Fermions on tori in uniform Abelian fields

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We study Fermi fields defined on tori in the presence of gauge backgrounds carrying a nontrivial topology. We show that $2k$-dimensional field space can alternatively be described by fields over a k-dimensional space. This dual description is particularly natural when the background is uniform and Abelian. The reduction in number of dimensions carries over to the lattice. The lattice ultraviolet regularization induces an infrared regularization of the lower dimensional representations. We focus on $k=1,2$.

I. INTRODUCTION

Dirac fields defined over even-dimensional tori can, for certain twists in the gauge background, be equivalently represented by fields defined over infinite flat spaces of half the number of dimensions. The two descriptions are related by a change of basis. The reduction in dimensionality comes from using Fourier space for half of the space-time variables. The integral Fourier momenta are used to string the remaining segments of continuous directions into infinite lines. This works for sections of nontrivial bundles. Although the two space-times are distinct in the topological sense, the fermionic index stays the same. For simple backgrounds, consisting of a constant Abelian field strength, the field equations are local in either dimensionality. Extra interactions would be required to single out one of the alternative descriptions as more natural.

The setup we are looking at has a long history. It has been analyzed before in various contexts [1–7]. Early studies in lattice field theory were carried out in [8]. Probably the main objection to developing a lattice version was the absence of an exact fermionic "viewpoint" of background topology. Fermions react to topological features in the background in ways strongly correlated to their chiral properties and without a lattice version of chirality only an approximate analysis was possible. This has changed with the domain wall formulation of lattice fermions [9] and, especially, with the overlap formulation [10]. Among recent studies on the response of fermionic fields to topology in the overlap regularization we mention [10–13].

In this paper we focus mainly on the aspect of the alternate description in terms of reduced dimensions. In the continuum, for our special backgrounds, locality is preserved and the equations of motion stay differential in either

description.[1] Our new contribution in this paper is to provide lattice analogues to the reduced dimension description. For example, we learn from this that the spectrum of lattice fermions moving on an $L_1 \times L_2$ toroidal lattice in a uniform Abelian gauge background depends only on the area $L_1 L_2$. There is no distinction between the arbitrary $L_1 \times L_2$ case and that on a degenerate lattice of dimensions $L_1 L_2 \times 1$.

In Secs. II and III we introduce the continuum version of the representations in two and four dimensions, respectively. The origin of the dimensional reduction is traced back to a natural Heisenberg algebra defined in this system. This algebra plays a central role in relating the developments of Refs. [16,17] to the general area of non-commutative field theories, as reviewed, for example, in [15]. In Sec. IV we focus on the Dirac equation in constant Abelian backgrounds, still in the continuum. Sections V and VI deal with the lattice version in two and four dimensions, respectively. We collect some numerical and asymptotic results in Sec. VII.

II. TWO DIMENSIONS—CONTINUUM

We consider a two-dimensional torus of size $l_1 \times l_2$. This can be viewed as the plane $\mathbf{R}^2$ modulo the lattice generated by e_1, e_2, chosen for simplicity as $e_1 = (l_1, 0)$ and $e_2 = (0, l_2)$. The fields $\psi(x) \equiv \psi(x_1, x_2)$ are complex fields satisfying certain periodicity properties. Our fields are coupled to a gauge field and are thus sections of a $U(1)$ bundle on the 2-torus. The bundle can be twisted, the twist being characterized by the first Chern class c_1. The first Chern number gives the topological charge: it measures the flux through the two torus in units of 2π (the charge e is set to 1).

Periodicity of fermion fields has to be taken modulo gauge transformations:

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[1]The description in terms of lower dimension fields is not restricted to Dirac fields: It would also be useful for analyzing the stability of the gauge backgrounds against small perturbations in the gauge fields [7,14].

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$$\psi(x+e_1)=\Omega_1(x)\,\psi(x) \qquad (1)$$

$$\psi(x+e_2)=\Omega_2(x)\,\psi(x). \qquad (2)$$

The $U(1)$ fields $\Omega_i(x)$ are the transition functions made consistent by

$$\Omega_1(x+e_2)\,\Omega_2(x)=\Omega_2(x+e_1)\,\Omega_1(x). \qquad (3)$$

By gauge transformations the transition functions are brought to the form

$$\Omega_1(x)=\exp\{\iota\pi qx_2/l_2\} \qquad (4)$$

$$\Omega_2(x)=\exp\{-\iota\pi qx_1/l_1\}. \qquad (5)$$

The consistency condition (3) forces q to take only integer values. q is the first Chern number (often referred to as a winding number in this setup).

Fields satisfying the boundary conditions (1), (2) make up a Hilbert space $\mathcal{H}_q$ with scalar product

$$\langle\phi|\psi\rangle=\int_{T^2}dx\,\phi^*(x)\psi(x). \qquad (6)$$

The integration is over the torus and because of the integrand's periodicity it can be performed over any fundamental cell.

$\mathcal{H}_q$ is left invariant by a ring of linear transformations generated by two unitary transformations $\mathbf{U}_i$,

$$\mathbf{U}_i\psi(x)=\exp\{2\pi\iota x_i/l_i\}\psi(x). \qquad (7)$$

Ordinary translations map out of $\mathcal{H}_q$ because the transition functions depend on x. This can be compensated by an appropriate gauge transformation leading to modified operations of translation by a_i:

$$\mathcal{T}_1(a_1)\psi(x)=\exp\{\iota\pi qa_1x_2/\mathcal{A}\}\psi[x+(a_1,0)] \qquad (8)$$

$$\mathcal{T}_2(a_2)\psi(x)=\exp\{-\iota\pi qa_2x_1/\mathcal{A}\}\psi[x+(0,a_2)] \qquad (9)$$

where $\mathcal{A}=l_1l_2$ is the area of the torus. The new translations along the two axes do not commute:

$$\mathcal{T}_1(a_1)\mathcal{T}_2(a_2)=\exp\{-2\pi\iota qa_1a_2/\mathcal{A}\}\mathcal{T}_2(a_2)\mathcal{T}_1(a_1). \qquad (10)$$

Elements of the non-commutative group of translations do not commute with the discrete Abelian group of transformations generated by $\mathbf{U}_i$. There exists only a finite subgroup of unitary transformations, generated by $K^{(1),(2)}$, which commutes with translations:

$$K^{(1)}=\mathbf{U}_1\mathcal{T}_2(l_2/q) \qquad (11)$$

$$K^{(2)}=\mathbf{U}_2\mathcal{T}_1(-l_1/q). \qquad (12)$$

The boundary conditions imply $(K^{(\mu)})^q=\mathbf{I}$ and the operators $K^{(\mu)}$ obey

$$K^{(1)}K^{(2)}=\exp\{2\pi\iota/q\}K^{(2)}K^{(1)}. \qquad (13)$$

Any one of the $K^{(\mu)}$ generates a Z_q group under which $\mathcal{H}_q$ decomposes into q orthogonal subspaces:

$$\mathcal{H}_q=\oplus_{n=1}^q\mathcal{H}_{q,n}. \qquad (14)$$

Decomposing under $K^{(2)}$, fermion fields in $\mathcal{H}_{q,n}$ satisfy

$$K^{(2)}\psi(x)=\exp\left\{-2\pi\iota\frac{n}{q}\right\}\psi(x). \qquad (15)$$

$K^{(1)}$ maps these subspaces into each other, but the non-commutative translations leave the subspaces invariant.

The generators of translations along each axis are given by

$$D_i^{(0)}=\partial_i+\pi\iota q\,\epsilon_{ij}x_j/\mathcal{A} \qquad (16)$$

where ϵ_{ij} is the antisymmetric tensor with two indices ($\epsilon_{12}=1$). The generators coincide with ordinary covariant derivatives in a constant magnetic field:

$$A_i(x)=-\pi q\,\epsilon_{ij}x_j/\mathcal{A} \qquad (17)$$

$$F=\partial_1A_2-\partial_2A_1=\frac{2\pi q}{\mathcal{A}}. \qquad (18)$$

The non-commutative translation group is made out of parallel transporters in this field. q is given by the total flux. The anti-self-adjoint generators of parallel transport do not commute:

$$[D_1^{(0)},D_2^{(0)}]=-\iota F=\frac{-2\pi\iota q}{\mathcal{A}}. \qquad (19)$$

Up to a rescaling, we have a Heisenberg algebra associated with one degree of freedom and we are led to a quantum mechanical view of the space as a Hilbert space of a particle in one dimension. For this particle, the covariant derivatives play the role of position and momentum operators. For $q>0$, we take $\mathbf{P}\leftrightarrow-\iota D_2^{(0)}$ and $\mathbf{Q}\leftrightarrow-\iota D_1^{(0)}/F$ producing a traditional position space representation in terms of functions of a single real variable $h(z)$. The space $\mathcal{H}_q$ is identified with $\oplus_q L_2(\mathbf{R})$ with the $D_i^{(0)}$ action given by

$$D_1^{(0)}h_n(z)=iFz\,h_n(z) \qquad (20)$$

$$D_2^{(0)}h_n(z)=\frac{dh_n(z)}{dz}. \qquad (21)$$

The fields get mapped into the quantum mechanical *position space* basis as follows. Since the function

$$\varphi(x_1,x_2)=\exp\left\{-\iota\pi q\frac{x_1x_2}{\mathcal{A}}\right\}\psi(x_1,x_2) \qquad (22)$$

is periodic in the variable x_1 with period l_1 we can expand it in a discrete Fourier series:

$$\varphi(x_1,x_2)=\sum_k\exp\left\{2\pi\iota\frac{x_1}{l_1}k\right\}h_k(x_2). \qquad (23)$$

The periodicity requirement in x_2 imposes a constraint on $h_k(x_2)$:

$$h_{k+q}(x_2) = h_k(x_2 + l_2). \tag{24}$$

The constraint is solved by

$$h_k(x_2) = h_n(x_2 + kl_2/q) \tag{25}$$

with $n = k \bmod q$. The expression for the field becomes

$$\psi(x_1, x_2) = \exp\left\{ \iota \pi q \frac{x_1 x_2}{\mathcal{A}} \right\} \sum_{n=1}^{q} \sum_{s \in \mathbf{Z}} \exp\left\{ 2 \pi \iota \frac{x_1}{l_1}(n+sq) \right\}$$
$$\times h_n\left(x_2 + \frac{n+qs}{q} l_2 \right). \tag{26}$$

The *wave function* $h_n(z)$ transforms as required under $D_1^{(0)}$ and $K^{(2)}$ and the index n labels the subspace $\mathcal{H}_{q,n}$ to which the component belongs.

