

Domain wall fermion and chiral gauge theories on the lattice with exact gauge invariance

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We discuss how to construct anomaly-free chiral gauge theories on the lattice with exact gauge invariance in the framework of a domain wall fermion. Chiral gauge coupling is realized by introducing a five-dimensional gauge field which interpolates between two different four-dimensional gauge fields at boundaries. The five-dimensional dependence is compensated by a local and gauge-invariant counterterm. The cohomology problem to obtain the counterterm is formulated in $(5+1)$ -dimensional space, using the Chern-Simons current induced from the five-dimensional Wilson fermion. We clarify the connection to the invariant construction based on the Ginsparg-Wilson relation using the overlap Dirac operator. The formulas for the measure and the effective action of Weyl fermions are obtained in terms of five-dimensional lattice quantities.

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I. INTRODUCTION

Through the Ginsparg-Wilson relation [1] and the exact chiral symmetry based on it [2], the Weyl fermion can naturally be introduced on the lattice. The chiral constraint imposed on the Weyl fermion is gauge-field dependent and, by introducing the basis of the Weyl fermion, the path integral can be set up [3,4]. In fact, it has been shown by Lüscher that the functional measure of the Weyl fermion can be constructed in anomaly-free Abelian chiral gauge theories so that it satisfies the requirements of smoothness, locality, and gauge invariance [3,5–7]. A similar construction has also been argued for generic non-Abelian chiral gauge theories, where, to treat the exact cancellation of gauge anomaly, a local cohomology problem in $4+2$ dimensions is formulated [4].¹

This construction is generic and applies to any local lattice fermion theory with the Dirac operator satisfying the Ginsparg-Wilson relation. In the case using the overlap Dirac operator [14–16], the path integral formalism for the Weyl fermion reproduces the overlap formula for the chiral deter-

minant by Narayanan and Neuberger [17].² The above invariant construction of the functional measure provides the method to fix the phase factor of the chiral determinant in the overlap formalism in a gauge-invariant manner.

It was also suggested [4,28] that there is a close relationship between the interpolation procedure in Lüscher's construction and the five-dimensional setup in Kaplan's domain wall fermion [29]. The purpose of this paper is to pursue this close connection and to show how to construct four-dimensional lattice chiral gauge theories with exact gauge invariance from the five-dimensional lattice framework of domain wall fermion.

For this purpose, we adopt the domain wall fermion in the vectorlike formalism by Shamir [30]. But two different four-dimensional gauge fields are introduced at the boundaries and they are interpolated by a five-dimensional gauge field. This inevitably causes the five-dimensional dependence of the partition function of the domain wall fermion. To take account of this five-dimensional dependence, we formulate an integrability condition. It turns out that the dependence is

²The author refers the reader to [18] for recent review of the overlap formalism. In the overlap formalism, reflecting chiral anomaly, the phase factor of the chiral determinant is not fixed in general and any reasonable choice of the phase factor should lead to the gauge anomaly for single Weyl fermion. The Wigner-Brillouin phase convention has been adopted for perturbative studies [19] and has also been tested numerically in a nonperturbative formulation of chiral gauge theories [20,21]. Geometrical treatment of the gauge anomaly in the overlap formalism has been discussed in detail in Abelian theories [22] and non-Abelian theories [23]. The $SU(2)$ global anomaly has been examined in [24]. An adiabatic phase choice has been proposed in [25] and used in the construction of noncompact abelian chiral gauge theories. The overlap formalism in odd dimensions has been considered in [16,26,27].

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¹The author refers the reader to [8,9] for recent review of this approach. In this approach, the exact cancellation of gauge anomalies in non-Abelian chiral gauge theories has been shown in all orders of the expansion in lattice perturbation theory [10,11]. For $SU(2)$ doublet, it has been shown that Witten's global anomaly is reproduced [12]. For $SU(2)_L \times U(1)_Y$ electroweak theory, the local cohomology problem in $4+2$ dimensions has to be solved in infinite volume lattice and the exact cancellation of gauge anomalies, including the mixed type, has been shown nonperturbatively [13].

governed by the lattice Chern-Simons term induced from the five-dimensional Wilson-Dirac fermion (with a negative mass) [31].

In order to compensate the five-dimensional dependence, we require a five-dimensional counter term. The counter term should be given by a smooth, local and gauge invariant functional of gauge field, in order to satisfy the requirement of the smoothness, the locality and the gauge invariance of the low energy effective action. We will argue that such local, gauge-invariant field can be obtained in anomaly-free chiral gauge theories, through the local cohomology problem in $(5+1)$ -dimensional space formulated with the lattice Chern-Simons current. Thus the reduction from the five-dimensional lattice to four-dimensional lattice is achieved in a local and gauge invariant manner.

The locality of the lattice Chern-Simons current is essential for the cohomological argument in $(5+1)$ -dimensional space and for this we require the so-called admissibility condition [3–5,32] extended to five-dimensional gauge fields (cf. [35]). With this condition, several properties of the Chern-Simons current are discussed. The earlier studies of the properties of the Chern-Simons current in the context of domain wall fermion can be found in [36,37].

Trying to formulate four-dimensional chiral gauge theories from the five-dimensional framework of domain wall fermion, our approach resembles to the wave-guide model [38,39] and the formalism proposed by Creutz *et al.* [40,41]. However, our approach is different from the wave-guide model in that we are considering the smooth (discrete, but smooth in lattice scale) interpolation in the fifth direction. The issue related to the disordered gauge degrees of freedom is taken account by the five-dimensional admissibility condition, which assures the existence of the chiral zero modes even with five-dimensional gauge fields (cf. [42–44]). Our approach is also different from the formalism by Creutz *et al.* in that we are isolating the chiral zero modes at one boundary as physical degrees of freedom, regarding the other boundary as reference.

This paper is organized as follows. In Sec. II we formulate the domain wall fermion for chiral gauge theories with the interpolating five-dimensional gauge field. Then we derive the integrability condition for the partition function of the domain wall fermion and state a sufficient condition to obtain the five-dimensional counterterm with the required properties. In Sec. III we examine the properties of the lattice Chern-Simons current. In Sec. IV we argue how to reconstruct the counterterm from the Chern-Simons current and formulate the local cohomology problem in $(5+1)$ -dimensional space. Section V is devoted to the discussions on the connection to the gauge-invariant construction based on the Ginsparg-Wilson relation.

II. DOMAIN WALL FERMION FOR CHIRAL GAUGE THEORIES

A. Interpolation with five-dimensional gauge field

Domain wall fermion, in its simpler vectorlike formulation, is defined by the five-dimensional Wilson-Dirac fermion with a negative mass in a finite extent fifth dimension.

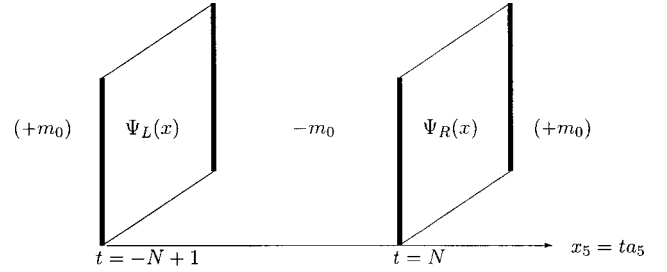


FIG. 1. Domain wall fermion in the simpler vector-like setup.

(See Fig. 1.) The four-dimensional lattice spacing a and the five-dimensional one a_5 are both set to unity. The fifth coordinate is denoted by t and takes integer values in the interval, $t \in [-N+1, N]$. In four dimensions, the lattice is assumed to have a finite volume L^4 and the periodic boundary condition is assumed for both fermion and gauge fields. The mass term is set to the negative value $-m_0$ where $0 < m_0 < 2$:

$$S_{\text{DW}} = \sum_{t=-N+1}^N \sum_x \bar{\psi}(x,t) (D_{5\text{w}} - m_0) \psi(x,t). \quad (2.1)$$

This setup is equivalent to impose the Dirichlet boundary condition at the boundaries in the fifth dimension as

$$\psi_R(x,t)|_{t=-N}=0, \quad \psi_L(x,t)|_{t=N+1}=0. \quad (2.2)$$

In order to introduce chiral-asymmetric gauge interaction for the chiral zero modes at the two boundaries $t = -N+1$ and N , the gauge field is assumed to be five dimensional:

$$U_\mu(z) = \{U_k(x,t), U_5(x,t)\}, \quad z = (x,t), \quad (2.3)$$

where $\mu = 1, \dots, 5$ and $k = 1, \dots, 4$. It is regarded to be interpolating a four-dimensional gauge field at $t = -N+1$, say $U_k^0(x)$, to another four-dimensional gauge field at $t = N$, say $U_k^1(x)$. We assume that outside the finite interpolation region $t \in [-\Delta, \Delta]$ ($\Delta < N$) the gauge field does not depend on t and $U_5(x,t) = 1$ (Fig. 2). Δ should be chosen large enough in order to make sure that the interpolation is smooth enough. The precise condition for this will be discussed below.

The gauge fields at the boundaries, $U_k^0(x)$ and $U_k^1(x)$, are chosen so that their field strengths are small enough and satisfy the following bound:

$$\|1 - P_{kl}(x)\| < \epsilon, \quad \epsilon < \frac{1}{30} \{1 - |1 - m_0|^2\}, \quad (2.4)$$

where $P_{kl}(x)$ is the four-dimensional plaquette variable. This is the so-called admissibility condition [3–5,32]. With the

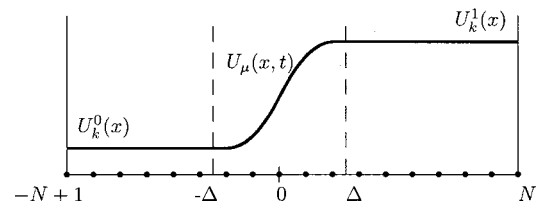


FIG. 2. Interpolating five-dimensional gauge field on the lattice.

admissibility condition, the space of the gauge fields is divided into topological sectors [33]. We assume that $U_k^0(x)$ and $U_k^1(x)$ are in the same topological sector. Furthermore, it ensures that the Hamiltonian defined through the transfer matrix of the five-dimensional Wilson-Dirac fermion, $H = -\ln T$, has a finite gap and the overlap Dirac operator (the effective four-dimensional Dirac operator of the boundary chiral modes) is local within the exponentially suppressed tail [32,34,35]. It also makes the limit that the size of the fifth dimension goes to infinity, $N \rightarrow \infty$, well-defined in vectorlike case [30,35].

When the gauge fields on the boundary walls, $U_k^0(x)$ and $U_k^1(x)$, are in the same topological sector, there exists a smooth path in the space of the admissible gauge fields. This one-parameter family of the gauge fields can be regarded as a five-dimensional gauge field and we may map this configuration onto the five-dimensional lattice. Then the difference of the four-dimensional gauge fields at the boundaries is estimated as

$$\|U_k^0(z) - U_k^1(z)\| \approx 2\Delta \|1 - P_{5k}(z)\|, \quad z = (x, t), \quad (2.5)$$

where $P_{\mu\nu}(z)$ is the five-dimensional plaquette variable. By taking Δ large enough, we can make $1 - P_{5k}(z)$ small enough. Therefore we may assume that the whole five-dimensional gauge field $U_\mu(x, t)$, interpolating between $U_k^0(x)$ and $U_k^1(x)$, is smooth enough and satisfies the five-dimensional bound on the field strength as follows:

$$\|1 - P_{\mu\nu}(z)\| < \epsilon', \quad \epsilon' < \frac{1}{50} \{1 - |1 - m_0|^2\}. \quad (2.6)$$

This condition, as we will see below, ensures that the Chern-Simons current induced from the five-dimensional Wilson-Dirac fermion is a local functional of the gauge field. The locality of the Chern-Simons current allows us the cohomological analysis of the gauge anomaly in the context of domain wall fermion.

The above argument is not complete in the sense that it does not provide the constructive proof of the existence of an interpolating five-dimensional gauge field for any admissible gauge fields at the boundaries. The locality properties of such an interpolating five-dimensional field should also be clarified. These questions are beyond the scope of this paper.

We also note a symmetry property of the five-dimensional Wilson-Dirac fermion. By the reflection of the gauge field in the fifth direction,

$$U_\mu(x, t) \rightarrow U'_\mu(x, t) = \begin{cases} U'_k(x, t) = U_k(x, -t+1) \\ U'_5(x, t) = U_5(x, -t+1)^{-1}, \end{cases} \quad (2.7)$$

the five-dimensional Wilson-Dirac operator transforms as follows:

$$D_{5w} \rightarrow D'_{5w} = P \gamma_5 D_{5w}^\dagger \gamma_5 P, \quad (2.8)$$

where P is the parity transformation operator in the fifth dimension:

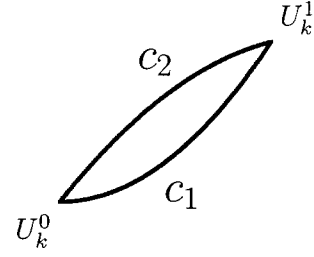


FIG. 3. A loop in the space of gauge fields.

$$P: t \rightarrow t' = -t + 1. \quad (2.9)$$

Therefore the interpolation in the reversed order implies the complex conjugation.

B. Integrability condition for domain wall fermion

Now we consider the partition function of the domain wall fermion,

$$\det(D_{5w} - m_0)|_{\text{Dir}}. \quad (2.10)$$

and examine its dependence on the path of the interpolation. Let us denote by c_1 the original choice of the path of the interpolation, which is represented by the five-dimensional gauge field $U_\mu(x, t)$. In order to examine the dependence on the path, we introduce another path, say c_2 and consider the difference of the logarithm of the partition function:

$$\ln \det(D_{5w} - m_0)|_{\text{Dir}}^{c_2} - \ln \det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}. \quad (2.11)$$

Let us assume first that the five-dimensional gauge field representing the path c_2 can be deformed to that representing the path c_1 , while satisfying the constraint on the five-dimensional plaquette variables Eq. (2.6). Whether this is always possible for any two paths depends on the topological structure of the space of the admissible gauge fields in consideration. Since the two paths can be regarded to form a loop in the space of the gauge fields (Fig. 3), the above condition is equivalent to whether any loops in the space of the gauge fields can be contractible to the point, or not. The case with the noncontractible loops will be discussed later.

