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Noncompact gauge fields on a lattice: SU(n) theories

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Recently, a noncompact regularization of the SU(2) gauge theory on a lattice has been investigated numerically. The results have been interpreted as an indication that the physical volume is larger than in the Wilson theory with the same number of sites. In its original formulation the noncompact regularization is directly applicable to U(n) theories for any n but to SU(n) theories for $n=2$ only. In this paper we extend it to SU(n) for any n and investigate some of its properties.

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I. INTRODUCTION

One of the present problems of lattice gauge theories is how to increase the physical volume where the numerical simulations are performed. The physical size of the lattice is indeed a major limitation in the study of hadronic structure functions [1] and light hadron spectroscopy [2], and in the evaluation of the ratio ϵ'/ϵ [3].

Recently, the size of the physical volume was investigated in a noncompact regularization and compared to that in the Wilson regularization for the gauge group SU(2) [4]. The comparison was made by determining in the two cases (with lattices of the same number of sites) the physical value of the lattice spacing in regions of the scaling window with the same physical properties. The authors concluded that the noncompact regularization provides a larger volume. It therefore appears interesting to repeat the simulation for the physically relevant SU(3) theory, but in its original formulation this regularization is directly applicable to U(n) theories for any n but to SU(n) theories for $n=2$ only. It is the purpose of the present paper to extend it to SU(n) for any n and to investigate some of its properties.

As is well known, a formal discretization of gauge theories breaks gauge invariance. To avoid this inconvenience Wilson assumed [5] as dynamical variables elements of the gauge group instead of the gauge fields that exist in the group algebra. In this way one gets a theory with an exact symmetry which has the desired formal continuum limit. This theory is said to be compact because the dynamical variables are compact.

The success of Wilson's regularization is by now celebrated in textbooks. But one might wonder whether its exact lattice symmetry can also be realized without compactifying the variables, and if this might have some advantages with respect to specific issues, reducing the artifacts of the lattice. In addition to the possibility of having larger volumes, the importance of noncompact gauge fields, especially in their coupling with matter fields, has been advocated in the investigation of a possible fixed point of QED at finite coupling [6]. Moreover, perturbative calculations should be easier since one does not have to expand the link variables of the Wilson theory in terms of the gauge fields. Perturbative calculations are at least necessary to make contact with the continuum formulation, but other applications like the study of renormalons should also be mentioned [7]. Finally, in numerical simulations one might expect a faster approach to the scaling, the more so the more important is the summation of the tadpoles [8] generated by the expansion of the link variables.

If one defines the covariant derivative in close analogy to the continuum as an ordinary discrete derivative plus the appropriate element of the algebra of the group, the lattice symmetry is broken, but it can be maintained by introducing compensating auxiliary fields which decouple in the continuum limit [9,10]. Such a regularization has been studied exhaustively in the case of SU(2). Specifically, the renormalization group parameter has been evaluated, and the perturbative properties have been shown to agree with Lüscher's calculation in Wilson regularization [11,12]. Moreover, Monte Carlo simulations [13,4] gave results compatible with Wilson's theory; in particular, they reproduced the correct value of the string tension, thus proving confinement with such a regularization [13]. What makes the issue worth further study is the conclusion by the authors of [4] that the physical volume is larger than in the Wilson theory with the same number of sites. This is what we expect heuristically for a regularization closer to the continuum.

In the original formulation of this noncompact regularization, with the exception of the case $n=2$, invariance with respect to SU(n) implies invariance with respect to U(n). The aim of this paper is to construct a potential that breaks the U(n) invariance down to SU(n) and to investigate some of its properties. For simplicity, explicit formulas will be given for $n=3$, but the generalization is obvious. The coupling to matter fields is also obvious and will therefore not be discussed.

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In Sec. II we report, for the convenience of the reader, the regularization for U(n). In Sec. III we show how to construct a SU(n) invariant theory. In Secs. IV and V we derive the Ward identities and the formulation in a background gauge, which might be useful in perturbative calculations. In the Appendix we report the explicit expression for the U(3) breaking potential.

II. THE NONCOMPACT REGULARIZATION FOR U(3)

We first consider the regularization of U(n) gauge theories. For $n=1$ we get a truly noncompact QED, namely, noncompact also in the coupling with the matter fields.

