Nested braneworlds and strong brane gravity

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We find the gravitational field of a "nested" domain wall living entirely within a brane-universe, or, a *localized* vortex within a wall. Using two illustrative examples, a vortex living on a critical Randall-Sundrum brane universe and a nested Randall-Sundrum scenario, we show that the induced gravitational field on the brane is identical to that of an $(n-1)$-dimensional Einstein domain wall. We comment on the absence of any "nonconventional" interactions and the definition of the braneworld Newton constant.

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It is intriguing to think that the universe in which we live might be a "defect" in some higher dimensional spacetime [1]. Although first advanced within the context of topological domain walls, the rather more general notion of a "braneworld" as being some four-dimensional submanifold of a higher dimensional spacetime has gained momentum recently after the observation [2,3] that it may provide a resolution to the hierarchy problem. Unlike a standard Kaluza-Klein (KK) compactification, physics is not averaged over these extra dimensions, but strongly localized on the brane—with only gravity propagating in the "bulk." Such a setup occurs quite naturally within the context of string and M theory [4], of which the phenomenological models of Arkani-Hamed, Dimopoulos, and Dvali (ADD) [2] and Randall and Sundrum (RS) [3] can be viewed as a simplification. (See [5] for $n\geqslant 2$ codimension gravitational solutions.) The key observation of these authors is that one can generate a hierarchy between the gravitational and other interactions via the volume factor (effective or real) of the internal extra dimensions. In particular, the RS model, in which our braneworld is a domain wall at the edge of anti–de Sitter (adS) spacetime, has the interesting feature of having a possibly infinite extra dimension, while giving a finite contribution to the volume factor.

Although these models provide a nice resolution to an old problem, and an interesting arena in which to explore new predictions of string theory, they may only be taken seriously if they reproduce what we see in our universe. In particular, gravity should behave as Einstein described:

$$R_{ab}-\frac{1}{2}Rg_{ab}=8\pi G T_{ab}, \qquad (1)$$

i.e., a massless spin two interaction (at suitably low energy scales).

In the RS scenario, it has been shown at least perturbatively [3,6] that we reproduce Einstein gravity at low energies on the brane; but if we are to seriously explore braneworld scenarios (as string and M-theory suggest we should) then it is crucial to have an understanding of the *nonperturbative* aspects of gravity on the brane. Previous attempts to do this fall into one of three categories: cosmological solutions [7–10], "zero-mode" solutions which are translationally invariant orthogonal to the brane [11], and gravitational wave solutions [12]. All of these generically contain some sort of singularity, e.g. the cosmological solutions tend to have initial or final singularities, and some, such as the zero mode Schwarzschild black string solution, are unstable [13]. A nonperturbative solution, such as a black hole bound to the brane, for example, requires a five-dimensional C metric, which is so far unforthcoming. (See [14] for a lower, $3+1$, dimensional analogue.)

The examples cited above explore certain features of nonperturbative gravity, the nonstandard Friedman cosmological equations [7] being a particularly celebrated issue. However, these represent a spatially homogeneous and isotropic braneworld, and to our knowledge there are no nonperturbative exact anisotropic solutions. We propose a successful alternative to the above tests by placing an extended source on the brane.

In this paper we focus on a cosmic "domain wall" living entirely *within*, and totally localized upon, the brane universe (a braneworld "cosmic string" was considered in linearized gravity [15], and found to exhibit departures from the four-dimensional Einstein result). From the higher dimensional perspective, our configuration is that of a vortex (by which we mean a codimension-2 object rather than some solitonic solution of a field theory) lying totally within the wall which constitutes the brane universe. Unlike the black hole problem, no C metric is required here. This configuration turns out to be directly and completely integrable, and represents a genuinely *fully localized* "intersection" of the two "branes." In the same way as one can view the RS scenario as a limit of a thick domain wall [16], one can view this solution as a zero-thickness limit of a nested topological defect [17], which can occur when one has condensates of other fields in the presence of a topological domain wall background, poetically called a *domain ribbon*.

We begin by deriving the solution for our domain ribbon. We do not restrict ourselves to critical braneworlds with a finely tuned tension, but consider noncritical walls as well. We find that the induced gravitational field on the brane is identical to what one would expect from a braneworld observer ignorant of the bulk, using Einstein gravity in one dimension less. This is a startling result—it states that at least for these highly symmetric setups, the gravitational interactions on the brane are Einsteinian in nature, even at the nonperturbative level. We first show how to derive this result before commenting on the absence of "non-conventional" terms.

