Monopole condensation in $SU(2)$ QCD

Y. M. Cho*
School of Physics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea

D. G. Pak†
Asia Pacific Center for Theoretical Physics, Seoul 130-012, Korea
and Department of Theoretical Physics, Tashkent State University, Tashkent 700-095, Uzbekistan
(Received 25 May 2001; revised manuscript received 7 December 2001; published 29 March 2002)

Based on the gauge independent decomposition of the non-Abelian gauge field into the dual potential and the valence potential, we calculate the one-loop effective action of $SU(2)$ QCD in an arbitrary constant monopole background, using the background field method. Our result provides strong evidence for dynamical symmetry breaking through the monopole condensation, which can induce the dual Meissner effect and establish the confinement of color in the non-Abelian gauge theory. The result is obtained by separating the topological degrees which describe the non-Abelian monopoles from the dynamical degrees of the gauge potential, and integrating out all the dynamical degrees of QCD.

DOI: 10.1103/PhysRevD.65.074027 PACS number(s): 12.38.Aw, 11.10.Lm, 11.15.−q

I. INTRODUCTION

One of the most outstanding problems in theoretical physics is the confinement problem in QCD. It has long been argued that monopole condensation could explain the confinement of color through the dual Meissner effect [1,2]. Indeed, if one assumes monopole condensation, one could easily argue that the ensuing dual Meissner effect guarantees confinement [3,4]. In this direction there has been remarkable progress in lattice simulation during the past decade. In fact, the recent numerical simulations have provided unmistakable evidence which supports the idea of magnetic confinement through monopole condensation [5,6]. Unfortunately so far there have been few satisfactory field theoretic proofs of monopole condensation in QCD.

The purpose of this paper is to reexamine the non-Abelian dynamics and establish monopole condensation in QCD from first principles. Utilizing a gauge independent parametrization of the non-Abelian gauge potential which emphasizes its topological character, we construct the one-loop effective action of QCD in the presence of a monopole background by integrating out all the dynamical degrees of the non-Abelian potential except the monopole background, using the background field method. *Remarkably the effective action generates a dynamical symmetry breaking made of monopole condensation, which strongly indicates that the physical confinement mechanism in QCD is indeed the magnetic confinement through the dual Meissner effect.* Our analysis makes it clear that it is precisely the magnetic moment interaction of the gluons which was responsible for the asymptotic freedom that generates the monopole condensation in QCD. We demonstrate our result with $SU(2)$ for simplicity, although the result should be applicable to any non-Abelian gauge theory.

To prove the magnetic confinement it is instructive for us to remember how the magnetic flux is confined in the superconductor through the Meissner effect. In the macroscopic Ginzburg-Landau description of superconductivity, the Meissner effect is triggered by the effective mass of the electromagnetic potential, which determines the penetration (confinement) scale of the magnetic flux. In the microscopic BCS description, this effective mass is generated by electron-pair (the Cooper pair) condensation. This suggests that, for confinement of the color electric flux, one needs the condensation of the monopoles. Equivalently, in the dual Ginzburg-Landau description, one needs the dynamical generation of the effective mass for the monopole potential. To demonstrate this, one must first identify the monopole potential, and separate it from the generic QCD connection, in a gauge-independent manner. This can be done with the ''Abelian'' projection [2,3], which provides a natural reparametrization of the non-Abelian connection in terms of the restricted connection (i.e., the dual potential) of the maximal Abelian subgroup H of the gauge group G and the gauge covariant vector field (i.e., the valence potential) of the remaining G/H degrees. With this separation, one can show that monopole condensation takes place in a one-loop correction, after one integrates out all the dynamical degrees of the non-Abelian gauge potential.