We want to construct a covariant derivative $\mathcal{D}_\mu$ which transforms according to

$$\mathcal{D}'_\mu(x)=g(x)\mathcal{D}_\mu(x)g^\dagger(x+\mu), \tag{1}$$

when $g(x)$ is an element of U(n). A simple discretization of the continuum would give

$$\mathcal{D}_\mu=\nabla_\mu+i(\chi_\mu \mathbb{1}+A^a_\mu T_a), \tag{2}$$

where ∇_μ is the ordinary right discrete derivative, and χ_μ and A^a_μ are the Abelian and non-Abelian gauge fields, respectively. We adopt for the generators of SU(3) in the fundamental representation the normalizations[1]

$$\{T_a,T_b\}=\frac{4}{3}\delta_{ab}\mathbb{1}+2d^c_{ab}T_c, \quad [T_a,T_b]=2if^c_{ab}T_c. \tag{3}$$

As is well known, with such a definition of the covariant derivative it is impossible to satisfy the transformation rule of Eq. (1). The way out that we reconsider here is based on the use of auxiliary compensating fields. It turns out that the lattice covariant derivative transforms in the right way if it acts on a field ψ in the fundamental representation according to

$$(\mathcal{D}_\mu\psi)(x)=D_\mu(x)\psi(x+\mu)-\frac{1}{a}\psi(x), \tag{4}$$

where D_μ has the following form:

$$D_\mu(x)=\left[\frac{1}{a}-\sigma_\mu(x)+i\chi_\mu(x)\right]\mathbb{1}+[iA^a_\mu-\alpha^a_\mu(x)]T_a. \tag{5}$$

In the above equation a is the lattice spacing, and σ_μ and α_μ are the additional fields necessary to enforce the lattice gauge invariance. With a little abuse of language we will also call D_μ a covariant derivative. The action of U(3) on the fields, for $g(x)\simeq\mathbb{1}-iT_a\theta^a(x)-i\mathbb{1}\theta^0(x)$, is

$$\begin{aligned}(A^a_\mu(x))'=&A^a_\mu(x)+\Delta_\mu\theta^a(x)+2f^a_{bc}\theta^b(x)A^c_\mu(x)\\&-a\sigma_\mu(x)\Delta_\mu\theta^a(x)-af^a_{bc}A^b_\mu(x)\Delta_\mu\theta^c(x)\\&-ad^a_{bc}\alpha^b_\mu(x)\Delta_\mu\theta^c(x)-a\alpha^a_\mu(x)\Delta_\mu\theta^0(x),\end{aligned}$$

$$\begin{aligned}(\alpha^a_\mu(x))'=&\alpha^a_\mu(x)+2f^a_{bc}\theta^b(x)\alpha^c_\mu(x)\\&-af^a_{bc}\alpha^b_\mu(x)\Delta_\mu\theta^c(x)+a\chi_\mu(x)\Delta_\mu\theta^a(x)\\&+ad^a_{bc}A^b_\mu(x)\Delta_\mu\theta^c(x)+aA^a_\mu(x)\Delta_\mu\theta^0(x),\end{aligned}$$

$$(6)$$

$$\begin{aligned}(\chi_\mu(x))'=&\chi_\mu(x)+\Delta_\mu\theta^0(x)-\frac{2}{3}a\alpha^a_\mu(x)\Delta_\mu\theta^a(x)\\&-a\sigma_\mu(x)\Delta_\mu\theta^0(x),\end{aligned}$$

$$\begin{aligned}(\sigma_\mu(x))'=&\sigma_\mu(x)+\frac{2}{3}aA^a_\mu(x)\Delta_\mu\theta^a(x)\\&+a\chi_\mu(x)\Delta_\mu\theta^0(x).\end{aligned}$$

Since all the fields are mixed with one another by the gauge transformations, we cannot say at this point which are the physical fields. They are selected by the action as we will see by studying the Ward identities.

A lattice action invariant under the above transformations is

$$\mathcal{L}_{YM}(x)=\frac{1}{4}\beta\,\mathrm{Tr}\,F^+_{\mu\nu}F_{\mu\nu}, \tag{7}$$

where $F_{\mu\nu}$ is the stress tensor

$$F_{\mu\nu}(x)=D_\mu(x)D_\nu(x+\mu)-D_\nu(x)D_\mu(x+\nu). \tag{8}$$

We notice that in such a formulation the measure in the partition function is flat.

In the formal continuum limit the field σ_μ becomes invariant and decouples together with α_μ, so that these seem to be the auxiliary fields. But the situation can be different at the quantum level. To control the decoupling of the redundant fields in the presence of quantum effects we use the fact that in a noncompact regularization, as well as $\mathcal{L}_{YM}$, there are other local invariants, which can be used to construct an appropriate potential and to give divergent masses to the fields that must stay decoupled. One such potential is

[1] These normalizations are slightly different from those used in [10].

