $r_+ \rightarrow \infty$ , while it starts from infinity at  $r_+=0$  as  $d \geq 6$ . In Fig. 3 we show the inverse temperatures of black holes with  $\tilde{\alpha}/l^2=0.001$  in different dimensions  $d=5$ ,  $6$ , and  $d=10$ , respectively. The behavior of the temperature of black holes with the Gauss-Bonnet term in  $d \geq 6$  dimensions is similar to that of AdS black holes without the Gauss-Bonnet term. But the case of  $d=5$  (see Fig. 4) is quite different from the corresponding one without

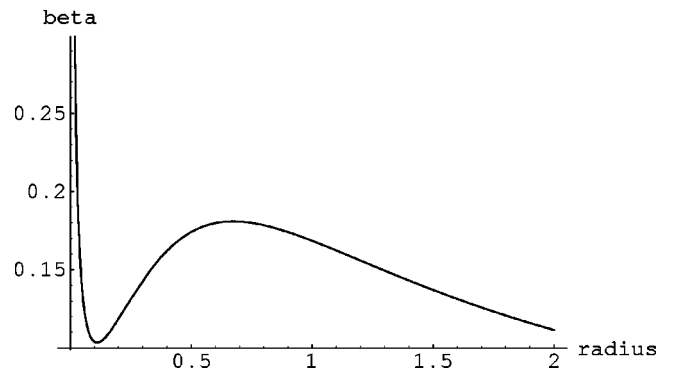


FIG. 4. The inverse temperature of the  $k=1$  Gauss-Bonnet black holes in  $d=5$  dimensions with  $\tilde{\alpha}/l^2=0.0056$ .

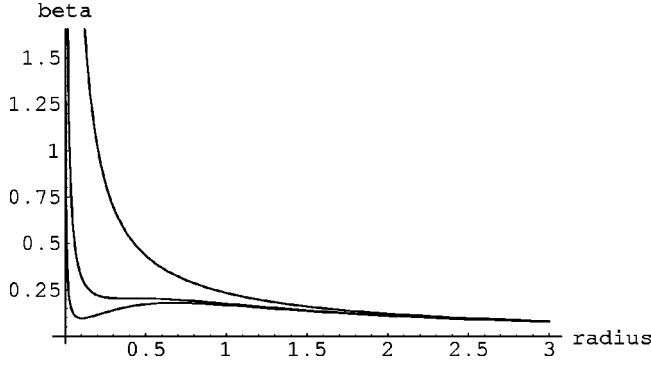


FIG. 5. The inverse temperature of the  $k=1$  Gauss-Bonnet black holes in  $d=5$  dimensions. The three curves from up to down correspond to the cases with the supercritical  $\tilde{\alpha}/l^2=0.20$ , critical  $1/36\approx 0.0278$ , and subcritical 0.005, respectively.

the Gauss-Bonnet term (see Fig. 1). Comparing Fig. 4 with Fig. 1, we see that a new phase of stably small black hole occurs in the Gauss-Bonnet black holes.

When  $d=5$ , we have from Eq. (2.8) the black hole horizon

$$r_+^2 = \frac{l^2}{2} \left( -1 + \sqrt{1 + \frac{4(m - \tilde{\alpha})}{l^2}} \right), \quad (2.22)$$

where  $m = 16\pi GM/3\Sigma_k$ . Therefore, in this case there is a mass gap  $M_0 = 3\Sigma_k \tilde{\alpha}/(16\pi G)$ : all black holes have a mass  $M \geq M_0$ . Using the horizon radius, from Fig. 4 we can see that the black holes can be classified to three branches:

$$\begin{aligned} \text{branch 1: } & 0 < r_+ < r_1, \quad C > 0, \\ \text{branch 2: } & r_1 < r_+ < r_2, \quad C < 0, \\ \text{branch 3: } & r_2 < r_+ < \infty, \quad C > 0, \end{aligned} \quad (2.23)$$

