Polarized Dirac fermions in de Sitter spacetime

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The tetrad gauge invariant theory of the free Dirac field in two moving frames of de Sitter spacetime is investigated, pointing out the operators that commute with the Dirac one. These are the generators of the symmetry transformations corresponding to isometries that give rise to conserved quantities according to the Noether theorem. With their help the plane wave spinor solutions of the Dirac equation with given momentum and helicity are derived and the final form of the quantum Dirac field is established. It is shown that the canonical quantization leads to a correct physical interpretation of the massive or massless fermion quantum fields.

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I. INTRODUCTION

The recent astrophysical investigations showing that the expansion of the universe is accelerating [1] may increase interest in de Sitter spacetime which could represent the far future limit of the actual universe. On the other hand, the Dirac fermions (leptons and quarks) are the principal components of matter because their gauge symmetries determine the main features of the physical picture. For these reasons, we believe that study of the tetrad gauge invariant theory of a free Dirac field in a de Sitter background may be important for understanding the influence of an external gravitational field minimally coupled with the fermion fields.

In general, the Dirac equation is studied in the so called diagonal tetrad gauge [2] which is preferred by many authors [3] since it gives simple equations and allows one to write those from central backgrounds directly in spherical coordinates. However, the Cartesian gauge mentioned in [2] seems to be more productive but less used up to now. The Dirac equation in de Sitter spacetime (of radius $R=1/\omega=\sqrt{3/\Lambda_c}$, produced by the cosmological constant Λ_c) has been studied in moving or static local charts (i.e., natural frames) suitable for separation of variables, leading to significant analytical solutions [4–6]. The first spinor solutions on this background were obtained in a static central chart using the diagonal tetrad gauge in spherical coordinates [4]. A few years later, with a new method [7], spherical wave solutions of the Dirac equation were derived in the moving local chart $\{t,r,\theta,\phi\}$ associated with the Cartesian one $\{t,\vec{x}\}$ with the line element

$$ds^2=dt^2-e^{2\omega t}d\vec{x}^2, \tag{1}$$

where a Cartesian tetrad gauge was considered [5]. Moreover, in [5] possible plane wave solutions in Cartesian coordinates were mentioned without writing them down explicitly. Since in these moving charts the operator $i\partial_t$ is no longer a Killing vector field, the quantum modes corresponding to all these particular solutions have no well-determined energies. Obviously, this is not an impediment but, in addition, there are some integration constants whose physical meaning remains obscure [5]. An alternative might be the particular solutions resulting from the separation of variables in static central charts $\{t_s,\vec{x}_s\}$, and the Cartesian tetrad gauge where the Dirac field transforms manifestly covariantly under time translations, generated by the Hamiltonian operator $i\partial_{t_s}$, and the rotations of the Cartesian space coordinates $\vec{x}_s$ [8]. Recently, we found that these solutions are energy eigenspinors whose integration constants are completely determined by the usual quantum numbers of the total angular momentum [6]. Unfortunately, because the energy spectrum is continuous and the solutions are too complicated, these cannot be normalized in the energy scale. Thus, actually we do not have yet a complete system of particular solutions that may be used for writing the general form of the quantum Dirac field in de Sitter spacetime.

We continue these investigations here, looking for a set of normalized particular solutions of the free Dirac field in the chart with the line element (1). Our aim is to write down the plane wave solutions suggested in [5], derived now as common eigenspinors of a complete set of commuting observables whose eigenvalues should determine the constants arising from the separation of variables. The main purpose of the present article is to show that these solutions are suitable for expressing the canonically quantized [9] Dirac field in terms of creation and annihilation operators of fermions with well-defined physical properties.

To this end we exploit the results of our previously constructed theory of external symmetry [10] which explains the relations among the geometric symmetries and the operators commuting with the Dirac one that were written with the help of the Killing vectors some time ago [11]. In fact these operators are nothing other than the generators of the spinor representation of the universal covering group of the isometry group [10] and, therefore, they represent the main physical observables among which we can choose different sets of commuting operators defining quantum modes. This method is efficient especially in the case of de Sitter spacetime where the high symmetry given by the SO(4,1) isometry group [12,13] offers the opportunity for a rich algebra of operators able to receive a physical meaning.

Our basic idea is that the significance of these observables is independent of the choice of the local chart and tetrad gauge, even though their form is strongly dependent on both these elements. Therefore, we interpret the generators of the subgroup $E(3)\subset SO(4,1)$ as the three-dimensional momentum and (orbital) angular momentum operators [14]. Then the corresponding generators of the spinor representation are the momentum and the total angular momentum operators. From this algebra we select the momentum components and, in addition, we construct a one-component Pauli-Lubanski (or helicity) operator [15], thus obtaining the set of commuting observables that defines quantum modes with given momentum and helicity. We show that the common eigenspinors of these operators are the desired plane wave solutions of the Dirac equation which can be easily normalized in the momentum scale. Moreover, we demonstrate that the system of these solutions is complete (in a generalized sense).

This set is used for expanding the free Dirac field in terms of creation and annihilation operators of fermions characterized by momentum and helicity, pointing out that the canonical quantization requires us to adopt the standard anticommutation rules in the momentum representation. In this way the conserved quantities predicted by the Noether theorem become the one-particle operators of the quantum field theory; among these the diagonal ones are the momentum, helicity, and charge operators. All the other one-particle operators corresponding to the remaining SO(4,1) generators, including the Hamiltonian operator, are not diagonal in this basis since they do not commute with the momentum components. The conclusion is that in our approach the second quantization can be done in a canonical manner obtaining new results specific to the de Sitter geometry. However, the free fermions on this background have some properties similar to those in Minkowski spacetime such as, for example, the well-known law of neutrino polarization [9].

We start in the second section with a brief review of the main results of our theory of external symmetry, presenting the form of the symmetry generators and the conservation laws due to the Noether theorem. The moving charts we use here are introduced in Sec. III where, in addition, we define a suitable basis of the SO(4,1) generators helping us to identify the momentum, angular momentum, and Hamiltonian operators. The plane wave solutions of the Dirac equation with fixed momentum and helicity are written in the next section, while Sec. V is devoted to the canonical quantization of the Dirac field. After we present our concluding remarks, in the Appendixes we give the form of the isometries in the parametrization used here and a few other useful formulas.

We specify that the definition of the physical observables in Sec. III, the form of the normalized plane wave solutions of the Dirac equation, as well as all the other results of Secs. IV and V are original. These are presented in natural units with $\hbar=c=1$.