III. FOUR DIMENSIONS—CONTINUUM

We consider now fields defined over a four-dimensional torus of size $l_1 \times l_2 \times l_3 \times l_4$. If the gauge group is $U(N_c)$ the bundles are classified by the instanton number Q and by the first Chern numbers $c_{\mu\nu}$ associated with the individual planes. If the gauge group is $SU(N_c)$ we only have the instanton number, but if all the fields are insensitive to the center $Z(N_c)$ the true gauge group becomes $SU(N_c)/Z(N_c)$ and there is a remnant of the first Chern numbers, now defined modulo N_c, labeling the so-called twist sectors. Excluding the case $N_c = 2$ with zero twist and odd instanton number the topological character can be encoded in transition functions $\Omega_\mu(x)$ restricted to the $U(1)^{N_c-1}$ subgroup of $SU(N_c)$ [7,19]. The matter fields obey the boundary conditions:

$$\psi(x + e^{(\mu)}) = \Omega_\mu(x)\psi(x). \tag{27}$$

The (not necessarily orthogonal) 4-vectors $e^{(\mu)}$ have lengths l_μ and span a non-degenerate four-dimensional lattice Λ. The dual lattice is generated by $\widetilde{e}_\nu^{(\mu)}$. Denoting by $R_{\mu\nu} = e_\nu^{(\mu)}$ the components of the invertible matrix R, $\widetilde{e}_\nu^{(\mu)} = (R^{-1})_{\nu\mu}$.

In the Abelian description we consider matter fields in the fundamental representation with all N_c color components satisfying decoupled boundary conditions:

$$\psi^a(x + e^{(\mu)}) = \Omega_\mu^{aa}(x)\psi^a(x). \tag{28}$$

We again can pick transition matrices of the form

$$\Omega_\mu^{aa}(x) = \exp\{\iota \pi Q_{\mu\nu}^a \widetilde{e}_\rho^{(\nu)} x_\rho\}. \tag{29}$$

The integers $Q_{\mu\nu}^a$ are antisymmetric in μ, ν and can be assembled into the matrix Q^a. For $U(N_c)$ the topological invariants are

$$c_{\mu\nu} = \sum_a Q_{\mu\nu}^a \tag{30}$$

$$Q = \sum_a \text{Pf}(Q^a). \tag{31}$$

Pf stands for Pfaffian. For $SU(N_c)$, $c_{\mu\nu} = 0$.

We study the space of fields subject to the boundary conditions (28) and (29). We deal with one of the components only and drop the color index; the number Q and the matrix Q^a will become distinguishable only by context. We assume that $\text{Pf}(Q) \neq 0$.

We again analyze classes of unitary operators that act in this space, starting with operations of multiplication by periodic functions. The generators of this discrete infinite Abelian group are

$$U^{(\mu)} = \exp\{2\pi \iota \widetilde{e}_\nu^{(\mu)} x_\nu\}. \tag{32}$$

A translation by b, suitably modified to respect the boundary conditions, is given by

$$\mathcal{T}(b)\psi(x) = \exp\left\{ \frac{\iota}{2} F_{\mu\nu}^{(0)} b_\mu x_\nu \right\} \psi(x+b). \tag{33}$$

The antisymmetric matrix $F_{\mu\nu}^{(0)}$ is given by

$$F_{\mu\nu}^{(0)} = 2\pi Q_{\rho\tau} \widetilde{e}_\mu^{(\rho)} \widetilde{e}_\nu^{(\tau)}. \tag{34}$$

Translations only commute up to a phase

$$\mathcal{T}(b)\mathcal{T}(d) = \exp\{\iota F_{\mu\nu}^{(0)} b_\nu d_\mu\} \mathcal{T}(d)\mathcal{T}(b). \tag{35}$$

As in the two-dimensional case, we modify the $U^{(\mu)}$'s to achieve commutativity with translations. The generators of the resulting non-Abelian group are given by (Q is invertible since by assumption its determinant does not vanish)

$$K^{(\mu)} = U^{(\mu)} \mathcal{T}(-(Q^{-1})_{\mu\nu} e^{(\nu)}). \tag{36}$$

They satisfy

$$K^{(\mu)} K^{(\nu)} = \exp\{-2\pi \iota (Q^{-1})_{\mu\nu}\} K^{(\nu)} K^{(\mu)}. \tag{37}$$

The representations of this group, directly related to the Heisenberg group, have been studied in [18] (see [19] for a review). The irreducible representations have dimensions $\text{Pf}(Q)$. Our boundary conditions make the group not only finite, but also irreducible (elements commuting with all the generators are multiples of the identity) and the space of fields satisfying the boundary conditions (for a fixed color component) decomposes into a direct sum of $\text{Pf}(Q)$ spaces. Translations do not mix these subspaces. As we shall see later, there is exactly one zero-mode in each of these subspaces, giving a total of Q, as expected (31).

The infinitesimal generators of translations are given by

$$D_\mu^{(0)} = \partial_\mu + \frac{\iota}{2} F_{\mu\nu}^{(0)} x_\nu, \tag{38}$$

that is, covariant derivatives in a gauge field of uniform field strength:

$$[D_\mu^{(0)}, D_\nu^{(0)}] = -\iota F_{\mu\nu}^{(0)}. \tag{39}$$

Before we derive a parametrization of the space in terms of fields living in only two dimensions we exploit symmetries to simplify the problem. By transformations in $SL(4,\mathbf{Z})$ we bring $Q_{\mu\nu}$ to a canonical form with $Q_{12} = -Q_{21} = q$ and $Q_{03} = -Q_{30} = q'$ and all other entries vanishing. The Pfaffian is equal to qq'.

We now perform space-time coordinate transformations:

$$x_\mu \to x'_\mu = l_\mu \tilde{e}_\nu^{(\mu)} x_\nu. \tag{40}$$

In general, the transformation is non-orthogonal changing the metric tensor and the spectrum of the Dirac operator. However, the previous formulas remain valid, and the matrix $F_{\mu\nu}^{(0)}$ is simpler now, with only $F_{03}^{(0)} = -F_{30}^{(0)} = F'$ $= 2\pi q'/(l_0 l_3)$ and $F_{12}^{(0)} = -F_{21}^{(0)} = F = 2\pi q/(l_1 l_2)$ as non-zero entries. As a result, the algebra of covariant derivatives decouples, becoming the Heisenberg algebra for a quantum particle in two dimensions and the problem factorizes over the two planes $1-2$ and $0-3$; so does the group generated by the operators $K^{(\mu)}$. One can choose $K^{(2)}$ and $K^{(3)}$ as members of a complete set of commuting operators. Since $(K^{(2)})^q = (K^{(3)})^{q'} = \mathbf{I}$ the Hilbert space can be decomposed into subspaces labeled by integers $1 \le n \le q, 1 \le n' \le q'$.

Selecting $-\iota D_1^{(0)}/F$ and $-\iota D_0^{(0)}/F'$ as position operators, we express our fields as functions of two real variables, $h_{n,n'}(z,z')$. The indices n,n' label the appropriate subspace. The covariant derivatives act as follows:

$$D_1^{(0)} h_n(z,z') = \iota F z h_n(z,z') \tag{41}$$

$$D_0^{(0)} h_n(z,z') = \iota F' z' h_n(z,z') \tag{42}$$

$$D_2^{(0)} h_n(z,z') = \partial_z h_n(z,z') \tag{43}$$

$$D_3^{(0)} h_n(z,z') = \partial_{z'} h_n(z,z'). \tag{44}$$

The expression relating the initial four-dimensional fields $\psi(x)$ and $h_{n,n'}(z,z')$ can be obtained from Eq. (26).

IV. UNIFORM ABELIAN FIELDS

We can use less dimensions for the fields, but, except for special backgrounds, the interactions will be non-local. If we are only concerned with fluctuations around one of the Abelian backgrounds which played a role earlier, not only do the equations stay local, but, in addition, the Dirac operator becomes evidently exactly diagonalizable. As a result, the famous relations between topology and fermionic zero modes is made very explicit.

We begin in two dimensions with a gauge potential given by

$$A_i(x) = -\pi q \epsilon_{ij}(x_j - 2x_j^{(0)})/\mathcal{A}. \tag{45}$$

The corresponding field is $F = F_{12} = 2\pi q/(l_1 l_2)$. The coordinates $x_i^{(0)}$ in Eq. (45) specify the location where Polyakov loops around the torus take value 1 and represent nontrivial parameters of the gauge background. For $x_i^{(0)} = 0$, the covariant derivative coincides with $D_i^{(0)}$ of Sec. II. In general,

$$D_i \equiv \partial_i - \iota A_i(x)$$
$$= \mathcal{T}_1(-x_1^{(0)}) \mathcal{T}_2(-x_2^{(0)}) D_i^{(0)} \mathcal{T}_2(x_2^{(0)})$$
$$\times \mathcal{T}_1(x_1^{(0)}). \tag{46}$$

Thus, the eigenvectors can always be obtained from those at $x_i^{(0)} = 0$.

The massless Dirac operator $D = \tau_1 D_1^{(0)} + \tau_2 D_2^{(0)}$ should have q zero modes. Let τ_i denote the Pauli matrices used to express the Dirac matrices γ_i in the chiral basis. The eigenvalue equation

$$\begin{pmatrix} 0 & D_1^{(0)} - \iota D_2^{(0)} \\ D_1^{(0)} + \iota D_2^{(0)} & 0 \end{pmatrix} \begin{pmatrix} \psi_1^{(\lambda)} \\ \psi_2^{(\lambda)} \end{pmatrix} = \iota \lambda \begin{pmatrix} \psi_1^{(\lambda)} \\ \psi_2^{(\lambda)} \end{pmatrix} \tag{47}$$

in the one-dimensional *position space* basis of Sec. II is built out of

$$D_1^{(0)} \pm \iota D_2^{(0)} \to \iota(\partial_z \pm Fz) = \begin{cases} \iota\sqrt{2F}a \\ \iota\sqrt{2F}a^\dagger \end{cases}. \tag{48}$$

∂_z stands for the derivative with respect to z and a is a standard harmonic oscillator annihilation operator:

$$a = \frac{1}{2}\sqrt{\frac{l_1 l_2}{\pi q}}\partial_z + \sqrt{\frac{\pi q}{l_1 l_2}}z$$

$$a^\dagger = -\frac{1}{2}\sqrt{\frac{l_1 l_2}{\pi q}}\partial_z + \sqrt{\frac{\pi q}{l_1 l_2}}z. \tag{49}$$

Dirac's eigenvalue equation takes the form

$$\iota\sqrt{2F}\begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \begin{pmatrix} h_{1,n}^{(\lambda)} \\ h_{2,n}^{(\lambda)} \end{pmatrix} = \iota\lambda \begin{pmatrix} h_{1,n}^{(\lambda)} \\ h_{2,n}^{(\lambda)} \end{pmatrix} \tag{50}$$

with $h_{i,n}^{(\lambda)}$ representing the eigenmode wave function in the one-dimensional basis. The index n takes q values and refers to the internal degree of freedom representing eigenstates of $K^{(2)}$ showing that spectrum is q-fold degenerate.