$$\mathcal{L}_1=\beta_1\sum_\mu \operatorname{Tr}\left[D_\mu^\dagger(x)D_\mu(x)-\frac{\mathbb{1}}{a^2}\right]^2$$
$$=\beta_1\sum_\mu \Bigg\{\frac{12}{a^2}\sigma_\mu^2(x)+\frac{8}{a^2}\alpha_\mu^a(x)\alpha_\mu^a(x)-\frac{12}{a}\sigma_\mu(x)[\sigma_\mu^2(x)+\chi_\mu^2(x)]-\frac{8}{a}[3\sigma_\mu(x)\alpha_\mu^a(x)\alpha_\mu^a(x)+2\sigma_\mu(x)A_\mu^a(x)A_\mu^a(x)$$
$$+2\chi_\mu(x)A_\mu^a(x)\alpha_\mu^a(x)]+3[\sigma_\mu^2(x)+\chi_\mu^2(x)]^2+\frac{4}{3}[A_\mu^2(x)+\alpha_\mu^2(x)]^2+4A_\mu^a(x)A_\mu^a(x)[\sigma_\mu^2(x)+3\chi_\mu^2(x)]$$
$$+4\alpha_\mu^a(x)\alpha_\mu^a(x)[3\sigma_\mu^2(x)+\chi_\mu^2(x)]+16\sigma_\mu(x)\chi_\mu(x)A_\mu^a(x)\alpha_\mu^a(x)+8d_{bc}^a\bigg[[A_\mu^a(x)A_\mu^b(x)+\alpha_\mu^a(x)\alpha_\mu^b(x)]\bigg(\sigma_\mu(x)\alpha_\mu^c(x)$$
$$+\chi_\mu(x)A_\mu^c(x)-\frac{1}{a}\alpha_\mu^c(x)\bigg)\bigg]+8f_{ab}^hf_{cd}^hA_\mu^a(x)\alpha_\mu^b(x)A_\mu^c(x)\alpha_\mu^d(x)+2d_{ab}^hd_{cd}^h[A_\mu^a(x)A_\mu^b(x)A_\mu^c(x)A_\mu^d(x)$$
$$+\alpha_\mu^a(x)\alpha_\mu^b(x)\alpha_\mu^c(x)\alpha_\mu^d(x)+2A_\mu^a(x)A_\mu^b(x)\alpha_\mu^c(x)\alpha_\mu^d(x)]+8d_{ab}^hf_{cd}^hA_\mu^a(x)\alpha_\mu^b(x)[A_\mu^c(x)A_\mu^d(x)+\alpha_\mu^c(x)\alpha_\mu^d(x)]\Bigg\}. \quad (9)$$

We see that $\mathcal{L}_1$ provides the desired divergent masses to the auxiliary fields in the trivial vacuum. A more general analysis of the mass spectrum will be given in Sec. IV. There are other invariant terms, which can be used, for instance, to make the propagator of some of the auxiliary fields strictly local on the lattice [11], but we will ignore them for simplicity.

The effect of $\mathcal{L}_1$ can be well understood by adopting a definition of the covariant derivative where the Abelian fields are, in a polar representation,

$$D_\mu(x)=\hat{D}_\mu(x)\exp i\phi_\mu(x), \quad (10)$$

where

$$\hat{D}_\mu=\rho_\mu\mathbb{1}+[i(A')_\mu^a-(\alpha')_\mu^a]T_a. \quad (11)$$

Because of $\mathcal{L}_1$, the ρ field acquires a nonvanishing expectation value $\langle\rho_\mu\rangle=1/a$. The U(3) symmetry is "spontaneously" broken, and the components of ϕ_μ are the Goldston bosons.[2] As we will see by studying the Ward identities, the physical fields are ϕ_μ and A'_μ.

It is worth noticing that in the absence of spontaneous symmetry breaking there is not even a discrete derivative, the term $1/a$ being absent in the definition of D_μ. The present definition of gauge theories on a lattice can then be regarded as a matrix model where the space-time dynamics is generated by a spontaneous breaking of the gauge symmetry.

III. THE NONCOMPACT REGULARIZATION FOR SU(n)

A derivative covariant with respect to SU(n) transformations only must in general contain all the fields of the U(n) theory, the only difference being that both Abelian fields σ_μ and χ_μ become auxiliary. So we cannot restrict ourselves to the SU(n) symmetry by changing the covariant derivative, and at the same time the potential $\mathcal{L}_1$ does not generate a mass for the χ field. Moreover, as will be confirmed in the next section by the Ward identities, no U(n) invariant potential can generate a mass for both Abelian fields. We must therefore explicitly break the U(3) symmetry in order to give the would-be Goldstone bosons a mass, actually a divergent mass.

The case $n=2$ is exceptional, because for SU(2) transformations, namely, for $\theta_0=0$, Eqs. (6) do not mix the multiplet A_μ,σ_μ with the multiplet α_μ,χ_μ. Therefore we can break U(2) by omitting the latter fields to get a SU(2) invariant theory. This case has already been exhaustively studied [10–13].