We then parametrize the smooth deformation of the path by the parameter $s \in [0, 1]$ as $U_\mu^s(z)$ where $U_\mu^{s=0}(z) = U_\mu(x)^{c_1}$ and $U_\mu^{s=1}(z) = U_\mu(z)^{c_2}$. By differentiating the partition function with respect to the parameter s and integrating back, we obtain an expression for the difference of the partition function,

$$\begin{aligned} & \ln \det(D_{5w} - m_0)|_{\text{Dir}}^{c_2} - \ln \det(D_{5w} - m_0)|_{\text{Dir}}^{c_1} \\ &= \int_0^1 ds \sum_x \sum_{t=-\Delta+1}^{\Delta} \{ \eta_\mu^a(xt, s) J_\mu^a(xt, s) |_{\text{Dir}} \}, \end{aligned} \quad (2.12)$$

where $\eta_\mu^a(z, s) T^a = \partial_s U_\mu^s(z) U_\mu^s(z)^{-1}$ and $J_\mu^a(z) |_{\text{Dir}}$ is given by

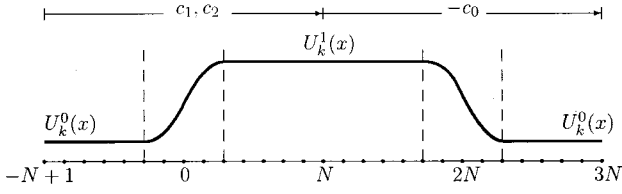


FIG. 4. Five-dimensional gauge field representing the loop.

$$J_\mu^a(z)|_{\text{Dir.}} = \text{Tr} \left(V_\mu^a(z) \frac{1}{D_{5\text{w}} - m_0} \right) \Big|_{\text{Dir.}}, \quad (2.13)$$

$$V_\mu^a(z) = \left\{ \frac{1}{2} (\gamma_\mu - 1) T^a U_\mu(z) \delta_{z z_1} \delta_{z_1 + \hat{\mu}, z_2} + \frac{1}{2} (\gamma_\mu + 1) U_\mu(z)^{-1} \times T^a \delta_{z z_2} \delta_{z_1, z_2 + \hat{\mu}} \right\}. \quad (2.14)$$

Since the variation of the gauge field $\eta_\mu^a(z, s)$ is restricted in the interpolation region $t \in [-\Delta + 1, \Delta]$, the summation over t in the above expression is restricted in the same finite region.

Because of this fact, as will be shown in detail in Appendix D, the difference Eq. (2.12) is well-defined and finite in the limit $N \rightarrow \infty$, as long as the gauge fields at the boundaries, U_k^0 and U_k^1 , are topologically trivial and do not cause any accidental zero modes of the low energy effective Dirac operator. In particular, the inverse five-dimensional Wilson-Dirac operator in Eq. (2.13), satisfying the Dirichlet boundary condition, can be replaced by the inverse five-dimensional Wilson-Dirac operator defined in the infinite extent of the fifth dimension,

$$\frac{1}{D_{5\text{w}} - m_0} \Big|_{\text{Dir.}}(xt, yt') \Rightarrow \frac{1}{D_{5\text{w}} - m_0}(xt, yt'), \quad (2.15)$$

where $t, t' \in [-\Delta + 1, \Delta]$. This implies that the path dependence of the partition function of the domain wall fermion actually does not depend on the specific choice of the Dirichlet boundary condition, which supports the chiral zero modes at the boundaries.

In order to make this point clear and to formulate an integrability condition for the domain wall fermion, we introduce the five-dimensional Wilson-Dirac fermion defined in the interval $t \in [-N + 1, 3N]$ in the fifth dimension and with the antiperiodic boundary condition. The five-dimensional gauge field which couples to this Wilson-Dirac fermion is assumed to form a loop in the space of the gauge field so that it goes along the path c_1 (or c_2) and comes back along a certain path c_0 (Fig. 4). Then we can infer that, in the limit $N \rightarrow \infty$,

$$\begin{aligned} & \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{Dir.}}^{c_2} - \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{Dir.}}^{c_1} \\ &= \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_2 + (-c_0)} - \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_1 + (-c_0)} \end{aligned}$$

$$(N \rightarrow \infty). \quad (2.16)$$

In the right-hand side (RHS) of this identity, we may choose any paths for c_0 . By choosing c_2 and c_1 , respectively, we can further obtain two identities as follows:

$$\begin{aligned} \text{Eq. (2.16)} &= \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_2 + (-c_2)} \\ &\quad - \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_1 + (-c_2)} \\ &= \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_2 + (-c_1)} \\ &\quad - \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_1 + (-c_1)}. \end{aligned} \quad (2.17)$$

By averaging these two expressions and using the property of the five-dimensional Wilson-Dirac operator under the reflection $t \rightarrow -t + 1$, we finally obtain

$$\begin{aligned} & \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{Dir.}}^{c_2} - \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{Dir.}}^{c_1} \\ &= \frac{1}{2} \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_2 + (-c_2)} \\ &\quad - \frac{1}{2} \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_1 + (-c_1)} \\ &\quad + i \text{Im} \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_2 + (-c_1)} \quad (N \rightarrow \infty). \end{aligned} \quad (2.18)$$

This identity may be regarded as the integrability condition for the domain wall fermion.

Several comments about this result are in order. First of all, in the RHS of this identity, the partition functions are defined with the antiperiodic boundary condition and it is clear that there is no contribution from the low-lying chiral modes. They are defined well for any topological sector of U_k^0 and U_k^1 . Secondly, the first and second terms in the RHS are turned out to be real by the reflection property and they depend on the single paths c_2 and c_1 , respectively. These terms then can be used to subtract the bulk contribution in the partition function of the domain wall fermion as

$$\begin{aligned} & \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{Dir.}}^{c_i} - \frac{1}{2} \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_i + (-c_i)} \\ & \quad (i = 1, 2). \end{aligned} \quad (2.19)$$

Thirdly, an important observation is that the imaginary part of the RHS is the complex phase induced from the five-dimensional Wilson-Dirac fermion with a negative mass, which has been known to reproduce the Chern-Simons term in the classical continuum limit [31]. We denote this lattice Chern-Simons term associated with the loop $c_2 + (-c_1)$ as $Q_{5\text{w}}^{c_2 + (-c_1)}$:

$$Q_{5\text{w}}^{c_2 + (-c_1)} \equiv \lim_{N \rightarrow \infty} \text{Im} \ln \det(D_{5\text{w}} - m_0) \Big|_{\text{AP}}^{c_2 + (-c_1)}. \quad (2.20)$$

C. A sufficient condition for the path independence of the domain wall fermion

Now we discuss the requirement to obtain the partition function of the domain wall fermion which does not depend on the path of the interpolation and which is gauge invariant. A sufficient condition for this can be stated as follows:

It is possible to construct a local counterterm which makes the partition function of the domain wall fermion independent of the path of the interpolation in a gauge-invariant manner, if the lattice Chern-Simons term in the RHS of the integrability condition Eq. (2.18) can be expressed in the form

$$Q_{5w}^{c_2+(-c_1)} = \lim_{N \rightarrow \infty} \sum_{t=-N+1}^{3N} \sum_x q_{5w}(z), \quad (2.21)$$

where $q_{5w}(z)$ is a smooth, local, and gauge-invariant functional of the five-dimensional gauge field, for all possible loops in the space of gauge fields.

We first assume that this is the case and discuss the consequences of this condition. In the next sections, we will discuss how to obtain the local and gauge-invariant functional $q_{5w}(z)$.

1. Local counterterm

The immediate consequence of the localization property of $q_{5w}(z)$ is that $Q_{5w}^{c_2+(-c_1)}$ can be decomposed into two parts $C_{5w}^{c_2}$ and $C_{5w}^{(-c_1)} = -C_{5w}^{c_1}$, which are associated with the paths c_1 and c_2 , respectively, and do not depend on other paths. Namely,

$$Q_{5w}^{c_2+(-c_1)} = C_{5w}^{c_2} - C_{5w}^{c_1}, \quad (2.22)$$

where

$$C_{5w}^{c_1} = \lim_{N \rightarrow \infty} \sum_{t=-N+1}^N \sum_x q_{5w}(z), \quad (2.23)$$

$$-C_{5w}^{c_2} = \lim_{N \rightarrow \infty} \sum_{t=+N+1}^{3N} \sum_x q_{5w}(z). \quad (2.24)$$

Then from the integrability condition Eq. (2.18) we can infer that in the limit $N \rightarrow \infty$

$$\begin{aligned} & \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_2}}{|\det(D_{5w} - m_0)|_{\text{AP}}^{c_2+(-c_2)}|^{1/2}} e^{-iC_{5w}^{c_2}} \\ &= \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}}{|\det(D_{5w} - m_0)|_{\text{AP}}^{c_1+(-c_1)}|^{1/2}} e^{-iC_{5w}^{c_1}}. \end{aligned} \quad (2.25)$$

This holds for any two paths. Therefore the (subtracted) partition function of the domain wall fermion plus the local counter term does not depend on the path of the interpolation and is determined uniquely by the four-dimensional gauge

fields, U_k^0 and U_k^1 . Thus the reduction from the five-dimensional lattice to the four-dimensional lattice is achieved.

We will see in Sec. V that this result holds for all topological sectors, through the more direct calculation of the partition function of the domain wall fermion.

2. Gauge invariance

Let us examine the gauge transformation property of the (subtracted) partition function of the domain wall fermion with the local counter term under the gauge transformation for U_k^1 ,

$$U_k^1(x) \rightarrow {}^g U_k^1(x) = g(x) U_k^1(x) g(x + \hat{k})^{-1}. \quad (2.26)$$

Since the partition function does not depend on the interpolation between ${}^g U^1$ and U^0 , one may choose any interpolation path between them. In this case, it turns out to be convenient to choose the interpolation as follows: we first interpolate the gauge function as

$$G(z) = \begin{cases} g(x), & t \in [\Delta + 1, N], \\ g(x, t), & t \in [-\Delta + 1, \Delta], \\ 1, & t \in [-N + 1, -\Delta], \end{cases} \quad (2.27)$$

and then apply it as a five-dimensional gauge transformation to the gauge field representing the original interpolation between U_k^1 and U_k^0

$$U_\mu(z) \rightarrow {}^G U_\mu(z) = G(z) U_\mu(z) G(z + \hat{\mu})^{-1}. \quad (2.28)$$

In Eq. (2.25), the partition functions of the five-dimensional Wilson-Dirac fermions with both boundary conditions are invariant under such five-dimensional gauge transformation. Then the gauge transformation is given solely by the gauge variation of the counterterm associated with the path c_1 :

$$\delta_G C_{5w}^{c_1} = \lim_{N \rightarrow \infty} \sum_{t=-N+1}^N \sum_x \delta_G q_{5w}(z). \quad (2.29)$$

Thus the question of the gauge invariance of the system reduces to the question of the gauge invariance of the counterterm associated with the path c_1 . When $q_{5w}(z)$ is a gauge invariant local field, it is gauge invariant.

III. PROPERTIES OF LATTICE CHERN-SIMONS CURRENT

In this section, we discuss properties of the Chern-Simons current which is obtained from the lattice Chern-Simons term by the variation with respect to gauge field. We will argue that the lattice Chern-Simons current is a smooth and local functional of the gauge field.

A. Chern-Simons current

Let us consider the smooth deformation of the five-dimensional gauge field $U_\mu(z)$ representing the loop in the Chern-Simons term. Under the deformation

$$\delta U_\mu(z) U_\mu(z)^{-1} = T^a \eta_\mu^a(z), \quad \text{Tr}\{T^a T^b\} = -\frac{1}{2} \delta^{ab}, \quad (3.1)$$

the variation of the lattice Chern-Simons term is given by

$$\delta \text{Im Tr Ln}(D_{5w} - m_0)|_{AP} = \text{Im Tr} \left(\delta D_{5w} \frac{1}{D_{5w} - m_0} \right) \Big|_{AP} \quad (3.2)$$

$$= \sum_z \eta_\mu^a(z) J_\mu^a(z), \quad (3.3)$$

where

$$J_\mu^a(z) = \text{Im Tr} \left(V_\mu^a(z) \frac{1}{D_{5w} - m_0} \right) \Big|_{AP}, \quad (3.4)$$

$$V_\mu^a(z) = \left\{ \frac{1}{2} (\gamma_\mu - 1) T^a U_\mu(z) \delta_{z_1, z_2 + \hat{\mu}} + \frac{1}{2} (\gamma_\mu + 1) \right. \\ \left. \times U_\mu(z)^{-1} T^a \delta_{z_2, z_1 + \hat{\mu}} \right\}. \quad (3.5)$$

We refer to the current $J_\mu^a(z)$ as the Chern-Simons current.

Since the Chern-Simons current is defined by the variation of the Chern-Simons term which is the gauge-invariant functional of the five-dimensional gauge field, it is gauge-covariant, integrable and conserved:³

$$\sum_z \eta_\mu^a(z) \delta_\zeta J_\mu^a(z) - \sum_z \zeta_\mu^a(z) \delta_\eta J_\mu^a(z) \\ - \sum_z \{ [\eta_\mu(z), \zeta_\mu(z)] \}^a J_\mu^a(z) = 0, \quad (3.6)$$

$$\{ D_\mu^* J_\mu \}^a(z) = 0. \quad (3.7)$$

B. Locality of the Chern-Simons current

Next we argue that the Chern-Simons current is a local functional of the five-dimensional gauge field, as long as the constraint on the five-dimensional plaquette variables Eq. (2.6) is satisfied. This fact follows from the following consideration. With the bound Eq. (2.6), we infer that the five-dimensional Wilson-Dirac operator is bounded from below by a positive constant [32,35],

$$(D_{5w} - m_0)^\dagger (D_{5w} - m_0) \geq \{ (1 - 50\epsilon)^{1/2} - |1 - m_0| \}^2. \quad (3.8)$$

³ ∂_μ and ∂_μ^* denote the forward and backward difference operators, respectively, $\partial_\mu f(x) = \{ f(x + \hat{\mu}) - f(x) \}$ and $\partial_\mu^* f(x) = \{ f(x) - f(x - \hat{\mu}) \}$, while D_μ and D_μ^* denote the covariant counterparts, $D_\mu f(z) = \{ U_\mu(z) f(z + \hat{\mu}) U_\mu(z)^{-1} - f(z) \}$ and $D_\mu^* f(z) = \{ f(z) - U_\mu(z - \hat{\mu}) f(z - \hat{\mu}) U_\mu(z - \hat{\mu})^{-1} \}$.