where

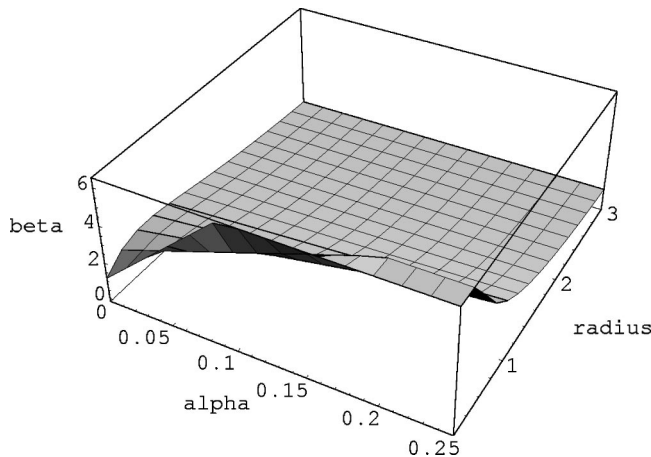


FIG. 6. The inverse temperature of the  $k=1$  Gauss-Bonnet black holes in  $d=5$  dimensions.

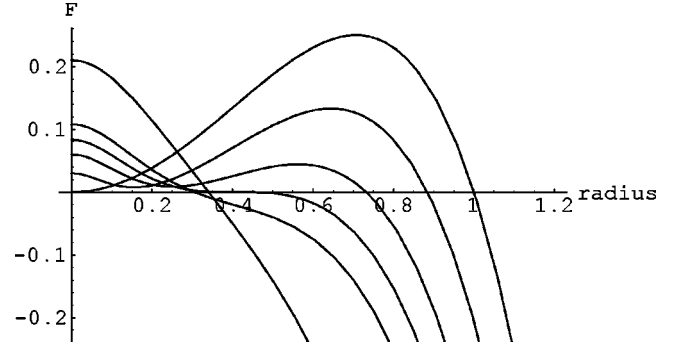


FIG. 7. The free energy of the  $k=1$  Gauss-Bonnet black holes in  $d=5$  dimensions. The curves counting up to down on the  $F$  axis correspond to the cases  $\tilde{\alpha}/l^2=0.070, 0.036, 1/36, 0.020, 0.010$ , and 0, respectively.

$$\begin{aligned} r_{1,2}^2 = & \frac{l^2}{4} \left( 1 - \frac{12\tilde{\alpha}}{l^2} \right) \\ & \times \left( 1 \mp \sqrt{1 - \frac{16\tilde{\alpha}}{l^2} \left( 1 - \frac{12\tilde{\alpha}}{l^2} \right)^{-2}} \right), \end{aligned} \quad (2.24)$$

with the assumption  $36\tilde{\alpha}/l^2 < 1$ . In branches 1 and 3, the heat capacity is positive, while it is negative in branch 2. Therefore the black holes are locally stable in branches 1 and 3, and unstable in branch 2. At the joint points of branches, namely,  $r_+ = r_{1,2}$ , the heat capacity diverges. Comparing with the case without the Gauss-Bonnet term, one can see that branch 1 is new.

When  $\tilde{\alpha}$  increases to the value,  $\tilde{\alpha}/l^2 = 1/36$ , we find that branch 2 with negative heat capacity disappears. Beyond this value, the heat capacity is always positive and the Gauss-Bonnet black holes are always locally stable. In Fig. 5, we show the inverse temperatures of the Gauss-Bonnet black hole with the parameter  $\tilde{\alpha}/l^2$ , subcritical value 0.001, critical value  $1/36$ , and supercritical value 0.20, respectively. In Fig. 6, the continuous evolution of the inverse temperature is plotted with the parameter  $\tilde{\alpha}/l^2$  from zero to 0.25, from