II. GAUGE AND EXTERNAL SYMMETRY

In a curved spacetime M the choice of the local charts $\{x\}$ of coordinates x^μ $(\mu,\nu,\ldots=0,1,2,3)$ is important from the observer point of view. In addition, the tetrad gauge covariant theory of fields with spin requires one to explicitly use the tetrad fields $e_{\hat\mu}(x)$ and $\hat e^{\hat\mu}(x)$, fixing the local frames and the corresponding coframes. These are labeled by the local indices $\hat\mu,\hat\nu,\ldots=0,1,2,3$, and have the orthonormalization properties $e_{\hat\mu}\cdot e_{\hat\nu}=\eta_{\hat\mu\hat\nu}$, $\hat e^{\hat\mu}\cdot\hat e^{\hat\nu}=\eta^{\hat\mu\hat\nu}$, and $\hat e^{\hat\mu}\cdot e_{\hat\nu}=\delta^{\hat\mu}_{\hat\nu}$, with respect to the Minkowski metric $\eta=\mathrm{diag}(1,-1,-1,-1)$. The one-forms $d\hat x^{\hat\mu}=\hat e^{\hat\mu}_\nu\, dx^\nu$ allow one to write the line element

$$ds^2=\eta_{\hat\mu\hat\nu}\,d\hat x^{\hat\mu}\,d\hat x^{\hat\nu}=g_{\mu\nu}(x)dx^\mu\,dx^\nu, \tag{2}$$

defining the metric tensor $g_{\mu\nu}$ (which raises or lowers the greek indices while for the hated greek ones we have to use the Minkowski metric). The derivatives in local frames are the vector fields $\hat\partial_{\hat\nu}=e^\mu_{\hat\nu}\partial_\mu$ which satisfy the commutation rules $[\hat\partial_{\hat\mu},\hat\partial_{\hat\nu}]=C_{\hat\mu\hat\nu\cdot}^{\cdot\cdot\hat\sigma}\hat\partial_{\hat\sigma}$, giving the Cartan coefficients that help us to write the connection components in local frames.

Let ψ be a Dirac free field of mass m, defined on the space domain D, and $\bar\psi=\psi^+\gamma^0$ its Dirac adjoint. The tetrad gauge invariant action of the Dirac field minimally coupled with the gravitational field is

$$\mathcal{S}[e,\psi]=\int d^4x\,\sqrt{g}\left\{\frac{i}{2}[\bar\psi\gamma^{\hat\alpha}D_{\hat\alpha}\psi-(\overline{D_{\hat\alpha}\psi})\gamma^{\hat\alpha}\psi]-m\bar\psi\psi\right\} \tag{3}$$

where $g=|\det(g_{\mu\nu})|$ and the Dirac matrices $\gamma^{\hat\alpha}$ satisfy $\{\gamma^{\hat\alpha},\gamma^{\hat\beta}\}=2\eta^{\hat\alpha\hat\beta}$. The covariant derivatives in local frames, $D_{\hat\alpha}=e^\mu_{\hat\alpha}D_\mu=\hat\partial_{\hat\alpha}+\hat\Gamma_{\hat\alpha}$, are expressed in terms of the spin connections

$$\hat\Gamma_{\hat\mu}=\hat\Gamma_{\hat\mu\hat\nu\hat\lambda}S^{\hat\nu\hat\lambda}=\frac{i}{4}(C_{\hat\mu\hat\nu\hat\lambda}-C_{\hat\mu\hat\lambda\hat\nu}-C_{\hat\nu\hat\lambda\hat\mu})S^{\hat\nu\hat\lambda} \tag{4}$$

given by the basis generators in covariant parametrization $S^{\hat\alpha\hat\beta}=i[\gamma^{\hat\alpha},\gamma^{\hat\beta}]/4$ of the usual spinor representation $\rho\sim(1/2,0)\oplus(0,1/2)$ of the $SL(2,\mathbb{C})$ group [15,16] (i.e., the universal covering group of the Lorentz group $L^\uparrow_+$, which is the gauge group of the metric η). The Dirac operator of the equation $E_D\psi=m\psi$ derived from the action (3) reads $E_D=i\gamma^{\hat\alpha}D_{\hat\alpha}$. In other respects, from conservation of the electric charge one deduces that when $e^0_i=0$ $(i,j,\ldots=1,2,3)$ the time-independent relativistic scalar product of two spinors [17],

$$\langle\psi,\psi'\rangle=\int_D d^3x\,\mu(x)\bar\psi(x)\gamma^0\psi'(x), \tag{5}$$

has the weight function $\mu=\sqrt{g}\,e^0_0$.

The action (3) is gauge invariant in the sense that it remains unchanged when one performs a gauge transformation

$$\psi(x)\to\psi'(x)=\rho[A(x)]\psi(x) \tag{6}$$

$$e_{\hat\alpha}(x)\to e'_{\hat\alpha}(x)=\Lambda_{\hat\alpha\cdot}^{\cdot\hat\beta}[A(x)]e_{\hat\beta}(x) \tag{7}$$

produced by $A(x)\in SL(2,\mathbb{C})$ and $\Lambda[A(x)]\in L^\uparrow_+$. Based on this symmetry, we have defined the group of external symmetry $S(M)$ corresponding to the isometry group $I(M)$. The transformations of $S(M)$ are isometries of $I(M)$,

$x \to x' = \phi_\xi(x)$ (depending on the parameters ξ^a, $a = 1,2,\ldots,n$), combined with appropriate gauge transformations in such a manner as to preserve the tetrad gauge. In a fixed gauge, one associates with each isometry ϕ_ξ the section $A_\xi(x) \in \mathrm{SL}(2,\mathbb{C})$ defined by

$$\Lambda^{\hat\alpha}_{\cdot\,\hat\beta}[A_\xi(x)] = \hat e^{\hat\alpha}_\mu[\phi_\xi(x)] \frac{\partial \phi^\mu_\xi(x)}{\partial x^\nu} e^\nu_{\hat\beta}(x) \tag{8}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in \mathrm{SL}(2,C)$. Then the transformations of the group $\mathrm{S}(M)$ are

$$(A_{xi},\phi_{xi}): \quad \begin{aligned} x &\to x' = \phi_\xi(x), \\ e(x) &\to e'(x') = e[\phi_\xi(x)], \\ \hat e(x) &\to \hat e'(x') = \hat e[\phi_\xi(x)], \\ \psi(x) &\to \psi'(x') = \rho[A_\xi(x)]\psi(x). \end{aligned} \tag{9}$$

In [10] we presented arguments that $\mathrm{S}(M)$ is the universal covering group of $\mathrm{I}(M)$ remarking that the representation defined by the last of Eqs. (9) is not the usual linear one of $\mathrm{S}(M)$. In fact this is *induced* by the representation ρ of the group $\mathrm{SL}(2,\mathbb{C})$ which, in general, differs from $\mathrm{S}(M)$. For this reason we say that ψ transforms according to the spinor representation of $\mathrm{S}(M)$ induced by ρ.