There have been many attempts to prove the monopole condensation in QCD [7,8]. Unfortunately, the effective action of QCD obtained from these earlier attempts has failed to establish the desired magnetic condensation, because the magnetic condensation was unstable. This instability of the magnetic condensation has been widely accepted and has never been convincingly revoked. In retrospect, there are many reasons why the earlier attempts have not been so successful. First, the attempts to calculate the effective action of QCD were gauge-dependent. In fact, the separation of the magnetic background from the quantum fields was not gauge-independent. So there is no way of knowing whether the desired magnetic condensation is indeed a gauge-independent phenomenon. Moreover, the origin of the magnetic background in the earlier attempts was completely ob-

*Electronic address: ymcho@yongmin.snu.ac.kr
†Electronic address: dmipak@mail.apctp.org

scure, and could not be associated to the non-Abelian monopoles. Consequently, magnetic condensation could not be interpreted as monopole condensation. But the most serious defect was the appearance of an imaginary part in the effective action, which was due to improper infrared regularization. This improper infrared regularization was the critical defect which really destroyed the magnetic condensation in the earlier attempts [7,8]. In this paper, we start from the gauge-independent separation of the monopole background from the quantum fields in our calculation of the effective action. More importantly, we make a proper infrared regularization which respects the causality, and show that the causality makes our monopole condensation stable.

Recently, Faddeev and Niemi have discovered the knot-like topological solitons in the Skyrme-type nonlinear sigma model, and made an interesting conjecture that the Skyrme-Faddeev action could be interpreted as an effective action for QCD in the low-energy limit [9,10]. With the effective action at hand we discuss the possible connection between Skyrme-Faddeev theory and QCD. We show that indeed the two theories are closely related, and demonstrate that we can derive a generalized Skyrme-Faddeev action from the effective action of QCD.

The paper is organized as follows. In Sec. II, we review the Abelian projection and the gauge-independent decomposition of the non-Abelian potential into the restricted potential and the valence potential. In Sec. III, we derive the integral expression of the one-loop effective action of $SU(2)$ QCD in the presence of a pure monopole background, using the background field method. In Sec. IV, we derive the integral expression of the effective action for an arbitrary constant (color) electromagnetic background, which we need to establish the stability of the monopole condensation. In Sec. V, we obtain the effective action for the pure monopole background, and demonstrate the existence of the monopole condensation which generates a dynamical symmetry breaking in QCD. In Sec. VI, we obtain the effective action for a pure electric background, and show that the electric background generates the pair annihilation of the valence gluons. In Sec. VII, we demonstrate the stability of the monopole condensation. We provide three independent arguments (the causality, the duality, and the perturbative expansion) which support the stability of the vacuum condensation. In Sec. VIII, we establish a deep connection between the Skyrme-Faddeev theory and QCD, and derive a generalized Skyrme-Faddeev action from our effective action as an effective action of QCD in the infrared limit. Finally, in Sec. IX we discuss the physical implications of our results.

II. ABELIAN PROJECTION AND VALENCE GLUON: A REVIEW

Consider $SU(2)$ QCD for simplicity. A natural way to identify the monopole potential is to introduce an isotriplet unit vector field $\hat{n}$ which selects the "Abelian" direction (i.e., the color charge direction) at each space-time point, and to decompose the connection into the restricted potential (called the Abelian projection) $\hat{A}_\mu$ which leaves $\hat{n}$ invariant and the valence potential $\vec{X}_\mu$ which forms a covariant vector field [2,3]

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n}\times\partial_\mu\hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu$$
$$(\hat{n}^2=1, \quad \hat{n}\cdot\vec{X}_\mu = 0), \tag{1}$$

where $A_\mu = \hat{n}\cdot\vec{A}_\mu$ is the "electric" potential. Notice that the restricted potential is precisely the connection which leaves $\hat{n}$ invariant under the parallel transport,

$$\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g\hat{A}_\mu \times \hat{n} = 0. \tag{2}$$

Under the infinitesimal gauge transformation

$$\delta\hat{n} = -\vec{\alpha}\times\hat{n}, \quad \delta\vec{A}_\mu = \frac{1}{g}D_\mu\vec{\alpha}, \tag{3}$$

one has

$$\delta A_\mu = \frac{1}{g}\hat{n}\cdot\partial_\mu\vec{\alpha}, \quad \delta\hat{A}_\mu = \frac{1}{g}\hat{D}_\mu\vec{\alpha},$$
$$\delta\vec{X}_\mu = -\vec{\alpha}\times\vec{X}_\mu . \tag{4}$$

This shows that $\hat{A}_\mu$ by itself describes an $SU(2)$ connection which enjoys the full $SU(2)$ gauge degrees of freedom. Furthermore $\vec{X}_\mu$ transforms covariantly under the gauge transformation. Most importantly, the decomposition is gauge-independent. Once the color direction $\hat{n}$ is selected, the decomposition follows independent of the choice of a gauge.