Given the positive lower and upper bounds for the five-dimensional Wilson-Dirac operator,

$$\tilde{\alpha} \leq (D_{5w} - m_0)^\dagger (D_{5w} - m_0) \leq \tilde{\beta}, \quad (3.9)$$

it follows that the inverse five-dimensional Wilson-Dirac operator in the infinite lattice decays exponentially at large distance in the five dimensions [32,35]:

$$\| \{ D_{5w}^\dagger D_{5w} \}^{-1}(z, w) \| \leq C \exp \left\{ -\frac{\tilde{\theta}}{2} d_5(z, w) \right\}, \quad (3.10)$$

where $d_5(z, w) = |z - w| = |x - y| + |s - t|$ and

$$C = \frac{4t}{\tilde{\beta} - \tilde{\alpha}} \left(\frac{1}{1-t} \frac{d_5(z, w)}{2} + \frac{t}{(1-t)^2} \right), \quad (3.11)$$

$$t = e^{-\tilde{\theta}}, \quad \cosh \tilde{\theta} = \frac{\tilde{\beta} + \tilde{\alpha}}{\tilde{\beta} - \tilde{\alpha}}. \quad (3.12)$$

Similar exponential bound can be established for the differentiation with respect to the gauge field.

The Chern-Simons current in consideration is defined in the finite volume lattice. It can be expressed in terms of the inverse five-dimensional Wilson-Dirac operator in the infinite lattice as

$$J_\mu^a(z) = \sum_{n \in \mathbb{Z}^5} (-)^{n_5} \text{Im tr} \left(\frac{1}{2} (\gamma_\mu - 1) T^a U_\mu(z) \frac{1}{D_{5w} - m_0} \right. \\ \left. \times (z + \hat{\mu}, z + n \cdot L) + \frac{1}{2} (\gamma_\mu + 1) U_\mu(z)^{-1} T^a \right. \\ \left. \times \frac{1}{D_{5w} - m_0} (z, z + \hat{\mu} + n \cdot L) \right), \quad (3.13)$$

where $n \cdot L = (\sum_k n_k) L + n_5 (4N)$. Then we see that the dependence of the Chern-Simons current on the gauge field is exponentially suppressed at large distance in the leading contribution with $n=0$, while the remaining dependences on the gauge field are at most of order $\mathcal{O}(\exp(-\theta L/2), \exp(-\theta 2N))$ and exponentially small. In this sense, the Chern-Simons current in the finite volume lattice can be regarded as a local functional of gauge field.

C. The Chern-Simons current and gauge anomaly

If we consider the variation of the Chern-Simons term under the gauge transformation, we obtain

$$\delta \text{Im Tr Ln}(D_{5w} - m_0)|_{AP} = \sum_z \{ D_\mu \omega(z) \}^a J_\mu^a(z) \\ = \sum_z \partial_\mu \{ \omega^a(z) J_\mu^a(z) \}. \quad (3.14)$$

The flow of the Chern-Simons current should be responsible for the gauge anomaly associated with the chiral zero mode

at the boundaries [45]. In fact, it has been shown by Golterman, Jansen, and Kaplan [36] that the asymptotic value of the fifth component of the Chern-Simons current reproduces the known result of the gauge anomaly in the classical continuum limit:

$$\lim_{a \rightarrow 0} \lim_{N \rightarrow \infty} J_5^a(x, N) = -\frac{1}{32\pi^2} \epsilon_{klmn} \text{Tr}\{T^a F_{kl}^1(x) F_{mn}^1(x)\}. \quad (3.15)$$

IV. RECONSTRUCTION OF THE CHERN-SIMONS TERM FROM THE CHERN-SIMONS CURRENT

In this section, we discuss how to obtain the local and gauge invariant field of the Chern-Simons term. Using the Chern-Simons current obtained in the previous section, we introduce a local topological field on $(5+1)$ -dimensional space and formulate a local cohomology problem. We will see that the trivial solution of the cohomology problem leads to the local field with the required properties.

A. Contractible loops

Let us first assume that a loop l in the space of the gauge fields can be contractible to a point. Namely, we assume that the five-dimensional gauge field $U_\mu(z)$ representing the loop l can be deformed to the uniform gauge field $U_\mu(z) = U_k^0(x)$ (for all t), while satisfying the constraint on the five-dimensional plaquette variables Eq. (2.6). It depends on the topological structure of the space of the admissible gauge fields in consideration, whether all possible loops are contractible or not. The case with the noncontractible loops will be discussed later.

Let us parametrize the smooth deformation of the loop l by the parameter $s \in [0, 1]$ as $U_\mu^s(z)$, where $U_\mu^{s=0}(z) = U_k^0(x)$ and $U_\mu^{s=1}(z) = U_\mu(z)$. The trivial interpolation (the point) at $s=0$ is denoted by l^0 . By differentiating the Chern-Simons term with respect to the parameter s and then integrating back, we obtain an expression for the Chern-Simons term as

$$\text{Im Tr Ln}(D_{5w} - m_0)|_{\text{AP}}(l) = \sum_z \left\{ \int_0^1 ds \eta_\mu^a(z, s) J_\mu^a(z) \Big|_{U_\mu^s} \right\}, \quad (4.1)$$

where $\eta_\mu^a(z, s) T^a = \partial_s U_\mu^s(z) U_\mu^s(z)^{-1}$. Here we took account of the fact that for the point l^0 represented by the uniform gauge field $U_\mu^0(x)$, the Chern-Simons term vanishes identically, $\text{Im Tr Ln}(D_{5w} - m_0)|_{\text{AP}}(l^0) = 0$, because of the reflection property of the five-dimensional Wilson-Dirac operator discussed in Sec. II.

The Chern-Simons term so reconstructed is now given by the sum of the local field in the curly bracket, because the Chern-Simons current is a local functional of the gauge field. We note, however, that the local field is not gauge invariant. In fact, under an infinitesimal gauge transformation, $\delta U_\mu^s \{U_\mu^s\}^{-1} = -D_\mu \omega(z, s)$, the variation of the local field is given by

$$\delta \left\{ \int_0^1 ds \eta_\mu^a(z, s) J_\mu^a(z) \Big|_{U_\mu^s} \right\} = - \int_0^1 ds \{D_\mu \partial_s \omega(z, s)\}^a J_\mu^a(z) \Big|_{U_\mu^s}. \quad (4.2)$$

The clue to obtain the gauge invariant local field is to note that the field in Eq. (4.2) may be corrected by the total divergence of a certain local current $K_\mu(z)$ without affecting the Chern-Simons term:

$$\text{Im Tr Ln}(D_{5w} - m_0)|_{\text{AP}}(l) = \sum_z \int_0^1 ds \{ \eta_\mu^a(z, s) J_\mu^a(z) \Big|_{U_\mu^s} - \partial_\mu^* K_\mu(z) \Big|_{U_\mu^s, \eta_\mu^s} \}. \quad (4.3)$$

The local field can be made gauge invariant if $J_\mu^a(z)$ would satisfy the relation

$$\begin{aligned} \delta \{ \eta_\mu^a(z, s) J_\mu^a(z) \Big|_{U_\mu^s} \} &= - \{D_\mu \partial_s \omega(z, s)\}^a J_\mu^a(z) \Big|_{U_\mu^s} \\ &= \partial_\mu^* \delta K_\mu(z) \Big|_{U_\mu^s, \eta_\mu^s} \end{aligned} \quad (4.4)$$

with a certain local current $K_\mu(z)$, under the infinitesimal gauge transformation $\delta U_\mu^s \{U_\mu^s\}^{-1} = -D_\mu \omega(z, s)$. As we can see from Eq. (4.4), the question whether $J_\mu^a(z)$ would satisfy the above relation and how to find the local current $K_\mu(z)$ defines a local cohomology problem.

This cohomology problem can be reformulated as a local cohomology problem in higher dimensions [4]. In the next subsection, we formulate the cohomology problem in five-dimensional lattice plus one dimensional continuum space.

B. Cohomology problem in $(5+1)$ -dimensional space

In order to reformulate the local cohomology problem in five-dimensional lattice plus one dimensional continuum space, let us introduce a gauge field on the $(5+1)$ -dimensional space as

$$(U_\mu(z, s), A_6(z, s)), \quad (4.5)$$

where $A_6 = T^a A_6^a$ and its gauge transformation property is specified as

$$\begin{aligned} A_6(z, s) &\rightarrow^G A_6(z, s) \\ &= G(z, s) A_6(z, s) G(z, s)^{-1} - \partial_s G(z, s) G(z, s)^{-1}. \end{aligned} \quad (4.6)$$

Accordingly, the covariant derivative in the continuum dimension can be defined by

$$\begin{aligned} \mathcal{D}_s U_\mu(z, s) &= \partial_s U_\mu(z, s) + A_6(z; s) U_\mu(z, s) \\ &\quad - U_\mu(z, s) A_6(z + \hat{\mu}, s). \end{aligned} \quad (4.7)$$

We now introduce a gauge invariant and local field in the $(5+1)$ -dimensional space by

$$q(z, s) \equiv \{ \mathcal{D}_s U_\mu(z, s) U_\mu(z, s)^{-1} \}^a J_\mu^a(z) \Big|_{U_\mu^s} \quad (4.8)$$

$$= \eta_\mu^a(z,s) J_\mu^a(z) |_{U_\mu^s} - \{D_\mu A_6(z,s)\}^a J_\mu^a(z) |_{U_\mu^s}, \quad (4.9)$$

where $T^a \eta_\mu^a(z,s) = \partial_s U_\mu(z,s) U_\mu(z,s)^{-1}$. This local field is topological. Namely, the summation of the field over the $(5+1)$ -dimensional space is invariant under the local variation of the $(5+1)$ -dimensional gauge field:

$$\sum_z \int_0^1 ds \delta q(z,s) = 0. \quad (4.10)$$

In fact, denoting $\delta U_\mu U_\mu^{-1} = T^a \xi_\mu^a$ and using Eq. (3.6), it follows from the second expression of the topological field in Eq. (4.9) that

$$\sum_z \int_0^1 ds \delta q(z,s) = \sum_z \int_0^1 ds (\partial_s \{\xi_\mu^a J_\mu^a\} - \partial_\mu \delta \{A_6^a J_\mu^a\}). \quad (4.11)$$

Now let us assume that this topological field is cohomologically trivial, that is, it can be written in the form

$$q(z,s) = \partial_\mu^* k_\mu(z,s) + \partial_s k_6(z,s), \quad (4.12)$$

where $(k_\mu(z,s), k_6(z,s))$ is a local current which is gauge invariant under the $(5+1)$ -dimensional gauge transformation. Then we can see that $k_6(z,s)$ provides the desired local, gauge invariant field with the required properties. In fact, from the $(5+1)$ -dimensional gauge invariance and the fact that $q(z,s)$ is a linear functional of $\mathcal{D}_s U_\mu U_\mu^{-1}$, we infer that $k_6(z,s)$ cannot depend on $A_6(z,s)$ and its s -dependence comes from that of the link variable U_μ^s . Then we can set

$$\check{Q}_{5w} \equiv \sum_z k_6(z,s=1) \quad (4.13)$$

$$= \sum_z \int_0^1 ds \{ \eta_\mu^a(z,s) J_\mu^a(z) |_{U_\mu^s} - \partial_\mu^* k_\mu(z,s) |_{A_6=0} \}, \quad (4.14)$$

assuming $\sum_z k_6(z,s=0) = 0$.

From the $(5+1)$ -dimensional gauge invariance of $k_\mu(z,s)$ and the fact that $q(z,s)$ is a linear functional of $\mathcal{D}_s U_\mu U_\mu^{-1}$, we also infer that $k_\mu(z,s)$ must be a linear functional of $\mathcal{D}_s U_\mu U_\mu^{-1}$ written as

$$k_\mu(z,s) = \sum_w j_{\mu\nu}^a(z,s;w) \{ \mathcal{D}_s U_\nu(w,s) U_\nu(w,s)^{-1} \}^a. \quad (4.15)$$

This in turn implies that

$$\delta \{ k_\mu(z,s) |_{U_\mu^s, A_6=0} \} + \sum_w \frac{\delta k_\mu(z,s)}{\delta A_6(w,s)} \delta A_6(w,s) = 0, \quad (4.16)$$

where $\delta U_\mu^s \{ U_\mu^s \}^{-1} = -D_\mu \omega(z,s)$ and $\delta A_6 = [\omega(z,s), A_6] - \partial_s \omega(z,s)$. On the other hand, by setting the link variables s -independent in Eq. (4.12), we obtain

$$- \{ D_\mu A_6(z,s) \}^a J_\mu^a(z) |_{U_\mu} = \partial_\mu^* \left\{ \sum_w \frac{\delta k_\mu(z,s)}{\delta A_6(w,s)} A_6(w,s) \right\} |_{U_\mu}. \quad (4.17)$$

By setting $A_6(z,s) = -\partial_s \omega(z,s)$ in this equation and using Eq. (4.16), we obtain

$$- \{ D_\mu \partial_s \omega(z,s) \}^a J_\mu^a(z) |_{U_\mu^s} = \partial_\mu^* \delta k_\mu(z) |_{U_\mu^s, A_6=0}. \quad (4.18)$$

Thus Eq. (4.4) is indeed satisfied.

In this manner, we can reconstruct the Chern-Simons term with the required properties for all contractible loops, through the cohomology problem with the topological field Eq. (4.8).

C. The ansatz for noncontractible loops

When the topological field Eq. (4.8) is shown to be cohomologically trivial, the remaining issue is to show the condition Eq. (2.21) for all possible noncontractible loops. One may try $k_6(z,s)$ as the ansatz for the Chern-Simons term:

$$\check{Q}_{5w} = \sum_z k_6(z,s=1). \quad (4.19)$$

The question is then to show

$$\exp(iQ_{5w}) = \exp(i\check{Q}_{5w}) \quad (4.20)$$

for all noncontractible loops.

This problem requires first to figure out the topological structure of the space of the gauge fields in consideration, which is constrained by the admissibility condition. So far, the topological structure of the space of the admissible gauge field is known only for abelian gauge theories [3]. In this case, it is indeed possible to show that the topological field Eq. (4.8) is cohomologically trivial and to establish Eq. (4.20) for all loops in the space of the gauge field, as shown in the approach of the Ginsparg-Wilson relation [3].