The transformations (9) leave invariant the form of the operator E_D in local frames. Consequently, each Killing vector $k_a = (\partial_{\xi^a}\phi_\xi)_{\xi=0}$ defines a basis generator of the spinor representation [11,10]

$$X_a = -ik^\mu_a D_\mu + \frac{1}{2} k_{a\ \mu;\nu} e^\mu_{\hat\alpha} e^\nu_{\hat\beta} S^{\hat\alpha\hat\beta}, \tag{10}$$

which *commutes* with E_D (the notation $_{;\nu}$ stands for the usual covariant derivatives). We must specify that this important result was obtained for the Dirac field in [11] without taking into account the symmetry transformations. In [10] we showed that the generators (10) satisfy the commutation relations

$$[X_a, X_b] = i c_{abc} X_c, \quad a,b,c = 1,2,\ldots,n \tag{11}$$

given by the structure constants of the isometry group $\mathrm{I}(M)$. On the other hand, each generator can be split into an orbital and a spin part as $X_a = L_a + S_a$, where the orbital terms

$$L_a = -ik^\mu_a(x)\partial_\mu \tag{12}$$

are the basis generators of the natural representation of $\mathrm{I}(M)$ carried by the space of the scalar functions over M. The spin terms

$$S_a(x) = \frac{1}{2}\Omega^{\hat\alpha\hat\beta}_a(x) S_{\hat\alpha\hat\beta} \tag{13}$$

are defined with the help of the functions

$$\Omega^{\hat\alpha\hat\beta}_a = (\hat e^{\hat\alpha}_\mu \partial_\nu k^\mu_a + k^\mu_a \partial_\mu \hat e^{\hat\alpha}_\nu) e^\nu_{\hat\lambda} \eta^{\hat\lambda\hat\beta}, \tag{14}$$

which are antisymmetric if and only if k_a is a Killing vector. Thus we see that the spin terms of the generators X_a generally depend on x and, therefore, they do not commute with the orbital terms. When L_a and S_a commute between themselves we say that the Dirac field transforms *manifestly* covariantly under the symmetry transformations parametrized by ξ^a.

Our theory of external symmetry offers us the framework we need to calculate the conserved quantities predicted by the Noether theorem. Starting with the infinitesimal transformations of the one-parameter subgroup of $\mathrm{S}(M)$ generated by X_a, we find that there exists a conserved current $\Theta^\mu[X_a]$ that satisfies $\Theta^\mu[X_a]_{;\mu} = 0$. For the action (3) this is

$$\Theta^\mu[X_a] = -\tilde T^\mu_{\cdot\,\nu} k^\nu_a + \frac{1}{4}\bar\psi\{\gamma^{\hat\alpha}, S^{\hat\beta\hat\gamma}\}\psi e^\mu_{\hat\alpha}\Omega_{a\hat\beta\hat\gamma} \tag{15}$$

where

$$\tilde T^\mu_{\cdot\,\nu} = \frac{i}{2}[\bar\psi\gamma^{\hat\alpha} e^\mu_{\hat\alpha}\partial_\nu\psi - (\overline{\partial_\nu\psi})\gamma^{\hat\alpha} e^\mu_{\hat\alpha}\psi] \tag{16}$$

is a notation for a part of the stress-energy tensor of the Dirac field [12,17]. Finally, it is clear that the corresponding conserved quantity is the real number

$$\int_D d^3x \sqrt{g}\,\Theta^0[X_a] = \frac{1}{2}[\langle\psi, X_a\psi\rangle + \langle X_a\psi,\psi\rangle]. \tag{17}$$

We note that it is premature to interpret this formula as an expectation value or to speak about Hermitian conjugation of the operators X_a with respect to the scalar product (5), before specifying the boundary conditions on D. What is important here is that this result is useful in quantization, giving directly the one-particle operators of quantum field theory.

III. OBSERVABLES IN de SITTER SPACETIME

Let us now consider M to be the de Sitter spacetime, defined as a hyperboloid of radius R in the five-dimensional flat spacetime M^5 of coordinates $Z^A (A, B, \ldots = 0,1,2,3,5)$ and metric $\eta^5 = \mathrm{diag}(1,-1,-1,-1,-1)$ [12]. The hyperboloid equation

$$\eta^5_{AB} Z^A Z^B = -R^2 \tag{18}$$

defines M as the homogeneous space of the pseudo-orthogonal group SO(4,1) which is at the same time the gauge group of the metric η^5 and the isometry group $\mathrm{I}(M)$ of the de Sitter spacetime. For this reason it is convenient to use the covariant real parameters $\xi^{AB} = -\xi^{BA}$ since in this case the orbital basis generators of the representation of SO(4,1), carried by the space of the scalar functions over M^5, have the standard form

$$L^5_{AB} = i[\eta^5_{AC} Z^C \partial_B - \eta^5_{BC} Z^C \partial_A]. \tag{19}$$

They will give us directly the orbital basis generators $L_{(AB)}$ of the scalar representations of $\mathrm{I}(M)$. Indeed, starting with the functions $Z^A(x)$ that solve Eq. (18) in the chart $\{x\}$, one can write down the operators (19) in the form (12), thus finding the generators $L_{(AB)}$ and implicitly the components $k^{\mu}_{(AB)}(x)$ of the Killing vectors associated with the parameters ξ^{AB} [10]. Furthermore, one has to calculate the spin parts $S_{(AB)}$, according to Eqs. (13) and (14), arriving at the final form of the basis generators $X_{(AB)}=L_{(AB)}+S_{(AB)}$ of the spinor representation of S(M) induced by ρ.