There are two terms (whose expression will be spelled out in the Appendix) that break the U(3) invariance of the action explicitly:

$$\mathcal{L}_2=\beta_2\frac{1}{a}\sum_\mu[\det D_\mu(x)+\det D_\mu^\dagger(x)], \quad (12)$$

$$\mathcal{L}'_2=\beta'_2\frac{i}{a}\sum_\mu[\det D_\mu(x)-\det D_\mu^\dagger(x)]. \quad (13)$$

But we can always get rid of one of them by the global transformation

$$D_\mu=D'_\mu\exp i\alpha_\mu. \quad (14)$$

For instance, we can get rid of $\mathcal{L}'_2$ by setting in the above equation $\alpha=(1/3)\arctan(\beta_2/\beta'_2)$. We assume this to be the case.

We now determine the minima of the action at constant fields in the presence of $\mathcal{L}_2$. We assume that the color sym-

[2] Needless to say, the U(3) symmetry remains exact. While for $\langle\rho_\mu\rangle=0$ it is realized linearly, for $\langle\rho_\mu\rangle\neq0$ it is realized nonlinearly.

metry is not spontaneously broken. As a consequence the colored fields cannot develop a nonvanishing expectation value, neither can they mix with the auxiliary Abelian fields. By adopting the Abelian polar representation of Eq. (10) we minimize $\mathcal{L}_2$ with respect to $\overline{\phi_\mu}$ at fixed $\overline{\rho_\mu}$, and then minimize the resulting action with respect to ρ_μ.

By noticing that

$$\overline{\mathcal{L}_2}=\beta_2\frac{2}{a}\overline{\rho_\mu^3}\cos(3\overline{\phi_\mu}) \tag{15}$$

we obtain the stationarity condition

$$\sin 3\overline{\phi_\mu}=0. \tag{16}$$

Assuming $\beta_2<0$, the minimum of $\mathcal{L}_2$ occurs at $\overline{\phi_\mu}=0,2\pi/3,4\pi/3$, namely, the covariant derivative at the minimum belongs to the center of SU(3).[3]

Next we require, as a normalization condition, that the total action have one and only one minimum at $\overline{\rho_\mu}=1$. To achieve this result we find it necessary to add another potential term:

$$\begin{aligned}\mathcal{L}_3&=\beta_3\frac{1}{a^2}\sum_\mu \mathrm{Tr}\left[D_\mu^\dagger(x)D_\mu(x)-\frac{\mathbb{1}}{a^2}\right]\\&=\beta_3\frac{1}{a^2}\sum_\mu\left[-\frac{6}{a}\sigma_\mu(x)+3\sigma_\mu^2(x)+3\chi_\mu^2(x)\right.\\&\quad\left.+2\alpha_\mu^a(x)\alpha_\mu^a(x)+2A_\mu^a(x)A_\mu^a(x)\right].\end{aligned} \tag{17}$$

This term seems to give a mass also to all the colored fields, but it has already been shown that this is not the case for SU(2), and the proof will be generalized in the next section.

Taking into account that at the minimum

$$\overline{\mathcal{L}_2}=-2\frac{|\beta_2|}{a}\overline{\rho_\mu^3}, \tag{18}$$

we then have, omitting some constant terms,

$$\overline{\mathcal{L}}=\sum_\mu\left\{3\beta_1\left(\overline{\rho_\mu^2}-\frac{1}{a^2}\right)^2-2\frac{|\beta_2|}{a}\overline{\rho_\mu^3}\right\}+\frac{3\beta_3}{a^2}\overline{\rho_\mu^2}. \tag{19}$$

This Lagrangian density is stationary for

$$\overline{\rho_\mu^{(0)}}=0,\quad \overline{\rho_\mu^{(\pm)}}=\frac{1}{4a\beta_1}\left[|\beta_2|\pm\sqrt{\beta_2^2+8\beta_1(2\beta_1-\beta_3)}\right]. \tag{20}$$

Since the potential diverges for divergent ρ, $\overline{\rho^{(+)}}$ is certainly a minimum. If $\overline{\rho^{(-)}}<0$, $\rho=0$ is a maximum; if $\overline{\rho^{(-)}}>0$, $\rho=0$ is a minimum which must be discarded; therefore in both cases we must require $\overline{\rho^{(+)}}=1/a$, which gives $|\beta_2|=\beta_3$, $4\beta_1>\beta_3$.