D. Some results in the infinite four-dimensional volume

In the infinite four-dimensional volume, the cohomology problem in $(5+1)$ -dimensional space defined in the previous subsections can be solved in certain cases (cf. [5,7,10,11]). In this subsection, we describe how to construct the local counterterms for the theories in the infinite four-dimensional volume.

1. Abelian chiral gauge theories

In the Abelian gauge theories, the lattice Chern-Simons current is a gauge-invariant conserved current, which is a local functional of the gauge fields:

$$\delta J_\mu(z) = 0, \quad \partial_\mu^* J_\mu(z) = 0. \quad (4.21)$$

With these two conditions we can directly apply the cohomological method using the Poincaré lemma on the lattice [5,7] to $J_\mu(z)$, to obtain

$$J_\mu(z) = \alpha_\mu + \beta_{\mu\nu\sigma} F_{\nu\sigma}(z - \hat{\nu} - \hat{\sigma}) + \gamma \epsilon_{\mu\nu\sigma\rho\tau} F_{\nu\sigma}(z - \hat{\nu} - \hat{\sigma}) \\ \times F_{\rho\tau}(z - \hat{\nu} - \hat{\sigma} - \hat{\rho} - \hat{\tau}) + \partial_\nu^* \chi_{\nu\mu}(z), \quad (4.22)$$

where $\chi_{\nu\mu}(z)$ is a gauge-invariant and local antisymmetric tensor field and

$$F_{\mu\nu}(z) = \frac{1}{i} \ln \{ U_\mu(z) U_\nu(z + \hat{\mu}) U_\mu^{-1}(z + \hat{\nu}) U_\nu^{-1}(z) \}. \quad (4.23)$$

From the lattice symmetries, we infer that α_μ and $\beta_{\mu\nu\rho}$ vanish identically. Through explicit calculations in the weak coupling expansion, we can also verify that α_μ and $\beta_{\mu\nu\rho}$ vanish identically and $\gamma = -1/32\pi^2$ [28]. From this result, we can reconstruct the lattice Chern-Simons term as⁴

$$Q_{5w} = \sum_z \int_0^1 ds A_\mu(z) J_\mu(z) |_{A \rightarrow sA} \quad [U_\mu(z; s) = e^{isA_\mu(z)}] \\ = \sum_z \frac{1}{3} \gamma \epsilon_{\mu\nu\sigma\rho\tau} A_\mu(z) F_{\nu\sigma}(z - \hat{\nu} - \hat{\sigma}) F_{\rho\tau} \\ \times (z - \hat{\nu} - \hat{\sigma} - \hat{\rho} - \hat{\tau}) + \sum_z \partial_\nu^* \hat{\chi}_{\nu\mu}(z) A_\mu(z), \quad (4.24)$$

where

$$\hat{\chi}_{\nu\mu}(z) = \int_0^1 ds \chi_{\nu\mu}(z) |_{A \rightarrow sA}. \quad (4.25)$$

If we consider an anomaly free Abelian chiral gauge theory, the charges of the Weyl fermions should satisfy the condition

$$\sum_\alpha e_\alpha^3 = 0. \quad (4.26)$$

Then, by rescaling the gauge field as $A_\mu \rightarrow e_\alpha A_\mu$ in each Weyl fermion contributions, we can see that the first term of the lattice Chern-Simons term vanishes identically. As to the second term, we may add the following total divergence term without affecting the lattice Chern-Simons term:

$$- \sum_z \partial_\nu^* (\hat{\chi}_{\nu\mu}(z) A_\mu(z)). \quad (4.27)$$

⁴Here $A_\mu(z)$ is the vector potential which represents the original admissible (five-dim.) link variable $U_\mu(z)$ and the field strength as follows [5]: $U_\mu(z) = e^{iA_\mu(z)}$; $|A_\mu(z)| \leq \pi(1 + 10\|z\|)$; $F_{\mu\nu}(z) = \partial_\mu A_\nu(z) - \partial_\nu A_\mu(z)$. This vector potential itself is not a local functional of the original link variable $U_\mu(z)$, but its local, gauge invariant functional becomes a local functional of $U_\mu(z)$.

Then we obtain the local expression of the lattice Chern-Simons term (the local counterterm) which is manifestly gauge invariant:

$$Q_{5w} = \sum_z \frac{1}{2} \hat{\chi}_{\nu\mu}(z - \hat{\nu}) F_{\nu\mu}(z - \hat{\nu}). \quad (4.28)$$

2. Non-Abelian chiral gauge theories in lattice perturbation theory

In the anomaly-free non-Abelian chiral gauge theories, it is possible to show that the topological field Eq. (4.8) is cohomologically trivial to any orders of the lattice perturbation theory, as shown in the approach based on the Ginsparg-Wilson relation [10,11]. In particular, one can construct the local field of the Chern-Simons term directly from the lattice Chern-Simons current, to any orders of the perturbation theory.

In the lattice perturbation theory, non-Abelian lattice gauge fields are treated in the expansion of the gauge coupling constant,

$$U_\mu(z) = 1 + \sum_{l=1}^{\infty} (igA_\mu(z))^l. \quad (4.29)$$

Accordingly, the local field of the lattice Chern-Simons term may be assumed to have the expansion in the gauge coupling constant:

$$Q_{5w} = \sum_z \sum_{l=1}^{\infty} q_{5w}^{(l)}(z). \quad (4.30)$$

Let us assume that the local fields $q_{5w}^{(l)}(z)$ are constructed to the order $l=n$ and consider how to construct the local field of the order $l=n+1$.

Since the Chern-Simons term should produce the Chern-Simons current under the local variation of the gauge field,

$$U_\mu(z) \rightarrow U_\mu(z) + \eta_\mu(z) U_\mu(z), \quad (4.31)$$

we should have

$$\sum_{l=1}^{\infty} \delta_\eta q_{5w}^{(l)}(z) = \eta_\mu^a(z) J_\mu^a(z). \quad (4.32)$$

Then we may expand the variation of the Chern-Simons current which is subtracted by the local fields up to the order $l=n$:

$$\eta_\mu^a(z) J_\mu^a(z) - \sum_{l=1}^n \delta_\eta q_{5w}^{(l)}(z) = \eta_\mu^a(z) \check{J}_\mu^{a(n)}(z) + \mathcal{O}(g^{n+2}). \quad (4.33)$$

The leading term in this expansion, $\check{J}_\mu^{a(n)}(z)$, is a local field of the vector potential of order $l=n$. Since the LHS is gauge invariant up to $\mathcal{O}(g^n)$, $\check{J}_\mu^{a(n)}(z)$ is invariant under the linearized gauge transformation,

$$A_\mu^a(z) \rightarrow A_\mu^a(z) + \partial_\mu \omega^a(z) \quad (4.34)$$

and the global gauge transformation. It also satisfies the conservation law,

$$\partial_{\mu}^* \check{J}_{\mu}^{a(n)}(z) = 0. \quad (4.35)$$

From these conditions, we can directly apply the cohomological method for the Abelian gauge theories [5,7] to $\check{J}_{\mu}^{a(n)}(z)$, to obtain

$$\check{J}_{\mu}^{a(n)}(z) = \partial_{\nu}^* \chi_{\nu\mu}^{a(n)}(z) \quad (n \neq 2), \quad (4.36)$$

where $\chi_{\nu\mu}^{a(n)}(z)$ is a local field which is invariant under the linearized gauge transformation and the global gauge transformation. When and only when $n=2$, a cohomologically non-trivial term can appear as

$$\check{J}_{\mu}^{a(2)}(z) = d^{abc} \gamma \epsilon_{\mu\nu\rho\sigma\tau} F_{\nu\rho}^b(z - \hat{\nu} - \hat{\rho}) F_{\sigma\tau}^c(z - \hat{\nu} - \hat{\rho} - \hat{\sigma} - \hat{\tau}) + \partial_{\nu}^* \chi_{\nu\mu}^{a(2)}(z), \quad (4.37)$$

which vanishes identically by the anomaly-free condition

$$\sum d^{abc} = \sum \text{Tr}(T^a \{T^b, T^c\}) = 0. \quad (4.38)$$

From this result, we can obtain the local field of order $l = n+1$, which reproduces $\check{J}_{\mu}^{a(n)}(z)$ under the local variation of the gauge field, as

$$\check{q}_{5w}^{(n+1)}(z) = \frac{1}{2} \check{\chi}_{\nu\mu}^{a(n)}(z - \hat{\nu}) \check{F}_{\nu\mu}^a(z - \hat{\nu}), \quad (4.39)$$

where

$$\check{F}_{\nu\mu}^a(z) = \partial_{\nu} A_{\mu}^a(z) - \partial_{\mu} A_{\nu}^a(z) \quad (4.40)$$

and

$$\check{\chi}_{\nu\mu}^{a(n)}(z) = \int_0^1 ds \chi_{\nu\mu}^{a(1)}(z) \Big|_{A \rightarrow sA} = \frac{1}{n+1} \chi_{\nu\mu}^{a(n)}(z). \quad (4.41)$$

Here we have used the fact that $\chi_{\nu\mu}^{a(n)}(z)$ has the following structure in the vector potentials:

$$\chi_{\nu\mu}^{a(n)}(z) = \sum \chi_{\nu\mu}^{a(n); a_1 a_2 \dots a_n}(z; z_1, z_2, \dots, z_n)_{\mu_1 \mu_2 \dots \mu_n} \times A_{\mu_1}^{a_1}(z_1) A_{\mu_2}^{a_2}(z_2) \dots A_{\mu_n}^{a_n}(z_n). \quad (4.42)$$

The above field $\check{q}_{5w}^{(n+1)}(z)$ is invariant under the linearized gauge transformation and the global gauge transformation. Next step is to construct $q_{5w}^{(n+1)}(z)$ which is gauge invariant under the full non-Abelian gauge transformation, while its leading term in the expansion of the gauge coupling constant remains to coincide with $\check{q}_{5w}^{(n+1)}(z)$. For this purpose, we follow the method adopted in [11]. Namely, this step can be achieved by the replacement of the field strength as

$$\check{F}_{\nu\mu}^a(z) \rightarrow \frac{2}{a} \text{Tr}\{T^a[1 - U_{\nu\mu}(z)]\}, \quad (4.43)$$

and by the replacement of the vector potentials in Eq. (4.42) as

$$\begin{aligned} A_{\mu}^a(z_k) &\rightarrow \hat{A}_{\mu}^a(z, z_k) \\ &= \frac{2}{a} \text{Tr}\{T^a[1 - W(z, z_k) U_{\mu}(z_k) \\ &\quad \times W(z, z_k + \hat{\mu})^{-1}]\}, \end{aligned} \quad (4.44)$$

where $W(z, z_k)$ is defined as the ordered product of the link variables from z_k to z along the shortest path that first goes in direction 1, then direction 2, and so on. Since it coincides with the original vector potential up to the linearized gauge transformation in the expansion of the gauge coupling constant as

$$\hat{A}_{\mu}^a(z, z_k) = g \{A_{\mu}(z_k) + \partial_{\mu}^{z_k} \omega(z, z_k)\} + \mathcal{O}(g^2), \quad (4.45)$$

where $\omega(z, z_k)$ is the oriented line sum of the gauge potential from z_k to z along the same path as defined for $W(z, z_k)$, the leading term of $q_{5w}^{(n+1)}(z)$ so defined actually coincides with $\check{q}_{5w}^{(n+1)}(z)$.

V. CONNECTION TO THE LATTICE THEORY OF WEYL FERMION BASED ON THE GINSPARG-WILSON RELATION

In this section, we discuss the connection of the chiral domain wall fermion discussed so far to the gauge-invariant construction of chiral gauge theories based on the Ginsparg-Wilson relation [3,4]. We will establish an identity relating the partition function of the domain wall fermion (subtracted and with the local counterterm) and the partition function of the Weyl fermion defined with the overlap Dirac operator satisfying the Ginsparg-Wilson relation.

A. Weyl fermion defined through the overlap Dirac operator

The lattice Dirac fermion theory defined with the Dirac operator which satisfies the Ginsparg-Wilson relation

$$\gamma_5 D + D \hat{\gamma}_5 = 0, \quad \hat{\gamma}_5 = \gamma_5 (1 - 2D), \quad (5.1)$$

possesses the exact symmetry under the chiral transformation

$$\delta\psi(x) = \hat{\gamma}_5 \psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x) \gamma_5. \quad (5.2)$$

Based on this exact chiral symmetry, the lattice Weyl fermion can be defined by imposing the constraint with the chiral operators $\hat{\gamma}_5$ and γ_5 ,

$$\hat{\gamma}_5 \psi_R(x) = +\psi_R(x), \quad \bar{\psi}_R(x) \gamma_5 = -\bar{\psi}_R(x). \quad (5.3)$$

By introducing the orthonormal basis for the Weyl fermion, $\{v_i(x)|i=1,2,\dots\}$ and $\{\bar{v}_k(x)|k=1,2,\dots\}$ ⁵ as

$$\hat{\gamma}_5 v_i(x) = +v_i(x), \quad (v_i, v_j) = \delta_{ij}, \quad (5.4)$$

$$\bar{v}_k(x) \gamma_5 = -\bar{v}_k(x), \quad (\bar{v}_k, \bar{v}_l) = \delta_{kl}, \quad (5.5)$$

the functional measure of the Weyl fermion can be set up and the path-integral formula of the partition function can be defined:

$$Z_w = \int \mathcal{D}[\psi_R] \mathcal{D}[\bar{\psi}_R] e^{-\sum_x \bar{\psi}_R(x) D \psi_R(x)} = \det(\bar{v}_k, D v_j). \quad (5.6)$$

The choice of the basis $\{v_i(x)|i=1,2,\dots\}$ may be different by a unitary transformation,

$$v_i(x) \rightarrow \tilde{v}_i(x) = \sum_j v_j(x) Q_{ji}, \quad (5.7)$$

which depends on the gauge field in general because $\hat{\gamma}_5$ does so. Then the measure is changed by the phase factor

$$\mathcal{D}[\psi_R] \mathcal{D}[\bar{\psi}_R] \rightarrow \mathcal{D}[\psi_R] \mathcal{D}[\bar{\psi}_R] \det\{Q_{ji}\}. \quad (5.8)$$

Accordingly, the partition function is changed by

$$\det(\bar{v}_k, D v_j) \rightarrow \det(\bar{v}_k, D \tilde{v}_j) = \det(\bar{v}_k, D v_j) \det\{Q_{ji}\}. \quad (5.9)$$

In the gauge-invariant construction of the chiral gauge theories based on the Ginsparg-Wilson relation [3,4], a general method to fix the phase of the functional measure (and the partition function) has been described so that it satisfies the requirements of the smoothness, the locality and the gauge invariance.