In de Sitter spacetime there are many static or moving charts of physical interest. Among the moving ones a special role is played by the chart $\{t_c,\vec{x}\}$ with the conformal time t_c and Cartesian spaces coordinates x^i defined by

$$Z^0=-\frac{1}{2\omega^2 t_c}[1-\omega^2(t_c^2-r^2)],$$

$$Z^5=-\frac{1}{2\omega^2 t_c}[1+\omega^2(t_c^2-r^2)], \tag{20}$$

$$Z^i=-\frac{1}{\omega t_c}x^i$$

with $r=|\vec{x}|$. Even if this chart covers only one-half of the manifold M, for $t_c\in(-\infty,0)$ and $\vec{x}\in D\equiv\mathbb{R}^3$, it has the advantage of a simple conformal flat line element [17]

$$ds^2=\frac{1}{\omega^2 t_c^2}(dt_c^2-d\vec{x}^2). \tag{21}$$

Moreover, the conformal time t_c is related through

$$\omega t_c=-e^{-\omega t} \tag{22}$$

to the proper time $t\in(-\infty,\infty)$ of an observer at $\vec{x}=0$ in the chart $\{t,\vec{x}\}$ with the line element (1). In what follows we study the Dirac field in the chart $\{t,\vec{x}\}$ using the conformal time as a helpful auxiliary ingredient. The form of the line element (21) allows one to choose the simple Cartesian gauge with the nonvanishing tetrad components [5]

$$e^0_0=-\omega t_c, \quad e^i_j=-\delta^i_j\omega t_c, \quad \hat{e}^0_0=-\frac{1}{\omega t_c}, \quad \hat{e}^i_j=-\delta^i_j\frac{1}{\omega t_c}. \tag{23}$$

In this gauge the Dirac operator reads

$$E_D=-i\omega t_c(\gamma^0\partial_{t_c}+\gamma^i\partial_i)+\frac{3i\omega}{2}\gamma^0$$

$$=i\gamma^0\partial_t+ie^{-\omega t}\gamma^i\partial_i+\frac{3i\omega}{2}\gamma^0 \tag{24}$$

and the weight function of the scalar product (5) is

$$\mu=(-\omega t_c)^{-3}=e^{3\omega t}. \tag{25}$$

The next step is to calculate the basis generators $X_{(AB)}$ of the spinor representation of S(M) in this gauge since these are the main operators that commute with E_D. The group SO(4,1) includes the subgroup E$(3)=$T(3)Ⓢ SO(3) which is just the isometry group of the three-dimensional Euclidean space of our moving charts $\{t_c,\vec{x}\}$ and $\{t,\vec{x}\}$ formed by $\mathbb{R}^3$ translations $x^i\to x^i+a^i$ and proper rotations $x^i\to R^i_{\cdot j}x^j$ with $R\in$ SO(3) [15]. Therefore, the basis generators of its universal covering group $\tilde{\mathrm{E}}(3)=$T(3)Ⓢ SU$(2)\subset$ S(M) can be interpreted as the components of the momentum $\vec{P}$ and total angular momentum $\vec{J}$ operators. The problem of the Hamiltonian operator seems to be more complicated, but we know that in the mentioned static central charts with the static time t_s this is $H=\omega X_{(05)}=i\partial_{t_s}$ [10]. Thus the Hamiltonian operator and the components of the momentum and total angular momentum operators (P^i and $J^i=\varepsilon_{ijk}J_{jk}/2$, respectively) can be identified as being the following basis generators of S(M):

$$H\equiv\omega X_{(05)}=-i\omega(t_c\partial_{t_c}+x^i\partial_i), \tag{26}$$

$$P^i\equiv\omega(X_{(5i)}-X_{(0i)})=-i\partial_i, \tag{27}$$

$$J_{ij}\equiv X_{(ij)}=-i(x^i\partial_j-x^j\partial_i)+S_{ij}, \tag{28}$$

after which one is left with the three basis generators

$$N^i\equiv X_{(5i)}+X_{(0i)}=\omega(t_c^2-r^2)P^i+2x^iH+2\omega(S_{i0}t_c+S_{ij}x^j), \tag{29}$$

which do not have an immediate physical significance. The SO(4,1) transformations corresponding to these basis generators and the associated isometries of the chart $\{t_c,\vec{x}\}$ are briefly presented in Appendix A.

In the other local chart $\{t,\vec{x}\}$, we have the same operators $\vec{P}$ and $\vec{J}=\vec{L}+\vec{S}$ (with $\vec{L}=\vec{x}\times\vec{P}$) whose components are the $\tilde{\mathrm{E}}(3)$ generators, while the Hamiltonian operator takes the form

$$H=i\partial_t+\omega\vec{x}\cdot\vec{P}, \tag{30}$$

where the second term, due to the external gravitational field, leads to the commutation rules

$$[H,P^i]=i\omega P^i. \tag{31}$$

We observe that in this chart the operators $K^i\equiv X_{(0i)}$ are the analogue of the basis generators of the Lorentz boosts of SL$(2,\mathbb{C})$ since in the limit of $\omega\to 0$, when Eq. (1) equals the Minkowski line element, the operators $H=P^0$, P^i, J^i, and K^i become the generators of the spinor representation of the group T(4)Ⓢ SL$(2,\mathbb{C})$ (i.e., the universal covering group of the Poincaré group [15,16]).

In both charts used here the generators (26)–(29) are self-adjoint with respect to the scalar product (5) with the weight function (25) if we consider the usual boundary conditions on $D\equiv\mathbb{R}^3$. Therefore, for any generator X we have $\langle X\psi,\psi'\rangle=\langle\psi,X\psi'\rangle$ if (and only if) ψ and ψ' are solutions of the Dirac equation that behave as tempered distributions or square integrable spinors with respect to the scalar product (5). Moreover, all these generators commute with the Dirac operator E_D. If, in addition, we take into account the algebra freely generated by them, then we get a large collection of observables among which we can choose suitable sets of commuting operators defining the fermion quantum modes at the level of relativistic quantum mechanics.

IV. POLARIZED PLANE WAVE SOLUTIONS

As suggested in [5], the plane wave solutions of the Dirac equation with $m\neq 0$ must be eigenspinors of the momentum operators P^i corresponding to the eigenvalues p^i, with the same time modulation as the spherical waves. Therefore, we have to look for particular solutions in the chart $\{t_c,\vec{x}\}$ involving either positive or negative frequency plane waves. Bearing in mind that these must be related among themselves through charge conjugation, we assume that, in the standard representation of the Dirac matrices (with diagonal γ^0 [16]), they have the form

$$\psi^{(+)}_{\vec{p}}=\begin{pmatrix} f^+(t_c)\alpha(\vec{p}) \\ g^+(t_c)\dfrac{\vec{\sigma}\cdot\vec{p}}{p}\alpha(\vec{p})\end{pmatrix} e^{i\vec{p}\cdot\vec{x}}, \tag{32}$$

$$\psi^{(-)}_{\vec{p}}=\begin{pmatrix} g^-(t_c)\dfrac{\vec{\sigma}\cdot\vec{p}}{p}\beta(\vec{p}) \\ f^-(t_c)\beta(\vec{p})\end{pmatrix} e^{-i\vec{p}\cdot\vec{x}}, \tag{33}$$

where $p=|\vec{p}|$ and σ_i denotes the Pauli matrices while α and β are arbitrary Pauli spinors depending on $\vec{p}$. Replacing these spinors in the Dirac equation given by Eq. (24) and denoting $k=m/\omega$ and $\nu_\pm=\frac{1}{2}\pm ik$, we find equations of the form (B2) whose solutions can be written in terms of Hankel functions as

$$f^+=(-f^-)^*=Ct_c^2e^{\pi k/2}H^{(1)}_{\nu_-}(-pt_c), \tag{34}$$

$$g^+=(-g^-)^*=Ct_c^2e^{-\pi k/2}H^{(1)}_{\nu_+}(-pt_c). \tag{35}$$

The integration constant C will be calculated from the orthonormalization condition in the momentum scale.