If $\overline{\rho^{(-)}}<0$ we must further impose that $\overline{\mathcal{L}}(\rho=\overline{\rho^{(+)}})<\overline{\mathcal{L}}(\rho=0)$. This strengthens the above inequality to $3\beta_1>\beta_3$. In conclusion we have

$$|\beta_2|=\beta_3,\quad 3\beta_1>\beta_3. \tag{21}$$

The masses of the auxiliary fields turn out to be

$$m_\rho^2=\frac{6}{a^2}(4\beta_1-\beta_3),\quad m_\phi^2=\frac{18}{a^2}\beta_3,\quad m_\alpha^2=\frac{8}{a^2}(2\beta_1+\beta_3). \tag{22}$$

In conclusion the full classical Lagrangian is

$$\mathcal{L}_G=\mathcal{L}_{YM}+\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3 \tag{23}$$

and the partition function is

$$Z=\int \mathcal{D}\sigma_\mu(x)\mathcal{D}\chi_\mu(x)\mathcal{D}\alpha_\mu(x)\mathcal{D}A_\mu(x)\times\exp\left\{-a^4\sum_x \mathcal{L}_G(x)\right\}. \tag{24}$$

We emphasize that the integration measure is flat.

IV. WARD IDENTITIES

To determine the mass spectrum and identify the physical fields we investigate the Ward identities.

We start with U(3) invariance and we assume that the color symmetry is not spontaneously broken. Therefore the effective action Γ must be stationary,

$$\frac{\partial\Gamma}{\partial A_\mu^a(x)}=\frac{\partial\Gamma}{\partial\alpha_\mu^a(x)}=\frac{\partial\Gamma}{\partial\chi_\mu(x)}=\frac{\partial\Gamma}{\partial\sigma_\mu(x)}=0 \tag{25}$$

for

$$A_\mu^a(x)=\alpha_\mu^a(x)=0,\quad \chi_\mu=\overline{\chi_\mu},\quad \sigma_\mu(x)=\overline{\sigma_\mu}. \tag{26}$$

Because of gauge invariance we have

$$\begin{aligned}\delta\Gamma&=\sum_{\mu,x}\left[\delta\chi_\mu(x)\frac{\partial\Gamma}{\partial\chi_\mu(x)}+\delta\sigma_\mu(x)\frac{\partial\Gamma}{\partial\sigma_\mu(x)}\right]\\&\quad+\sum_{a,\mu,x}\left[\delta A_\mu^a(x)\frac{\partial\Gamma}{\partial A_\mu^a(x)}+\delta\alpha_\mu^a(x)\frac{\partial\Gamma}{\partial\alpha_\mu^a(x)}\right]=0.\end{aligned} \tag{27}$$

Introducing the explicit expressions for the variations and integrating by parts we obtain

[3] All the minima are therefore in one-to-one correspondence with those of the Wilson theory, and the difficulty raised in Ref. [11] in connection with this degeneracy can then be overcome as in the compact regularization.

$$\delta\Gamma=\sum_{\mu,x}\theta^a(x)\Bigg\{\frac{2}{3}a\Delta_\mu^{(-)}\left[\alpha_\mu^a(x)\frac{\partial\Gamma}{\partial\chi_\mu(x)}-A_\mu^a(x)\frac{\partial\Gamma}{\partial\sigma_\mu(x)}\right]-\Delta_\mu^{(-)}\frac{\partial\Gamma}{\partial A_\mu^a(x)}+2f_{bc}^aA_\mu^b(x)\frac{\partial\Gamma}{\partial A_\mu^c(x)}+a\Delta_\mu^{(-)}\Bigg[\sigma_\mu(x)\frac{\partial\Gamma}{\partial A_\mu^a(x)}$$
$$-f_{bc}^aA_\mu^b(x)\frac{\partial\Gamma}{\partial A_\mu^c(x)}+d_{bc}^a\alpha_\mu^b(x)\frac{\partial\Gamma}{\partial A_\mu^c(x)}\Bigg]+2f_{bc}^a\alpha_\mu^b(x)\frac{\partial\Gamma}{\partial\alpha_\mu^c(x)}+a\Delta_\mu^{(-)}\Bigg[-\chi_\mu(x)\frac{\partial\Gamma}{\partial\alpha_\mu^a(x)}-f_{bc}^a\alpha_\mu^b(x)\frac{\partial\Gamma}{\partial\alpha_\mu^c(x)}$$
$$-d_{bc}^aA_\mu^b(x)\frac{\partial\Gamma}{\partial\alpha_\mu^c(x)}\Bigg]\Bigg\}-\sum_{\mu,x}\theta^0(x)a\Delta_\mu^{(-)}\left\{(1-a\sigma_\mu(x))\frac{\partial\Gamma}{\partial\chi_\mu(x)}+a\chi_\mu(x)\frac{\partial\Gamma}{\partial\sigma_\mu(x)}-\alpha_\mu^a(x)\frac{\partial\Gamma}{\partial A_\mu^a(x)}+A_\mu^a(x)\frac{\partial\Gamma}{\partial\alpha_\mu^a(x)}\right\}$$
$$=0. \tag{28}$$

