

Vortex waistlines and long range fluctuations

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We examine the manner in which a linear potential results from fluctuations due to vortices linked with the Wilson loop. Our discussion is based on exact relations and inequalities between the Wilson loop and the vortex and electric flux order parameters. We show that, contrary to the customary naive picture, only vortex fluctuations of a thickness of the order of the spatial linear size of the loop are capable of producing a strictly linear potential. An effective theory of these long range fluctuations emerges naturally in the form of a strongly coupled $Z(N)$ lattice gauge theory. We also point out that dynamical fermions introduced in this medium undergo chiral symmetry breaking.

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I. INTRODUCTION

Center vortices are widely believed to be the most important degrees of freedom for confinement in Yang-Mills theories. (For a recent review see [1].) Attempts to isolate and compute the vortex content in the path integral at large β have been a very active area of study in the past two years. (For a general view of the various approaches and issues involved, see [2]).

A popular plausibility argument for confinement by vortices goes as follows. To estimate the vortex contribution to the expectation of a given Wilson loop, one assumes that typically many thin vortices—i.e. thin compared to the physically large ($\gg 1$ fm²) Wilson loop area A —link with the Wilson loop. A simple estimate is obtained by subdividing the surface spanned by the loop into $n=A/a$ pieces of equal area a and assuming that a vortex piercing a given small area piece is present with a probability p . Provided that the vortices linking with the loop at different places are *independent*, the Wilson loop expectation is then given by

$$\sum_{k=0}^n \binom{n}{k} (-1)^k p^k (1-p)^{n-k} = (1-2p)^{A/a}, \quad (1)$$

which is the desired area law.

In the present paper we point out that while this simple picture indeed gives the correct area law, it is fundamentally incorrect because it is not supported by the underlying full non-Abelian gauge theory. The response of the vortex distribution in the vacuum to the introduction of the external probe represented by a Wilson loop should be such as to minimize the effective quantum action (free energy) of the system. To examine this we make the above arguments more precise by introducing a set of vortex containers linking with the Wilson loop in the spirit of [3]. Any set of vortex con-

tainers can then be used to obtain an upper bound to the Wilson loop by making use of rigorous inequalities mostly based on reflection positivity. We show that thick vortex containers of thickness of the order of the (shorter) side length of the Wilson loop yield the “best” and in fact the only strict area law upper bound. This shows that vortices do not have some fixed characteristic thickness but the most important vortex fluctuations disordering a Wilson loop of a given size have a core thickness of the order of the linear size of the Wilson loop. It also means that generally vortices cannot be unambiguously located as individual objects in every given single gauge field configuration. The only meaningful quantity is the number of vortices (mod N) linking with a given Wilson loop, and this is a well-defined gauge invariant quantity. In the remainder of the paper we show how the physical picture implied by the previous discussion leads to a simple effective $Z(N)$ gauge theory for the long distance center degrees of freedom. We then observe that the introduction of dynamical fermions in this effective strongly coupled system results in chiral symmetry breaking.

The paper is organized as follows. In Sec. II we review and extend various relations and inequalities relating the electric free energy (Fourier transform of the vortex free energy) to the Wilson loop. These relations form the basis for our discussion of vortices and the emergence of a linear potential in Sec. III. Section IV further discusses the physical picture and introduces the effective theory. Section V contains some conclusions.

II. ELECTRIC FLUX INEQUALITIES

After a brief description of our notation, in this section we introduce the basic electric flux inequalities. These are already interesting in their own right since they yield a rigorous upper bound on the Wilson loop in terms of the electric flux free energy.

We work on a hypercubic lattice Λ of length L_μ in space-time direction $\mu = 1, \dots, d$. We assume the standard Wilson formulation of lattice gauge theory with $SU(2)$ group-valued link variables, the Wilson action and periodic boundary con-

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ditions in all directions. Expectation values of observables are defined as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int d[U] \mathcal{O} \exp\left(\beta \sum_p \frac{1}{2} \text{tr} U_p\right), \quad (2)$$

where the integration is over all the group-valued link variables and Z , the partition function, is the same integral without the operator insertion \mathcal{O} .

Let us denote by $\mathcal{O}[\mathcal{V}_{\mu\nu}]$ the operator that flips the sign of the coupling [introduces a $Z(2)$ “twist”] on a coclosed ($d-2$)-dimensional set of plaquettes $\mathcal{V}_{\mu\nu}$ winding around the periodic lattice in the directions perpendicular to the $\mu\nu$ directions. The expectation value of this operator defines the vortex free energy:

$$\exp(-F_v[\mu\nu]) = \langle \mathcal{O}[\mathcal{V}_{\mu\nu}] \rangle. \quad (3)$$