The plane wave solutions are determined up to the significance of the Pauli spinors α and β. For $\vec{p}\neq 0$ these can be treated as in the flat case [9,16] since, in the tetrad gauge (23), the spaces of these spinors carry unitary linear representations of the $\tilde{\mathrm{E}}(3)$ group. Indeed, the transformation (9) produced by $(A,\phi_{A,\vec{a}})\in\tilde{\mathrm{E}}(3)\subset \mathrm{S}(M)$ where $A\in \mathrm{SU}(2)$ and $\vec{a}\in\mathbb{R}^3$ involves the usual linear isometry of E(3), $x^i\to x'^i=\phi^i_{A,\vec{a}}(\vec{x})\equiv\Lambda^{i}_{\cdot\, j}(A)x^j+a^i$ with $\Lambda(A)\in \mathrm{SO}(3)$, and the global transformation $\psi(t,\vec{x})\to\psi'(t,\vec{x}')=\rho(A)\psi(t,\vec{x})$. Consequently, the Pauli spinors transform according to the unitary (linear) representation

$$\alpha(\vec{p})\to e^{-i\vec{a}\cdot\vec{p}}A\alpha[\Lambda(A)^{-1}\vec{p}] \tag{36}$$

(and similarly for β) which preserves orthogonality. This means that any pair of orthogonal spinors $\tilde{\xi}_\sigma(\vec{p})$ with polarizations $\sigma=\pm 1/2$ (obeying $\tilde{\xi}^+_\sigma\tilde{\xi}_{\sigma'}=\delta_{\sigma\sigma'}$) represents a good basis in the space of Pauli spinors

$$\alpha(\vec{p})=\sum_\sigma \tilde{\xi}_\sigma(\vec{p})a(\vec{p},\sigma) \tag{37}$$

whose components $a(\vec{p},\sigma)$ are the particle wave functions in the momentum representation. According to the standard interpretation of the negative frequency terms [9,16], the corresponding basis of the space of β spinors is defined by

$$\beta(\vec{p})=\sum_\sigma \tilde{\eta}_\sigma(\vec{p})[b(\vec{p},\sigma)]^*,\quad \tilde{\eta}_\sigma(\vec{p})=i\sigma_2[\tilde{\xi}_\sigma(\vec{p})]^* \tag{38}$$

where $b(\vec{p},\sigma)$ are the antiparticle wave functions. It remains to choose a specific basis, using supplementary physical assumptions. Since it is not certain that the so called spin basis [9] can be correctly defined in de Sitter geometry, we prefer the *helicity* basis. This is formed by orthogonal Pauli spinors of helicity $\lambda=\pm 1/2$ which satisfy

$$\vec{\sigma}\cdot\vec{p}\,\tilde{\xi}_\lambda(\vec{p})=2p\lambda\,\tilde{\xi}_\lambda(\vec{p}),\quad \vec{\sigma}\cdot\vec{p}\,\tilde{\eta}_\lambda(\vec{p})=-2p\lambda\,\tilde{\eta}_\lambda(\vec{p}). \tag{39}$$

The desired particular solutions of the Dirac equation with $m\neq 0$ result from our starting formulas (32) and (33) where we insert the functions (34) and (35) and the spinors (37) and (38) written in the helicity basis (39). It remains to calculate the normalization constant C with respect to the scalar product (5) with the weight function (25). After a few manipulations, in the chart $\{t,\vec{x}\}$, it turns out that the final form of the fundamental spinor solutions of positive and negative frequencies with momentum $\vec{p}$ and helicity λ is

$$U_{\vec{p},\lambda}(t,\vec{x})=iN\begin{pmatrix} \dfrac{1}{2}e^{\pi k/2}H^{(1)}_{\nu_-}(qe^{-\omega t})\tilde{\xi}_\lambda(\vec{p}) \\ \lambda e^{-\pi k/2}H^{(1)}_{\nu_+}(qe^{-\omega t})\tilde{\xi}_\lambda(\vec{p})\end{pmatrix} e^{i\vec{p}\cdot\vec{x}-2\omega t}, \tag{40}$$

$$V_{\vec{p},\lambda}(t,\vec{x})=iN\begin{pmatrix} -\lambda e^{-\pi k/2}H^{(2)}_{\nu_-}(qe^{-\omega t})\tilde{\eta}_\lambda(\vec{p}) \\ \dfrac{1}{2}e^{\pi k/2}H^{(2)}_{\nu_+}(qe^{-\omega t})\tilde{\eta}_\lambda(\vec{p})\end{pmatrix} e^{-i\vec{p}\cdot\vec{x}-2\omega t}, \tag{41}$$

where we introduced the new parameter $q=p/\omega$ and

$$N=\frac{1}{(2\pi)^{3/2}}\sqrt{\pi q}. \tag{42}$$

Using Eqs. (B1) and (B3), it is not hard to verify that these spinors are charge conjugated to each other,

$$V_{\vec{p},\lambda}=(U_{\vec{p},\lambda})^c=\mathcal{C}(\overline{U}_{\vec{p},\lambda})^T,\quad \mathcal{C}=i\gamma^2\gamma^0, \tag{43}$$

satisfy the orthonormalization relations

$$\langle U_{\vec{p},\lambda}, U_{\vec{p}',\lambda'}\rangle = \langle V_{\vec{p},\lambda}, V_{\vec{p}',\lambda'}\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}'), \quad (44)$$

$$\langle U_{\vec{p},\lambda}, V_{\vec{p}',\lambda'}\rangle = \langle V_{\vec{p},\lambda}, U_{\vec{p}',\lambda'}\rangle = 0, \quad (45)$$

and represent a *complete* system of solutions in the sense that

$$\int d^3p \sum_{\lambda} [U_{\vec{p},\lambda}(t,\vec{x})U^{+}_{\vec{p},\lambda}(t,\vec{x}') + V_{\vec{p},\lambda}(t,\vec{x})V^{+}_{\vec{p},\lambda}(t,\vec{x}')] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}'). \quad (46)$$