Our target in this paper is the Weyl fermion theory which is defined with the overlap Dirac operator satisfying the Ginsparg-Wilson relation [14,15]. The overlap Dirac operator, D , is given by the explicit formula [14]

$$D = \frac{1}{2} \left(1 + \gamma_5 \frac{H}{\sqrt{H}} \right). \quad (5.10)$$

Here H is chosen as the Hermitian operator obtained through the transfer matrix of the five-dimensional Wilson-Dirac fermion:⁶

$$H = -\ln T. \quad (5.11)$$

In this case $\hat{\gamma}_5$ is given by the spectrum asymmetry of H ,

⁵ (v_i, v_j) is the inner product of the spinor field defined by $(v_i, v_j) = \sum_x v_i(x)^\dagger v_j(x)$.

⁶Boriçi has pointed out that the transfer matrix can be expressed by a simpler four-dimensional Hermitian matrix as $T = (1 + \mathcal{H}) / (1 - \mathcal{H})$, $\mathcal{H} = \gamma_5 (D_w - m_0) (1 / [1 + a_5 (D_w - m_0)])$, which leads to the same spectrum asymmetry operator, $\mathcal{H} / \sqrt{\mathcal{H}^2} = H / \sqrt{H^2}$ [46].

$$\hat{\gamma}_5 = \gamma_5 (1 - 2D) = -\frac{H}{\sqrt{H^2}}, \quad (5.12)$$

and the chiral basis, $\{v_i(x)|i=1,2,\dots\}$, can be chosen as the eigenvectors of H belonging to the negative eigenvalues (up to the global phase choice). The partition function resulted from the path-integral formula Eq. (5.6), reproduces the overlap formula of the chiral determinant [17]:

$$\det(\bar{v}_k, D v_j) = \det(\bar{v}_k, v_j). \quad (5.13)$$

Here the phase of the chiral basis can be chosen following the gauge-invariant method of [3,4] in anomaly-free chiral gauge theories.

B. Gauge-invariant partition function of Weyl fermions from the domain wall fermion

We now argue the connection of the (subtracted) partition function of the domain wall fermion with the local counter-term,

$$\lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}}{|\det(D_{5w} - m_0)|_{\text{AP}}^{c_1 + (-c_1)|_{1/2}}} e^{-iC_{5w}(c_1)} \quad (5.14)$$

to the partition function of the Weyl fermion given by Eq. (5.13).⁷

1. Partition functions of the 5-dimensional Wilson fermions

The partition functions of the five-dimensional Wilson fermions with the Dirichlet boundary condition and with the antiperiodic boundary condition can be expressed explicitly in terms of the transfer matrix, as shown in Appendix B. The results are given as follows:

$$\begin{aligned} \det(D_{5w} - m_0)|_{\text{Dir}}^{c_1} &= \det \left(P_R + P_L T_1^{(N-\Delta)} \right) \\ &\times \left\{ U_5^{-1} \prod_{c_1} T_t U_{5,t-1}^{-1} \right\} T_0^{N-\Delta} \\ &\times \det \left(P_R + P_L \prod_{c_1} U_{5,t} \right) \prod_{t=-N+1}^N N_t, \end{aligned} \quad (5.15)$$

⁷It may be worthwhile to recall the situation in the vectorlike case. In the vectorlike theories [30], the connection between the domain wall fermion and the overlap Dirac operator is directly seen in the fact that the determinant of domain wall fermion (subject to Dirichlet boundary condition in the fifth dimension) is factorized into four-dimensional part for the low-lying massless mode and five-dimensional part for the remaining massive modes [47,48]: $\lim_{N \rightarrow \infty} [\det(D_{5w} - m_0)|_{\text{Dir}} / \det(D_{5w} - m_0)|_{\text{AP}}] = \det D$. In the LHS, N is the size of the fifth dimension and the contribution of the massive modes is factorized in the determinant of the five-dimensional Wilson fermion subject to the antiperiodic boundary condition in the fifth dimension [48].

$$\begin{aligned}
& \det(D_{5w} - m_0)|_{\text{AP}}^{c_1+(-c_0)} \\
&= \det \left(1 + T_0^{N-\Delta} \left\{ U_5^{-1} \prod_{-c_0} T_t U_{5,t-1}^{-1} \right\} \right) \\
& \quad \times T_1^{2(N-\Delta)} \left\{ U_5^{-1} \prod_{c_1} T_t U_{5,t-1}^{-1} \right\} T_0^{N-\Delta} \\
& \quad \times \det \left(P_R + P_L \prod_{c_1+(-c_0)} U_{5,t} \right) \prod_{t=-N+1}^{3N} N_t,
\end{aligned} \tag{5.16}$$

where T_t is the transfer matrix with $U_k(x,t)$, the explicit form of which is given in Appendix A, and $N_t = \det(P_L + P_R B_t)$. The subscript 0 and 1 denote the quantities with $U_k^0(x)$ and $U_k^1(x)$, respectively. $U_5(x,t)$ appears in between the product of the transfer matrices so that the five-dimensional gauge covariance is maintained. Note that for the Abelian gauge group, the extra phase factor, which consists of the product of $U_5(x,t)^{-1}$, appears.

In these formulas, the dominant term in $T_0^{(N-\Delta)}$ in the limit $N \rightarrow \infty$ is estimated as follows:

$$\begin{aligned}
T_0^{(N-\Delta)} &= P_0 T_0^{(N-\Delta)} + (1 - P_0) T_0^{(N-\Delta)} \\
&= \sum_i v_i^0 \otimes v_i^{0\dagger} e^{(N-\Delta)|\lambda_i^0|} \\
& \quad + \mathcal{O}(e^{-(N-\Delta)\lambda_+^0}),
\end{aligned} \tag{5.17}$$

where the projection operator P_0 is introduced by

$$P_0 = \frac{1}{2} \left(1 - \frac{H_0}{\sqrt{H_0^2}} \right), \quad H_0 = -\ln T_0, \tag{5.18}$$

and $\{v_i^0(x)\}$ are chosen as eigenfunctions of $H_0 = -\ln T_0$ belonging to the negative eigenvalues λ_i^0 , while λ_+^0 is the smallest positive eigenvalue of H_0 . Similar estimation holds for $T_1^{(N-\Delta)}$. With these results, we can infer that in the limit $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}}{\left\{ \prod_{t=-N+1}^N N_t \right\} N_1^{N-\Delta} N_0^{N-\Delta}} = \det \left(P_R + P_L P_1 \left\{ U_{5,\Delta}^{-1} \prod_{c_1} T_t U_{5,t-1}^{-1} \right\} P_0 \right) \det \left(P_R + P_L \prod_{c_1} U_{5,t} \right), \tag{5.19}$$

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{\text{AP}}^{c_1+(-c_0)}}{\left\{ \prod_{t=-N+1}^{3N} N_t \right\} N_1^{2(N-\Delta)} N_0^{2(N-\Delta)}} &= \det \left(1 - P_0 + P_0 \left\{ U_5^{-1} \prod_{-c_0} T_t U_{5,t-1}^{-1} \right\} P_1 \left\{ U_5^{-1} \prod_{c_1} T_t U_{5,t-1}^{-1} \right\} P_0 \right) \\
& \quad \times \det \left(P_R + P_L \prod_{c_1+(-c_0)} U_{5,t} \right),
\end{aligned} \tag{5.20}$$

where $N_1 = \det(1 - P_1 + P_1 T_1)$ and $N_0 = \det(1 - P_0 + P_0 T_0)$. (See Appendix C for the derivation of these results.)

From Eqs. (5.19) and (5.20), it follows immediately that

$$\lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}}{|\det(D_{5w} - m_0)|_{\text{AP}}^{c_1+(-c_1)}|^{1/2}} e^{-iC_{5w}(c_1)} = \frac{\det \left(P_R + P_L P_1 \left\{ U_{5,\Delta}^{-1} \prod_{c_1} T_t U_{5,t-1}^{-1} \right\} P_0 \right) \times \det \left(P_R + P_L \prod_{c_1} U_{5,t} \right)}{\left| \det \left(1 - P_0 + P_0 \left\{ U_5^{-1} \prod_{-c_1} T_t U_{5,t-1}^{-1} \right\} P_1 \left\{ U_5^{-1} \prod_{c_1} T_t U_{5,t-1}^{-1} \right\} P_0 \right) \right|^{1/2}} e^{-iC_{5w}(c_1)}. \tag{5.21}$$

2. Factorization of the partition function of the domain wall fermion

Now we introduce the chiral basis $\{v_i^0(x)\}$ associated with the gauge fields $U_k^0(x)$ as

$$P_0 v_i^0(x) = v_i^0(x), \quad (v_i^0, v_j^0) = \delta_{ij} \quad (i, j = 1, 2, \dots). \tag{5.22}$$

$\{v_i^0(x)\}$ may be chosen as the eigenfunctions of $H_0 = -\ln T_0$ belonging to the negative eigenvalues. Similarly, we introduce the chiral basis $\{v_i^1\}$ associated with the gauge fields $U_k^1(x)$. We also introduce the chiral basis for the antifield $\{\bar{v}_k\}$ as

$$\bar{v}_k(x) P_L = \bar{v}_k(x) \quad (k = 1, 2, \dots). \tag{5.23}$$

In terms of these chiral bases, the formula Eq. (5.21) of the (subtracted) partition function of the domain wall fermion with the local counterterm can be rewritten further as

$$\lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}}{|\det(D_{5w} - m_0)|_{\text{AP}}^{c_1 + (-c_1)|/2}} e^{-iC_{5w}(c_1)} = \det(\bar{v}_k, v_i^1) e^{i\phi(c_1)} \det(\bar{v}_k, v_j^0)^* e^{-iC_{5w}(c_1)}, \quad (5.24)$$

where

$$e^{i\phi(c_1)} = \frac{\det\left(v_i^1, \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} v_j^0\right)}{\left| \det\left(v_i^1, \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} v_j^0\right) \right|} \times \det\left(P_R + P_L \prod_{t=-\Delta+1}^{\Delta} U_{5,t}\right). \quad (5.25)$$

The first factor in the right-hand side (RHS) of Eq. (5.24) is nothing but the overlap formula, which gives the partition function of the right-handed Weyl fermion at $t=N$ coupled to the gauge field $U_k^1(x)$,

$$\det(\bar{v}_k, v_i^1) = \det(\bar{v}_k, Dv_i^1). \quad (5.26)$$

Similarly, the third factor in the RHS of Eq. (5.24) reproduces the partition function of the left-handed Weyl fermion at $t=-N+1$, which couples to the gauge field $U_k^0(x)$. On the other hand, the second factor in the RHS of Eq. (5.24), which is the phase factor defined by Eq. (5.25),

$$e^{i\phi(c_1)} \quad (5.27)$$

comes from the interpolation between the gauge fields $U_k^0(x)$ and $U_k^1(x)$.⁸ It depends on the choices of the chiral bases $\{v_i^0\}$, $\{v_i^1\}$ and the path c_1 , but the path dependence is to be compensated by the local counter term,

$$e^{-iC_{5w}(c_1)}. \quad (5.28)$$

From these results, it is quite natural to choose the chiral basis of the Weyl fermion coupled to the gauge field U_k^1 as follows:

$$v_i(x) = \begin{cases} v_i^1(x) e^{i\phi(c_1)} e^{-iC_{5w}(c_1)} & (i=1), \\ v_i^1(x) & (i \neq 1). \end{cases} \quad (5.29)$$

With this choice, the (subtracted) partition function of the domain wall fermion with the local counterterm is factorized into two chiral determinants:

$$\lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{\text{Dir}}^{c_1}}{|\det(D_{5w} - m_0)|_{\text{AP}}^{c_1 + (-c_1)|/2}} e^{-iC_{5w}(c_1)} = \det(\bar{v}_k, Dv_j) \det(\bar{v}_k, Dv_j^0)^*. \quad (5.30)$$

The path-independence and the gauge invariance of $\det(\bar{v}_k, Dv_j)$ are obvious from this identity.⁹

C. The functional measure of the Weyl fermion from the domain wall fermion

The choice of the chiral basis in Eq. (5.29) indeed leads to the functional measure of the Weyl fermion which is independent of the path of the interpolation. To see this, we choose another path, say c_2 , then we get another basis:

$$\tilde{v}_i(x) = \begin{cases} v_i^1(x) e^{i\phi(c_2)} e^{-iC_{5w}(c_2)} & (i=1), \\ v_i^1(x) & (i \neq 1). \end{cases} \quad (5.31)$$

These two bases are related by the unitary transformation Q_{ij} ,

⁸In the original derivation of the overlap formula in [17], this term was not considered because the gauge field was assumed to be four-dimensional. It was also true in the wave-guide model [38,39].

⁹The complex phase part of this identity can be regarded as the lattice counterpart of the relation between the η -invariant and the effective action for the chiral fermions [28,49–52]. Note also that this result is a generalization of the identity in vectorlike theories [47,48] to the case of chiral gauge theories.

$$\tilde{v}_i(x) = Q_{ij}^{-1} v_j(x), \quad (5.32)$$

which determinant turns out to be

$$\det Q = e^{-i\phi(c_2)} e^{iC_{5w}(c_2)} \cdot e^{i\phi(c_1)} e^{-iC_{5w}(c_1)}. \quad (5.33)$$

But it is not difficult to see that the phase factors along the paths c_1 and c_2 multiply to give

$$\begin{aligned} e^{i\phi(-c_2)} \cdot e^{i\phi(c_1)} &= \frac{\det \left(1 - P_0 + P_0 \left\{ U_5^{-1} \prod_t^{-c_2} T_t U_{5,t-1}^{-1} \right\} P_1 \left\{ U_5^{-1} \prod_t^{c_1} T_t U_{5,t-1}^{-1} \right\} \right)}{\left| \det \left(1 - P_0 + P_0 \left\{ U_5^{-1} \prod_t^{-c_2} T_t U_{5,t-1}^{-1} \right\} P_1 \left\{ U_5^{-1} \prod_t^{c_1} T_t U_{5,t-1}^{-1} \right\} \right) \right|} \det \left(P_R + P_L \prod_t^{c_1+(-c_2)} U_{5,t} \right) \\ &= \lim_{N \rightarrow \infty} \frac{\det(D_{5w} - m_0)|_{AP}^{c_1+(-c_2)}}{|\det(D_{5w} - m_0)|_{AP}^{c_1+(-c_2)}} \\ &= e^{iQ_{5w}^{c_1+(-c_2)}}, \end{aligned} \quad (5.34)$$

and this identity can be regarded as the integrability condition for the phase factor $\exp(i\phi(c_i))$.