The twist amounts to a discontinuous gauge transformation with multivaluedness in $Z(2)$, i.e., forces the presence of a $\pi_1(SU(2)/Z(2))$ vortex wrapped around the periodic lattice. As indicated by the notation, the expectation depends only on the directions in which \mathcal{V} winds through the lattice, not the exact shape or location of \mathcal{V} . This expresses the mod 2 conservation of flux. Indeed, the twist -1 on the plaquettes forming \mathcal{V} can be moved to the plaquettes forming any other homologous coclosed set \mathcal{V}' by the change of variables $U_b \rightarrow -U_b$ in the numerator in Eq. (3) for each bond b in a set of bonds cobounded by $\mathcal{V} \cup \mathcal{V}'$. By the same token Eq. (3) is invariant under changes mod 2 in the number of homologous twisted coclosed sets introduced in Λ . A simple consequence of this is that

$$\langle \mathcal{O}[\mathcal{V}_{\mu\nu}] \mathcal{O}[\mathcal{V}'_{\mu\nu}] \rangle = 1. \quad (4)$$

We will assume that, for sufficiently large $|A_{\mu\nu}|$, and dimension $d \leq 4$, the vortex free energy (3) behaves as

$$F_v[\mu\nu] \sim \left(\prod_{\lambda \neq \mu\nu} L_\lambda \right) \exp(-\rho(\beta) |A_{\mu\nu}|), \quad (5)$$

where $|A_{\mu\nu}| = L_\mu L_\nu$. This is the optimal behavior under exponential transverse spreading (creation of mass gap) of the flux introduced by the twist on \mathcal{V} , with ρ approaching, at least asymptotically, the exact linear potential string tension. This behavior is expected by physical reasoning [4], and explicitly seen in the strong coupling expansion [5]. Recently, it became possible to demonstrate this in numerical simulations at large β [6,7]. The behavior (5) for a “vortex in a box” is essential for our argument in the following.

The $Z(2)$ Fourier transform of Eq. (3),

$$\exp(-F_{\text{el}}) = \left\langle \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}_{\mu\nu}]) \right\rangle, \quad (6)$$

gives the corresponding dual (with respect to the center) color-electric free energy. The mod 2 conservation of the magnetic flux is now expressed by the projection property

$$\left\langle \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}_{\mu\nu}]) \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}'_{\mu\nu}]) \right\rangle = \left\langle \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}_{\mu\nu}]) \right\rangle, \quad (7)$$

as is easily seen by using Eq. (4).

Consider now a rectangular Wilson loop C placed, say, in the $[12]$ plane. Let \mathcal{V} be a coclosed stack of plaquettes winding around the periodic lattice in the perpendicular $\mu = 3, \dots, d$ directions and through C , and insert unity in the numerator in the Wilson loop expectation $W[C] = \langle \frac{1}{2} \text{tr} U[C] \rangle$ in the form

$$1 = \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}]) + \frac{1}{2} (1 + \mathcal{O}[\mathcal{V}]).$$

Then

$$\begin{aligned} W[C] &= \left\langle \frac{1}{2} \text{tr} U[C] \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}]) \right\rangle \\ &\quad + \left\langle \frac{1}{2} \text{tr} U[C] \frac{1}{2} (1 + \mathcal{O}[\mathcal{V}]) \right\rangle \\ &= \left\langle \frac{1}{2} \text{tr} U[C] \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}]) \right\rangle \\ &\quad + \left\langle \frac{1}{2} \text{tr} U[C] \frac{1}{2} (1 - \mathcal{O}[\mathcal{V}']) \right\rangle. \end{aligned} \quad (8)$$

Here \mathcal{V}' is another coclosed stack of plaquettes winding around the lattice in the perpendicular $\mu = 3, \dots, d$ directions but not threading through C . The second equality in Eq. (8) is obtained by making the change of variables $U_b \rightarrow -U_b$ in the second term in the first equality in Eq. (8) for each bond b in a set of bonds cobounded by \mathcal{V} and \mathcal{V}' . This “moves” the twist (-1) on the plaquettes forming \mathcal{V} to those forming \mathcal{V}' . This set of bonds necessarily involves one (or an odd number of) bond(s) on C , which results in the minus sign in the second term in the second equality. Equation (8) is represented graphically as

$$\left\langle \square \right\rangle = \left\langle \square \blacksquare \right\rangle + \left\langle \square \blacksquare' \right\rangle \quad (9)$$

where a filled square stands for the operator $\frac{1}{2}(1 - \mathcal{O}[\mathcal{V}])$, with \mathcal{V} crossing the two-dimensional plane containing the loop at the location of the filled square, and winding around the lattice in the remaining $d-2$ perpendicular directions.