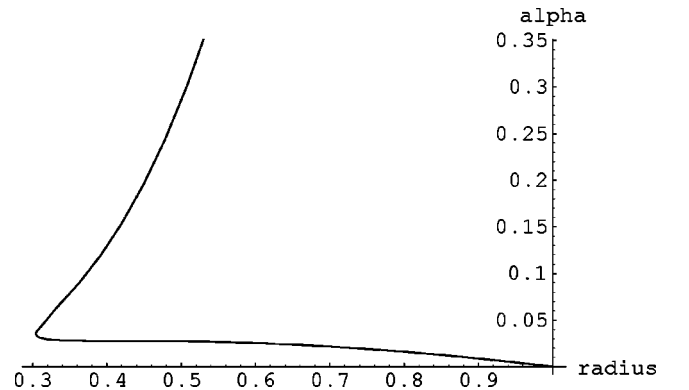


FIG. 8. The curves  $\tilde{\alpha}_2$  (the upper one) and  $\tilde{\alpha}_1$  (the lower one) for the Gauss-Bonnet black holes in  $d=5$  dimensions. They join at  $r_+/l=0.3043$  and  $\tilde{\alpha}/l^2=0.0360$ . In the region between  $\tilde{\alpha}_2$  and  $\tilde{\alpha}_1$  black holes have a negative free energy and are globally preferred.

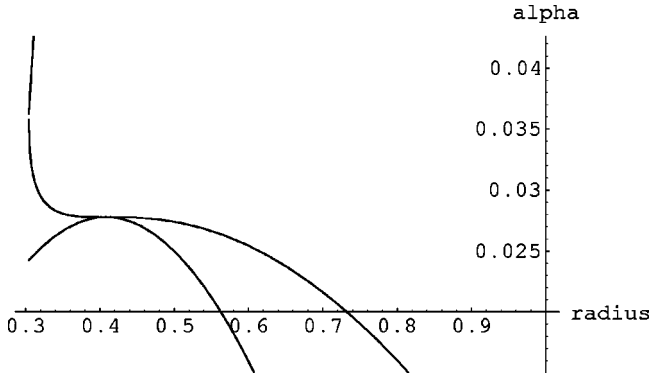


FIG. 9. The curves  $\tilde{\alpha}_2$ ,  $\tilde{\alpha}_1$ , and  $\tilde{\alpha}_0$  (the lowest one) for the Gauss-Bonnet black holes in  $d=5$  dimensions. The region above the curve  $\tilde{\alpha}_0$  is locally stable. The curve  $\tilde{\alpha}_0$  touches the curve  $\tilde{\alpha}_1$  at  $r_+/l=0.4082$  and  $\tilde{\alpha}/l^2=0.0278$ . The separated one is the curve  $\tilde{\alpha}_2$ .

which one can see clearly that the black holes evolve from two branches to one branch via three branches.

However, inspecting the free energy (2.14) reveals that these stably small black holes are not globally preferred: The free energy always starts from some positive value at  $r_+=0$  and then goes to negative infinity as  $r_+\rightarrow\infty$ . In Fig. 7 the free energy of black holes with different parameter  $\tilde{\alpha}/l^2$  is plotted. We see that all curves cross the horizontal axis (horizon radius) one time only, where  $F=0$ . In Fig. 8 we plot the region where the free energy is negative. The region is

$$\tilde{\alpha}_1 < \tilde{\alpha} < \tilde{\alpha}_2, \quad (2.25)$$

where

$$\tilde{\alpha}_{2,1} = \frac{r_+^2}{4} + \frac{3r_+^4}{2l^2} \pm \frac{r_+^2}{2} \sqrt{\frac{9r_+^4}{l^4} + \frac{11r_+^2}{3l^2} - \frac{5}{12}}. \quad (2.26)$$

The joint point of the two curves is at  $\tilde{\alpha}/l^2=0.0360$  and  $r_+/l=0.3043$ . Beyond this region, the thermal AdS space is

$$\tilde{\alpha}_{2,1} = \frac{r_+^2}{4(d-2)l^2} [6(d-2)r_+^2 - (d-8)l^2 \pm \sqrt{36(d-2)^2r_+^4 - 4(d-2)(d-16)l^2r_+^2 + d(32-7d)l^4}]. \quad (2.28)$$

And as in the case of  $d=5$ , these two curves connect at

$$r_+^2 = \frac{l^2}{18(d-2)} (d-16 + \sqrt{(d-16)^2 + 9d(7d-32)}),$$

$$\tilde{\alpha} = \frac{r_+^2}{4(d-2)l^2} (6(d-2)r_+^2 - (d-8)l^2) \quad (2.29)$$

globally preferred. We see that there is a smallest horizon radius  $r_+/l=0.3043$ : there will not exist the Hawking-Page phase transition when the black hole horizon is smaller than the value  $r_+/l=0.3043$ . When black holes cross the curves  $\tilde{\alpha}_2$  and  $\tilde{\alpha}_1$ , a Hawking-Page phase transition happens.