Let us begin by noting that the gravitational field of a vortex wall will have dependence on only two spacetime coordinates, r and z say, with z roughly representing the direction orthogonal to the domain wall representing our brane universe, and r the direction orthogonal to the vortex or domain ribbon within our brane universe. We therefore expect that, schematically, the energy-momentum tensor of the system will have the form

$$T_{ab}=\sigma h_{ab}\frac{\delta(z)}{\sqrt{g_{zz}}}+\mu\gamma_{ab}\frac{\delta(z)\,\delta(r)}{\sqrt{g_{zz}g_{rr}}} \tag{2}$$

where h_{ab} is the induced metric on the brane universe, and γ_{ab} the induced metric on the vortex. The most general metric consistent with these symmetries can (generalizing [10]) in n dimensions be reduced to the form

$$ds^2=A^{2/(n-2)}d\mathbf{x}_\kappa^2-e^{2\nu}A^{-(n-3)/(n-2)}(dr^2+dz^2), \tag{3}$$

where $d\mathbf{x}_\kappa^2$ represents the "unit" metric on a constant curvature spacetime ($\kappa=0$ corresponds to an $(n-2)$-dimensional Minkowski spacetime, $\kappa=\pm1$ to $(n-2)$-dimensional de Sitter and anti–de Sitter spacetimes), and the brane universe sits at $z=0$, the vortex at $r=z=0$. This is basically a double analytic continuation of the cosmological metric in [10], where it is the time translation symmetry ∂_t which is broken, rather than ∂_r. The key result of that paper needed here was to show that the conformal symmetry of the t,z plane meant that the gravity equations were completely integrable in the bulk, and the brane universe was simply a boundary $(T(\tau),Z(\tau))$ of that bulk (identified with another boundary of another general bulk). The dynamical equations of the embedding of the boundary reduced to pseudo-cosmological equations for $Z(\tau)$. We may therefore use the results of [10] (appropriately modified) to deduce that our solution must be a section, $(R(\zeta),Z(\zeta))$ of the general bulk metric

$$ds^2=Z^2d\mathbf{x}_\kappa^2-\left(k_n^2Z^2+\kappa-\frac{c}{Z^{(n-3)}}\right)dR^2-\frac{dZ^2}{\left(k_n^2Z^2+\kappa-\frac{c}{Z^{(n-3)}}\right)} \tag{4}$$

where $d\mathbf{x}_\kappa^2$ is now a constant curvature Lorentzian spacetime, and $k_n^2=-2\Lambda/(n-1)(n-2)$. If $c<0$, the metric becomes singular at the adS horizon, $Z=0$. However, if $c>0$, the metric is analogous to a Euclidean black hole, and R becomes an angular coordinate—the spacetime closing off before the adS horizon.

For simplicity, we will assume our brane universe is Z_2 symmetric (i.e., spacetime is reflection symmetric around the wall) and that the integration constant, c, vanishes. This gives the equations of motion for the source (2) as

$$Z'^2(\zeta)=(k_n^2-\sigma_n^2)Z^2+\kappa \tag{5a}$$

$$Z''(\zeta)=(k_n^2-\sigma_n^2)Z-\frac{\mu_n}{2}\sigma_nZ\,\delta(\zeta) \tag{5b}$$

$$R'(\zeta)=\frac{\sigma_nZ}{(k_n^2Z^2+\kappa)} \tag{5c}$$

where $\sigma_n=8\pi G_n\sigma/2(n-2)$, and $\mu_n=8\pi G_n\mu$. For example, the Randall-Sundrum domain wall (in n dimensions) is given by setting $\kappa=\mu=0$ (flat, no vortex) and $\sigma_n=k_n$. In this case, we have the solution $Z=Z_0$ a constant, and $kR=\zeta/Z_0$. Letting $Z_0=1$, and $Z=e^{-k_nz}$ gives the usual RS coordinates. Replacing the Minkowski metric (spanned by (t, x_i and ζ) by an arbitrary $(n-1)$-dimensional metric gives the usual relation between Newton's constant in n and $n-1$ dimensions for the RS universe:

$$G_{n-1}=\frac{(n-3)}{2}k_nG_n=\frac{(n-3)}{2}\sigma_nG_n, \tag{6}$$

a relationship confirmed by the perturbative analysis of [3,6,18].

In general, the Z equation (5a) can be integrated away from $R=0$ to give

$$Z=\begin{cases}\dfrac{1}{2\sqrt{a}}[e^{\pm\sqrt{a}(\zeta-\zeta_0)}-\kappa e^{\mp\sqrt{a}(\zeta-\zeta_0)}], & a>0,\\ Z_0\pm\kappa\zeta, & a=0,\kappa=0,1,\\ \dfrac{1}{\sqrt{|a|}}\cos\sqrt{|a|}(\pm\zeta-\zeta_0), & a<0,\kappa=1\ \text{only},\end{cases} \tag{7}$$

where $a=k_n^2-\sigma_n^2$, which is zero for a critical wall, and is positive (negative) for a sub- (super-) critical wall, respectively. In the absence of the vortex, a critical wall is one with a Minkowski induced metric, and is the original RS scenario [3]; a supercritical wall is one which has a de Sitter induced metric, and can be regarded as an inflating cosmology [9], whereas the subcritical wall has an adS induced metric, and has only recently been considered from the phenomenological point of view [19].