Simple identities like Eq. (9), or Eqs. (16),(17) below, serve as the starting point for deriving relations between the Wilson loop and vortex free energies by use of reflection positivity. Given a reflection r about a $(d-1)$ -dimensional lattice hyperplane, one defines an antilinear mapping θ on functions of the bond variables by $\theta F[\{U_b\}] = F^*[\{U_{rb}\}]$. By the reflection positivity properties of the lattice gauge theory (LGT) action [8], this induces a positive semidefinite inner product on the space of configurations allowing the use of the corresponding Schwarz inequalities. Thus, starting

with Eq. (9), consider a reflection about a $d-1$ -dimensional hyperplane π perpendicular to the “vertical” loop legs and containing, say, the top “horizontal” leg of the loop C . One then has the inequality

$$\langle \left[\text{loop} \right] \rangle \leq \langle \left[\text{reflected loop} \right] \rangle^{1/2} \langle \left[\text{original loop} \right] \rangle^{1/2} = \langle \left[\text{doubled loop} \right] \rangle \quad (10)$$

for the first term, and a similar statement for the second term in Eq. (9). In Eq. (10) we made use of Eq. (7)¹ and $\frac{1}{2}\text{tr}\mathbf{1} = 1$. The loop has now doubled in size along the direction of one of its legs. Proceeding now by repeated reflections in hyperplanes containing one of the legs of the loop resulting from the previous reflection, and use of Eq. (7), one may eventually completely eliminate all $\text{tr}U$ factors from the loop operator by virtue of the lattice periodicity. Applying this procedure to both terms on the right-hand side (RHS) of Eq. (9) one obtains [9]

$$\langle W[C] \rangle \leq 2(\exp(-F_{\text{el}}))^{|A_C|/|A_{12}|}, \quad (11)$$

where $|A_C|$ is the minimal area bounded by the Wilson loop. If then the vortex free energy behaves as in Eqs. (5), (11) implies area law for the Wilson loop.

Note that the result (11) manifestly incorporates mod 2 conservation since of course Eqs. (6) and (3) do. This is an important point that we now explore a bit further. Any multiple factors of $\frac{1}{2}(1 - \mathcal{O}[\mathcal{V}])$ occurring in the derivation above were eliminated by Eq. (7). Suppose instead that we keep a number of such factors to make contact with the naive picture of a Wilson loop pierced by several independent vortices. So imagine that we subdivide the 2-dimensional plane A containing C into large squares of side length l , i.e. we view A from a coarse lattice of spacing l . We denote A by A' when viewed from the coarse lattice. Rewrite the identity (9) using Eq. (7) in the equivalent form

$$W[C] = \left\langle \frac{1}{2} \text{tr}U[C] \prod_i \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}_i]) \right\rangle + \left\langle \frac{1}{2} \text{tr}U[C] \prod_j \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}'_j]) \right\rangle \quad (12)$$

where the product in the first term includes one factor (indexed by i) for every large square in A (every plaquette in A') tiling C (Fig. 1), and the identical arrangement translated outside C in the second term. One now applies reflection positivity to repeatedly reflect about $(d-1)$ -dimensional hyperplanes π'_1 , or π'_2 , perpendicular to the $\mu=1$, or 2-direction, respectively, containing bonds in A' (Fig. 1). In this way one derives

¹Notice that Eq. (7) remains valid in the presence of additional operators in the expectation, as long as moving \mathcal{V}' to \mathcal{V} by a change of variables does not affect the additional operator. In particular, this is the case for a Wilson loop when neither or both \mathcal{V} and \mathcal{V}' link with the loop.

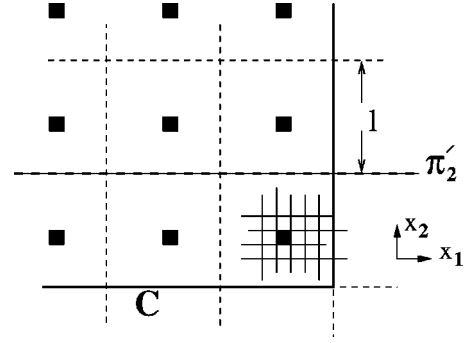


FIG. 1. Arrangement for reflections in first term in Eq. (12) to obtain Eq. (13) (see text).

$$W[C] \leq 2 \prod_i \left\langle \prod_{k \in A'} \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}_k]) \right\rangle^{1/|A'|} \quad (13)$$

where in the product inside the expectation there is one factor for each plaquette in A' . Equation (13) now has the form of a typical “chessboard estimate” inequality (see e.g. [8,9]). It is of course equivalent to Eq. (11) by (7) since $|A'| = |A|/l^2$, and the number of factors in the outside product equals $|A'_C| = |A_C|/l^2$.

Returning to Eq. (8) and noting that

$$\begin{aligned} & \left\langle \frac{1}{2} \text{tr}U[C] \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}]) \right\rangle + \left\langle \frac{1}{2} \text{tr}U[C] \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}']) \right\rangle \\ &= \left\langle \frac{1}{2} \text{tr}U[C] \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}]) \frac{1}{2}(1 + \mathcal{O}[\mathcal{V}']) \right\rangle \\ &+ \left\langle \frac{1}{2} \text{tr}U[C] \frac{1}{2}(1 + \mathcal{O}[\mathcal{V}]) \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}']) \right\rangle \\ &= \left\langle \frac{1}{2} \text{tr}U[C] \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}]\mathcal{O}[\mathcal{V}']) \right\rangle \end{aligned} \quad (14)$$

$$= \left\langle \frac{1}{2} \text{tr}U[C] \frac{1}{2}(1 - \mathcal{O}[\mathcal{V}'']) \right\rangle \quad (15)$$

one obtains the alternative identities represented graphically by

$$W[C] = \frac{1}{2} \left\langle \left[\text{loop} \right] - \left[\text{reflected loop} \right] \right\rangle \quad (16)$$

$$= \frac{1}{2} \left\langle \left[\text{loop} \right] - \left[\text{reflected loop} \right] \right\rangle \quad (17)$$