Let us observe that the factor $e^{-3\omega t}$ is exactly the quantity necessary to compensate the weight function (25). Other important properties are

$$P^i U_{\vec{p},\lambda} = p^i U_{\vec{p},\lambda}, \quad P^i V_{\vec{p},\lambda} = -p^i V_{\vec{p},\lambda}, \quad (47)$$

$$W U_{\vec{p},\lambda} = p\lambda U_{\vec{p},\lambda}, \quad W V_{\vec{p},\lambda} = -p\lambda V_{\vec{p},\lambda}, \quad (48)$$

where

$$W = \vec{J}\cdot\vec{P} = \vec{S}\cdot\vec{P} \quad (49)$$

is the helicity operator, which is analogous to the timelike component of the four-component Pauli-Lubanski operator of the Poincaré algebra [15]. Thus, we arrive at the conclusion that the fundamental solutions (40) and (41) form a complete system of common eigenspinors of the operators P^i and W. Since the spin was fixed *a priori* by choosing the representation ρ, we consider that the complete set of commuting operators that determines separately each of the sets of the particle or antiparticle eigenspinors is $\{E_D, \vec{S}^2, P^i, W\}$. Finally, we note that these solutions can be redefined at any time with other momentum-dependent phase factors as

$$U_{\vec{p},\lambda} \to e^{i\chi(\vec{p})}U_{\vec{p},\lambda}, \quad V_{\vec{p},\lambda} \to e^{-i\chi(\vec{p})}V_{\vec{p},\lambda}, \quad \chi(\vec{p}) \in \mathbb{R}, \quad (50)$$

without affecting the above properties.

In the case $m=0$ (when $k=0$) it is convenient to consider the chiral representation of the Dirac matrices (with diagonal γ^5 [9]) and the chart $\{t_c, \vec{x}\}$. We find that the fundamental solutions in the helicity basis of the left-handed massless Dirac field,

$$U^0_{\vec{p},\lambda}(t_c,\vec{x}) = \lim_{k\to 0}\frac{1-\gamma^5}{2}U_{\vec{p},\lambda}(t_c,\vec{x}) = \left(\frac{-\omega t_c}{2\pi}\right)^{3/2}\begin{pmatrix}(\frac{1}{2}-\lambda)\tilde{\xi}_\lambda(\vec{p})\\ 0\end{pmatrix}e^{-ipt_c+i\vec{p}\cdot\vec{x}}, \quad (51)$$

$$V^0_{\vec{p},\lambda}(t_c,\vec{x}) = \lim_{k\to 0}\frac{1-\gamma^5}{2}V_{\vec{p},\lambda}(t_c,\vec{x}) = \left(\frac{-\omega t_c}{2\pi}\right)^{3/2}\begin{pmatrix}(\frac{1}{2}+\lambda)\tilde{\eta}_\lambda(\vec{p})\\ 0\end{pmatrix}e^{ipt_c-i\vec{p}\cdot\vec{x}}, \quad (52)$$

are nonvanishing only for positive frequency and $\lambda=-1/2$ or negative frequency and $\lambda=1/2$, as in Minkowski spacetime. Obviously, these solutions have similar properties to Eqs. (43)–(48).

V. QUANTIZATION

The quantization can be done by considering that the wave functions in momentum representation, $a(\vec{p},\lambda)$ and $b(\vec{p},\lambda)$, become field operators (so that $b^*\to b^\dagger$) [9]. Then the quantum field that satisfies the Dirac equation with $m\neq 0$ in the chart $\{t,\vec{x}\}$ reads

$$\psi(t,\vec{x}) = \psi^{(+)}(t,\vec{x}) + \psi^{(-)}(t,\vec{x}) = \int d^3p \sum_{\lambda}[U_{\vec{p},\lambda}(x)a(\vec{p},\lambda) + V_{\vec{p},\lambda}(x)b^\dagger(\vec{p},\lambda)]. \quad (53)$$

We assume that the particle $(a, a^\dagger)$ and antiparticle $(b, b^\dagger)$ operators must satisfy the standard anticommutation relations in the momentum representation, from which the nonvanishing ones are

$$\{a(\vec{p},\lambda), a^\dagger(\vec{p}',\lambda')\} = \{b(\vec{p},\lambda), b^\dagger(\vec{p}',\lambda')\} = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}'), \quad (54)$$

since then the equal-time anticommutator takes the *canonical* form

$$\{\psi(t,\vec{x}), \bar{\psi}(t,\vec{x}')\} = e^{-3\omega t}\gamma^0\delta^3(\vec{x}-\vec{x}'), \quad (55)$$

as follows from Eq. (46). In general, the partial anticommutator functions

$$\tilde{S}^{(\pm)}(t,t',\vec{x}-\vec{x}') = i\{\psi^{(\pm)}(t,\vec{x}), \bar{\psi}^{(\pm)}(t',\vec{x}')\} \quad (56)$$

and the total one $\tilde{S}=\tilde{S}^{(+)}+\tilde{S}^{(-)}$ are rather complicated since for $t\neq t'$ we no longer have identities like Eq. (B3) which would simplify their time-dependent parts. In any event, these are solutions of the Dirac equation in both their sets of coordinates and help one to write the Green functions in usual manner. For example, from the standard definition of the Feynman propagator [9],

$$\tilde{S}_F(t,t',\vec{x}-\vec{x}') = i\langle 0|T[\psi(x)\bar{\psi}(x')]|0\rangle = \theta(t-t')\tilde{S}^{(+)}(t,t',\vec{x}-\vec{x}') - \theta(t'-t)\times\tilde{S}^{(-)}(t,t',\vec{x}-\vec{x}'), \quad (57)$$

we find that

$$[E_D(x)-m]\tilde{S}_F(t,t',\vec{x}-\vec{x}\,')=-e^{-3\omega t}\delta^4(x-x'). \quad (58)$$

Another argument for this quantization procedure is that the one-particle operators given by the Noether theorem have similar structures and properties to those of the quantum theory of free fields in flat spacetime. Indeed, starting with the form (17) of the conserved quantities, we find that for any self-adjoint generator X of the spinor representation of the group $\mathrm{S}(M)$ there exists a *conserved* one-particle operator of quantum field theory, which can be calculated simply as

$$\mathbf{X}=:\langle\psi,X\psi\rangle:, \quad (59)$$

respecting the normal ordering of the operator products [9]. Hereby we recover the standard algebraic properties

$$[\mathbf{X},\psi(x)]=-X\psi(x), \quad [\mathbf{X},\mathbf{X}']=:\langle\psi,[X,X']\psi\rangle: \quad (60)$$

because of the canonical quantization adopted here.