Our decomposition, which has recently become known as the Cho decomposition [10] or the Cho-Faddeev-Niemi decomposition [11], was first introduced a long time ago in an attempt to demonstrate the monopole condensation in QCD [2,3]. But only recently the importance of the decomposition in clarifying the non-Abelian dynamics has become appreciated by many authors [10,11]. Indeed it is this decomposition which has played a crucial role to establish the Abelian dominance in Wilson loops in QCD [12], and the possible connection between the Skyrme-Faddeev action and the effective action of QCD [13,14].

To understand the physical meaning of our decomposition, notice that the restricted potential $\hat{A}_\mu$ actually has a dual structure. Indeed the field strength made of the restricted potential is decomposed as

$$\hat{F}_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu})\hat{n},$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{5}$$
$$H_{\mu\nu} = -\frac{1}{g}\hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n}) = \partial_\mu\tilde{C}_\nu - \partial_\nu\tilde{C}_\mu ,$$

where $\tilde{C}_\mu$ is the "magnetic" potential [2,3]. Notice that we can always introduce the magnetic potential (at least locally sectionwise), because $H_{\mu\nu}$ is closed,

$$\partial_\mu \tilde{H}_{\mu\nu}=0 \quad (\tilde{H}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}H_{\rho\sigma}). \tag{6}$$

This allows us to identify the non-Abelian magnetic potential by

$$\vec{C}_\mu=-\frac{1}{g}\hat{n}\times\partial_\mu\hat{n}, \tag{7}$$

in terms of which the magnetic field is expressed as

$$\vec{H}_{\mu\nu}=\partial_\mu\vec{C}_\nu-\partial_\nu\vec{C}_\mu+g\vec{C}_\mu\times\vec{C}_\nu=-g\vec{C}_\mu\times\vec{C}_\nu=-\frac{1}{g}\partial_\mu\hat{n}\times\partial_\nu\hat{n}=H_{\mu\nu}\hat{n}. \tag{8}$$

Another important feature of $\hat{A}_\mu$ is that, as an $SU(2)$ potential, it retains all the essential topological characteristics of the original non-Abelian potential. This is because the topological field $\hat{n}$ can naturally describe the non-Abelian topology $\pi_2(S^2)$ and $\pi_3(S^2)\simeq\pi_3(S^3)$. Clearly the isolated singularities of $\hat{n}$ define $\pi_2(S^2)$ which describes the non-Abelian monopoles. Indeed $\hat{A}_\mu$ with $A_\mu=0$ and $\hat{n}=\hat{r}$ (or equivalently $\vec{C}_\mu$ with $\hat{n}=\hat{r}$) describes precisely the Wu-Yang monopole [15,16]. Besides, with the S^3 compactification of R^3, $\hat{n}$ characterizes the Hopf invariant $\pi_3(S^2)\simeq\pi_3(S^3)$ which describes the topologically distinct vacua [17,18]. This tells us that the restricted gauge theory made of $\hat{A}_\mu$ could describe the dual dynamics which should play an essential role in $SU(2)$ QCD [2,12,19].

With Eq. (1) we have

$$\vec{F}_{\mu\nu}=\hat{F}_{\mu\nu}+\hat{D}_\mu\vec{X}_\nu-\hat{D}_\nu\vec{X}_\mu+g\vec{X}_\mu\times\vec{X}_\nu, \tag{9}$$

so that the Yang-Mills Lagrangian is expressed as

$$\mathcal{L}=-\tfrac{1}{4}\vec{F}^2_{\mu\nu}=-\tfrac{1}{4}\hat{F}^2_{\mu\nu}-\tfrac{1}{4}(\hat{D}_\mu\vec{X}_\nu-\hat{D}_\nu\vec{X}_\mu)^2-\frac{g}{2}\hat{F}_{\mu\nu}\cdot(\vec{X}_\mu\times\vec{X}_\nu)-\frac{g^2}{4}(\vec{X}_\mu\times\vec{X}_\nu)^2+\lambda(\hat{n}^2-1)+\lambda_\mu\hat{n}\cdot\vec{X}_\mu, \tag{10}$$