Since we are interested in having a domain ribbon on our brane universe, we require solutions with nonzero μ_n, and hence a discontinuity in Z'. To achieve this, we simply patch together different branches of the solutions (7) for $\zeta>0$ and $\zeta<0$; the R coordinate is given by integrating Eq. (5c). From Eq. (7) we see that critical and supercritical walls can only support a vortex if $\kappa=1$, i.e., if the induced metric on the vortex itself is a de Sitter universe. A subcritical wall on the other hand can support all induced geometries on the vortex.

This procedure gives a general domain ribbon solution, but in order to investigate strong brane gravity we focus on two specific solutions: A domain ribbon in a Randall-Sundrum (critical) wall; and a "nested RS scenario," i.e., a flat Minkowski domain ribbon living on a subcritical adS domain wall. We wish to test that the induced gravity on the brane is Einsteinian in nature. If so, the braneworld metric in the former case would be that of a vacuum domain wall, and in the latter case an RS metric in $(n-1)$ dimensions (one might also expect a localized ribbon graviton zero mode). We now demonstrate for each example in turn that we do in fact get the induced Einstein gravity described.

Domain ribbon on an RS wall. The Randall-Sundrum universe is a critical $(\sigma_n = k_n)$ domain wall in adS spacetime; this means that a domain ribbon on this wall *must* have $\kappa = 1$, i.e., a "spherical" spatial geometry. We can therefore read off the trajectory from Eq. (7) with $a=0$ and $Z_0 = 4/\mu_n k_n$ from Eq. (5b). Integrating Eq. (5c) gives $e^{\mp 2k_nR} = (1+k_n^2Z^2)/(1+k_n^2Z_0^2)$, with $Z<Z(\zeta)$ giving the bulk. At first sight neither the trajectory nor bulk looks like the original RS scenario; however, the coordinate transformation

$$k_n u = e^{k_nR}/\sqrt{1+k_n^2Z^2} \tag{8a}$$

$$(\tilde{t},\tilde{\mathbf{x}}) = k_n u Z(\sinh t, \cosh t\,\mathbf{n}_{n-2}) \tag{8b}$$

[where $\mathbf{n}_{n-2}$ is the unit vector in $(n-2)$ dimensions] gives the bulk in planar coordinates:

$$ds^2 = \frac{1}{k_n^2u^2}[d\tilde{t}^2 - d\tilde{\mathbf{x}}^2 - du^2]. \tag{9}$$

The trajectory then becomes

$$u = u_0 = \frac{\mu_n}{k_n\sqrt{16+\mu_n^2}}, \quad \zeta<0,$$
$$\tilde{\mathbf{x}}^2 - \tilde{t}^2 + \left(u - \frac{1}{2k_n^2u_0}\right)^2 = \frac{1}{4k_n^4u_0^2}, \quad \zeta>0. \tag{10}$$

The change of coordinates means that the trajectory is no longer manifestly Z_2 symmetric; however, the $\zeta<0$ branch now becomes a subset of the RS planar domain wall, specifically, the interior of the hyperboloid

$$\frac{\tilde{\mathbf{x}}^2-\tilde{t}^2}{k_n^2u_0^2} = \frac{16}{k_n^2\mu_n^2} = [2\pi G_{n-1}\mu]^{-2} \tag{11}$$

[using Eq. (6)]. However, recall that the global spacetime structure of a vacuum domain wall is that of two identified copies of the interior of a hyperboloid in Minkowski spacetime of proper radius $1/2\pi G_{n-1}\mu$ [20], therefore Eq. (11) corresponds identically with what we would expect from $(n-1)$-dimensional Einstein gravity. The $\zeta>0$ branch is a hyperboloid in the bulk centered on $u = 1/2k_n^2u_0$ with comoving radius $1/2k_n^2u_0$. As μ increases, more and more of the hyperboloid is removed, with the spacetime "disappearing" only as $\mu\to\infty$. Interestingly, while this is par for the course for a domain wall, it is completely different to the behavior one would expect from a vortex.

The induced metric on the brane universe (setting $\hat{t} = 4t/\mu_nk_n$)

$$ds_{n-1}^2 = \left(1 - \frac{\mu_nk_n|\zeta|}{4}\right)^2\left[d\hat{t}^2 - \left(\frac{4}{\mu_nk_n}\right)^2\cosh^2\frac{\mu_nk_n\hat{t}}{4}d\Omega_{II}^2\right] - d\zeta^2 \tag{12}$$

is *precisely* the metric of a self-gravitating Einstein domain wall of tension μ written in Gaussian normal coordinates [21]. This can be seen from Eq. (6) and the Israel equations in $(n-1)$ dimensions for a wall of tension μ:

$$\Delta K_{ab} = -\frac{8\pi G_{n-1}\mu}{(n-3)}h_{ab} = -\frac{k_n\mu_n}{2}h_{ab} \tag{13}$$

which is clearly the correct expression for the jump in extrinsic curvature at $\zeta=0$ in Eq. (12).