We express the expected positive chirality zero mode in each sector in terms of $|p\rangle$, the eigenstate of the number operator $a^\dagger a$:

$$\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}. \tag{51}$$

The rest of the eigenstates are given by

$$h_n^{(\pm p)} = \frac{1}{\sqrt{2}}\begin{pmatrix} |p\rangle \\ \pm|p-1\rangle \end{pmatrix} \tag{52}$$

where $p = 1,2 \ldots$. The corresponding eigenvalues are

$$\iota\lambda^{(\pm p)} = \pm\iota\sqrt{2Fp}. \qquad (53)$$

$h^{(\pm p)}$ map into each other under chirality: $\tau_3 h^{(\pm p)} = h^{(\mp p)}$.

In the original two-dimensional representation we obtain a well known form of the zero mode. Starting from

$$h_{1,n}^{(0)}(z) = \left(\frac{2q}{l_1 l_2}\right)^{1/4} e^{-Fz^2/2} \qquad (54)$$

we get, with $(x,y) \in \mathbf{T}^2$,

$$\psi_1^{(0)}(x_1,x_2) = \left(\frac{2q}{l_1 l_2}\right)^{1/4} \exp\left\{\iota\pi q\frac{x_1 x_2}{\mathcal{A}}\right\} \sum_{n=1}^{q}\sum_{s\in\mathbf{Z}} c_n$$

$$\times \exp\left(-\frac{F\left(x_2 + \frac{n+qs}{q}l_2\right)^2}{2} + \iota F x_1 l_2(n+sq)\right). \qquad (55)$$

The c_n are the components of an arbitrary q-dimensional vector of unit modulus.

The zero mode is not spatially homogeneous in terms of either (x_1,x_2) or z. In particular, for $q=1$, the solution is proportional to a theta function. It vanishes at the point $\bar{x}_1 = l_1/2, \bar{x}_2 = l_2/2$: Eq. (55) gives

$$\psi_1^{(0)}(\bar{x}_1,\bar{x}_2) = e^{\iota\pi/4}\left(\frac{2q}{l_1 l_2}\right)^{1/4}$$

$$\times\sum_s e^{-(Fl_2^2/2)[s+(1/2)]^2}(-1)^s. \qquad (56)$$

The exponential is even under $s\rightarrow -1-s$ while $(-1)^s$ is odd. The inhomogeneity of the zero mode reflects the existence of the two extra physical parameters $(x^{(0)},y^{(0)})$, beyond the magnitude of the quantized magnetic flux, identifying where the Polyakov loops winding around the torus take value 1. At the coordinates of the zero of $\psi_1^{(0)}$, the two Polyakov loops take value -1. These coordinates could be used as parameters as well.

According to Eqs. (52), (53) the spectrum for $q>1$ is the same as the spectrum of q independent fermions, each defined over a region of area $\mathcal{A}/q$. This can be understood as follows: Dirac's equation for $q>1$ could be solved by using the $q=1$ solution over the region $0\leq x_1\leq l_1/q$, $0\leq x_2\leq l_2$ (which carries one unit of magnetic flux) and extending it via the boundary condition $\psi(x_1+l_1/q,x_2)=\exp[\iota\pi x_2/l_2]\psi(x_1,x_2)$ to the rest of the x_1 domain $l_1/q\leq x_1\leq l_1$. This gives the solution corresponding to the sector $n=0$, an eigenspace of $K^{(2)}$. Applying $K^{(1)}$ one obtains the remaining $q-1$ other solutions. All of them will satisfy the Dirac equation and the boundary condition

$$\psi(x_1+l_1/q,x_2) = e^{\iota\pi x_2/l_2 + \iota 2\pi n/q}\psi(x_1,x_2). \qquad (57)$$

Thus, the spectrum is that of q independent fermions defined over an area with a single unit of magnetic flux.

Let us turn to the massive Dirac operator $D+m$. For $q=1$ we get

$$\begin{pmatrix} m & \iota\sqrt{2F}a^\dagger \\ \iota\sqrt{2F}a & m \end{pmatrix}\begin{pmatrix} \tilde{h}_1^{(\lambda)} \\ \tilde{h}_2^{(\lambda)} \end{pmatrix} = \lambda\begin{pmatrix} \tilde{h}_1^{(\lambda)} \\ \tilde{h}_2^{(\lambda)} \end{pmatrix}. \qquad (58)$$

Since in the presence of mass $D+m$ is neither Hermitian nor anti-Hermitian it is better to consider the Hermitian Dirac operator $H=\tau_3 D$:

$$\begin{pmatrix} m & \iota\sqrt{2F}a^\dagger \\ -\iota\sqrt{2F}a & -m \end{pmatrix}\begin{pmatrix} \tilde{h}_1^{(E)} \\ \tilde{h}_2^{(E)} \end{pmatrix} = E\begin{pmatrix} \tilde{h}_1^{(E)} \\ \tilde{h}_2^{(E)} \end{pmatrix}. \qquad (59)$$

H has one isolated eigenvalue at $E=m$ with m-independent eigenfunction $\binom{|0\rangle}{0}$ and an infinite set of paired eigenvalues, each pair labeled by $p=1,2,3,\ldots,$

$$E^{(\pm p)} = \pm\sqrt{m^2+2pF}, \qquad (60)$$

$$\tilde{h}^{(\pm p)} = \frac{1}{\sqrt{1+(z^{(\pm p)})^2}}\begin{pmatrix} |p\rangle \\ z^{(\pm n)}|p-1\rangle \end{pmatrix}$$

$$z^{(+p)}z^{(-p)} = -1 \qquad (61)$$

with

$$z^{(\pm p)} = -\frac{m}{2}\sqrt{\frac{l_1 l_2}{p\pi}} \pm \sqrt{\frac{m^2}{4}\frac{l_1 l_2}{p\pi}+1}. \qquad (62)$$

As in all other cases, the states and eigenvalues depend only on the area $\mathcal{A}=l_1 l_2$ but not on the ratio l_1/l_2. One can view the one-dimensional version as the limit when the ratio goes to zero or infinity while the area is kept fixed.

When we vary m from positive values to negative ones isolated state crosses zero energy at $m=0$. The paired eigenvalues never get to zero and are symmetric under $m\rightarrow -m$. The paired eigenstates depend on m, but the signs of $z^{(\pm p)}$ are m independent. On the lattice there is full dependence on m, but the number of zero level crossings as m is varied stays the same as in continuum.

Consider now four dimensions with gauge group $SU(2)$ and fields in the fundamental, defined over an $l_0\times l_1\times l_2 \times l_3$ torus. The boundary conditions are taken in the canonical form from the beginning:

$$Q_{\mu\nu}^\pm = \pm\begin{pmatrix} 0 & 0 & 0 & q' \\ 0 & 0 & q & 0 \\ 0 & -q & 0 & 0 \\ -q' & 0 & 0 & 0 \end{pmatrix}. \qquad (63)$$

$\pm$ denotes the two color components. Note that the first Chern class invariants vanish. The topological charge is given by $Q=2qq'$ where the factor 2 arises from the sum over color components. Odd topological charges cannot be obtained this way—as mentioned before, $SU(2)$ is somewhat exceptional. The following constant field strength is compatible with the boundary conditions:

$$F_{\mu\nu}^{\pm}=\pm\begin{pmatrix} 0 & 0 & 0 & F' \\ 0 & 0 & F & 0 \\ 0 & -F & 0 & 0 \\ -F' & 0 & 0 & 0 \end{pmatrix} \quad (64)$$

where $F=2\pi q/(l_1 l_2)$ and $F'=2\pi q'/(l_0 l_3)$. The upper color component coincides with the matrix $F_{\mu\nu}^{(0)}$ of the previous section in canonical form.

The fermion field is in the fundamental representation of $SU(2)$ denoted by

$$\Psi=\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}. \quad (65)$$

The index theorem tells us that there are $Q=2qq'$ zero modes of positive chirality more than zero modes of negative chirality. The eigenvalue equations decouple into color components. Because of the pseudoreality of $SU(2)$ the upper color component determines the lower one by the symmetry

$$\Psi\rightarrow\tau_2^S\tau_2^C\Psi^*. \quad (66)$$

$\tau_2^{S,C}$ act in spin and color space respectively, while Ψ^* is the complex conjugate vector. The spectrum is doubly degenerate. We choose a chiral representation for the Dirac matrices:

$$\gamma_\mu=\begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad (67)$$

$$\gamma_5=\begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}. \quad (68)$$

The matrices $\sigma_\mu=(\mathbf{I},-\iota\vec{\tau})$ are the Weyl matrices and $\bar{\sigma}_\mu$ are the adjoints. They satisfy

$$\bar{\sigma}_\mu\sigma_\nu=\bar{\eta}_{\mu\nu}^\alpha\sigma_\alpha. \quad (69)$$

$\bar{\eta}_{\mu\nu}^\alpha$ is the 't Hooft symbol; $\bar{\eta}_{\mu\nu}^0=\delta_{\mu\nu}$ and the $\bar{\eta}_{\mu\nu}^i$ are a basis of anti-self-dual tensors. The eigenvalue problem is

$$D\psi(x)\equiv D_\mu\gamma_\mu\psi(x)=\begin{pmatrix} 0 & \bar{D} \\ \hat{D} & 0 \end{pmatrix}\psi(x)=\iota\lambda\psi(x). \quad (70)$$

The symbol $\bar{D}=D_\mu\bar{\sigma}_\mu(\hat{D}=D_\mu\sigma_\mu)$ denotes the positive (negative) chirality Weyl operator restricted to the upper color component. The covariant derivative D_μ coincides with the generator of translations studied in Sec. III, $D_\mu^{(0)}$. Choosing a new origin of coordinates gives the vector potential another form. The different choices are gauge inequivalent, as evidenced by the Polyakov loops, but are related by translations $\mathcal{T}(b)$.

In position space representation $\bar{D}$ becomes ($\hat{D}=-\bar{D}^\dagger$):

$$\bar{D}=\begin{pmatrix} D_0+\iota D_3 & \iota(D_1-\iota D_2) \\ \iota(D_1+\iota D_2) & D_0-\iota D_3 \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} \iota\sqrt{2F'}a' & -\sqrt{2F}a^\dagger \\ -\sqrt{2F}a & \iota\sqrt{2F'}a'^\dagger \end{pmatrix}. \quad (71)$$

a,a' are two independent annihilation operators. a is the same as in Eq. (49). To get a' replace the $1-2$ plane by the $0-3$ plane and z by z'.