where λ and λ_μ are the Lagrangian multipliers. From the Lagrangian we have

$$\partial_\mu(F_{\mu\nu}+H_{\mu\nu}+X_{\mu\nu})=-g\hat{n}\cdot[\vec{X}_\mu\times(\hat{D}_\mu\vec{X}_\nu-\hat{D}_\nu\vec{X}_\mu)],$$
$$\hat{D}_\mu(\hat{D}_\mu\vec{X}_\nu-\hat{D}_\nu\vec{X}_\mu)=g(F_{\mu\nu}+H_{\mu\nu}+X_{\mu\nu})\hat{n}\times\vec{X}_\mu, \tag{11}$$

where

$$X_{\mu\nu}=g\hat{n}\cdot(\vec{X}_\mu\times\vec{X}_\nu). \tag{12}$$

Notice that here $\hat{n}$ has no equation of motion even though the Lagrangian contains it explicitly. This implies that it is not a local degree of freedom, but a topological degree of freedom [19]. From this we conclude that the non-Abelian gauge theory can be viewed as a restricted gauge theory made of the restricted potential, which has an additional colored source made of the valence gluon.

Obviously the Lagrangian (10) is invariant under the active gauge transformation (3). But notice that the decomposition introduces a new gauge symmetry that we call the passive gauge transformation [13,19],

$$\delta\hat{n}=0, \quad \delta\vec{A}_\mu=\frac{1}{g}D_\mu\vec{\alpha}, \tag{13}$$

under which we have

$$\delta A_\mu=\frac{1}{g}\hat{n}\cdot D_\mu\vec{\alpha}, \quad \delta\hat{A}_\mu=\frac{1}{g}(\hat{n}\cdot D_\mu\vec{\alpha})\hat{n},$$
$$\delta\vec{X}_\mu=\frac{1}{g}[D_\mu\vec{\alpha}-(\hat{n}\cdot D_\mu\vec{\alpha})\hat{n}]. \tag{14}$$

This is because, for a given $\vec{A}_\mu$, one can have infinitely many different decompositions of Eq. (1), with different $\hat{A}_\mu$ and $\vec{X}_\mu$ by choosing different $\hat{n}$. Equivalently, for a fixed $\hat{n}$, one can have infinitely many different $\vec{A}_\mu$ which are gauge-equivalent to each other. So it must be clear that with our decomposition we automatically have another type of gauge invariance which comes from different choices of decomposition. This extra gauge invariance plays a crucial role in quantizing the theory [19].

Another advantage of the decomposition (1) is that it can actually "Abelianize" (or more precisely "dualize") the non-Abelian dynamics [2,12,19]. To see this, let $(\hat{n}_1,\hat{n}_2,\hat{n})$ be a right-handed orthonormal basis and let

$$\vec{X}_\mu=X^1_\mu\hat{n}_1+X^2_\mu\hat{n}_2,$$
$$(X^1_\mu=\hat{n}_1\cdot\vec{X}_\mu, \quad X^2_\mu=\hat{n}_2\cdot\vec{X}_\mu)$$

and find

$$\hat{D}_\mu\vec{X}_\nu=[\partial_\mu X^1_\nu-g(A_\mu+\tilde{C}_\mu)X^2_\nu]\hat{n}_1+[\partial_\mu X^2_\nu+g(A_\mu+\tilde{C}_\mu)X^1_\nu]\hat{n}_2. \tag{15}$$

So with

$$B_\mu=A_\mu+\tilde{C}_\mu, \quad G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu,$$
$$X_\mu=\frac{1}{\sqrt{2}}(X^1_\mu+iX^2_\mu), \tag{16}$$

one could express the Lagrangian explicitly in terms of the dual potential B_μ and the complex vector field X_μ,

$$\mathcal{L}=-\frac{1}{4}G^2_{\mu\nu}-\frac{1}{2}|\tilde{D}_\mu X_\nu-\tilde{D}_\nu X_\mu|^2+igG_{\mu\nu}X^*_\mu X_\nu-\frac{1}{2}g^2[(X^*_\mu X_\mu)^2-(X^*_\mu)^2(X_\nu)^2], \tag{17}$$

where now

$$\tilde{D}_\mu = \partial_\mu + igB_\mu .$$

Clearly this describes an Abelian gauge theory coupled to the charged vector field X_μ. But the important point here is that the Abelian potential B_μ is given by the sum of the electric and magnetic potentials $A_\mu + \tilde{C}_\mu$. In this form, the equations of motion (11) are reexpressed as