We first assume $\theta_a=0$. By taking the derivative with respect to χ_ν and to σ_ν we get at the minimum

$$(1-a\overline{\sigma}_\mu)\frac{\partial^2\Gamma}{\partial\chi_\nu(y)\partial\chi_\mu(x)}+a\overline{\chi}_\mu\frac{\partial^2\Gamma}{\partial\chi_\nu(y)\partial\sigma_\mu(x)}=0,$$
$$(1-a\overline{\sigma}_\mu)\frac{\partial^2\Gamma}{\partial\sigma_\nu(y)\partial\chi_\mu(x)}+a\overline{\chi}_\mu\frac{\partial^2\Gamma}{\partial\sigma_\nu(y)\partial\sigma_\mu(x)}=0. \tag{29}$$

Analogously, if we assume $\theta^0=0$ and take the derivatives with respect to A_μ,α_μ we get

$$(1-a\overline{\sigma}_\mu)\frac{\partial^2\Gamma}{\partial A_\nu(y)\partial A_\mu(x)}+a\overline{\chi}_\mu\frac{\partial^2\Gamma}{\partial A_\nu(y)\partial\alpha_\mu(x)}=0,$$
$$(1-a\overline{\sigma}_\mu)\frac{\partial^2\Gamma}{\partial\alpha_\nu(y)\partial A_\mu(x)}+a\overline{\chi}_\mu\frac{\partial^2\Gamma}{\partial\alpha_\nu(y)\partial\alpha_\mu(x)}=0. \tag{30}$$

These equations show that in general there is a combination of the fields χ_μ,σ_μ

$$\chi'_\mu(x)=\frac{1}{a}\{-s_\mu[1-a\sigma_\mu(x)]+ac_\mu\chi_\mu(x)\} \tag{31}$$

and a combination of the fields A_μ,α_μ

$$A'_\mu(x)=-s_\mu\alpha_\mu(x)+c_\mu A_\mu(x), \tag{32}$$

with

$$c_\mu=\frac{1-a\overline{\sigma}_\mu}{[(1-a\overline{\sigma}_\mu)^2+a^2\overline{\chi}_\mu^2]^{1/2}},\quad s_\mu=\frac{a\overline{\chi}_\mu}{[(1-a\overline{\sigma}_\mu)^2+a^2\overline{\chi}_\mu^2]^{1/2}}, \tag{33}$$

which are massless. These are the physical fields. The actual auxiliary fields are the orthogonal combinations

$$\sigma'_\mu(x)=\frac{1}{a}\{1-c_\mu[1-a\sigma_\mu(x)]-as_\mu\chi_\mu(x)\},$$
$$\alpha'_\mu(x)=c_\mu\alpha_\mu(x)+s_\mu A_\mu(x). \tag{34}$$

The rotation to the primed fields is obtained by multiplying D_μ by an element of the center of SU(3).

In SU(3) invariant theories we have only Eq. (30), so that a mass for both Abelian fields is no longer forbidden. In this case both Abelian fields are auxiliary.

V. THE BACKGROUND GAUGE AND THE BECCHI-ROUET-STORA SYMMETRY

Even though the very motivation of the lattice regularization is to perform numerical simulations, perturbative calculations are nevertheless needed, at least to determine the asymptotic scaling region through the comparison with continuum theory. Since the background gauge is particularly suitable for such a calculation [14], in this section we formulate the noncompact regularization in a background gauge.

A background field can be introduced in close analogy with the continuum (see, for example, [15] and references therein) by performing a shift of the gauge fields. We define a background covariant derivative, which depends solely on the background fields, and the quantum fluctuation with respect to these fields,

$$D_\mu(x)=D_{B,\mu}(x)+Q_\mu(x), \tag{35}$$

where

$$D_{B,\mu}(x)=\left[\frac{1}{a}-\sigma_{B,\mu}(x)+i\chi_{B,\mu}(x)\right]\mathbb{1}+[iA_{B,\mu}^a(x)-\alpha_{B,\mu}^a(x)]T_a,$$
$$Q_\mu(x)=[-\sigma_{Q,\mu}(x)+i\chi_{Q,\mu}(x)]\mathbb{1}+[iA_{Q,\mu}^a(x)-\alpha_{Q,\mu}^a(x)]T_a. \tag{36}$$

A gauge transformation of the covariant derivative D_μ,

$$D'_\mu(x)=[D_{B,\mu}(x)+Q_\mu(x)]'=g(x)[D_{B,\mu}(x)+Q_\mu(x)]g^\dagger(x+\mu), \tag{37}$$

can be interpreted, among others, in the two following ways.