A positive chirality eigenstate of the Dirac operator with eigenvalue $\iota\lambda$ also is an eigenstate of the operator $\hat{D}\bar{D}$ with eigenvalue $-\lambda^2$:

$$\hat{D}\bar{D}=-2\begin{pmatrix} F'a'^\dagger a'+Fa^\dagger a & 0 \\ 0 & F'a'a'^\dagger+Faa^\dagger \end{pmatrix}. \quad (72)$$

Let $|p,p'\rangle$, with non-negative integers p and p', be simultaneous eigenvectors of the number operators. There is a positive chirality zero mode given by

$$h_{n,n'}^{(0)}=\begin{pmatrix} |0,0\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (73)$$

There is an identical zero mode in each of the subspaces labeled by n-n' making up qq' zero modes for each color component and confirming the index theorem. Actually, all operators act in the same way in each subspace, so the entire spectrum is replicated Q times.

The set of eigenvalues of the Dirac operator consists of

$$\iota\lambda^{(\pm,p,p')}=\pm\iota\sqrt{2Fp+2F'p'}. \quad (74)$$

The corresponding eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} |p,p'\rangle \\ 0 \\ \pm\sqrt{\dfrac{F'p'}{Fp+F'p'}}|p,p'-1\rangle \\ \pm\iota\sqrt{\dfrac{Fp}{Fp+F'p'}}|p-1,p'\rangle \end{pmatrix}. \quad (75)$$

For $p,p'>0$ there exist additional eigenstates with the same eigenvalue:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ |p-1,p'-1\rangle \\ \pm\iota\sqrt{\dfrac{Fp}{Fp+F'p'}}|p,p'-1\rangle \\ \pm\sqrt{\dfrac{F'p'}{Fp+F'p'}}|p-1,p'\rangle \end{pmatrix}. \quad (76)$$

The spectrum and eigenstates of the massive Hermitian Dirac operator $H = \gamma_5(D+m)$ can be easily obtained in terms of those of the massless Dirac operator. The unchanged zero modes become eigenstates of H with eigenvalue m. The massive Dirac operator mixes other eigenstates of the massless Dirac operator, ψ_λ, with $\gamma_5\psi_\lambda$. The eigenvalues and eigenvectors of H result from diagonalizing the corresponding 2×2 matrix giving eigenvalues equal to $\pm\sqrt{m^2+\lambda^2}$. The eigenvectors are of the form $(1/\sqrt{2})(e^{-\iota\delta/2}\psi_\lambda \pm e^{\iota\delta/2}\gamma_5\psi_\lambda)$, where δ is the phase of the complex number $m+\iota\lambda$.

V. TWO DIMENSIONS—LATTICE

We now turn to the lattice and examine a constant Abelian background in two dimensions. We wish to determine how an ultraviolet cutoff interacts with the "dimensional reduction" we saw before.

The fermion fields $\psi(n_1,n_2)$ are complex functions of sites on an $L_1\times L_2$ square lattice. If we visualize the computer memory storing these functions, there is only a finite list of complex numbers and no mention of boundary conditions. No continuum features are lost because the link gauge variables are parallel transporters to a finite distance and therefore in the continuum limit include both the vector potential and the transition functions. The link variables thus keep track also of what we normally view as boundary conditions. To keep our lattice description formally close to the continuum formulation we choose to view our lattice fields slightly differently, as defined over an infinite lattice but restricted by

$$\psi(n+E^{(i)})=\exp\left\{\iota\pi\sum_j q_{ij}\frac{n_j}{L_j}\right\}\psi(n). \tag{77}$$

$E^{(i)}$ is a vector of length L_i in the i direction, $q_{ij}=q\epsilon_{ij}$ and the indices i,j take the values 1,2.

The space of fields satisfying the boundary conditions (77) can be described in a way equivalent to the continuum one-dimensional quantum mechanical formulation. First, we gauge-transform the function $\psi(n)$:

$$\kappa(n)=\exp\left\{-\pi\iota q\frac{n_1n_2}{L_1L_2}\right\}\psi(n). \tag{78}$$

The new function, κ, is periodic in n_1 with period L_1 and can be parametrized as

$$\kappa(n_1,n_2)=\sum_{p_1=1}^{L_1}\exp\{\iota2\pi p_1n_1/L_1\}c(n_2,p_1). \tag{79}$$

The momentum p_1 is defined modulo L_1. Imposing the boundary condition in n_2, we get

$$c(n_2+L_2,p_1)=c(n_2,p_1+q). \tag{80}$$

Defining $q_1=\gcd(q,L_1)$, we can write $p_1=sq+n\bmod L_1$ where n is an integer-defined modulo q_1 and s is an integer defined modulo L_1/q_1:

$$c(n_2,p_1)=h(n_2+L_2s,n). \tag{81}$$

We have arrived at the following expression for $\psi(n)$:

$$\psi(n_1,n_2)=\exp\left\{\pi\iota q\frac{n_1n_2}{L_1L_2}\right\}\sum_{s=1}^{L_1/q_1}\sum_{n=1}^{q_1}$$
$$\times\exp\{\iota2\pi(sq+n)n_1/L_1\}h(n_2+L_2s,n). \tag{82}$$

For $q=1$, $q_1=1$, and our representation is given in terms of a function h defined on a one-dimensional lattice of length $N\equiv L_1L_2$. For general q, it is still true that the space of fermions fields is N dimensional, but now one has q_1 functions living on a periodic lattice of size N/q_1.

The lattice fields $\psi(n)$ can be viewed as the collection of values of the continuum fields $\psi(x)$ (appearing in Sec. II) at the lattice points $x=na$, where a is the lattice spacing and $l_\mu=L_\mu a$. Thus, the transformations acting on the space of lattice fields are given by the restriction of the continuum transformations to the lattice points. In particular, we have a group of modified translations by vectors whose components are integer multiples of the lattice spacing a. The generators of the group are $\mathcal{T}_i$:

$$\mathcal{T}_i\psi(n)=\exp\left\{\iota\pi\sum_j\frac{q_{ij}n_j}{N}\right\}\psi(n+\hat{i}). \tag{83}$$

N equals L_1L_2 and $\hat{i}$ is a vector of unit length in the i direction. We obtained a non-commutative discrete subgroup of the continuum translation group:

$$\mathcal{T}_1\mathcal{T}_2=\exp\left\{-2\pi\iota\frac{q}{N}\right\}\mathcal{T}_2\mathcal{T}_1. \tag{84}$$

The one-dimensional representation is obtained in a basis that diagonalizes $\mathcal{T}_1$:

$$\mathcal{T}_1h(k,n)=\exp\left\{2\pi\iota\left(\frac{qk}{N}+\frac{n}{L_1}\right)\right\}h(k,n). \tag{85}$$

The translation operator in the 2 direction acts as follows:

$$\mathcal{T}_2h(k,n)=h(k+1,n). \tag{86}$$

If $N_{12}\equiv\gcd(N,q)\neq1$ the group generated by $\mathcal{T}_i$ acts reducibly on our space of lattice fields decomposing it into N_{12} subspaces, each invariant under translations. This constitutes the lattice remnant of the group generated by $K^{(i)}$ in the continuum. These operators contain translations by l_i/q and can be extended to the lattice only if L_i is divisible by q. Otherwise, one obtains only a subgroup of the group found in the continuum.

We pick lattice gauge fields having a constant plaquette value:

$$U_i(n)=\alpha_i\exp\left\{\iota\pi\sum_j\frac{q_{ij}n_j}{N}\right\}. \tag{87}$$

The α_i are complex numbers of unit modulus. The plaquettes are given by

$$U_{12}(n)=\exp\left\{-\imath 2\pi\frac{q}{N}\right\}\equiv\zeta^{-q} \tag{88}$$

where $\zeta=\exp\{\imath 2\pi/N\}$. The values of the plaquettes do not depend on α_i, but the latter are physical parameters because they influence the Polyakov loops. The α_i are related to the continuum parameters $x_i^{(0)}$ (taken in units of the lattice spacing):

$$\alpha_i=e^{-2\pi\imath q\Sigma_j\epsilon_{ij}x_j^{(0)}/N}. \tag{89}$$

In the continuum the spectrum of the Dirac operators was independent of $x_i^{(0)}$, by unitary equivalence generated by appropriate translations. On the lattice translations are discrete and can only be used to restrict the ranges of the $x_i^{(0)}:0\leqslant x_i^{(0)}<1$.

The lattice Dirac operators contain the covariant shift operators T_i:

$$\mathcal{T}_i=\alpha_i T_i. \tag{90}$$

For the rest of the section we restrict ourselves to the $q=1$ case. The T_i operators take the following matrix form when acting on $h(k)$'s:

$$T_2=\alpha_2\begin{pmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ 0 & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & 0 & 1\\ 1 & 0 & \ldots & 0 & 0\end{pmatrix} \tag{91}$$

$$T_1=\alpha_1\begin{pmatrix}1 & 0 & 0 & \ldots & 0\\ 0 & \zeta & 0 & \ldots & 0\\ 0 & 0 & \zeta^2 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & 0 & \zeta^{N-1}\end{pmatrix}. \tag{92}$$

It is convenient to choose a different basis of Dirac matrices for the lattice analysis:

$$\gamma_1=\tau_2,\gamma_2=\tau_3,\gamma_3=-i\gamma_1\gamma_2=\tau_1. \tag{93}$$

The Wilson Dirac operator D_W is

$$D_W=m+\sum_\mu (r-V_\mu) \tag{94}$$

where the unitary matrices V_μ are

$$V_\mu=\frac{r-\gamma_\mu}{2}\mathcal{T}_\mu+\frac{r+\gamma_\mu}{2}\mathcal{T}_\mu^\dagger. \tag{95}$$

In what follows we will fix the Wilson parameter r to 1. From now on periodicity in the "site" index n is implied by

the convention that it is always viewed modulo N and this extends to operations involving site indices. For simplicity, let us start with $x_i^{(0)}=0$.

The Wilson Dirac lattice Hamiltonian is

$$H_W=\tau_1 D_W$$
$$=\begin{pmatrix}-\sin\Theta & 2+m-\cos\Theta-T\\ 2+m-\cos\Theta-T^\dagger & \sin\Theta\end{pmatrix}. \tag{96}$$

Here, Θ is an $N\times N$ diagonal matrix:

$$\Theta=\mathrm{diag}[1,\theta_1,\theta_2,\ldots,\theta_{N-1}] \tag{97}$$

with

$$\theta_n=\frac{2\pi n}{N},\quad T_{kn}=\delta_{k+1,n}. \tag{98}$$

H_W is a local one-dimensional Hamiltonian defined on a discretised circle. The calculation of eigenvalues and eigenvectors requires fewer operations than in the basis corresponding to a two-dimensional grid.