$$\partial_\mu(G_{\mu\nu}+X_{\mu\nu}) = igX^*_\mu(\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu) - igX_\mu(\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu)^*,$$

$$\tilde{D}_\mu(\tilde{D}_\mu X_\nu - \hat{D}_\nu X_\mu) = igX_\mu(G_{\mu\nu}+X_{\mu\nu}), \tag{18}$$

where now

$$X_{\mu\nu} = -ig(X^*_\mu X_\nu - X^*_\nu X_\mu).$$

This shows that one can indeed Abelianize the non-Abelian theory with our decomposition. The remarkable change in this "Abelian" formulation is that here the topological field $\hat{n}$ is replaced by the magnetic potential $\tilde{C}_\mu$.

But notice that here we have never fixed the gauge to obtain this Abelian formalism, and one might ask how the non-Abelian gauge symmetry is realized in this "Abelian" theory. To discuss this, let

$$\vec{\alpha} = \alpha_1 \hat{n}_1 + \alpha_2 \hat{n}_2 + \theta \hat{n},$$

$$\alpha = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2), \tag{19}$$

$$\vec{C}_\mu = -\frac{1}{g}\hat{n}\times\partial_\mu \hat{n} = -C^1_\mu \hat{n}_1 - C^2_\mu \hat{n}_2,$$

$$C_\mu = \frac{1}{\sqrt{2}}(C^1_\mu + iC^2_\mu).$$

Then the Lagrangian (17) is invariant not only under the active gauge transformation (3) described by

$$\delta A_\mu = \frac{1}{g}\partial_\mu\theta - i(C^*_\mu\alpha - C_\mu\alpha^*), \quad \delta\tilde{C}_\mu = -\delta A_\mu ,$$

$$\delta X_\mu = 0, \tag{20}$$

but also under the passive gauge transformation (13) described by

$$\delta A_\mu = \frac{1}{g}\partial_\mu\theta - i(X^*_\mu\alpha - X_\mu\alpha^*), \quad \delta\tilde{C}_\mu = 0,$$

$$\delta X_\mu = \frac{1}{g}\tilde{D}_\mu\alpha - i\theta X_\mu . \tag{21}$$

This tells us that the "Abelian" theory not only retains the original gauge symmetry, but actually has enlarged (both the active and passive) gauge symmetries. But we emphasize that this is not the "naive" Abelianization of the $SU(2)$ gauge theory which one obtains by fixing the gauge. Our Abelianization is a gauge-independent Abelianization. Besides, here the Abelian gauge group is actually made of $U(1)_e \otimes U(1)_m$, so that the theory becomes a dual gauge theory [2,12,19]. This is evident from Eqs. (20) and (21).

III. MONOPOLE BACKGROUND

With this preparation, we will now derive the integral expression of the one-loop effective action in the presence of the pure monopole background $\vec{C}_\mu$. To do this, we resort to the background field method [20,21]. So we first divide the gauge potential $\vec{A}_\mu$ into two parts, the slow-varying classical part $\vec{A}^{(c)}_\mu$ and the fluctuating quantum part $\vec{A}^{(q)}_\mu$, and identify the magnetic potential $\vec{C}_\mu$ as the classical background [13,19]:

$$\vec{A}_\mu = \vec{A}^{(c)}_\mu + \vec{A}^{(q)}_\mu ,$$

$$\vec{A}^{(c)}_\mu = \vec{C}_\mu , \quad \vec{A}^{(q)}_\mu = A_\mu \hat{n} + \vec{X}_\mu . \tag{22}$$

With this we introduce two types of gauge transformations, namely the background gauge transformation and the physical gauge transformation. Naturally, we identify the background gauge transformation as

$$\delta\vec{C}_\mu = \frac{1}{g}\bar{D}_\mu\vec{\alpha},$$

$$\delta(A_\mu\hat{n} + \vec{X}_\mu) = -\vec{\alpha}\times(A_\mu\hat{n} + \vec{X}_\mu), \tag{23}$$

where now $\bar{D}_\mu$ is defined with only the background potential $\vec{C}_\mu$,

$$\bar{D}_\mu = \partial_\mu + g\vec{C}_\mu . \tag{24}$$

As for the physical gauge transformation which leaves the background potential invariant, we must have

$$\delta\vec{C}_\mu = 0, \quad \delta(A_\mu\hat{n} + \vec{X}_\mu) = \frac{1}{g}D_\mu\vec{\alpha}. \tag{25}$$