Equation (16) is the graphical representation of Eq. (14). Equation (15), graphically depicted in Eq. (17), is then ob-

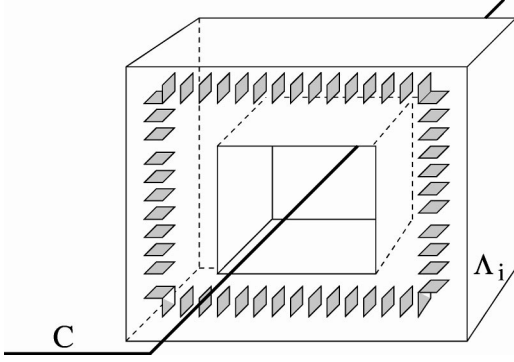


FIG. 2. Vortex container Λ_i enclosing coclosed set of plaquettes \mathcal{V}_i (shaded) linking with Wilson loop C . $d=3$, or 3-dimensional section in $d=4$.

tained from Eq. (14) by simply merging \mathcal{V} and \mathcal{V}' in the coclosed set \mathcal{V}'' linking with C by another shift in integration variables and mod 2 flux conservation.²

Alternatively, Eqs. (16),(17) may be directly obtained as follows. Insert $1 = \int_{Z(2)} d\gamma$, where $\gamma \in Z(2)$, in the numerator in the expectation $W[C]$, and make a shift of integration variables $U_b \rightarrow \gamma U_b$ for all $b \in B$, where B is a set of bonds whose coboundary is $\mathcal{V} \cup \mathcal{V}'$ or \mathcal{V}'' . The result is Eq. (16), or Eq. (17), respectively.

III. THE WILSON LOOP IN TERMS OF VORTEX CONTAINERS

In this section we introduce a set of vortex containers linking with a given Wilson loop. Using the relations derived in the previous section, we obtain an upper bound on the Wilson loop in terms of vortex fluctuations occurring in the given set of vortex containers. We then ask the question, which set of vortex containers gives the “best” (i.e. the lowest) upper bound to the Wilson loop. It turns out that the favored set of vortex containers has only one single thick container utilizing an area of the order of the area spanned by the Wilson loop. This yields a strictly linear lower bound to the heavy quark potential, whereas a collection of many thin vortex containers results in a suppression of the potential by a logarithmic factor.

By Eq. (7) and footnote 1, one can insert multiple $\frac{1}{2}(1 - \mathcal{O}[\mathcal{V}])$ factors in Eqs. (15),(17) corresponding to a collection of coclosed sets $\{\mathcal{V}_i\}$ linking with the loop C . Imagine enclosing each \mathcal{V}_i in a “vortex container” [3], i.e., a sublattice $\Lambda_i \subset \Lambda$ containing \mathcal{V}_i and wrapping around C (Fig. 2). Imagine integrating over the bond variables in the interior of each container, keeping bond variables on its boundary $\partial\Lambda_i$ fixed, and define

$$f_{\Lambda_i}(U_{\partial\Lambda_i}) = \frac{1}{2} \left(1 - \frac{z_{\Lambda_i}^{(-)}(U_{\partial\Lambda_i})}{z_{\Lambda_i}^{(+)}(U_{\partial\Lambda_i})} \right), \quad (18)$$

² $\mathcal{V} \cup \mathcal{V}' \sim \mathcal{V}''$ is topologically trivial with respect to the lattice T^d topology, but not with respect to the obstruction of the loop C .

where

$$z_{\Lambda_i}^{(\mp)}(U_{\partial\Lambda_i}) \equiv \int \prod_{b \in \Lambda_i - \partial\Lambda_i} dU_b \times \prod_{p \in \Lambda_i - \partial\Lambda_i} \exp\left(\frac{\beta}{2} (\mp 1)^{\mathcal{V}_i[p]} \text{tr} U_p\right) \quad (19)$$

with the characteristic function $\mathcal{V}_i[p] \equiv 1$ if $p \in \mathcal{V}_i$, 0 otherwise. $z_{\Lambda_i}^{(+)}(U_{\partial\Lambda_i})$ is of course simply the partition function for (the interior of) Λ_i . Then one obtains [3]

$$W[C] = \left\langle \frac{1}{2} \text{tr} U[C] \prod_i f_{\Lambda_i}(U_{\partial\Lambda_i}) \right\rangle. \quad (20)$$

Equation (13) can be rewritten in the same way:

$$W[C] \leq 2 \prod_i \left\langle \prod_{k \in A'} f_{\Lambda_k}^i(U_{\partial\Lambda_k}) \right\rangle^{|A'|}, \quad (21)$$

where now each container Λ_k containing the coclosed set \mathcal{V}_k is of transverse area l^2 and wraps around the lattice in the longitudinal directions.