Since we are assuming that the lattice Chern-Simons term can be decomposed into the two parts $C_{5w}^{c_1}$ and $C_{5w}^{-c_2}$ locally and gauge-invariantly,

$$e^{iQ_{5w}^{c_1+(-c_2)}} = e^{iC_{5w}^{c_1}} \cdot e^{-iC_{5w}^{c_2}}, \quad (5.35)$$

the determinant of the unitary transformation, $\det Q$, turns out to be unity. This holds for any two paths. Therefore the measure does not depend on the path of the interpolation and is determined uniquely.

We should note that from the integrability condition Eq. (5.34) and the expression for the (subtracted) partition function Eq. (5.24), we can directly infer that Eq. (2.25) holds and the (subtracted) partition function with the local counter-term is independent on the path of the interpolation for all topological sectors.

D. The connection to the gauge-invariant construction based on the Ginsparg-Wilson relation

Let us now look closely at the connection to the gauge-invariant construction by Lüscher [3,4]. We note first that our result Eq. (5.30) should be compared with Eq. (7.1) in [4]:¹⁰

$$\det(1 - P_L + P_L D Q_1 D_0^\dagger) W^{-1} = \det(\bar{v}_k, D v_j) \det(\bar{v}_k, D v_j^0)^\dagger, \quad (5.36)$$

where Q_1 is the evolution operator of the chiral projector

$$P_1 = Q_1 P_0 Q_1^{-1}, \quad (5.37)$$

¹⁰Since we are considering the right-handed Weyl fermions, instead of the left-handed Weyl fermions, P_R in the original equation (7.1) of [4] is replaced by P_L .

while W is the Wilson line defined by the line integral of the measure term current,

$$W = e^{i \sum_x \int_0^1 dt \{ \partial_t U_k(x,t) U_k(x,t)^{-1} \} j_k^a(x,t)}, \quad (5.38)$$

where t is the continuous parameter of the interpolation.

The determinant in the LHS of Eq. (5.36) may be expressed in terms of the chiral bases $\{v_j^1\}$, $\{v_j^0\}$ and $\{\bar{v}_k\}$ as

$$\begin{aligned} \det(1 - P_L + P_L D Q_1 D_0^\dagger) &= \det(\bar{v}_k, D v_j^1) \det(v_i, Q_1 v_j^0) \\ &\quad \times \det(\bar{v}_k, D v_j^0)^*. \end{aligned} \quad (5.39)$$

Then we note the correspondence as

$$\det(v_i^1, Q_1 v_j^0) \Leftrightarrow e^{i\phi(c_1)} = \frac{\det \left(v_i^1, \left\{ \prod_{t=-\Delta+1}^{\Delta(c_1)} T_t \right\} v_j^0 \right)}{\left| \det \left(v_i^1, \left\{ \prod_{t=-\Delta+1}^{\Delta(c_1)} T_t \right\} v_j^0 \right) \right|} \quad (5.40)$$

[in the gauge $U_5(z) = 1$], which implies that in the domain wall fermion the evolution of the basis of the Weyl fermion is realized by the successive multiplications of the transfer matrices, like time development along the fifth dimension.¹¹ For the loop in the space of the gauge fields, this leads to the correspondence of the integrability conditions:

¹¹We note that the use of the transfer matrix here is in the same spirit as the use of the Hamiltonian for the evolution of the (second quantized) vacuum states of the overlap formalism, adopted in the adiabatic phase choice [25]. Our result then provides a discretization method of the continuous evolution.

$$\begin{aligned}
 e^{-\int_0^1 dt \sum_i (v_i, \partial_t v_i)} &= \det(1 - P_0 + P_0 Q_1) \\
 &\Leftrightarrow e^{i\phi(c_1)} \cdot e^{-i\phi(c_2)} = e^{iQ_{5w}^{c_1+(-c_2)}}.
 \end{aligned} \tag{5.41}$$

The above correspondences become exact in the continuum limit of the fifth dimension, which should be taken as $a_5 \rightarrow 0$ with Δa_5 fixed. In taking the continuum limit, one may smooth the interpolation further by replacing T_t with $T_t^{N_t}$ and take the limit $N_t \rightarrow \infty$ first as an intermediate step, to get

$$e^{i\phi(c_1)} = \frac{\det\left(v_i^1, \left\{ \prod_{t=-\Delta+1}^{\Delta(c_1)} P_t \right\} v_j^0\right)}{\left| \det\left(v_i^1, \left\{ \prod_{t=-\Delta+1}^{\Delta(c_1)} P_t \right\} v_j^0\right) \right|}. \tag{5.42}$$

Then, using $P_t = Q_t P_0 Q_t^{-1}$, we can see that this becomes identical to $\det(v_i^1, Q_1 v_j^0)$ in the limit $a_5 \rightarrow 0$ with Δa_5 fixed.

On the other hand, the dependence of the determinant in the LHS of Eq. (5.36) on the path of the interpolation is compensated by the Wilson line W . Here the measure term $\sum_x \{\partial_t U_k(x, t) U_k(x, t)^{-1}\}^a j_k^a(x, t)$ is obtained from the topological field in $(4+2)$ -dimensional space. W corresponds to the local counter term $\exp(iC_{5w}(c_1))$ in the domain wall fermion, where the local field $k_6(x, t)$ is obtained from the topological field in $(5+1)$ -dimensional space:

$$\begin{aligned}
 W &= e^{i\int_0^1 dt \{\partial_t U_k(x, t) U_k(x, t)^{-1}\}^a j_k^a(x, t)} \\
 &\Leftrightarrow e^{iC_{5w}(c_1)} = e^{i \sum_{t=-\infty}^{\infty} \sum_x k_6(x, t)}.
 \end{aligned} \tag{5.43}$$

In this respect, it is possible to see that the topological field in $(5+1)$ -dimensional space reduces to the topological field in $(4+2)$ -dimensional space introduced by Lüscher [4], in the continuum limit. For this, we recall that the $(5+1)$ -dimensional topological field is defined from the local variation of the Chern-Simons term. From Eq. (5.42) and Eq. (5.34) the Chern-Simons term can be expressed as follows, in the same intermediate step as above:

$$\begin{aligned}
 Q_{5w}^{c_1+(-c_2)} &= \text{Im} \ln \det \left(1 - P_0 + P_0 \right. \\
 &\quad \left. \times \left\{ \prod_{t=-\Delta+1}^{\Delta(-c_2)} P_t \right\} P_1 \left\{ \prod_{t=-\Delta+1}^{\Delta(c_1)} P_t \right\} P_0 \right).
 \end{aligned} \tag{5.44}$$

In this equation, one may express each P_t using the chiral basis as $P_t = \sum_i v_i^t \otimes v_i^{t\dagger}$ and factorize the determinant. Then we consider the minimal deformation of the loop $c_1 + (-c_2)$ at a certain point $t_0 + 1$ in the path c_1 as

$$\begin{aligned}
 U_k(x, t_0 + 1) &\equiv U_k^{s=0}(x, t_0 + 1) \\
 &\Rightarrow U_k(x, t_0 + 1)' \equiv U_k^{s=\Delta s}(x, t_0 + 1),
 \end{aligned} \tag{5.45}$$

and evaluate the variation of the Chern-Simons term, to obtain

$$\begin{aligned}
 \Delta Q_{5w}^{c_1+(-c_2)} &= \text{Im} \ln \det(v_i^{t_0}, \{v_j^{t_0+1}\}') \\
 &\quad + \text{Im} \ln \det(\{v_i^{t_0+1}\}', v_j^{t_0}) \\
 &\quad - \text{Im} \ln \det(v_i^{t_0}, v_j^{t_0+1}) - \text{Im} \ln \det(v_i^{t_0+1}, v_j^{t_0}) \\
 &= \text{Im} \ln \det(1 - P^{t_0} \\
 &\quad + P^{t_0} P^{t_0+1} P^{t_0+2} \{P^{t_0+1}\}' P^{t_0}).
 \end{aligned} \tag{5.46}$$

In the limit $\Delta s \rightarrow 0$ and $a_5 \rightarrow 0$, this variation reduces to

$$i \text{Tr} P[\partial_s P, \partial_s P] a_5 \Delta s. \tag{5.47}$$

In the generic gauge $U_5(z) \neq 1$, this result reads

$$i \text{Tr} \{P[\partial_s P, \mathcal{D}_s P] + (\partial_s A_5) P\} a_5 \Delta s, \tag{5.48}$$

and if we replace ∂_s to the covariant derivative \mathcal{D}_s , it exactly coincides with the $(4+2)$ -dimensional topological field [4].

Finally, we note that the gauge anomaly obtained from the asymptotic value of the Chern-Simons current in the domain wall fermion is related to the gauge anomaly expressed by the overlap Dirac operator D [53]:

$$\lim_{N \rightarrow \infty} J_5^a(x, N) = -\text{Tr} \{T^a \gamma_5 D\}(x, x) \Big|_{U_k^1}, \tag{5.49}$$

$$\lim_{N \rightarrow \infty} J_5^a(x, -N+1) = -\text{Tr} \{T^a \gamma_5 D\}(x, x) \Big|_{U_k^0}. \tag{5.50}$$

To see this, we infer from the locality property of the Chern-Simons current that the fifth component at $t=N$

$$\begin{aligned}
 J_5^a(x, N) &= \text{Tr} \left\{ T^a \frac{1}{2} (\gamma_5 - 1) \left(\frac{1}{D_{5w} - m_0} \right) \Big|_{AP} (z + \hat{S}, z) \right. \\
 &\quad \left. + T^a \frac{1}{2} (\gamma_5 + 1) \left(\frac{1}{D_{5w} - m_0} \right) \Big|_{AP} (z, z + \hat{S}) \right\} \Big|_{t=N}
 \end{aligned} \tag{5.51}$$

becomes independent of the interpolation in the limit $N \rightarrow \infty$, depending only on the asymptotic value of the gauge field at $t=N$, $U_k^1(x)$. Then using the formula of the overlap Dirac operator expressed in terms of the inverse five-dimensional Wilson-Dirac operator [35,53]¹² given by

¹²In [35], domain wall fermion is defined vector-likely in the interval $t \in [-N+1, N]$. The interval should be extended to $t \in [-N+1, 3N]$ in our case.

$$\begin{aligned}
D = \lim_{N \rightarrow \infty} & \left\{ 1 - P_R \left(\frac{1}{D_{5w} - m_0} \right) \Big|_{AP} (N, N) P_L \right. \\
& - P_L \left(\frac{1}{D_{5w} - m_0} \right) \Big|_{AP} (-N + 1, -N + 1) P_R \\
& - P_R \left(\frac{1}{D_{5w} - m_0} \right) \Big|_{AP} (N, -N + 1) P_R \\
& \left. - P_L \left(\frac{1}{D_{5w} - m_0} \right) \Big|_{AP} (-N + 1, N) P_L \right\}, \quad (5.52)
\end{aligned}$$

we obtain Eqs. (5.49) and (5.50). The gauge anomalies Eqs. (5.49) and (5.50) can be evaluated in the classical continuum limit as in [28,54] (see also [55–58]) and the earlier calculation [36] is reproduced.

VI. CONCLUSION

The introduction of chirally asymmetric gauge couplings to the chiral zero modes of domain wall fermion, as the original proposal by Kaplan, inevitably makes the system five-dimensional. We have shown that the five-dimensional dependence can be compensated by the local and gauge-invariant counterterm in anomaly-free chiral gauge theories.

The chiral structure of the dimensionally reduced low energy effective action of the chiral zero modes can be understood again by the Ginsparg-Wilson relation. In fact, it provides a concrete example of the gauge-invariant construction of the chiral gauge theories based on the Ginsparg-Wilson relation, where the continuous interpolation in the space of the gauge fields is partly replaced by the discrete step-wise interpolation. Hope is that such discrete treatment of the interpolation of the gauge fields would be useful for a practical implementation of the gauge-invariant lattice chiral gauge theories.

We note that in the gauge-invariant construction with the domain wall fermion, the Ginsparg-Wilson relation is not used explicitly, in sharp contrast to the invariant construction by Lüscher based on the Ginsparg-Wilson relation. This is because one of the points of the invariant construction is the formulation of the integrability condition in the space of gauge fields and the idea behind it is generic. The local cohomology problem follows from the integrability condition, as long as the requirement of locality is satisfied.

In this respect, we also note that our construction is applicable to any domain wall fermion theory defined with a proper five-dimensional Dirac operator of the structure

$$D_5(-m_0) = (D_4(-m_0) + 1)\delta_{t,s} - P_L \delta_{t+1,s} - P_R \delta_{t,s+1}, \quad (6.1)$$

$$D_4(-m_0)^\dagger = \gamma_5 D_4(-m_0) \gamma_5. \quad (6.2)$$

Such a five-dimensional fermion theory can lead to a certain four-dimensional lattice Dirac operator satisfying the

Ginsparg-Wilson relation [35,46,60].¹³ In this sense, our construction partly shares the general applicability with the gauge-invariant construction based on the Ginsparg-Wilson relation.¹⁴

In the original proposal by Kaplan [29], the dynamical treatment of the five-dimensional gauge field was also intended. The question of this ambitious attempt is still open.

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APPENDIX A: TRANSFER MATRIX OF 5-DIMENSIONAL WILSON FERMION

The transfer matrix of the five-dimensional Wilson fermion is given in the chiral basis of gamma matrices as follows:

$$T = e^{-H} = \begin{pmatrix} \frac{1}{B} & -\frac{1}{B}C \\ -C^\dagger \frac{1}{B} & B + C^\dagger \frac{1}{B}C \end{pmatrix}, \quad (A1)$$

where C and B are two by two matrices in the spinor space which define the four-dimensional Wilson-Dirac operator D_w as

$$D_w - m_0 + 1 = \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix}. \quad (A2)$$

Explicitly, they are given as follows:

$$C = \sigma_\mu \frac{1}{2} (\nabla_\mu + \nabla_\mu^*), \quad (A3)$$

$$B = 1 + \left(-\frac{1}{2} \nabla_\mu \nabla_\mu^* - m_0 \right). \quad (A4)$$

¹³Boriçi's five-dimensional implementation of the overlap Dirac operator with the Hermitian Wilson-Dirac operator ($a_5 = 0$) has this structure.