Writing the eigenvector for energy E as

$$\psi_n=\begin{pmatrix}a_n\\ b_n\end{pmatrix} \tag{99}$$

where n ranges between 0 and $N-1$, we get the eigenvalue equations

$$Ea_n=-\sin\theta_n a_n+(2+m-\cos\theta_n)b_n-b_{n+1} \tag{100}$$

$$Eb_{n+1}=(2+m-\cos\theta_{n+1})a_{n+1}-a_n+\sin\theta_{n+1}b_{n+1}. \tag{101}$$

In matrix form we have

$$\begin{pmatrix}2+m-\cos\theta_{n+1} & \sin\theta_{n+1}-E\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_{n+1}\\ b_{n+1}\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\ -E-\sin\theta_n & 2+m-\cos\theta_n\end{pmatrix}\begin{pmatrix}a_n\\ b_n\end{pmatrix}. \tag{102}$$

Inverting the matrix on the left-hand side we get

$$\begin{pmatrix}a_{n+1}\\ b_{n+1}\end{pmatrix}=M_n\begin{pmatrix}a_n\\ b_n\end{pmatrix}. \tag{103}$$

The eigenvalue condition is that $M\equiv M_{N-1}M_{N-2}\ldots M_0$ have an eigenvalue equal to unity. Observe that $\det M=1$; thus if M has one eigenvalue unity the other also must be unity and the eigenvalue condition can also be written as

$$\mathrm{Tr}M(E)=2. \tag{104}$$

When this condition is fulfilled $M(E)$ has a single eigenvector, not two, because it is non-diagonalizable. The energies E are the roots of a polynomial of degree $2N$. Typically these roots are all simple and to each corresponds a single eigenstate of H_W—there are no degeneracies.

We have ended up with an exact formula for the characteristic polynomial of H_W,

$$\det(E - H_W) = (-1)^N K_N [\text{Tr} M(E) - 2]. \tag{105}$$

The proportionality factor K_N is calculated by comparing the terms E^{2N} in $\text{Tr} M(E)$ and in $\det(E - H_W)$,

$$K_N = \prod_{n=0}^{N-1} (2 + m - \cos \theta_n) \tag{106}$$

$$= \left[\frac{2 + m + \sqrt{(2+m)^2 - 1}}{2} \right]^N$$

$$+ \left[\frac{2 + m - \sqrt{(2+m)^2 - 1}}{2} \right]^N - \frac{1}{2^{N-1}}.$$

This formula is correct even when the square roots are imaginary, with the understanding that the second term on the right-hand side is the complex conjugate of the first. It is derived by setting $2 + m = 1/2[z + (1/z)]$ and noting that the finite Laurent series in z defining K_N is symmetric under $z \leftrightarrow 1/z$ and vanishes when z is an Nth root of unity.

We can now easily re-introduce arbitrary $x_1^{(0)}$ and $x_2^{(0)}$. All that happens is that Θ gets shifted to $\Theta - 2\pi x_1^{(0)}/N$ and the characteristic polynomial is now proportional to

$$\text{Tr} M(E) - 2 \cos(2\pi x_2^{(0)}).$$

The proportionality constant now depends on $x_1^{(0)}$.

Numerically, the finite precision of the machine will limit the usefulness of a direct implementation of the characteristic polynomial as a means of locating the eigenvalues of H_W when N gets large. A better way exists for $x_i^{(0)} = 0$ which has extra symmetry. T_1 and T_2 are then isospectral and there exists an intertwining map, F_N. (Actually, $T_1, T_1^\dagger, T_2, T_2^\dagger$ all have identical spectra.) F_N is the discrete Fourier transform:

$$F_{jk} = \frac{1}{\sqrt{N}} e^{(2\pi i/N)jk}. \tag{107}$$

F is symmetric and unitary,

$$T_2 F = F T_1 \tag{108}$$

$$F T_2 = T_1^\dagger F. \tag{109}$$

F can be combined with a rotation in spinor space to produce a unitary matrix that commutes with H_W,

$$U = \frac{1-i}{\sqrt{2}} \frac{1 - i\gamma_3}{\sqrt{2}} F. \tag{110}$$

One checks now that

$$D_W U = U D_W \tag{111}$$

$$U H_W = H_W U. \tag{112}$$

The simpler case of commutativity with U^2 is more obvious and can be checked directly:

$$F^2 e^{i\Theta} F^2 = e^{-i\Theta} \tag{113}$$

leading to

$$F^2 H_W F^2 = \tau_1 H_W \tau_1 \tag{114}$$

where we notationally ignore factors of unit matrices in tensor products.

The eigenvalues of U are $1, i, -1, -i$. Their degeneracies are computed from the ranks of the appropriate projectors. Since $U^4 = 1$ the projectors can be written in terms of $1, U, U^*, U^2$. For the multiplicities we only need the traces of $1, F, F^*, F^2$. The trace of F^2 is trivial because

$$F^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{115}$$

The trace of F is not trivial and is given by a famous sum of Gauss:

$$S_G(N) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{(2\pi i/N)k^2}. \tag{116}$$

$S_G(N)$ is determined by its values at $N = 1, 2, 3, 4$ and by (nontrivial) mod 4 periodicity: $S_G(1) = 1$, $S_G(2) = 0$, $S_G(3) = i$, $S_G(4) = 1 + i$. The reflection $U^2 = \gamma_3 F^2 = \tau_1 F^2$ reverses the order of sites with $n = 0$ as a fixed point for all N and flips a and b (in our γ-matrix basis).

The origin of the U symmetry is in the continuum. The harmonic oscillator case is unique in that all the eigenfunctions are eigenstates of the Fourier transform as a result of a discrete symmetry that exchanges the coordinate with its conjugate momentum. The eigenvalues of the Fourier transform are $1, i, -1, -i$, depending on the oscillator level mod 4. To get an operator that commutes with the Dirac operator in our case the Fourier transform needs to be combined with the action of a rotation in spinor space depending on the chirality matrix γ_3. On the lattice one can decompose H_W into four blocks each of sizes (roughly) of $N/2 \times N/2$. The exact sizes depend on $N \mod 4$. The U-symmetry is simpler to understand when also using the standard representation in terms of fields on a two-dimensional torus of equal sides. Then we have a symmetry under rotations by $\pi/2$, which corresponds to a switch between coordinate and momentum in the one-dimensional framework. The two-dimensional interpretation as a discrete rotation explains why an action on spinorial indices is also required. As only the area is relevant, the requirement that the torus have equal sides can be dropped. In practice we shall exploit only U^2, corresponding to a rotation by π, and the issue does not even

arise. In the one-dimensional basis the U^2 reflection symmetry contains the square of the Fourier transform combined with γ_3 in a simple manner.

The U^2 symmetry allows us to factorize $\mathrm{Tr}M(E)-2$. The trace contains the sum of the expectation of $M(E)$ in two states. These contributions are individually high degree polynomials in E and vary over many orders of magnitude. The contributions nearly cancel at the eigenvalues and are very sensitive to E. The symmetry makes it possible to work in only one sector, where a single state contributes. As a result one deals with a single expectation value, which can be renormalized iteratively, in effect dividing out the quantity that has to vanish by a positive function of E that tames the large variability. This provides a numerically stable method for locating the eigenvalues down to machine accuracy.

One needs to separate the case $N=$even from $N=$odd. For $N=2L$, L being integer, the reflection has two fixed points: $n=0$ and $n=L$. Thus, one can impose on ψ_0 and ψ_L to be eigenvectors of τ_1 with identical eigenvalue. Start from $\psi_0=(1/\sqrt{2})\binom{1}{1}\equiv\xi_+$ and let $\xi_-=(1/\sqrt{2})\binom{-1}{1}$. We need to have then

$$\xi_-^\dagger M_{L-1}(E)M_{L-2}(E)\ldots M_0(E)\xi_+=0. \qquad (117)$$

This gives a polynomial equation for E of degree $2L$; the roots are all the energies of the even states. Similarly, for the odd states we have the equation

$$\xi_+^\dagger M_{L-1}(E)M_{L-2}(E)\ldots M_0(E)\xi_-=0. \qquad (118)$$

Extending to $N=2L-1$ requires imposing $\psi_{L-1}=\pm\tau_1\psi_L$ coordinated with $\pm\tau_1\psi_0=\psi_0$. This implies

$$(1-\tau_1 M_{L-1})\psi_{L-1}=0 \qquad (119)$$

if $\tau_1\psi_0=\psi_0$, and

$$(1+\tau_1 M_{L-1})\psi_{L-1}=0 \qquad (120)$$

if $\tau_1\psi_0=-\psi_0$.

Explicit evaluation of M_{L-1} produces

$$M_{L-1}=\begin{pmatrix} \dfrac{1-(s+E)^2}{2+m+c} & s+E \\ -(s+E) & 2+m+c \end{pmatrix} \qquad (121)$$

where $s=\sin[\pi/(2L-1)]$ and $c=\cos[\pi/(2L-1)]$. The eigenvalues of $\tau_1 M_{L-1}$ are $\lambda=\pm1$, independently of E. The corresponding two eigenvectors do depend on E:

$$\hat{\xi}_\lambda(E)=\frac{1}{\sqrt{1+[(\lambda+E+s)/(2+m+c)]^2}}\begin{pmatrix} 1 \\ \dfrac{\lambda+E+s}{2+m+c} \end{pmatrix}. \qquad (122)$$

The eigenvalue conditions are that $(M_{L-2}M_{L-3}\ldots M_0)\xi_\sigma$ be parallel to $\hat{\xi}_\sigma(E)$ for $\sigma=\pm1$,

$$\hat{\xi}_{+1}^T(E)\tau_2(M_{L-2}M_{L-3}\ldots M_0)\xi_+=0 \qquad (123)$$

$$\hat{\xi}_{-1}^T(E)\tau_2(M_{L-2}M_{L-3}\ldots M_0)\xi_-=0. \qquad (124)$$

They look slightly different from the even case because now the vectors $\hat{\xi}_{\pm1}$ are not orthogonal to each other.

The above is useful for numerical searches for energy eigenvalues of H_W. The vectors obtained by the sequential action of M_n are normalized at each step and this eliminates accuracy problems. N is limited now only because for very large N the spacings between some eigenvalues may be below machine accuracy.

Looking at some examples the following patterns emerge. The odd and even states have energy sequences that separate (interlace with) each other. For even N the topological charge $-\frac{1}{2}\mathrm{Tr}H_W/\sqrt{H_W^2}$ is carried by the odd states. For odd N the topological charge is carried by the odd and even states equally—each sector having an imbalance equal to unity. When m is taken from positive values to ~-1 it is always an even state that does the zero crossing. There are no degeneracies.

Since there are no degeneracies one can calculate directly the eigenvalues λ of the symmetry operator U for each one of the eigenstates. A simple way to do this is to evaluate $F\psi$ at site 0 only. We distinguish the even and odd cases.