Notice that both Eqs. (23) and (25) respect the original gauge transformation,

$$\delta\vec{A}_\mu = \frac{1}{g}D_\mu\vec{\alpha}. \tag{26}$$

Now, we fix the gauge by imposing the following gauge condition to the quantum fields:

$$\vec{F} = \bar{D}_\mu(A_\mu\hat{n} + \vec{X}_\mu) = 0,$$

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}[(\partial_\mu A_\mu)^2 + (\bar{D}_\mu \vec{X}_\mu)^2]. \tag{27}$$

The corresponding Faddeev-Popov (FP) determinant is given by

$$M_{\rm FP}^{ab} = \frac{\delta F^a}{\delta \alpha^b} = (\bar{D}_\mu D_\mu)^{ab}. \tag{28}$$

With this gauge fixing the effective action takes the following form:

$$\exp[iS_{\rm eff}(\vec{C}_\mu)] = \int \mathcal{D}A_\mu \mathcal{D}\vec{X}_\mu \mathcal{D}\vec{c}\mathcal{D}\vec{c}^* \exp\left\{ i\int \left(-\frac{1}{4}\hat{F}_{\mu\nu}^2 - \frac{1}{4}(\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu)^2 - \frac{g}{2}\hat{F}_{\mu\nu}\cdot(\vec{X}_\mu \times \vec{X}_\nu) - \frac{g^2}{4}(\vec{X}_\mu \times \vec{X}_\nu)^2 + \vec{c}^*\bar{D}_\mu D_\mu \vec{c} - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \frac{1}{2\xi}(\bar{D}_\mu \vec{X}_\mu)^2 \right) d^4x \right\}, \tag{29}$$

where $\vec{c}$ and $\vec{c}^*$ are the ghost fields. Notice that the effective action (29) is explicitly invariant under the background gauge transformation (3), if we add the following gauge transformation of the ghost fields to Eq. (3):

$$\delta\vec{c} = -\alpha \times \vec{c}, \quad \delta\vec{c}^* = -\alpha \times \vec{c}^*. \tag{30}$$

This guarantees that the resulting effective action we obtain after the functional integral should be invariant under the remaining background gauge transformation which involves only $\vec{C}_\mu$. This, of course, is the advantage of the background field method which greatly simplifies the calculation of the effective action [20,21].

Now, we can perform the functional integral in Eq. (29). Remember that in the one-loop approximation only the terms quadratic in quantum fields become relevant in the functional integral. So the A_μ integration becomes trivial, and the $\vec{X}_\mu$ and ghost integrations result in the following functional determinants (with $\xi = 1$):

$$\det{}^{-1/2} K_{\mu\nu}^{ab} \simeq \det{}^{-1/2}[-g_{\mu\nu}(\overline{DD})^{ab} - 2gH_{\mu\nu}\epsilon^{abc}n^c],$$

$$\det M_{FP}^{ab} \simeq \det(-\overline{DD})^{ab}. \tag{31}$$

One can simplify the determinant K [13,22],

$$\ln\det{}^{-1/2} K = -\tfrac{1}{2}\ln\det[(-\overline{DD})^{ab} + i\sqrt{2}gH\epsilon^{abc}n^c] - \tfrac{1}{2}\ln\det[(-\overline{DD})^{ab} - i\sqrt{2}gH\epsilon^{abc}n^c] - \ln\det(-\overline{DD})^{ab}, \tag{32}$$

where

$$H = \sqrt{\vec{H}_{\mu\nu}^2}.$$

With this, the one-loop contribution of the functional determinants to the effective action can be written as

$$\Delta S = i\ln\det(-\bar{D}^2 + \sqrt{2}gH) + i\ln\det(-\bar{D}^2 - \sqrt{2}gH), \tag{33}$$

where now $\bar{D}_\mu$ acquires the following Abelian form:

$$\bar{D}_\mu = \partial_\mu + ig\tilde{C}_\mu. \tag{34}$$

Remarkably, the functional determinant (33) acquires the Abelian form. This, of course, is precisely due to the fact that our decomposition (1) Abelianizes QCD. But we emphasize again that this Abelianization is gauge-independent.