¹⁴The complex phase of the determinant of such a generic five-dimensional Dirac operator (with a negative mass) can also produce the Chern-Simons term. This class of lattice Chern-Simons terms would be understood in relation to the Ginsparg-Wilson relation in five dimensions recently discussed by Bietenholz and Nishimura [27], since it is straightforward to construct the five-dimensional overlap Dirac operator [16] from such a generic five-dimensional Dirac operator.

For the gauge field satisfying the admissibility condition Eq. (2.4), the Hamiltonian defined through the transfer matrix

$$H = -\ln T \quad (\text{A5})$$

has a finite gap [30,32,34].

APPENDIX B: EVALUATION OF THE PARTITION FUNCTIONS OF 5-DIMENSIONAL WILSON FERMIONS

In this appendix, we describe the calculation of the functional determinant of the five-dimensional Wilson-Dirac fermion, in the cases with the antiperiodic boundary condition and Dirichlet boundary condition in the fifth dimension. Here we follow the method given by Neuberger in [47], with a

slight extension to include the fifth component of the five-dimensional gauge field. A more generic method has been given by Lüscher [59,60].

Let us consider the five-dimensional Wilson-Dirac operator

$$W = (D_w - m_0 + 1) \delta_{ts} - P_L U_5(t)^{-1} \delta_{t+1,s} - P_R \delta_{t,s+1} U_5(s), \quad (\text{B1})$$

where $t, s \in [-T+1, T]$ for Dirichlet boundary condition and $t, s \in [-T+1, 3T]$ for anti-periodic boundary condition. We denote the size of the fifth dimension as N in both cases. (N is an even integer.)

In the chiral basis of the gamma matrices, W is written explicitly in the matrix form as follows:

$$W = \begin{pmatrix} \begin{pmatrix} B_1 & C_1 \\ -C_1^\dagger & B_1 \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -U_{5,1} \end{bmatrix} & \cdots & \begin{bmatrix} +Y & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -U_{5,1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B_2 & C_2 \\ -C_2^\dagger & B_2 \end{pmatrix} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \begin{bmatrix} 0 & 0 \\ 0 & -U_{5,N-1} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & +X \end{bmatrix} & \cdots & \begin{bmatrix} -U_{5,N-1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B_N & C_N \\ -C_N^\dagger & B_N \end{pmatrix} \end{pmatrix},$$

where $X=0$, $Y=0$ for the Dirichlet boundary condition and $X=U_{5,N}$, $Y=U_{5,N}^{-1}$ for the antiperiodic boundary condition. In order to make W almost lower tridiagonal, we first exchange the right- and left-handed component columns for each t . Then we move the leftmost column to the place of the rightmost column:

$$\begin{aligned} W &\Rightarrow \begin{pmatrix} \begin{pmatrix} C_1 & B_1 \\ B_1 & -C_1^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -U_{5,1} & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & +Y \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -U_{5,1}^{-1} \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C_2 & B_2 \\ B_2 & -C_2^\dagger \end{pmatrix} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \begin{bmatrix} 0 & 0 \\ -U_{5,N-1} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ +X & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & -U_{5,N-1}^{-1} \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C_N & B_N \\ B_N & -C_N^\dagger \end{pmatrix} \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} \begin{pmatrix} B_1 & 0 \\ -C_1^\dagger & -U_{5,1} \end{pmatrix} & \cdots & \cdots & \begin{bmatrix} +Y & C_1 \\ 0 & B_1 \end{bmatrix} \\ \begin{bmatrix} -U_{5,1}^{-1} & C_2 \\ 0 & B_2 \end{bmatrix} & \begin{pmatrix} B_2 & 0 \\ -C_2^\dagger & -U_{5,2} \end{pmatrix} & \cdots & \vdots \\ \vdots & \begin{bmatrix} -U_{5,2}^{-1} & C_3 \\ 0 & B_3 \end{bmatrix} & \ddots & \vdots \\ \vdots & \cdots & \ddots & \begin{pmatrix} B_N & 0 \\ -C_N^\dagger & +X \end{pmatrix} \end{pmatrix}. \end{aligned}$$

We then introduce the following abbreviations for the blocked matrix elements:

$$\alpha_t \equiv \begin{pmatrix} B_t & 0 \\ -C_t^\dagger & -U_{5,t} \end{pmatrix}, \quad \alpha_X \equiv \begin{pmatrix} B_N & 0 \\ -C_N^\dagger & X \end{pmatrix}$$

$$[X=0(\text{Dir.}), U_{5,N}(\text{AP})]$$

$$\beta_t \equiv \begin{bmatrix} -U_{5,t-1}^{-1} & C_t \\ 0 & B_t \end{bmatrix}, \quad \beta_Y \equiv \begin{bmatrix} Y & C_1 \\ 0 & B_1 \end{bmatrix}$$

$$[Y=0(\text{Dir.}), U_{5,N}^{-1}(\text{AP})]. \quad (\text{B2})$$

Using these, W assumes the following form:

$$W \Rightarrow \begin{pmatrix} \alpha_1 & \cdot & \cdot & \beta_Y \\ \beta_2 & \alpha_2 & \cdot & \cdot \\ \cdot & \ddots & \ddots & \cdot \\ \cdot & \cdot & \ddots & \alpha_N \end{pmatrix}.$$

In order to take account of the boundary element, β_Y , we assume the following factorization:

$$\begin{pmatrix} \alpha_1 & \cdot & \cdot & \beta_Y \\ \beta_2 & \alpha_2 & \cdot & \cdot \\ \cdot & \ddots & \ddots & \cdot \\ \cdot & \cdot & \ddots & \alpha_X \end{pmatrix} = \begin{pmatrix} \alpha_1 & \cdot & \cdot & \cdot \\ \beta_2 & \alpha_2 & \cdot & \cdot \\ \cdot & \ddots & \ddots & \cdot \\ \cdot & \cdot & \ddots & \alpha_X \end{pmatrix} \times \begin{pmatrix} 1 & \cdot & \cdot & -V_1 \\ \cdot & 1 & \cdot & -V_2 \\ \cdot & \cdot & \ddots & \vdots \\ \cdot & \cdot & \cdot & 1 - V_N \end{pmatrix},$$

and consider the recursion equations for the elements V_t :

$$\begin{aligned} -\alpha_1 V_1 &= \beta_Y, \\ -\beta_2 V_1 - \alpha_2 V_2 &= 0, \\ &\vdots \\ -\beta_{N-1} V_{N-2} - \alpha_{N-1} V_{N-1} &= 0, \\ -\beta_N V_{N-1} + \alpha_X (1 - V_N) &= \alpha_X. \end{aligned}$$

These equations can easily be solved to get

$$V_N = \alpha_X^{-1} \alpha_N \cdot \prod_{t=1}^N \{-\alpha_t^{-1} \beta_t\} \cdot \beta_1^{-1} \beta_Y. \quad (\text{B3})$$

Then the determinant of W is evaluated as follows:

$$\begin{aligned} \det(D_{5w} - m_0)_{X,Y} &= \prod_{t=1}^{N-1} \det \alpha_t \cdot \det \alpha_X \det(1 - V_N) \\ &= \prod_{t=1}^N \det \alpha_t \cdot \det(\alpha_N^{-1} \alpha_X - \alpha_N^{-1} \alpha_X V_N). \end{aligned}$$

Here we have omitted the sign factor given by $(-1)^{q(N+1)}$ where $q = 2N_c L^4$ and N_c is the dimension of the gauge group representation, because it turns out to be unity.

The products $-\alpha_t^{-1} \beta_t$, $\alpha_N^{-1} \alpha_X$ and $\beta_1^{-1} \beta_Y$ are evaluated as

$$\begin{aligned} -\alpha_t^{-1} \beta_t &= \begin{pmatrix} 1 & 0 \\ 0 & U_{5,t}^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{B_t} & -\frac{1}{B_t} C_t \\ -C_t^\dagger \frac{1}{B_t} & B_t + C_t^\dagger \frac{1}{B_t} C_t \end{pmatrix} \\ &\quad \times \begin{pmatrix} U_{5,t-1}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & U_{5,t}^{-1} \end{pmatrix} T_t \begin{pmatrix} U_{5,t-1}^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \\ \alpha_N^{-1} \alpha_X &= \begin{pmatrix} 1 & 0 \\ 0 & -U_{5,N}^{-1} X \end{pmatrix}, \\ \beta_1^{-1} \beta_Y &= \begin{pmatrix} -U_{5,0} Y & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Collecting these results, we finally obtain

$$\begin{aligned} \det(D_{5w} - m_0)|_{\text{Dir.}} &= \det \left(P_R + P_L T_N \prod_{t=1}^{N-1} \{U_{5,t}^{-1} T_t\} \right) \\ &\quad \times \det \left(P_R + P_L \prod_t U_{5,t} \right) \cdot \prod_{t=1}^N \\ &\quad \times \det(P_L + P_R B_t), \\ \det(D_{5w} - m_0)|_{\text{AP}} &= \det \left(1 + \prod_{t=1}^N \{U_{5,t}^{-1} T_t\} \right) \\ &\quad \times \det \left(P_R + P_L \prod_t U_{5,t} \right) \cdot \prod_{t=1}^N \\ &\quad \times \det(P_L + P_R B_t). \end{aligned}$$

APPENDIX C: THE PARTITION FUNCTIONS IN THE LIMIT $N \rightarrow \infty$

In this appendix, we evaluate the partition function of the five-dimensional Wilson-Dirac fermions in the limit $N \rightarrow \infty$ and derive Eqs. (5.19) and (5.20). Here we describe the case with the antiperiodic boundary in some detail. The case with the Dirichlet boundary condition can be evaluated in the same manner.

As shown in Appendix B, the partition function of the five-dimensional Wilson-Dirac fermion with the antiperiodic boundary condition is given by

$$\begin{aligned}
 & \det(D_{5w} - m_0)|_{AP} \Big/ \prod_{t=-N+1}^{3N} N_t \\
 &= \det \left(1 + \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \left\{ \prod_{c_1} T_t \right\} T_0^{2(N-\Delta)} \right). \tag{C1}
 \end{aligned}$$

Divided by $\det(1 + T_0^{2(N-\Delta)})$, the RHS can be rewritten as follows:

$$\begin{aligned}
 & \det \left(1 + \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \right. \\
 & \quad \times \left. \left\{ \prod_{c_1} T_t \right\} T_0^{2(N-\Delta)} \right) \Big/ \det(1 + T_0^{2(N-\Delta)}) \\
 &= \det \left(\frac{1}{1 + T_0^{2(N-\Delta)}} + \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \right. \\
 & \quad \times \left. \left\{ \prod_{c_1} T_t \right\} \frac{T_0^{2(N-\Delta)}}{1 + T_0^{2(N-\Delta)}} \right). \tag{C2}
 \end{aligned}$$

In the limit $N \rightarrow \infty$, the dominant term in $T_0^{2(N-\Delta)}$ is evaluated as

$$\begin{aligned}
 T_0^{2(N-\Delta)} &= P_0 T_0^{2(N-\Delta)} + (1 - P_0) T_0^{2(N-\Delta)} \\
 &= \sum_i v_i^0 \otimes v_i^{0\dagger} e^{2(N-\Delta)|\lambda_i|} \\
 & \quad + \mathcal{O}(e^{-2(N-\Delta)\lambda_+^0}) \tag{C3}
 \end{aligned}$$

where v_i^0 are eigenfunctions of $H_0 = -\ln T_0$ belonging to the negative eigenvalues λ_i^0 , while λ_+^0 is the smallest positive eigenvalue of $H_0 = -\ln T_0$. Then the factors involving T_0 in Eq. (C2) reduce to the projection operators

$$\lim_{N \rightarrow \infty} \frac{T_0^{2(N-\Delta)}}{1 + T_0^{2(N-\Delta)}} = P_0, \quad \lim_{N \rightarrow \infty} \frac{1}{1 + T_0^{2(N-\Delta)}} = 1 - P_0, \tag{C4}$$

where

$$P_0 = \sum_i v_i^0 \otimes v_i^{0\dagger} = \frac{1}{2} \left(1 - \frac{H_0}{\sqrt{H_0^2}} \right). \tag{C5}$$

Therefore, the RHS of Eq. (C2) reduces to the following expression:

$$(C2) = \det \left(1 - P_0 + \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \left\{ \prod_{c_1} T_t \right\} P_0 \right) \tag{C6}$$

$$= \det \left(v_i^0, \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \left\{ \prod_{c_1} T_t \right\} v_j^0 \right). \tag{C7}$$

In the last expression the determinant is taken about the indices i and j of basis $\{v_i^0\}$.

Moreover, the term $T_1^{2(N-\Delta)}$ is also evaluated as

$$\begin{aligned}
 T_1^{2(N-\Delta)} &= P_1 T_1^{2(N-\Delta)} + (1 - P_1) T_1^{2(N-\Delta)} \\
 &= \sum_i v_i \otimes v_i^\dagger e^{2(N-\Delta)|\lambda_i|} + \mathcal{O}(e^{-2(N-\Delta)\lambda_+^1}) \tag{C8}
 \end{aligned}$$

where v_i are eigenfunctions of $H_1 = -\ln T_1$ belonging to the negative eigenvalues λ_i , while λ_+^1 (λ_-^1) is the smallest positive (largest negative) eigenvalue of $H_1 = -\ln T_1$. Then, the dominant matrix element in Eq. (C7) is given by

$$\begin{aligned}
 & \left(v_i^0, \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \left\{ \prod_{c_1} T_t \right\} v_j^0 \right) \\
 &= \sum_k \left(v_i^0, \left\{ \prod_{-c_2} T_t \right\} v_k^1 \right) e^{2(N-\Delta)|\lambda_k|} \left(v_k^1, \left\{ \prod_{c_1} T_t \right\} v_j^0 \right) \\
 & \quad + \mathcal{O}(e^{-2(N-\Delta)\lambda_+}). \tag{C9}
 \end{aligned}$$

Accordingly, the dominant contribution of the determinant in Eq. (C7) is evaluated as

$$\begin{aligned}
 & \det \left(v_i^0, \left\{ \prod_{-c_2} T_t \right\} T_1^{2(N-\Delta)} \left\{ \prod_{c_1} T_t \right\} v_j^0 \right) \\
 &= \det \left(1 - P_0 + P_0 \left\{ \prod_{-c_2} T_t \right\} P_1 \left\{ \prod_{c_1} T_t \right\} \right) \\
 & \quad \times \det(1 - P_1 + P_1 T_1)^{2(N-\Delta)} \\
 & \quad \times (1 + \mathcal{O}(e^{-2(N-\Delta)(\lambda_+^1 + |\lambda_-^1|}))). \tag{C10}
 \end{aligned}$$

From this result, we immediately obtain Eq. (5.20).