Interpretation I

$$(D_{B,\mu}(x))'=D_{B,\mu}(x),$$

$$(Q_\mu(x))' = g(x)[D_{B,\mu}(x) + Q_\mu(x)]g^\dagger(x+\mu) - D_{B,\mu}(x). \tag{38}$$

Interpretation II

$$(D_{B,\mu}(x))' = g(x)D_{B,\mu}(x)g^\dagger(x+\mu),$$

$$(Q_\mu(x))' = g(x)Q_\mu(x)g^\dagger(x+\mu). \tag{39}$$

According to the first interpretation the background derivative is invariant, while following the second interpretation it transforms as the full covariant derivative. In the second case the quantum fluctuation undergoes a rotation like a matter field in the adjoint representation.

The presence of the background field enables us to introduce a gauge fixing term that breaks the symmetry with respect to the first interpretation, while preserving the symmetry according to the second one. The resulting effective action is a gauge invariant functional of the background field [15].

To define the gauge fixed theory we follow, for example, Ref. [17]. The fundamental fields of the quantum theory are

$$D_{B,\mu}, Q_\mu(x), c(x), \overline{c}(x), b(x) \tag{40}$$

where $c(x), \overline{c}(x)$ are scalar Grassmann fields with, respectively, positive and negative unit ghost number and canonical dimension equal to 1 while $b(x)$ is a real scalar c-number field with vanishing ghost number and canonical dimension equal to 2; the gauge quantum and background fields obviously have vanishing ghost number. The quantum Lagrangian is renormalizable by power counting, Becchi-Rouet-Stora (BRS) invariant, and with zero ghost number [16].

We now determine the equations for a BRS transformation of the various fields. It is worth noticing that the BRS symmetry corresponds to the gauge symmetry broken by the gauge fixing term; therefore we determine the BRS equations starting from those for an infinitesimal gauge transformation according to the first interpretation which are, for $g(x) \simeq \mathbb{1} - i\theta^a(x)T_a$,

$$\delta D_{B,\mu}(x) = 0,$$

$$\delta Q_\mu(x) = -i\theta^a(x)T_a D_\mu(x) + iD_\mu(x)\theta^a(x+\mu)T_a. \tag{41}$$

A BRS transformation is obtained by means of the s operator, whose action on the various fields is specified by the following equations:

$$s\, D_{B,\mu} = 0,$$

$$s\, Q_\mu(x) = -ic(x)Q_\mu(x) + iQ_\mu c(x+\mu) - ic(x)D_{B,\mu}(x) + iD_{B,\mu}c(x+\mu),$$

$$s\, c(x) = -iK(x), \tag{42}$$

$$s\, \overline{c}(x) = b(x),$$

$$s\, b(x) = 0.$$

$K(x)$ is determined so as to obtain the nilpotency of the s operator: namely,

$$K(x) = c(x)c(x). \tag{43}$$

The quantum theory is defined by the path integral

$$Z[D_{B,\mu}(x)] = \int \mathcal{D}Q_\mu(x)\mathcal{D}c(x)\mathcal{D}\overline{c}(x)\mathcal{D}b(x) \times \exp\left\{-\sum_{x,\mu}[\mathcal{L}_G(x) + \mathcal{L}_{BRS}(x)]\right\} \tag{44}$$

where

$$\begin{aligned}\mathcal{L}_{BRS}(x) &= -\lambda\beta\,\mathrm{Tr}\, s\{\overline{c}(x)[\mathcal{G}(x) - b(x)]\}\\ &= -\lambda\beta\,\mathrm{Tr}\{b(x)\mathcal{G}(x) - b(x)b(x)\}\\ &\quad + \lambda\beta\,\mathrm{Tr}\{\overline{c}(x)s\mathcal{G}(x)\}\\ &= \mathcal{L}_{gf}(x) + \mathcal{L}_{ghost}(x).\end{aligned} \tag{45}$$

The quantity $\mathcal{G}(x) = i\mathcal{G}^0(x)\mathbb{1} + \mathcal{G}^a(x)T_a$ is the gauge fixing constraint and λ is a real positive parameter. We can get rid of the $b(x)$ field with a Gaussian integration, so obtaining

$$\mathcal{L}_{gf}(x) = -\frac{\lambda\beta}{2}\mathcal{G}^a(x)\mathcal{G}^a(x). \tag{46}$$

A gauge fixing term that preserves the exact gauge symmetry for transformations of the background field is

$$\mathcal{G}(x) = \frac{i}{2}\left\{\sum_\mu [D^\dagger_{B,\mu}(x-\mu)Q_\mu(x-\mu) - Q_\mu(x)D^\dagger_{B,\mu}(x)] - \mathrm{H.c.}\right\}. \tag{47}$$

Following the second interpretation $\mathcal{G}(x)$ varies according to

$$(\mathcal{G}(x))' = g(x)\mathcal{G}(x)g^\dagger(x). \tag{48}$$

As a consequence the gauge fixing term is invariant under gauge transformations of the background field and the effective action is a gauge invariant functional of the latter.