If the state is even, $\gamma_3 F^2\psi=\psi$, then

$$(F\psi)_0=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\psi_n=\lambda\psi_0 \qquad (125)$$

and λ can be ±1. If the state is odd, $\gamma_3 F^2\psi=-\psi$, then

$$(F\psi)_0=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\psi_n=i\lambda\psi_0 \qquad (126)$$

and $i\lambda$ can be ±1.

We now turn to the naive lattice Dirac operator. In the one-dimensional version we now have

$$H_N=\begin{pmatrix} -\sin\Theta & \dfrac{1}{2}(T^\dagger-T) \\ \dfrac{1}{2}(T-T^\dagger) & \sin\Theta \end{pmatrix}. \qquad (127)$$

If the L_i are even there are two additional symmetries involving staggering operators along each direction:

$$S_i\psi(n)=(-1)^{n_i}\psi(n). \qquad (128)$$

The S_i commute among themselves, satisfy $S_i^2=1$ and also

$$S_i T_i=-T_i S_i. \qquad (129)$$

The two anticommuting matrices $\Pi_i=S_i\gamma_i$, $i=1,2$ commute with H_N. We diagonalize one of the Π_i's, Π_2 for example. The eigenspaces of the two ±1 eigenvalues correspond to two species of staggered fermions. Going over to the one-dimensional description, the operator S_2 becomes the corresponding one-dimensional staggering operator $\bar{S}$:

$$\bar{S}_{nm} = (-1)^n \delta_{nm}. \tag{130}$$

$\bar{S}^2 = 1$, $\bar{S}$ is Hermitian and $\bar{S} T \bar{S} = -T$.

We obtain a reduced Hamiltonian for naive fermions by introducing the following Hermitian-unitary matrix:

$$R = \frac{1 + \tau_1}{2} I + \frac{1 - \tau_1}{2} S_2. \tag{131}$$

It brings H_N into a simpler form:

$$R H_N R = -\tau_3 H_{NR} \tag{132}$$

where $H_{NR} = -\iota S_2 D_{12}$ and

$$D_{12} = \frac{1}{2} (T_1 - T_1^\dagger + \iota T_2 - \iota T_2^\dagger). \tag{133}$$

In the one-dimensional representation we get

$$H_{NR} = \bar{S} \left(\frac{1}{2} (T - T^\dagger) + \sin \Theta \right). \tag{134}$$

The operator $S_1 S_2$ anticonmutes with H_{NR}, implying that the spectrum of H_N is real, symmetric around zero and doubly degenerate. We can cast the eigenvalue equation in the following form:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \bar{M}_n(E) \begin{pmatrix} a_n \\ b_n \end{pmatrix} \tag{135}$$

where

$$\bar{M}_n(E) = \begin{pmatrix} 2E(-1)^n - 2\sin \theta_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{136}$$

For E to be an eigenvalue, $\bar{M}(E) = \bar{M}_N(E) \cdots \bar{M}_1(E)$ must have an eigenvalue equal to 1. However, since the determinant of $\bar{M}(E)$ is 1, the condition becomes again:

$$\mathrm{Tr}(\bar{M}(E)) = 2. \tag{137}$$

An analysis similar to the one performed for the Wilson-Dirac operator would apply also here.

Let us conclude this section by analyzing the existence and multiplicity of zero modes. We restrict our attention to positive chirality modes. There always exists an equal number of negative chirality modes; this is a reflection of the famous phenomenon of fermion doubling. The equation for the zero mode is

$$h_{n+1}^{(0)} - h_{n-1}^{(0)} = -2 h_n^{(0)} \sin \theta_n. \tag{138}$$

This is a second order recursion relation determining all the elements of the series in terms of $h_0^{(0)}$ and $h_1^{(0)}$. Our solutions must satisfy the periodicity condition $h_{n+N}^{(0)} = h_n^{(0)}$. It is necessary and sufficient that this condition be satisfied for two consecutive values of n. Since the problem is homogeneous this could lead to two conditions on the ratio $h_1^{(0)}/h_0^{(0)}$, so existence is not guaranteed. To prove existence

we make use of the fact that the solution, if it exists, satisfies $h_n^{(0)} = h_{-n}^{(0)}$. For $n = 1$ this relation follows from the vanishing of θ_0. For other values of n it can be proven by applying the iteration Eq. (138).

Most of our previous formulas have been derived for even N, so we will consider this case first. Then, $h_{N/2}^{(0)} = h_{-N/2}^{(0)}$, which is equivalent to periodicity at $n = -N/2$. From the vanishing of $\sin \theta_{N/2}$ we conclude $h_{N/2-1}^{(0)} = h_{N/2+1}^{(0)}$, which together with reflection around $n = 0$ yields $h_{-N/2+1}^{(0)} = h_{N/2+1}^{(0)}$ (periodicity at $-N/2 + 1$). We conclude that the iteration procedure is automatically periodic for arbitrary values of $h_0^{(0)}$ and $h_1^{(0)}$ and therefore there are two positive chirality and two negative chirality zero modes. Their shape can be automatically determined by applying the iteration to any pair of initial values of $h_0^{(0)}$ and $h_1^{(0)}$. Once one zero mode has been determined the other can be found by applying the symmetry operator:

$$h_n^{(0)} \to (-1)^n h_{N/2+n}^{(0)}. \tag{139}$$

In the odd N case, although staggering does not apply, the equation for the zero modes (138) and the reflection property ($h_n^{(0)} = h_{-n}^{(0)}$) still holds. This, however, is not enough to imply periodicity, but allows us to reduce the two equations to one. This fixes the ratio $h_1^{(0)}/h_0^{(0)}$: A single zero mode exists for each chirality. To conclude our analysis of zero modes we mention that for the $x_i^{(0)} \neq 0$ case, the reflection property and the existence of zero modes is lost.

In conclusion, the one-dimensional character of the lattice problem holds independently of which lattice fermions we use. Also, from the numerical point of view there are no hidden features and the problem truly has the complexity of a one-dimensional system.

VI. FOUR DIMENSIONS—LATTICE

In this section we look at an $SU(2)$ gauge theory on a four-dimensional lattice $\mathcal{L}$ of size $L_0 \times L_1 \times L_2 \times L_3$. We again take Abelian transition functions:

$$\psi(n + E^{(\mu)}) = \exp \left\{ \iota \pi \sum_\nu q_{\mu\nu} \frac{n_\nu}{L_\nu} \tau_3 \right\} \psi(n). \tag{140}$$

$q_{\mu\nu}$ is an antisymmetric tensor whose only nonzero components are $q_{03} = q'$ and $q_{12} = q$. The vector $E^{(\mu)}$ is a vector of length L_μ along the μ direction. Following the two-dimensional example we introduce shift operators $\mathcal{T}_\mu$ which act as follows:

$$\mathcal{T}_\mu \psi^+(n) = \exp \left\{ \iota \pi \sum_\nu q_{\mu\nu} \frac{n_\nu}{L_\mu L_\nu} \right\} \psi^+(n + \hat{\mu}). \tag{141}$$

Changing the sign of the exponent on the left-hand side, one obtains the translation operator acting on the lower color component. The translation operators are just the covariant shift operator in a background lattice gauge field with the links given by

$$U_\mu(n) = \exp\left\{ \iota\pi \sum_\nu q_{\mu\nu} \frac{n_\nu}{L_\mu L_\nu} \tau_3 \right\}. \qquad (142)$$

These links produce a homogeneous plaquette field.

The covariant shift operators define a finite group:

$$\mathcal{T}_\mu \mathcal{T}_\nu = \exp\left\{ -2\pi\iota \frac{q_{\mu\nu}}{L_\mu L_\nu} \right\} \mathcal{T}_\nu \mathcal{T}_\mu. \qquad (143)$$

For $q = q' = 1$ the entire $L_0 L_1 L_2 L_3$-dimensional space is irreducible under this group. A convenient basis is obtained by diagonalizing a maximal set of commuting matrices; we choose $\mathcal{T}_1$ and $\mathcal{T}_0$, leading to a two-dimensional labeling of the basis given by the corresponding eigenvalues of $\mathcal{T}_1$ and $\mathcal{T}_0$. This is the lattice equivalent of the continuum two-dimensional representation of four-dimensional lattice fields. Letting $N_{12} = L_1 L_2$ and $N_{03} = L_0 L_3$, the eigenvalues of $\mathcal{T}_1$ are given by $\exp\{\iota(m/N_{12})\}$ and those of $\mathcal{T}_0$ are given by $\exp\{\iota(m'/N_{03})\}$, where m (m') are integers defined modulo N_{12} (N_{03}). An arbitrary vector in the space of lattice fields can be expanded in this basis with coefficients $h_{m,m'}$. From the commutation relations the action of the remaining operators $\mathcal{T}_2$ and $\mathcal{T}_3$ can be derived by manipulations from Sec. V applied to each of the $1-2$ and $0-3$ planes.

For the general q, q' case, the centralizer of the translation group might be nontrivial, leading to different subspaces invariant under translations. The subspaces are labeled and mapped into each other by subgroups of the continuum group generated by the $K^{(\mu)}$. If L_1 and L_2 are divisible by q, and L_0 and L_3 are divisible by q', we obtain qq' subspaces and labellings, just as in the continuum. On the opposite extreme, when the lengths L_μ are coprime with the associated integers q or q', the subgroup is trivial and we have no invariant subspace. In the general case we work with that subgroup of the continuum group generated by $K^{(\mu)}$ which maps lattice points into other lattice points.

We set now $q = q' = 1$ and study the spectrum of four-dimensional lattice Dirac operators in the constant background field given by the links Eq. (142). We focus on the upper component, ψ^+, because, as in the continuum, the lower component can be obtained by charge conjugation.

The naive lattice Dirac operator is given by

$$D_N = \frac{1}{2} \sum_\mu \gamma_\mu(\mathcal{T}_\mu - \mathcal{T}_\mu^\dagger) = \begin{pmatrix} 0 & \hat{D}_L \\ \bar{D}_L & 0 \end{pmatrix} \qquad (144)$$

where we have defined the lattice Weyl operators in a way similar to the continuum. The nonzero spectrum of D_N can be obtained by studying the eigenvalues and eigenvectors of the $\hat{D}_L \bar{D}_L$ acting on Weyl spinor lattice fields. The spectrum of this operator is the same as that of $\bar{D}_L \hat{D}_L$. Unlike in the continuum, on the lattice the spectra are the same even including zero modes: for every positive chirality zero mode there is a negative chirality one.