The diagonal one-particle operators result directly from the definition (59) and the properties (44)–(48). In this way we obtain the momentum components

$$\mathbf{P}^i=:\langle\psi,P^i\psi\rangle:=\int d^3p\, p^i\sum_\lambda [a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda)+b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)] \quad (61)$$

and the helicity (or Pauli-Lubanski) operator

$$\mathbf{W}=:\langle\psi,W\psi\rangle:=\int d^3p\sum_\lambda p\lambda[a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda)+b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)]. \quad (62)$$

The definition (59) holds for the generators of internal symmetries too, including the particular case of $X=1$, when the bracket

$$\mathbf{Q}=:\langle\psi,\psi\rangle:=\int d^3p\sum_\lambda [a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda)-b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)] \quad (63)$$

gives just the charge operator corresponding to the internal U(1) symmetry of the action (3) [16,17]. It is obvious that all these operators are self-adjoint and represent the generators of the external or internal symmetry transformations of the quantum fields [9]. The conclusion is that, for fixed mass and spin, the helicity state vectors of the Fock space defined as common eigenvectors of the set $\{\mathbf{Q},\mathbf{P}^i,\mathbf{W}\}$ form a complete system of orthonormalized vectors in a generalized sense, i.e., the helicity basis.

The Hamiltonian operator $\mathbf{H}=:\langle\psi,H\psi\rangle:$ is conserved but is not diagonal in this basis since it does not commute with $\mathbf{P}^i$ and $\mathbf{W}$ as follows from the commutation relations (31) and the properties (60). Its form in momentum representation can be calculated using the identity

$$HU_{\vec{p},\lambda}(t,\vec{x})=-i\omega\left(p^i\partial_{p^i}+\frac{3}{2}\right)U_{\vec{p},\lambda}(t,\vec{x}) \quad (64)$$

and the similar one for $V_{\vec{p},\lambda}$, leading to

$$\mathbf{H}=\frac{i\omega}{2}\int d^3p\, p^i\sum_\lambda [a^\dagger(\vec{p},\lambda)\overleftrightarrow{\partial}_{p^i}a(\vec{p},\lambda)+b^\dagger(\vec{p},\lambda)\overleftrightarrow{\partial}_{p^i}b(\vec{p},\lambda)] \quad (65)$$

where the derivatives act as $f\overleftrightarrow{\partial}h=f\partial h-(\partial f)h$. The result is the expected behavior of $\mathbf{H}$ under the space translations of $\tilde{\mathrm{E}}(3)$ that transform the operators a and b according to Eq. (36). Moreover, it is worth pointing out that the change of the phase factors (50) associated with the transformations

$$a(\vec{p},\lambda)\to e^{-i\chi(\vec{p})}a(\vec{p},\lambda), \quad b(\vec{p},\lambda)\to e^{-i\chi(\vec{p})}b(\vec{p},\lambda) \quad (66)$$

leaves invariant the operators ψ, $\mathbf{Q}$, $\mathbf{P}^i$, and $\mathbf{W}$ as well as Eqs. (54), but transforms the Hamiltonian operator

$$\mathbf{H}\to\mathbf{H}+\omega\int d^3p[p^i\partial_{p^i}\chi(\vec{p})]\sum_\lambda [a^\dagger(\vec{p},\lambda)a(\vec{p},\lambda)+b^\dagger(\vec{p},\lambda)b(\vec{p},\lambda)]. \quad (67)$$

This remarkable property may be interpreted as a new type of gauge transformation depending on momentum instead of coordinates. Our preliminary calculations indicate that this gauge may be helpful for analyzing the behavior of the theory near $\omega\sim 0$.

In the simpler case of a left-handed massless field with the fundamental spinor solutions (51) and (52) we obtain similar results, and we recover the standard rule of neutrino polarization.

VI. CONCLUDING REMARKS

We have derived here a complete system of normalized plane wave solutions of the Dirac equation in the chart with the line element (1) of de Sitter spacetime. These describe the quantum modes of polarized free fermions (or antifermions), determined by the complete set of commuting operators $\{E_D, \vec{S}^2, P^i, W\}$. A crucial point was the choice of the Cartesian gauge [5] in which the Dirac field transforms manifestly covariantly under the $\tilde{\mathrm{E}}(3)$ subgroup, since in these conditions one can perform the second quantization in a canonical way as in special relativity. We recall that in the static central charts $\{t_s,\vec{x}_s\}$ there is another appropriate Cartesian gauge where the Dirac field transforms manifestly covariantly under the subgroup $\mathrm{T}(1)_{t_s}\otimes\mathrm{SU}(2)\subset\mathrm{S}(M)$ (involving the time translations generated by $H=i\partial_{t_s}$) [10]. Then the separation of variables can be done as in the central problems of special relativity, leading to common eigenspinors of the complete set $\{E_D,\vec{S}^2,H,\vec{J}^2,J_3,\mathcal{K}\}$ [6] which includes the usual spin-orbit operator $\mathcal{K}$ [16]. This method allowed us to obtain the solutions presented in [6] as well as the normalized energy eigenspinors of the Dirac field in anti–de Sitter spacetime [18]. Thus we draw the conclusion that one can reproduce similar conjectures as in the Minkowski flat spacetime if one exploits the manifest covariance with respect to a suitable subgroup of $S(M)$. All our examples [10] indicate that for each local chart there exists a specific subgroup of $S(M)$ the spinor representation of which can be brought into covariant form by an adequate tetrad gauge fixing. Obviously, this approach requires one to solve new problems, from purely mathematical ones up to those regarding the physical interpretation.