Note that $f_{\Lambda_i}(U_{\partial\Lambda_i})$, Eq. (18), is nothing but Eq. (6) [$Z(2)$ FT of vortex free energy] now defined on a lattice Λ_i (the vortex container) with fixed (instead of periodic) b.c. in the transverse directions, but still periodic b.c. in the longitudinal directions (in which the Λ_i 's have torus topology by construction). Vortex containers including integration over fluctuations (summation of entropy effects) in their interior are a convenient device for discussing scales larger than their thickness in terms of free energy costs rather than the action of individual configurations.

From Eq. (20) one can now, trivially, obtain the bound

$$W[C] \leq \prod_i \max_{U_{\partial\Lambda_i}} |f_{\Lambda_i}(U_{\partial\Lambda_i})|, \quad (22)$$

where the maximum is taken over all values of the bond variables on the boundary $\partial\Lambda_i$.

Equation (22) makes a direct connection with the naive notion of independent vortices winding through a large Wilson loop, resulting in disorder, and area law, provided they are able to grow sufficiently long to pierce through the loop at any point. For this to be possible the vortices must be allowed to grow sufficiently thick to keep their free energy cost fixed as their length increases with that of the Wilson loop. Let d_i be the size of Λ_i in each of the two directions transverse to the set \mathcal{V}_i used in its definition; its longitudinal size is given by $|\mathcal{V}_i|$. We must first assume that all the vortex containers are thick enough to reach the regime where

$$|\ln(z_{\Lambda_i}^{(-)}(U_{\partial\Lambda_i})/z_{\Lambda_i}^{(+)}(U_{\partial\Lambda_i}))| \sim |\mathcal{V}_i| \exp(-\rho d_i^2) \quad (23)$$

for all $U_{\partial\Lambda_i}$. So to keep the free energy cost of each vortex less than a fixed value f , we need

$$d_i^2 \geq \frac{1}{\rho} \ln \left(\frac{|\mathcal{V}_i|}{f} \right). \quad (24)$$

Then also $\max |f_{\Lambda_i}(U_{\partial\Lambda_i})| < \frac{1}{2}(e^f - 1) < \text{const}$. But $|\mathcal{V}_i|$ (in d dimensions) is of the order R^{d-2} for linkage through points away from the perimeter of a rectangular loop of side lengths T and $R, T \gg R$. Such a loop can then accommodate $\sim RT/\ln R$ containers wrapped around it. Thus Eq. (22) gives a confining but not quite purely linear potential

$$V(R) \geq \text{const } R/\ln R. \quad (25)$$

The same reasoning, and consequent failure to produce a purely linear potential, applies to the familiar argument for confinement by vortices, outlined in the Introduction, which tacitly underlies or is implied by many discussions in the literature. One assumes randomly distributed vortices of a certain thickness and basically arbitrary length. It is crucial that one assumes that the cost and hence the probability for vortices to link anywhere with the loop is fixed for any Wilson loop size [cp. Eq. (1)]. One then considers one vortex linked with a large Wilson loop. With the vortex thickness assumed much less than the loop's linear dimensions, one now sums over all positions of intersection with a surface spanning the loop. With the above assumptions, this produces a factor proportional to the loop area. One then sums over all intersection points for two *independent* vortices linking with the loop, and so on. This clearly exponentiates generating area law:

$$\begin{aligned} W[C] &\sim 1 + (-1)k|A_C| + \frac{((-1)k|A_C|)^2}{2!} + \dots \\ &= \exp(-k|A_C|). \end{aligned} \quad (26)$$

We now see that purely linear confinement is obtained this way only by adopting a non-interacting gas picture, and ignoring the actual free energy requirements for having vortices of sufficient length link anywhere with a large loop: the type of discussion just given above for Eq. (22) applies to each term in such a summation. Thus, for one vortex linking with the large loop of side lengths T and R ($T \gg R$), a vortex cross section area of order $\ln R$ is required; otherwise, linking anywhere far away from the perimeter for fixed, bounded vortex free energy cost f , as required by the argument, is not possible. This leads at best to Eq. (25), *not* Eq. (26).

The problem arises because one treats the vortices as localized and independent. For sufficiently thick vortices free energy costs are indeed correctly estimated in magnitude as above, i.e. Eq. (23). Thus, if one imagines each vortex enclosed in a vortex container of fixed, but sufficiently large, width d , the exponential transverse spreading $\sim \exp(-\rho d^2)$ renders the overall vortex bulk free energy cost inside insensitive to the exact values of the gauge fields on the container boundary. The vortex, however, is surrounded by the pure gauge long tail that encodes its nontrivial topology, and flux quantization. This tail incurs no additional action cost, but is of infinite range and communicates the presence of nontrivial topological flux inside the container to everywhere outside.