The region in which black holes are locally stable is determined by the curve  $\tilde{\alpha}_0$ ,

$$\tilde{\alpha}_0 = \frac{l^2r_+^2 - 2r_+^4}{2l^2 + 12r_+^2}. \quad (2.27)$$

In Fig. 9 the curve  $\tilde{\alpha}_0$  is plotted (the lowest one): the region is locally stable above this curve, namely,  $\tilde{\alpha} > \tilde{\alpha}_0$ , and locally unstable below this curve. This curve  $\tilde{\alpha}_0$  touches the curve  $\tilde{\alpha}_1$  at  $\tilde{\alpha}=1/36 \approx 0.0278$  and  $r_+/l=0.4082$ . Unfortunately, in Fig. 9 most of the curve  $\tilde{\alpha}_2$  is outside the plot. In Fig. 9 one can see that there is a large region where black holes are locally stable, but not globally preferred.

When  $d \geq 6$ , unlike the case  $d=5$ , there is no mass gap. The properties of Gauss-Bonnet black holes are qualitatively similar to those of black holes without the Gauss-Bonnet term. This can be seen from the behavior of the Hawking temperature of black holes in Fig. 3. This implies that the equation  $\partial T/\partial r_+=0$  has only one positive real root  $r_+=r_0(d, \tilde{\alpha}/l^2)$ . Using Eq. (2.13), one can obtain the positive real root. But its expression is complicated, so we do not present it here. Given a spacetime dimension  $d$  and a fixed parameter  $\tilde{\alpha}/l^2$ , when a black hole has a horizon  $r_+ > r_0$ , the black hole is locally stable. Otherwise, it is unstable.

The free energy (2.14) always starts from zero in the case  $d \geq 6$ , reaches a positive maximum at some  $r_+$ , and then goes to negative infinity as  $r_+\rightarrow\infty$ . This behavior is the same as the case without the Gauss-Bonnet term (see the curve of  $\tilde{\alpha}=0$  in Fig. 7). The region where the black hole is globally preferred is restricted by a relation like Eq. (2.25), but with

in the  $\tilde{\alpha}-r_+$  plane. Therefore the phase structure of black holes in  $d \geq 6$  dimensions is similar to the one in  $d=5$  dimensions (Fig. 8).

Finally let us mention that the temperature behavior (Fig. 4) of  $d=5$  Gauss-Bonnet black holes is quite similar to the one of the Reissner-Nordström (RN) black holes in AdS spaces in the canonical ensemble [34,35]. There under the critical value of charge, the phase of stably small black holes occurs as well. However, there is a big difference between

two cases: For the RN black holes, the small black hole is not only locally stable, but also globally preferred, while the small Gauss-Bonnet black hole is only locally stable and not globally preferred, instead a thermal AdS space is preferred.

### III. CONCLUSIONS AND DISCUSSIONS

We have presented exact topological black hole solutions in Einstein theory with a Gauss-Bonnet term and a negative cosmological constant, generalizing the spherically symmetric black hole solution found by Boulware and Deser [28] to the case where the event horizon of black holes is a positive, zero or negative constant curvature hypersurface. We have examined thermodynamic properties and analyzed phase structures of these topological black holes.

When the even horizon is a zero curvature hypersurface, we find that thermodynamic properties of Gauss-Bonnet black holes are completely the same as those without the Gauss-Bonnet term, although the two black hole solutions are quite different. As a result, these  $k=0$  Gauss-Bonnet black holes are not only locally thermodynamically stable, but also globally preferred. In particular, the entropy of these black holes satisfies the area formula. Note that usually black holes in gravity theories with higher derivative curvature terms do not obey the area formula.