The nested RS scenario. The other interesting setup we consider is a subcritical instead of critical brane universe. A subcritical brane universe is one for which the tension of the brane is not sufficient to cancel the negative bulk cosmological constant, $|\Lambda|$, and for which the "effective" cosmological constant on the brane, $-2\lambda = (n-2)(n-3)(k_n^2-\sigma_n^2)$, is still negative. From the point of view of an observer living on the brane, the $(n-1)$-dimensional universe is adS_{n-1}, and we can ask the question whether it is possible to have a Randall-Sundrum RSI wall living in this adS spacetime. Therefore, to set up this nested RS scenario, we look for a planar domain ribbon (which we expect to obey some sort of "criticality" condition analogous to $\sigma_n = k_n$ for the original RS wall) within a subcritical brane world, i.e., a $\kappa=0$ solution from Eq. (7). Defining $k_{n-1}^2 = k_n^2 - \sigma_n^2$,

$$Z = Z_0 e^{-k_{n-1}|\zeta|}, \quad R = \pm\frac{4}{k_n^2\mu_n}(Z^{-1}-Z_0^{-1}) \tag{14}$$

where $\mu_n = 4k_{n-1}/\sigma_n$. Rewriting this in terms of conformal coordinates gives $R = \pm(4/\mu_n)(u-u_0)$. Each branch of this trajectory is an adS wall, which, if it were not for the vortex at $(u_0,0)$ would reach the adS boundary at $R = \mp 4u_0/\mu_n$.

The induced metric on the braneworld

$$ds_{n-1} = Z_0^2 e^{-2k_{n-1}|\zeta|}[dt^2 - dx_i^2] - d\zeta^2 \tag{15}$$

is indeed that of an RS universe. However, the RS universe has a strict criticality relation between the tension of the brane and the bulk cosmological constant. Here, we have

$$k_{n-1} = 2\pi G_n\sigma_n\mu = \frac{4\pi G_{n-1}\mu}{(n-3)} \tag{16}$$

[using Eq. (6) in terms of σ_n rather than k_n] which is precisely the RS criticality condition $\sigma_n = 4\pi G_n\sigma/(n-2) = k_n$ adjusted for one dimension less.

Naturally, it would be interesting to know the full tensor structure of gravity, both on the domain wall braneworld, as well as on the lower-dimensional ribbon world. A full analysis of the graviton propagator is rather involved, not only because the domain wall braneworld now has a nontrivial trajectory through the adS bulk (which can in any case be remedied by a judicious–Gaussian normal–choice of gauge), but because this trajectory now contains a ''kink''—the vortex — and therefore no longer respects the bulk symmetries, thus making a simple eigenfunction expansion of the operator impossible with respect to the usual bases. However, it is easy to show that at least the vortex world volume does have something akin to the localized zero mode of the RS braneworld. To do this, one can either perform the usual perturbation analysis around the adS background with the relevant boundary (in which case very little changes from the original RS analysis), or one can simply replace the flat metric in the canonical bulk form by a general Einstein metric in the usual fashion.

This general approach can of course be modified to allow for more complicated braneworld-domain ribbon configurations, such as negative tension walls and ribbons, as well as patching between spacetimes with different cosmological constants, although the nontrivial trajectories induced by the presence of the vortex will, in general, mean that either a ''mirror'' vortex must be introduced on the negative tension brane, or the positive tension brane must match to the negative tension brane across the vortex.

To sum up: we have shown how to derive the spacetime of a braneworld with an extended nested wall source, and with illustrative examples shown that the braneworld metric is identical to that of an $(n-1)$-dimensional gravitating domain wall (either with or without induced cosmological constant) *provided* we identify the n and $(n-1)$-dimensional Newton's constants using the tension of the brane as indicated by the analysis of the de Sitter brane [18]. What therefore has happened to the nonconventional terms present in the cosmological braneworlds, which after all have effectively the same symmetries as the nested wall? The major reason for their absence is most probably the fact that we have an idealized source; however, an important second difference which must not be overlooked comes from the relation of the Newton's constants in four and five dimensions. The nonconventional cosmological terms come from using the critical RS relation (with k_n), whereas the Newton's constant ought to really be determined from perturbation theory. Our results suggest that perhaps part of gravity's ''unconventionality'' lies in our imperfect understanding of what it is.

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