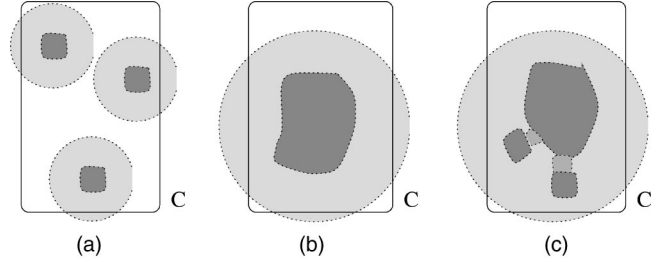


FIG. 3. Vortices linking with Wilson loop C . Cores shown in darker shading, and long range (infinite extent) pure gauge tails in light shading: (a) Well-separated vortices of approximately fixed width; (b) configurations of equivalent mod 2 flux having one thickened vortex lowering potential energy; (c) configurations contributing essentially as in (b) showing the possible ambiguities in counting vortices beyond mod N .

So even though the gauge field values on a thick container's boundaries are irrelevant for estimating the bulk free energy cost inside, they are very much relevant for signalling the presence of a vortex inside to other vortices or other topological obstructions outside.

This acts like an “irreducible” interaction between vortices that acts at all distances, and enforces flux conservation mod N . This interaction allows a system of vortex excitations to adjust the amount of flux spreading, i.e., adjust the thickness of vortices to minimize the free energy of the system. The thickness of vortex cores then is not fixed, but is adjusted relative to their length as required by the presence of other vortices and/or other obstructions (e.g. Wilson loop legs) sensitive to the presence of topological $Z(N)$ flux (Fig. 3). This means that in general vortices cannot be considered isolated, and a definite number of vortices, specified more precisely than mod N , cannot necessarily be unambiguously assigned to every configuration.

Thus, in the presence of the Wilson loop source, the optimal configurations for the system are not those of multiple isolated linked vortices, each of some fixed free energy cost [Fig. 3(a)], hence length $|\mathcal{V}| \sim R^{(d-2)}$ and fixed width $d^2 \sim \ln R \ll R, T$. It is more advantageous, in terms of free energy cost, for multiple linking vortices to thicken and merge, the total topological flux being conserved mod N [Fig. 3(b)]. Since the Wilson loop operator is affected by the topological flux through it only mod N , this should optimize the expectation. But then the picture and expansion in terms of groups of isolated independent vortices (26) is no longer applicable.

Similarly, Eq. (22) leads to Eq. (25) because it is obtained by assuming the vortex inside each container as completely isolated and independent of all the others. The exact expression (20) holds for any number of factors in the product inside the expectation (reflecting mod N conservation). In view of the above discussion, one may as well combine containers into ones as thick as possible by integrating over the boundary fields of neighboring containers. With $T \gg R$, this amounts to taking containers in the product in Eqs. (20),(22) having transverse area $\sim R^2$, and longitudinal extension $\sim (\text{const } R)^{(d-2)}$. Equation (22) now gives

$$V(R) \geq \text{const } R - \frac{\text{const}}{R} (\ln R + \text{const}) \quad (27)$$

replacing Eq. (25). For loops with $T \geq R$, basically only one vortex container is needed, which means that strict linear potential arises essentially from thick vortex fluctuations spanning the entire loop area. It is interesting to note that such thick vortices also produce nonperturbative $1/R$ contributions (at scales outside the short distance perturbation theory regime).

Inequality (22) is actually rather crude. The inequalities (11),(13) following from reflection positivity are much more powerful because of the exponents that allow estimates uniform in the lattice size. They give directly pure area law. As is easily seen, this is true even if one further crudely bounds the RHS of Eq. (13) from above as done in Eq. (22), since $\ln L_\mu/A' \rightarrow 0$ as $|\Lambda| \rightarrow \infty$. There is no real reason for doing this though.³

IV. LONG RANGE VORTEX FLUCTUATIONS—EFFECTIVE THEORY

The above discussion, based on exact relations and inequalities between the Wilson loop and the free energy order parameters, indicates that an effective picture of the long distance confining fluctuations as isolated, independent vortices winding over long distances—in other words, as some kind of an approximately dilute or weakly interacting vortex gas—is not generally applicable. It does not take properly into account the relation between length and thickness of a vortex implied by the cost in free energy necessary to create the vortex in the first place, nor the correlations between vortices caused by their long range (topologically nontrivial) pure gauge tails. These correlations are present irrespective of the separation and enforce the mod N conservation of topological flux. Even though the cost diminishes exponentially with the transverse thickness of a vortex (creation of mass gap), these effects must still be properly accounted for if vortices of basically arbitrary length are to be present in the vacuum. These effects then generally tend to cause neighboring thick vortices to thicken further and merge since this lowers the free-energy cost for the vortices (above any background of fluctuations that may be present) for the same mod N total flux in the system. Thus, our previous discussion implies that the long distance linear potential should not properly be viewed as arising from the fluctuations caused by a gas of independent vortices winding through the loop, each of thickness much less than the linear dimensions of the loop. Rather, the fluctuation is more accurately described as that due to vortices of thickness comparable to the (shorter) linear loop dimension. Negative values of the loop occur then with almost exactly equal measure weight as positive values [no vortex (mod 2)]. This optimizes the expectation to