In Sec. IV we related the spectrum of the four-dimensional Dirac operator to that of the two-dimensional case and this relation extends to the lattice. $\mathcal{T}_{1,2}$ commute with $\mathcal{T}_{0,3}$ and hence $\hat{D}_L \bar{D}_L$ becomes block-diagonal:

$$\hat{D}_L \bar{D}_L = -\begin{pmatrix} D_{12}^\dagger D_{12} + D_{03}^\dagger D_{03} & 0 \\ 0 & D_{12} D_{12}^\dagger + D_{03} D_{03}^\dagger \end{pmatrix} \qquad (145)$$

where

$$D_{12} = \frac{1}{2}[\mathcal{T}_1 - \mathcal{T}_1^\dagger + \iota(\mathcal{T}_2 - \mathcal{T}_2^\dagger)] \qquad (146)$$

$$D_{03} = \frac{1}{2}[\mathcal{T}_0 - \mathcal{T}_0^\dagger + \iota(\mathcal{T}_3 - \mathcal{T}_3^\dagger)]. \qquad (147)$$

D_{12} and D_{03} commute and our two-dimensional analysis applies [see Eq. (133)]. Let λ_{12} and λ_{03} be the eigenvalues of $D_{12}^\dagger D_{12}$ and $D_{03}^\dagger D_{03}$, respectively. Then the spectrum of the naive lattice Dirac operator is given by $\pm\iota\sqrt{\lambda_{12} + \lambda_{03}}$ and is twofold degenerate. For the corresponding naive Dirac Hamiltonian $H_N = \gamma_5 D_N$, the spectrum is given by the same formula without the ι factor. We emphasize that $D_{12}^\dagger D_{12} = H_{NR}^2$, where H_{NR} is the staggered reduced Hamiltonian appearing in Sec. V above Eq. (133). Eigenvalues and eigenvectors can be readily constructed in terms of those of H_{NR}.

The reduction of the four-dimensional naive lattice Dirac operator to two dimensions does not extend to the Wilson-Dirac or the overlap operator. Here, all components are coupled to each other. A complete analysis, which will not be included here, can be done along the lines of Sec. V.

VII. NUMERICAL AND ASYMPTOTIC RESULTS

In this section we will complement the results of the previous sections by providing some numerical and analytical results on the spectra of the naive lattice Dirac operator and the overlap Dirac operator [10,20] in a constant field strength gauge background for certain values of L_μ and topological charge. In particular, we will compare the eigenvectors of the lattice Dirac operators with their continuum counterparts. The numerical method used for the determination of the eigenvalues is an exact diagonalization procedure. This method, although precise and stable, is computationally costly, limiting what we can do.

We begin by describing our two-dimensional results. For the numerical determination we diagonalize the lattice Dirac operators (naive and overlap), with link variables given by

$$U_1(n) = \exp\left\{ \iota\pi q \frac{I(n_2, n_1)}{L_1 L_2} \right\} \qquad (148)$$

$$U_2(n) = \exp\left\{ -\iota\pi q \frac{I'(n_1, n_2)}{L_1 L_2} \right\} \qquad (149)$$

where $I(n_2, n_1) = n_2$ for $1 \leq n_1 < L_1$ and $I(n_2, L_1) = (L_1 + 1)n_2$ (and a similar relation for I' exchanging the 1-2 labels). The value of the plaquette is constant and equal to $\exp\{-\iota 2\pi(q/N)\}$, with $N = L_1 L_2$. The associated lattice Dirac operators (matrices) have dimension $2N$. The cases of very small and very large N can be studied analytically. For $N = 2$ the naive Dirac operator D_N vanishes. For $N = 4$ there

TABLE I. Eigenvalues of lattice Dirac operators for an $L_1=L_2=24$ lattice (see text for precise explanation of the entries).

Continuum	Naive	Ovp, $M=1$	Ovp, $M=0.75$	Ovp, $M=1.25$
0	0 [4]	0	0	0
1	0.9946 [4]	1.0018	0.9978	1.0027
2	1.9783 [4]	2.0071	1.9910	2.0108
10	9.4642 [4]	10.167	9.7723	10.265
20	17.893 [4]	20.615	19.080	21.040
50	37.371 [4]	52.902	44.316	56.123
100	57.805 [4]	106.45	78.832	121.57
200		199.11	128.617	259.15
500		350.07	202.842	523.46

are again 4 zero eigenvalues. In addition, there are two doubly degenerate pairs at $\pm\iota\sqrt{2}$. For $N=6$ there are three fourfold-degenerate eigenvalues at 0 and $\pm\iota\sqrt{3/2}$. We have numerically diagonalized the naive lattice Dirac operator D_N for several values of q and L_i. In Table I we display a selection of the eigenvalues for one typical case with $q=1$ and $L_1=L_2=24$. We dropped the complex factor ι from the eigenvalues and list only the positive branch on account of the spectral symmetry about zero. Since the continuum eigenvalues are given by $\sqrt{2Fn}$, where $F=2\pi q/N$ is the magnetic field, we display the square of the eigenvalues divided by $2F$, a dimensionless quantity. We see that the degeneracy of the eigenvalues (indicated inside square brackets) is equal to 4. Only a factor of 2 is exact at all even values of N. The extra twofold degeneracy is approximate, but extremely accurate for large values of N.

For large N the low-lying eigenvalues are well described by

$$\iota\lambda_n=\sqrt{2Fn}[1+CFn+O(F^2)] \tag{150}$$

with $C=-1/4$. The dependence on N and q only enters through F and we are close to the continuum limit. C was determined by expanding D_{12} to first subleading order in terms of continuum operators:

$$D_{12}=D_1+\iota D_2+\frac{1}{6}(D_1^3+\iota D_2^3)+\cdots. \tag{151}$$

In the one-dimensional quantum mechanical representation we get

$$D_{12}^\dagger D_{12}=2Fa^\dagger a-\frac{F^2}{6}[a^4+(a^\dagger)^4+6(a^\dagger a)^2]+\cdots. \tag{152}$$

The eigenvalues of $-D_{12}^\dagger D_{12}$ are the square of the eigenvalues of D_N, and perturbation theory predicts the observed value for C in Eq. (150).

We have also studied the eigenvalues and eigenvectors of the overlap operator D_O. D_O is constructed in terms of the unitary operator $\Omega=\tau_3 H_{WD}(-M,r)/|H_{WD}(-M,r)|$, where $H_{WD}(-M,r)$ is the Wilson Dirac Hamiltonian with mass $-M$ and Wilson parameter r. The eigenvalues of Ω are of

the form $e^{\iota\delta_O(M,r)}$ with the M and r dependence explicitly indicated. Chirality implies symmetric spectrum and only the positive values will be shown. The continuum eigenvalues are approached by $M\delta_O(M,r)$ and results are presented for the square of this quantity divided by $2F$. In terms of $\delta_O(M,r)$ the eigenvalues of the overlap operator are

$$M[1-\cos\delta_O(M,r)+\iota\sin\delta_O(M,r)]. \tag{153}$$

This is the operator whose inverse gives the propagator for internal fermions lines. External lines should be described instead by the exactly chiral invariant operator $D_I=2\iota M[(1+\Omega)/(1-\Omega)]$ [10,21]. The eigenvalues of the latter are

$$\iota 2M\tan[\delta_O(M,r)/2]. \tag{154}$$

Determining the eigenvalues for small values of N is somewhat more involved than for the naive operator, since now in addition we must keep track of the M and r dependence. For example, for $N=2, q=1$ there is a region of values of these parameters for which Ω is equal to I and another one in which it is equal to $-I$. For $2r^2-4Mr+M^2<0$ (which includes the point $r=M=1$) Ω has two eigenvalues equal to 1 and two equal to -1. Table I contains a selection of eigenvalues for $q=1$, $L_1=L_2=24$, $r=1$ and $M=0.75,1,1.25$. For large N, the behavior of the low-lying spectrum can be understood analytically. The eigenvalues are again given by Eq. (150) where the constant $C\equiv C_{\text{overlap}}(M,r)$ is

$$C_{\text{overlap}}(M,r)=-\frac{1}{4}+\frac{r}{M}-\frac{2}{3M^2}. \tag{155}$$

For $r=1$ and $M=0.75,1,1.25$ $C=-11/108,1/12,37/300$, respectively, which works well for the numbers in Table I.

Equation (155) is found by expanding the lattice operator in terms of the continuum one. We separate the Wilson-Dirac operator into additive terms as follows:

$$D_{WD}=-M+D_N+rW. \tag{156}$$

W stands for the Wilson term, which is of order a^2, while the naive operator D_N is order a. a is the lattice spacing; equivalently, we set the lattice spacing to unity and use powers of
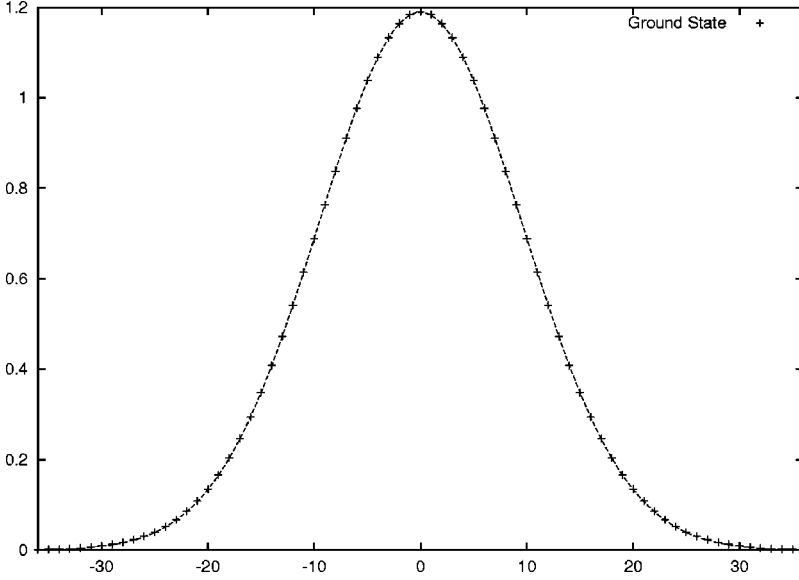
FIG. 1. The shape of the overlap (positive chirality) zero mode in the one-dimensional representation compared to the continuum Gaussian, for a two-dimensional constant field with $q=1$ and $L_1=L_2=24$.