APPENDIX D: THE INVERSE FIVE-DIMENSIONAL WILSON-DIRAC OPERATOR

In this appendix, we discuss the relation between the inverse five-dimensional Wilson-Dirac operator defined with the Dirichlet boundary condition and that defined in the infinite extent of the fifth dimension, and show that the replacement of Eq. (2.15) is allowed in the limit $N \rightarrow \infty$.

For this purpose, we first note that the Dirichlet boundary condition can be implemented by including the surface term in the infinite volume. Namely, if we consider the five-dimensional Dirac fermion defined in the infinite extent of fifth dimension, but with the couplings between the lattice sites $(-N, -N+1)$ and between the lattice sites $(N, N+a_5)$ omitted, then the field in the interval $[-N+1, N]$ does not have any coupling to those outside the region and it is nothing but the field defined with the Dirichlet boundary condition imposed at $t = -N$ and $t = N+1$. (See Fig. 5.)

We denote the lattice Dirac operator for this fermion by D_{5w}^∇ , which can be expressed with the surface interaction as follows:

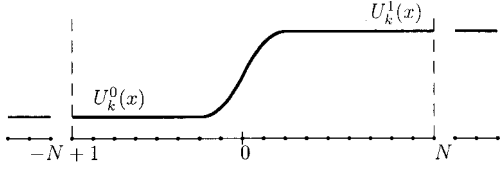


FIG. 5. Implementation of Dirichlet B.C. by surface interaction.

$$D_{5w}^{\vee} - m_0 = D_{5w} - m_0 - V_{(-N+1;N)}, \quad (\text{D1})$$

where

$$V_{(-N+1;N)} = \{-P_L \delta_{s,-N} \delta_{t,-N+1} - P_R \delta_{s,-N+1} \delta_{t,-N} \\ - P_L \delta_{s,N} \delta_{t,N+1} - P_R \delta_{s,N+1} \delta_{t,N}\}. \quad (\text{D2})$$

Then it follows immediately that

$$\frac{1}{D_{5w}^{\vee} - m_0} - \frac{1}{D_{5w} - m_0} = \frac{1}{D_{5w} - m_0} V_{(-N+1;N)} \frac{1}{D_{5w}^{\vee} - m_0}. \quad (\text{D3})$$

Since the inverse of $D_{5w}^{\vee} - m_0$ in the interval $[-N+1, N]$ is nothing but the inverse five-dimensional Wilson-Dirac operator defined with the Dirichlet boundary condition:

$$\frac{1}{D_{5w}^{\vee} - m_0}(xs, yt) = \frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}(xs, yt), \\ s, t \in [-N+1, N], \quad (\text{D4})$$

we can infer that

$$\frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}} - \frac{1}{D_{5w} - m_0} = \frac{1}{D_{5w} - m_0} V_{(-N+1;N)} \\ \times \frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}, \quad (\text{D5})$$

for $s, t \in [-N+1, N]$. Using the explicit form of the surface interaction, it can be rewritten further as

$$\frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}(xs, yt) - \frac{1}{D_{5w} - m_0}(xs, yt) \\ = -\frac{1}{D_{5w} - m_0}(xs; z, -N) P_L \frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}(z, -N+1; yt) \\ - \frac{1}{D_{5w} - m_0}(xs; z, N+1) P_R \frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}(z, N; yt), \quad (\text{D6})$$

where the summation over z is understood and $s, t \in [-N+1, N]$.

Now we consider the case where $s, t \in [-\Delta+1, \Delta]$. [See Eq. (2.15).] From the exponential bound Eq. (3.10), the inverse five-dimensional Wilson-Dirac operators defined in the infinite extent of the fifth dimension in the RHS of Eq. (D6) vanish identically in the limit $N \rightarrow \infty$. On the other hand, the inverse five-dimensional Wilson-Dirac operators defined with the Dirichlet boundary condition in the RHS of Eq. (D6) can be expressed in terms of the transfer matrix using the same technique used in Appendix B [53]. They are given by

$$P_L \frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}(z, -N+1; yt) = +P_L \Delta(z, yt), \quad (\text{D7})$$

$$P_R \frac{1}{D_{5w} - m_0} \Big|_{\text{Dir.}}(z, N; yt) = -P_R \Delta(z, yt), \quad (\text{D8})$$

where

$$\Delta(z, y; t) = \frac{1}{P_R + T_1^{(N-\Delta)} \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} T_0^{(N-\Delta)} P_L} \left\{ T_1^{(N-\Delta)} U_{5,\Delta}^{-1} \prod_{s=-\Delta+1}^t T_s U_{5,s-1}^{-1} \beta_t^{-1} \right\}, \quad t \in [-\Delta+1, \Delta]. \quad (\text{D9})$$

For large N , $\Delta(z, y; t)$ can be estimated as

$$\Delta(z, y; t) \simeq \frac{1}{(1 - P_1) P_R + P_1 \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} T_0^{(N-\Delta)} P_L} \left\{ P_1 U_{5,\Delta}^{-1} \prod_{s=-\Delta+1}^t T_s U_{5,s-1}^{-1} \beta_t^{-1} \right\}$$

$$\begin{aligned} & \simeq \frac{\text{cofactor of} \left\{ (1-P_1)P_R + P_1 \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} T_0^{(N-\Delta)} P_L \right\}}{\det(1-P_0+P_0 T_0)^{(N-\Delta)}} \\ & \times \frac{1}{\det \left((1-P_1)P_R + P_1 \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} P_0 P_L \right)} \left\{ P_1 U_{5,\Delta}^{-1} \prod_{s=-\Delta+1}^t T_s U_{5,t-1}^{-1} \beta_t^{-1} \right\}, \quad (\text{D10}) \end{aligned}$$

and we can infer that it vanishes identically in the limit $N \rightarrow \infty$, as long as

$$\det \left((1-P_1)P_R + P_1 \left\{ U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1} \right\} P_0 P_L \right) \neq 0. \quad (\text{D11})$$

Thus the RHS of Eq. (D6) vanishes in this limit.

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- [1] P.H. Ginsparg and K.G. Wilson, Phys. Rev. D **25**, 2649 (1982).
[2] M. Lüscher, Phys. Lett. B **428**, 342 (1998).
[3] M. Lüscher, Nucl. Phys. **B549**, 295 (1999).
[4] M. Lüscher, Nucl. Phys. **B568**, 162 (2000).
[5] M. Lüscher, Nucl. Phys. **B538**, 515 (1999).
[6] H. Suzuki, Prog. Theor. Phys. **101**, 1147 (1999).
[7] T. Fujiwara, H. Suzuki, and K. Wu, Phys. Lett. B **463**, 63 (1999); Nucl. Phys. **B569**, 643 (2000).
[8] M. Lüscher, Nucl. Phys. B (Proc. Suppl.) **83**, 34 (2000).
[9] M. Lüscher, lectures given at the International School of Sub-nuclear Physics, Erice, 2000, hep-th/0102028.
[10] H. Suzuki, Nucl. Phys. **B585**, 471 (2000); H. Igarashi, K. Okuyama, and H. Suzuki, hep-lat/0012018.
[11] M. Lüscher, J. High Energy Phys. **06**, 028 (2000).
[12] O. Bär and I. Campos, Nucl. Phys. **B581**, 499 (2000).
[13] Y. Kikukawa and Y. Nakayama, Nucl. Phys. **B597**, 519 (2001).
[14] H. Neuberger, Nucl. Phys. **B417**, 141 (1998).
[15] H. Neuberger, Phys. Lett. B **427**, 353 (1998).
[16] Y. Kikukawa and H. Neuberger, Nucl. Phys. **B513**, 735 (1998).
[17] R. Narayanan and H. Neuberger, Nucl. Phys. **B412**, 574 (1994); Phys. Rev. Lett. **71**, 3251 (1993); Nucl. Phys. **B443**, 305 (1995).
[18] H. Neuberger, hep-lat/0101006.
[19] S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B **348**, 543 (1995); Nucl. Phys. **B443**, 386 (1995); **B466**, 335 (1996); **B461**, 305 (1996); Phys. Lett. B **402**, 134 (1997); A. Yamada, Phys. Rev. D **57**, 1433 (1998); Nucl. Phys. **B514**, 399 (1998); **B529**, 483 (1998).
[20] R. Narayanan and H. Neuberger, Phys. Lett. B **348**, 549 (1995); Nucl. Phys. **B477**, 521 (1996); Nucl. Phys. B (Proc. Suppl.) **53**, 661 (1997); Phys. Lett. B **393**, 360 (1997); **402**, 320 (1997); Y. Kikukawa, R. Narayanan, and H. Neuberger, *ibid.* **399**, 105 (1997); Phys. Rev. D **57**, 1233 (1998).
[21] T. Izubuchi and J. Nishimura, J. High Energy Phys. **10**, 002 (1999).
[22] H. Neuberger, Phys. Rev. D **59**, 085006 (1999).
[23] D.H. Adams, Nucl. Phys. **B589**, 633 (2000); Phys. Rev. Lett. **86**, 200 (2001).
[24] H. Neuberger, Phys. Lett. B **437**, 117 (1998).
[25] H. Neuberger, Phys. Rev. D **63**, 014503 (2001).
[26] R. Narayanan and J. Nishimura, Nucl. Phys. **B508**, 371 (1997).
[27] W. Bietenholz and J. Nishimura, J. High Energy Phys. **07**, 015 (2001).
[28] T. Aoyama and Y. Kikukawa, “A lattice implementation of the η -invariant and effective action for chiral fermions on the lattice,” hep-lat/9905003.
[29] D.B. Kaplan, Phys. Lett. B **288**, 342 (1992).
[30] Y. Shamir, Nucl. Phys. **B406**, 90 (1993), V. Furman and Y. Shamir, *ibid.* **B439**, 54 (1995).
[31] H. So, Prog. Theor. Phys. **73**, 582 (1985); **74**, 585 (1985); A. Coste and M. Lüscher, Nucl. Phys. **B323**, 631 (1989).
[32] P. Hernández, K. Jansen, and M. Lüscher, Nucl. Phys. **B552**, 363 (1999).
[33] M. Lüscher, Commun. Math. Phys. **85**, 39 (1982).
[34] H. Neuberger, Phys. Rev. D **61**, 085015 (2000).
[35] Y. Kikukawa, Nucl. Phys. **B584**, 511 (2000).
[36] M. Golterman, K. Jansen, and D.B. Kaplan, Phys. Lett. B **301**, 219 (1993).
[37] K. Jansen, Phys. Rep. **273**, 1 (1996).
[38] D.B. Kaplan, Nucl. Phys. B (Proc. Suppl.) **20**, 597 (1993).
[39] M.F.L. Golterman, K. Jansen, D.N. Petcher, and J.C. Vink, Phys. Rev. D **49**, 1606 (1994); Nucl. Phys. B (Proc. Suppl.) **34**, 593 (1994).
[40] M. Creutz and I. Horvath, Phys. Rev. D **50**, 2297 (1994); M. Creutz, M. Tytgat, C. Rebbi, and She-Sheng Xue, Phys. Lett. B **402**, 341 (1997); M. Creutz, Phys. Rev. Lett. **83**, 2636 (1999).
[41] M. Creutz, Rev. Mod. Phys. **73**, 119 (2001).
[42] M. Golterman and Y. Shamir, Phys. Rev. D **51**, 3026 (1995); Phys. Lett. B **353**, 84 (1995); Nucl. Phys. B (Proc. Suppl.) **47**, 603 (1996).
[43] R. Narayanan and H. Neuberger, Phys. Lett. B **358**, 303 (1995).
[44] S. Aoki and K. Nagai, Phys. Rev. D **53**, 5058 (1996); Phys. Rev. D **56**, 1121 (1997); **58**, 045002 (1998).
[45] C.G. Callan, Jr. and J.A. Harvey, Nucl. Phys. **B250**, 427 (1985).
[46] A. Boriçip, Nucl. Phys. B (Proc. Suppl.) **83**, 771 (2000); hep-lat/9912040.

- [47] H. Neuberger, Phys. Rev. D **57**, 5417 (1998).
- [48] P. Vranas, Phys. Rev. D **57**, 1415 (1998).
- [49] L. Alvarez-Gaumé, S. Della Pietra, and V. Della Pietra, Phys. Lett. **166B**, 177 (1986); Commun. Math. Phys. **109**, 691 (1987).
- [50] R.D. Ball and H. Osborn, Phys. Lett. **165B**, 410 (1985); Nucl. Phys. **B263**, 243 (1986); R.D. Ball, Phys. Lett. B **171**, 435 (1986); Phys. Rep. **182**, 1 (1989).
- [51] M.F. Atiya, V.K. Patodi, and I.M. Singer, Math. Proc. Cambridge Philos. Soc. **77**, 43 (1975); **78**, 405 (1975); **79**, 71 (1976).
- [52] D.B. Kaplan and M. Schmaltz, Phys. Lett. B **368**, 44 (1996).
- [53] Y. Kikukawa and T. Noguchi, “Low energy effective action of domain-wall fermion and the Ginsparg-Wilson relation,” hep-lat/9902022.
- [54] Y. Kikukawa and A. Yamada, Phys. Lett. B **448**, 265 (1999).
- [55] D.H. Adams, hep-lat/9812003.
- [56] H. Suzuki, Prog. Theor. Phys. **102**, 141 (1999).
- [57] K. Fujikawa, Nucl. Phys. **B546**, 480 (1999).
- [58] T.-W. Chiu, Phys. Lett. B **445**, 371 (1999); T.-W. Chiu and T.-H. Hsieh, hep-lat/9901011.
- [59] M. Lüscher (private communication).
- [60] P. Hernández, K. Jansen, and M. Lüscher, hep-lat/0007015.