³Equation (13), however, might serve as a starting point for more sophisticated approximations where the behavior (5) for the vortex free energy under spreading of flux is at least partially derived rather than assumed.

be as small as possible, i.e. exhibit exact area law.⁴

This reflects the striking behavior revealed by the numerical simulations [6], that over sufficiently large scales there is “condensation” of vorticity carrying flux. Over a hypercube of side length of about 1 fm, the weighted probability at large β for nonzero (mod N) flux configurations goes to unity. The weighted probability then that one finds a vortex of at least this transverse size going through a 2-dim face on the hypercube boundary is approaching one. More generally, above this scale, vortices of any length, by corresponding appropriate adjustment in thickness, can occur at practically zero free energy cost. One may view this as percolation of vortices in the following sense. If one considers any two disjoint segments on the boundary of a large 4-dimensional simply connected region, the probability of being joined by a vortex of sufficient thickness (≥ 1 fm) is finite.

This picture of “percolated” vortices in all possible $[\mu\nu]$ orientations, with flux in intersections being conserved only mod 2, implies that in general it is difficult to unambiguously identify individual vortices.⁵ Rather, in the absence of obstructions or boundaries introduced by external probes, one can talk about an average nonvanishing vorticity field defined on the coarse scale of 1 fm, measured by the “circulation” (plaquettes, Wilson loops) above this scale.

What simple effective theory can describe this vacuum? We stress that we mean an effective theory strictly of only these long range vortex fluctuations (confiners) resulting from integrating out to an appropriate scale. Even at that scale there will of course be all kinds of other $SU(2)$ [$SU(N)$] fluctuations which we consider irrelevant for confinement. Let us list the minimal requirements for the effective theory:

(i) On a coarse scale of about $l = 1 - 1.2$ fm, the partition function should be expressible solely in terms of vortex excitations [coclosed (closed dual) surfaces of codim 2].

(ii) The mod 2 (mod N) property should be manifestly incorporated.

(iii) The vortex flux through each coarse scale plaquette should incur an action equal to the vortex free energy per unit length for thickness $\sim l$, as defined and computed from Eq. (3). This amount of free energy for the confiners (their action on the coarser scale) is the cost *above* the sea of all other vacuum fluctuations. [Again, note that this depends only mod 2 on the number of “vortex-introducing” singular gauge transformations injected in the box in Eq. (3).]

(iv) On this coarse lattice, the thick vortices should be in a “percolation phase.” Thus, despite (i), there should not be a useful expansion of the theory—i.e. a convergent, or, at least formally, systematic expansion scheme allowing computation of observables—such that each term in the expansion

⁴Recall that it is a rigorous result that the potential cannot rise faster than linearly.

⁵The way embedded non-self-interacting 2-dimensional surfaces (in our case, surfaces of a certain thickness) can grow to densely fill 4-dimensional space, the so called capped gropes and towers, has been extensively studied in manifold theory [10].

sion is characterized by a well-defined number of vortex excitations.

(v) The Wilson loop expectation should give strict area law as in the bound (11).

Let then Λ_c be the coarse lattice of lattice spacing $\sim l$, and $\chi_p \in Z(2)$ variables residing on its plaquettes. Then the simplest way to implement (i) above is by the partition sum:

$$Z_{\Lambda_c} = \int_{Z(2)} \prod_p d\chi_p \prod_c \frac{1}{2} \left(1 + \prod_{p \in \partial c} \chi_p \right) \exp A_{\text{eff}}. \quad (28)$$

The measure enforces the constraint

$$\prod_{p \in \partial c} \chi_p = 1 \quad (29)$$

on the plaquettes forming the boundary of every 3-dimensional cube c on Λ_c , so only excitations on coclosed sets are allowed. Equivalently, on the dual lattice, Eq. (29) assumes the form

$$\prod_{p \in \partial^* b} \chi_p^* = 1 \quad (30)$$

on the plaquettes forming the coboundary of every bond. The requirement (ii) is then automatically taken care of.

The general form of the effective action A_{eff}

$$\begin{aligned} A_{\text{eff}} = & \beta_{\text{eff}} \sum_p \chi_p + \beta_{2p} \sum_{(p,p') \in \partial^* b} \chi_p \chi_{p'} \\ & + \beta_{3p} \sum_{(p,p',p'') \in \partial^* b} \chi_p \chi_{p'} \chi_{p''} + \dots \end{aligned} \quad (31)$$

involves, in addition to the basic plaquette term, quasiloca interaction terms involving two or more plaquettes in the coboundary of each bond, etc. Now, from (iii), and Eq. (5), we must have

$$\beta_{\text{eff}} \sim \exp(-\rho(\beta)l^2) \quad (32)$$

giving, in principle, the coupling β_{eff} in terms of the coupling at the original lattice spacing, as ρ must approach the string tension for sufficiently large l . From the numerical simulations [6], $l \sim 1.1$ fm. This gives $\beta_{\text{eff}} \sim 0.002$. This very small value reflects of course the fact that at this choice of the physical length l vortex flux is found to become very ‘‘light.’’ Correspondingly, the terms involving products of two or more plaquettes must be of order β_{eff}^2 and higher, hence entirely negligible.