When the event horizon is a negative constant hypersurface, these black holes are qualitatively similar to those without the Gauss-Bonnet term. These  $k=-1$  Gauss-Bonnet black holes are always locally stable and globally preferred.

When the event horizon is a positive constant hypersurface, however, some interesting features occur. When  $d=5$ , a new phase of thermodynamically stable small black holes appears if the Gauss-Bonnet coefficient is under a critical value. Beyond the critical value, the black holes are always thermodynamically stable. Inspecting the free energy of black holes reveals these stable small black holes are not globally preferred, instead a thermal AdS space is preferred. The phase structures are plotted, from which we find that there is a smallest black hole radius. Beyond this radius, the Hawking-Page phase transition will not happen. From the phase diagram we see that there is a large region in which the black hole is locally stable, but not globally preferred. When  $d \geq 6$ , however, the new phase of stable small black holes disappears. Once again, the thermodynamic properties of the black holes are qualitatively similar to those of black holes without the Gauss-Bonnet term.

Now we discuss the case  $\alpha < 0$ . The vacuum solution is still the one (2.7). So in this case there is no restriction on the value of  $\tilde{\alpha}$ ; the solution (2.6) is still asymptotically AdS. Those expressions of thermodynamic quantities (2.8), (2.9), (2.11), (2.13), and (2.14) are applicable as well.

When  $k=0$ , since thermodynamic quantities are independent of the parameter  $\tilde{\alpha}$  in this case, the conclusion is the same as the case  $\tilde{\alpha} > 0$ , but with a new singularity at

$$r_s^{d-1} = \frac{4|\tilde{\alpha}|/l^2}{1+4|\tilde{\alpha}|/l^2} r_+^{d-1}, \quad (3.1)$$

which is always shielded by the event horizon  $r_+$ .

When  $k=-1$ , the situation is similar to the case  $\tilde{\alpha} > 0$ ; nothing special appears. In this case, the smallest radius is

$$r_{\min}^2 = \frac{(d-3)}{2(d-1)} l^2 \times \left( 1 + \sqrt{1 + \frac{(d-5)(d-1)}{(d-3)^2} \frac{4|\tilde{\alpha}|}{l^2}} \right). \quad (3.2)$$

The smallest black hole has a vanishing Hawking temperature. Inside the event horizon there is an additional singularity at

$$r_s^{d-1} = \frac{4|\tilde{\alpha}|/l^2}{1+4|\tilde{\alpha}|/l^2} r_+^{d-1} \left( 1 - \frac{l^2}{r_+^2} - \frac{|\tilde{\alpha}|l^2}{r_+^4} \right), \quad (3.3)$$

except for the one at  $r=0$ . The black holes are also always locally stable and globally preferred.

When  $k=1$ , there is also a smallest horizon radius

$$r_{\min}^2 = 2|\tilde{\alpha}|, \quad (3.4)$$

but this smallest black hole has a divergent Hawking temperature. In this case the event horizon coincides with an additional singularity at  $r^2 = 2|\tilde{\alpha}|$ . For larger black holes the additional singularity is located at

$$r_s^{d-1} = \frac{4|\tilde{\alpha}|/l^2}{1+4|\tilde{\alpha}|/l^2} r_+^{d-1} \left( 1 + \frac{l^2}{r_+^2} - \frac{|\tilde{\alpha}|l^2}{r_+^4} \right), \quad (3.5)$$

inside the black hole horizon. The inverse temperature of black holes starts from zero at the smallest radius (3.4), reaches its maximal at some  $r_+$  and goes to zero when  $r_+ \rightarrow \infty$ . The thermodynamic properties of black holes are qualitatively similar to the case with  $\tilde{\alpha} = 0$ . As a result, the new phase, which appears in the case  $d=5$  and  $0 < \tilde{\alpha}/l^2 \leq 1/36$ , does not occur in this case.

*Note added.* Perturbative AdS black hole solution with  $R^2$  curvature terms and its thermodynamic properties have been discussed recently in [36].

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