$1/\sqrt{N}$ instead. We write $\Omega=\exp\{-\mathcal{E}/M\}$, where $\mathcal{E}$ is an anti-Hermitian matrix. In the continuum limit we get

$$\mathcal{E}=D_N+\frac{1}{3M^2}D_N^3+\frac{r}{2M}(D_NW+WD_N)+\cdots. \quad (157)$$

The above holds to order a^3. In the continuum limit, W can be expressed in terms of the Dirac operator D:

$$W=\frac{-1}{2}(D^2+\iota F_{\mu\nu}\gamma^\mu\gamma^\nu). \quad (158)$$

Expressions (157) and (158) are true in any dimension and for arbitrary background. In two-dimensions the anticommutator of W and D is given by

$$WD+DW=-[D^3+\iota(D_\mu F_{\mu\nu})\gamma^\nu]. \quad (159)$$

The second term is proportional to the Euclidean equations of motion and hence vanishes for any classical solution, including the uniform fields we are considering. Hence, the behavior of the eigenvalues in the continuum limit can be extracted from our previously derived results for the naive lattice Dirac operator. In this way we obtain Eq. (150) with the C of Eq. (155). The second and third terms in Eq. (157) are proportional to D^3 and therefore affect the eigenvalues without modifying the eigenvectors.

For typical values of the parameters M,r (including $r=M=1$) the lattice corrections are smaller for the overlap than for the naive Dirac operator. For example, for $r=1$ and $M=2(1-1/\sqrt{3})$ the leading $O(a^2)$ correction to the eigenvalues vanishes. This also follows from the numerical data in Table I, where the spectrum for $r=M=1$ is found to be fairly close to continuum for the lowest several hundred eigenvalues.

To conclude the description of the two-dimensional case we will show some results on the low-lying eigenvectors of the overlap ($r=M=1$). In Figs. 1 and 2 we display the shape of the positive chirality part of the ground state and the second excited state, respectively. The data are obtained from the numerical diagonalization at $L_i=24$. The resulting two-dimensional array of points is transformed to the one-dimensional representation and then compared to the continuum result, given by eigenstates of the Harmonic oscillator.

We turn now to four dimensions. Our numerical work is done for $SU(2)$ and for gauge fields whose only nonzero components are $F_{12}=-F_{21}=F\tau_3$ and $F_{03}=-F_{30}=F'\tau_3$. The color components decouple so we essentially are dealing with $U(1)$ systems. As explained in Sec. VI, the eigenvalue problem for the naive Dirac operator reduces to the two-dimensional problem. Here we focus on the overlap operator, which does not simplify similarly.

Our objective is to find the leading corrections to the low-lying spectrum when continuum is approached. Equations (157), (158) are valid now, but Eq. (159) is replaced by

$$WD+DW=-[D^3+\iota(D_\mu F_{\mu\nu})\gamma^\nu+\iota\epsilon^{\alpha\rho\mu\nu}\gamma^5\gamma^\alpha F_{\mu\nu}D_\rho]$$

$$(160)$$

where we have made use of the Bianchi identities. The second term vanishes for gauge configurations that solve the classical equations of motion. The last term was absent in two dimensions. For a uniform field this term commutes with D. All the terms in $\mathcal{E}$ anticommute with γ_5 and are anti-Hermitian, so that in the chiral basis $\mathcal{E}$ has the form

$$\mathcal{E}=\begin{pmatrix} 0 & -X^\dagger \\ X & 0 \end{pmatrix} \quad (161)$$

with X given by

$$X=\bar{D}_N+\left(\frac{1}{3M^2}-\frac{r}{2M}\right)\bar{D}\hat{D}\bar{D}-\frac{r}{M}(F(D_0+\iota\tau_3D_3)$$

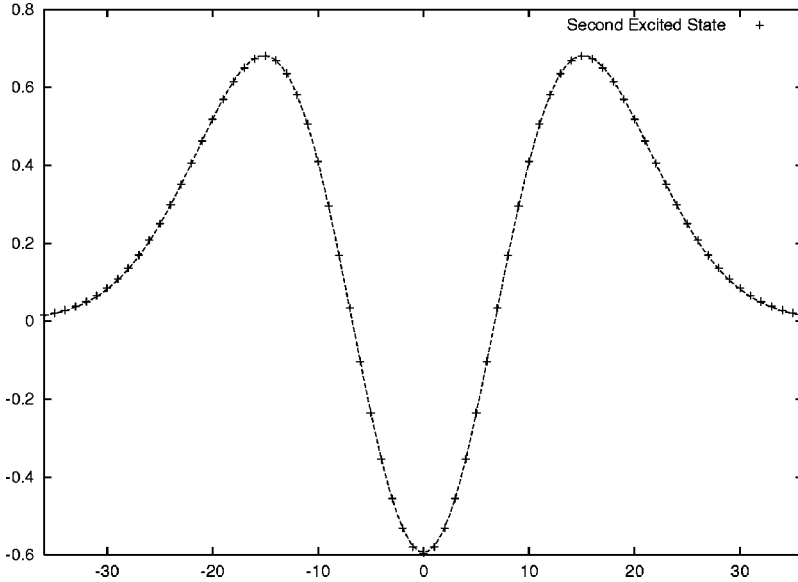$$+F'(\iota\tau_1D_1+\iota\tau_2D_2))\tau_3+\cdots. \quad (162)$$

FIG. 2. The shape of the positive chirality component of the second excited state of the overlap compared with the continuum prediction, for $q=1$ and $L_1=L_2=24$.

The eigenvalues of $X^\dagger X$ are obtained by writing the covariant derivatives in terms of creation and annihilation operators and using perturbation theory. With our choice of gamma matrices $X^\dagger X$ is already diagonal in the spin space. The $1-1$ component of $X^\dagger X$ is readily expressed in terms of the two number operators. The corresponding eigenvalue of $X^\dagger X$ becomes

$$2Fn_{12}+2F'n_{03}+4\left(-\frac{1}{4}+\frac{r}{M}-\frac{2}{3M^2}\right)(Fn_{12}+F'n_{03})^2$$

$$-\frac{4r}{M}(F^2n_{12}+F'^2n_{03}) \tag{163}$$

where n_{12}, n_{03} are non-negative integers. The first two terms give the continuum result, the third one is of a type familiar from the two-dimensional case, while the fourth one is genuinely four dimensional. The $2-2$ spin component of $X^\dagger X$ gives another set of eigenvalues given by a formula identical to Eq. (163) except that the last term has the opposite sign and n_{12}, n_{03} are now integers strictly larger than zero. The sign change in the last term lifts the degeneracy of the continuum eigenvalues. The eigenvalues of $\mathcal{E}$ are given by those of $\pm\imath\sqrt{X^\dagger X}$.

We have numerically determined the eigenvalues of the overlap operator in four dimensions. Our exact diagonalization methods cannot be carried out for very large sizes and a comparison with the previously derived formulas is difficult.

VIII. SUMMARY

With one exception, any $U(1)$ or $SU(N)$ gauge field on a two-dimensional or four-dimensional Euclidean torus can be smoothly deformed to an Abelian background. When the background is topologically nontrivial the Abelian field can be deformed to a non-vanishing uniform magnetic field. Any topological invariant is therefore captured by a representative with uniform Abelian background. The Dirac equation in these backgrounds can be viewed as an equation over infinite space but in half the dimension. This naturally leads to an exact diagonalization in terms of harmonic oscillator wave functions. In the massless case one gets explicit formulas for the expected zero modes. Much of this translates to the lattice; for example, the diagonalization of the Wilson Dirac operator and the associated overlap Dirac operator on a two-dimensional torus simplifies to an equation on a one-dimensional circle of length fixed by the area of the original torus. In these backgrounds the connection between fermions and topology becomes particularly transparent both in the continuum and on the lattice, once one employs overlap fermions there.

[1] L. Landau, Z. Phys. **64**, 629 (1930).

[2] A.A. Abrikosov, Sov. Phys. JETP **32**, 1442 (1957) [reprinted in *Solitons and Particles,* edited by C. Rebbi and G. Soliani (World Scientific, Singapore, 1984)].

[3] M.J. Duff and M. Ramón-Medrano, Phys. Rev. D **12**, 376 (1975); I.A. Batalin, S.G. Matinyan, and G.K. Saviddy, Yad. Fiz. **26**, 407 (1977); S.G. Matinyan and G.K. Saviddy, Nucl. Phys. **B134**, 539 (1978); G.K. Saviddy, Phys. Lett. **71B**, 133 (1977); H. Pagels and E. Tomboulis, Nucl. Phys. **B143**, 485 (1978); N.K. Nielsen and P. Olesen, *ibid.* **B144**, 376 (1978); Phys. Lett. **79B**, 304 (1978).

[4] H. Leutwyler, Phys. Lett. **96B**, 154 (1980); Nucl. Phys. **B179**, 129 (1981).

[5] G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).

[6] P. van Baal, Commun. Math. Phys. **94**, 397 (1984).

[7] P. van Baal, Commun. Math. Phys. **85**, 529 (1982).

[8] J. Smit and J. Vink, Nucl. Phys. **B286**, 485 (1987).

[9] D.B. Kaplan, Phys. Lett. B **288**, 342 (1992).

[10] R. Narayanan and H. Neuberger, Phys. Lett. B **302**, 62 (1993); Phys. Rev. Lett. **71**, 3251 (1993); Nucl. Phys. **B443**, 305 (1995); H. Neuberger, Phys. Rev. D **57**, 5417 (1998); Annu. Rev. Nucl. Part. Sci. **51**, 23 (2001).

[11] C.R. Gattringer, I. Hip, and C.B. Lang, Nucl. Phys. **B508**, 329 (1997).

[12] Ting-Wai Chiu, Phys. Rev. D **58**, 074511 (1998).

[13] C. Gattringer and I. Hip, Nucl. Phys. **B536**, 363 (1998).

[14] M. García Perez, A. González-Arroyo, and C. Pena, J. High Energy Phys. **09**, 033 (2000).

[15] M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2002).

[16] A. González-Arroyo and M. Okawa, Phys. Lett. **120B**, 174 (1983); Phys. Rev. D **27**, 2397 (1983); T. Eguchi and R. Nakayama, Phys. Lett. **122B**, 59 (1983); A. González-Arroyo and C.P. Korthals Altes, *ibid.* **131B**, 396 (1983).

[17] L. Susskind, hep-th/0101029; S. Hellerman and M. Van Raamsdonk, J. High Energy Phys. **10**, 039 (2001).

[18] J. Groeneveld, J. Jurkiewicz, and C.P. Korthals Altes, Phys. Scr. **23**, 1022 (1981); J. Ambjorn and H. Flyvberg, Phys. Lett. **97B**, 241 (1980); B. van Geemen and P. van Baal, J. Math. Phys. **27**, 455 (1986); D.R. Lebedev and M.I. Polikarpov, Nucl. Phys. **B269**, 285 (1986).

[19] A. González-Arroyo, *Gauge Fields on the Four-dimensional Torus*, Proceedings of the Peñiscola 1997 Advanced School on Non-Perturbative Quantum Field Physics (World Scientific, Singapore, 1998).

[20] H. Neuberger, Phys. Lett. B **417**, 141 (1998); **427**, 353 (1998).

[21] H. Neuberger, Nucl. Phys. B (Proc. Suppl.) **73**, 697 (1999).