The effective model (28) is now seen to simply be a $Z(2)$ LGT. Indeed, the constraint (29) can be explicitly solved by introducing $Z(2)$ bond variables γ_b by

$$\chi_p = \prod_{b \in \partial p} \gamma_b. \quad (33)$$

Then

$$Z_{\Lambda_c} = \int_{Z(2)} \prod_b d\gamma_b \exp \left(\beta_{\text{eff}} \sum_p \gamma_{\partial p} + \dots \right), \quad (34)$$

where the ellipses indicate the additional clover and higher loop terms corresponding to the additional terms in Eq. (31). The theory is in the deep strong coupling regime $\beta_{\text{eff}} \ll 1$. Thus (iv) above is indeed satisfied. The theory can be treated in the strong coupling expansion. It cannot, however, be meaningfully expanded in its vortex excitations—that would be appropriate in the weak coupling $\beta_{\text{eff}} \gg 1$ regime in the form of the usual weak coupling expansion for discrete groups (dilute vortex gas) [11].

It should perhaps be explicitly pointed out that the $Z(2)$ variables in Eqs. (28),(34), serve as an effective description of long distance fluctuations creating topological $Z(2)$ flux [elements of $\pi_1(SU(2)/Z(2))$] in the original theory [12]. They have nothing to do with the $Z(2)$ part of the original $SU(2)$ bond variables. Note that the $Z(2)$ gauge theory interaction, together with Eq. (32), correctly reproduce the effects of flux spreading and thickening of vortices while conserving flux mod 2. Correspondingly, the Wilson loop now automatically gives the correct area law.

A Wilson loop inserted in Eq. (34) represents the coupling of an external quark current to the long distance confining fluctuations. Its replacement by dynamical quarks introduces fermions in the medium of these fluctuations. Since the effective coupling is strong, it induces dynamical chiral symmetry breaking (CSB).

CSB in strongly coupled LGT has actually been demonstrated analytically in the superstrong gauge coupling limit (no plaquette action term) by expansion or mean field (large N or d) approximations [13,14], and rigorously by infrared bounds [15]. It is physically obvious that the result extends to a finite region in the strong coupling regime.⁶

We may try to use the results in [13–15] to get an estimate of the contribution to the quark condensate in the effective theory. Corrections from the plaquette term in the action are totally negligible due to the smallness of β_{eff} . One then has

$$\langle \bar{q}q \rangle = z(l)N \frac{1}{l^3} \sqrt{\frac{2}{d}} (1 - k(d))^{1/2}, \quad (35)$$

where $\frac{1}{8} \leq k(4) < 0.35$. $z(l)$ is some renormalization factor that, in a more sophisticated treatment, should depend on how fermions are introduced at the original lattice spacing. Here we naively set it equal to one—this is equivalent to simply taking staggered fermions on the coarse lattice. With $l = 1.1$ fm, this gives $\langle \bar{q}q \rangle = (195 \text{ MeV})^3$ for $N=2$, and $(223 \text{ MeV})^3$ for $N=3$. This indicates that the quark condensate may be entirely accounted for by the long range confining fluctuations.

⁶It should be possible to prove this by cluster expansion techniques around the $\beta=0$ point, though not so straightforward for technical reasons (bounding terms with Grassmann integrands).

Following our previous development, the effective $Z(N)$ theory appears to emerge rather naturally, and in fact in a fairly unique manner. The idea that an effective theory of long range vortex fluctuations must be a $Z(N)$ LGT is not new, but has not, we believe, been formulated in this way before. Recently, a model equivalent to Eq. (28), in the representation (30) and employing additive $Z(2)$ variables, was considered in [16], apparently without any reference to $Z(2)$ LGT.

V. CONCLUSIONS

In the present paper we studied the energetics of how vortices can disorder Wilson loops of different sizes. Vortices of any thickness smaller than the linear size of a given Wilson loop can link with it and contribute to disordering its average. Here we pointed out that for Wilson loops of any given size it is the vortices of “maximal” thickness, i.e., thickness of the order of the linear size of the loop, that give the most important contribution resulting in an area-law suppression of large Wilson loops and a linear heavy quark potential. On the other hand, vortices of any *fixed* thickness contribute only with a logarithmically suppressed term to the potential. This is in contradiction with the naive picture of

confinement by vortices which assumes that vortices of fixed thickness can link with a fixed probability with arbitrarily large Wilson loops. The correct picture must take properly into account the relation between the length and thickness of a vortex imposed by the free energy requirements for creating the vortex, as well as the interaction between vortices introduced by the constraint of mod N conservation of the vortex flux. This picture of vortices naturally yields a long distance effective $Z(N)$ gauge theory above the confinement scale of around 1 fm. The effective theory is deep in the strong coupling regime which makes it impossible to interpret it in terms of a simple vortex gas expansion. The only useful expansion one can consider is the strong coupling one. Being deep in the strong coupling regime, the effective $Z(N)$ gauge theory naturally produces chiral symmetry breaking in the presence of fermion fields.

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