In the de Sitter geometry, one has to look for an orbital analysis analogous to the Wigner theory of the induced representations of the Poincaré group [15,16]. This is necessary if we want to understand the meaning of the rest frames (of the massive particles) in de Sitter spacetime and to find the "booster" mechanisms changing the value of p or even giving rise to waves of arbitrary momentum from those with $\vec{p}=0$. We believe that this theory can be done starting with the orbital analysis in M^5 since this helps us to find SO(4,1)-covariant definitions for our basic operators on M. More precisely, for each momentum $q\in M^5_q$ we can write a five-dimensional momentum operator $P(q)$ of components $P^A(q)=\eta^{AC}q^B X_{(BC)}$ while a generalized five-dimensional Pauli-Lubanski operator in M has to be defined by $W_A=-\frac{1}{8}\varepsilon_{ABCDE}X^{(BC)}X^{(DE)}$ [19]. Thus it is clear that for the representative momentum $\hat{q}=(\omega,0,0,0,-\omega)$ of the orbit $q^2=0$, associated with the little group $E(3)\subset SO(4,1)$, we recover our operators $P(\hat{q})=(H,\vec{P},-H)$ and $W=\hat{q}^A W_A$ as given by Eqs. (26), (27), and (49), respectively. We hope that in this way one may construct generalized Wigner representations of the group $S(M)$ in the spaces of spinors depending on momentum.

Moreover, it is important to investigate the physical consequences of the transformation laws of the main observables of this theory and to point out the role and significance of the transformations (66) and (67). Of particular interest could be the study of the influence of the de Sitter gravitational field on the energy measurements since these are affected by the uncertainty relations $\Delta H\Delta P^i\geq\omega|\langle P^i\rangle|/2$ due to the commutation relations (31). Of course, for very small values of ω it is less probable that these produce observable effects in local measurements, the spacetime appearing then as a flat one.

Other problems that could appear in further investigations of the Dirac free field seem to be rather technical, e.g., the properties of commutators and Green functions, calculation of the action of more complicated conserved operators, evaluation of the inertial effects, etc. However, in our opinion, the next important step from the physical point of view would be to construct a similar theory for the free electromagnetic field, thus completing the framework one needs for developing perturbative QED in de Sitter spacetime.

The results obtained here show that, even though many particular features of the quantum theory in curved spacetimes depend on the choice of the local chart and tetrad gauge, there are covariance properties providing us with operators with invariant commutation relations. For this reason we hope that our approach based on external symmetries will be an argument for a general tetrad gauge covariant theory of quantum fields with spin in which the second quantization should be independent of the frames one uses.

APPENDIX A: SO(4,1) TRANSFORMATIONS AND ISOMETRIES

The spacetime M^5 is pseudo-Euclidean with a metric η^5 that is invariant under the coordinate transformations $Z^A\to{}^5\Lambda^{A\cdot}_{\cdot B}Z^B$ where ${}^5\Lambda\in SO(4,1)$. Each coordinate transformation gives rise to an isometry of M which can be calculated in the local chart $\{t_c,\vec{x}\}$ using Eqs. (20). We remind the reader that the basis generators ${}^5X_{AB}$ of the fundamental (linear) representation of SO(4,1), carried by M^5, have the matrix elements

$$({}^5X_{AB})^{C\cdot}_{\cdot D}=i(\delta^C_A\eta_{BD}-\delta^C_B\eta_{AD}). \tag{A1}$$

The transformations of $SO(3)\subset SO(4,1)$ are simple rotations of Z^i and x^i which transform alike since this symmetry is global. For the other transformations generated by H, P^i, and N^i the linear transformations in M^5 and the isometries are different. Those due to H,

$$e^{-i\xi^5_H H}:\quad \begin{aligned} &Z^0\to Z^0\cosh\alpha-Z^5\sinh\alpha,\\ &Z^5\to -Z^5\sinh\alpha+Z^0\cosh\alpha,\\ &Z^i\to Z^i,\end{aligned} \tag{A2}$$

where $\alpha=\omega\xi_H$, produce the dilatations $t_c\to t_c e^{\alpha}$ and $x^i\to x^i e^{\alpha}$, while the transformations

$$e^{-i\vec{\xi}_P\cdot{}^5\vec{P}}:\quad \begin{aligned} &Z^0\to Z^0+\omega\vec{\xi}_P\cdot\vec{Z}+\frac{1}{2}\omega^2\xi_P^2(Z^0+Z^5),\\ &Z^5\to Z^5-\omega\vec{\xi}_P\cdot\vec{Z}-\frac{1}{2}\omega^2\xi_P^2(Z^0+Z^5),\\ &Z^i\to Z^i+\omega\xi^i_P(Z^0+Z^5)\end{aligned} \tag{A3}$$

give the space translations $x^i\to x^i+\xi^i_P$ at fixed t_c. More interesting are the transformations

$$e^{-i\vec{\xi}_N\cdot{}^5\vec{N}}:\quad \begin{aligned} &Z^0\to Z^0-\vec{\xi}_N\cdot\vec{Z}+\frac{1}{2}\xi_N^2(Z^0-Z^5),\\ &Z^5\to Z^5-\vec{\xi}_N\cdot\vec{Z}+\frac{1}{2}\xi_N^2(Z^0-Z^5),\\ &Z^i\to Z^i-\xi^i_N(Z^0-Z^5),\end{aligned} \tag{A4}$$

which lead to the isometries

$$t_c\to\frac{t_c}{1-2\omega\vec{\xi}_N\cdot\vec{x}-\omega^2\xi_N^2(t_c^2-r^2)}, \tag{A5}$$

$$x^i \rightarrow \frac{x^i + \omega \xi_N^i (t_c^2 - r^2)}{1 - 2\omega \vec{\xi}_N \cdot \vec{x} - \omega^2 \xi_N^2 (t_c^2 - r^2)}. \tag{A6}$$

We denoted here $\xi_P^2 = (\vec{\xi}_P)^2$ and $\xi_N^2 = (\vec{\xi}_N)^2$.

APPENDIX B: SOME PROPERTIES OF HANKEL FUNCTIONS

According to the general properties of the Hankel functions [20], we deduce that those used here, $H_{\nu_\pm}^{(1,2)}(z)$, with $\nu_\pm = \frac{1}{2} \pm ik$ and $z \in \mathbb{R}$, are related among themselves through

$$[H_{\nu_\pm}^{(1,2)}(z)]^* = H_{\nu_\mp}^{(2,1)}(z), \tag{B1}$$

and satisfy the equations

$$\left(\frac{d}{dz} + \frac{\nu_\pm}{z}\right) H_{\nu_\pm}^{(1,2)}(z) = ie^{\pm \pi k} H_{\nu_\mp}^{(1,2)}(z) \tag{B2}$$

and the identities

$$e^{\pm \pi k} H_{\nu_\mp}^{(1)}(z) H_{\nu_\pm}^{(2)}(z) + e^{\mp \pi k} H_{\nu_\pm}^{(1)}(z) H_{\nu_\mp}^{(2)}(z) = \frac{4}{\pi z}. \tag{B3}$$

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