VI. SUMMARY

We have reconsidered a lattice regularization of gauge theories which makes use of auxiliary fields in order to enforce exact gauge invariance with noncompact fields. The form of the covariant derivative, for $n>2$, is the same for U(n) and SU(n) theories. This means that the physical Abelian field of the U(n) theory must become an additional auxiliary field in the SU(n) theory. This can be guaranteed at the quantum level by explicitly breaking the U(n) symmetry in such a way as to generate a divergent mass for this field. The terms of the Lagrangian that realize this condition have been exhibited and their effect investigated. The regularization can now be used on essentially the same footing for every n.

We have also investigated the Ward identities of the effective action, confirming that the mass spectrum has the desired properties. Finally, we have formulated the theory in the background gauge and written the BRS identities, showing that a perturbative treatment can be done in close analogy with the continuum, avoiding the cumbersome expansion of the link variables.

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APPENDIX

In this Appendix we report the explicit expression for $\mathcal{L}_2$:

$$\begin{aligned}\mathcal{L}_2 &= \beta_2 \frac{1}{a} \sum_\mu \left[-\det D_\mu(x) - \det D^\dagger_\mu(x) + \frac{2}{a^3} \right] \\ &= \beta_2 \frac{1}{a} \sum_\mu \Bigg\{ \frac{6}{a^2}\sigma_\mu(x) - \frac{6}{a}[\sigma^2_\mu(x) - \chi^2_\mu(x)] - \frac{2}{a}[A^a_\mu(x)A^a_\mu(x) - \alpha^a_\mu(x)\alpha^a_\mu(x)] + 2\sigma^3_\mu(x) - 6\sigma_\mu(x)\chi^2_\mu(x) \\ &\quad + 4\chi_\mu(x)A^a_\mu(x)\alpha^a_\mu(x) + 2\sigma_\mu(x)[A^a_\mu(x)A^a_\mu(x) - \alpha^a_\mu(x)\alpha^a_\mu(x)] - 4\sum_{a=1}^{8} d^8_{aa}\{2A^8_\mu(x)A^a_\mu(x)\alpha^a_\mu(x) \\ &\quad + \alpha^8_\mu(x)[A^a_\mu(x)A^a_\mu(x) - \alpha^a_\mu(x)\alpha^a_\mu(x)]\} - 4\sum_{a=4}^{7} d^3_{aa}\{2A^3_\mu(x)A^a_\mu(x)\alpha^a_\mu(x) + \alpha^3_\mu(x)[A^a_\mu(x)A^a_\mu(x) - \alpha^a_\mu(x)\alpha^a_\mu(x)]\} \\ &\quad - 8d^1_{57}\{A^1_\mu(x)[A^5_\mu(x)\alpha^7_\mu(x) + A^7_\mu(x)\alpha^5_\mu(x)] + \alpha^1_\mu(x)[A^5_\mu(x)A^7_\mu(x) + \alpha^5_\mu(x)\alpha^7_\mu(x)]\} - 8d^1_{46}\{A^1_\mu(x)[A^4_\mu(x)\alpha^6_\mu(x) \\ &\quad + A^6_\mu(x)\alpha^4_\mu(x)] + \alpha^1_\mu(x)[A^4_\mu(x)A^6_\mu(x) + \alpha^4_\mu(x)\alpha^6_\mu(x)]\} - 8d^2_{47}\{A^2_\mu(x)[A^4_\mu(x)\alpha^7_\mu(x) + A^4_\mu(x)\alpha^5_\mu(x)] + \alpha^2_\mu(x) \\ &\quad \times [A^4_\mu(x)A^7_\mu(x) + \alpha^4_\mu(x)\alpha^7_\mu(x)]\} - 8d^2_{56}\{A^2_\mu(x)[A^5_\mu(x)\alpha^6_\mu(x) + A^6_\mu(x)\alpha^5_\mu(x)] + \alpha^2_\mu(x)[A^5_\mu(x)A^6_\mu(x) \\ &\quad + \alpha^5_\mu(x)\alpha^6_\mu(x)]\} \Bigg\}. \end{aligned} \tag{A1}$$

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