One can evaluate the functional determinants in Eq. (33) with the Fock-Schwinger proper time method, and for a constant background H we find

$$\Delta\mathcal{L} = \frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\frac{gH/\sqrt{2}\mu^2}{\sinh(gHt/\sqrt{2}\mu^2)}[\exp(-\sqrt{2}gHt/\mu^2) + \exp(\sqrt{2}gHt/\mu^2)], \tag{35}$$

where μ is a dimensional parameter.

IV. ARBITRARY BACKGROUND

Before we evaluate the above integral and establish the monopole condensation, we now derive the integral expression of the one-loop effective action in the presence of arbitrary background $\hat{A}_\mu$, which we need to establish the stability of the monopole condensation. So we repeat the above procedure, but now replacing the monopole background $\vec{C}_\mu$ by the restricted potential $\hat{A}_\mu$. So we first divide the gauge potential $\vec{A}_\mu$ into two parts, and now identify the restricted potential $\hat{A}_\mu$ as the classical background,

$$\vec{A}_\mu = \vec{A}_\mu^{(c)} + \vec{A}_\mu^{(q)},$$

$$\vec{A}_\mu^{(c)} = \hat{A}_\mu, \quad \vec{A}_\mu^{(q)} = \vec{X}_\mu. \tag{36}$$

With this we now identify the gauge transformation (3) as the background gauge transformation. As for the physical gauge transformation, which leaves the background potential invariant, we must have

$$\delta\hat{A}_\mu = 0, \quad \delta\vec{X}_\mu = \frac{1}{g}D_\mu \vec{\alpha}. \tag{37}$$

Again notice that both Eqs. (3) and (37) respect the original gauge transformation (26). Now, we fix the gauge by imposing the following gauge condition to the quantum field [19,22]:

$$\vec{F}=\hat{D}_\mu\vec{X}_\mu=0,$$

$$\mathcal{L}_{\rm gf}=-\frac{1}{2\xi}(\hat{D}_\mu\vec{X}_\mu)^2. \tag{38}$$

The corresponding Faddeev-Popov determinant is given by

$$M^{ab}_{\rm FP}=\frac{\delta F^a}{\delta\alpha^b}=(\hat{D}_\mu D_\mu)^{ab}. \tag{39}$$

With this gauge fixing, the effective action takes the following form:

$$\begin{aligned}\exp[iS_{\rm eff}(\hat{A}_\mu)]=\int \mathcal{D}\vec{X}_\mu\mathcal{D}\vec{c}\mathcal{D}\vec{c}*\\ \times\exp\bigg\{i\int\bigg(-\frac{1}{4}\hat{F}^2_{\mu\nu}-\frac{1}{4}(\hat{D}_\mu\vec{X}_\nu-\hat{D}_\nu\vec{X}_\mu)^2\\ -\frac{g}{2}\hat{F}_{\mu\nu}\cdot(\vec{X}_\mu\times\vec{X}_\nu)-\frac{g^2}{4}(\vec{X}_\mu\times\vec{X}_\nu)^2\\ +\vec{c}*\hat{D}_\mu D_\mu\vec{c}-\frac{1}{2\xi}(\hat{D}_\mu\vec{X}_\mu)^2\bigg)d^4x\bigg\}.\end{aligned} \tag{40}$$

Notice again that, with Eq. (30), the effective action (40) is explicitly invariant under the background gauge transformation (3) which involves only $\hat{A}_\mu$.

Now, we can perform the functional integral. To do this, we let the background field $\hat{F}_{\mu\nu}$ be

$$\hat{F}_{\mu\nu}=G_{\mu\nu}\hat{n},$$

$$G_{\mu\nu}=F_{\mu\nu}+H_{\mu\nu},$$

and in the one-loop approximation find that the $\vec{X}_\mu$ and ghost integrations result in the following functional determinants (with $\xi=1$):

$$\det{}^{-1/2}K^{ab}_{\mu\nu}\simeq\det{}^{-1/2}[-g_{\mu\nu}(\hat{D}\hat{D})^{ab}-2gG_{\mu\nu}\epsilon^{abc}n^c],$$

$$\det M_{\rm FP}=\det[-(\hat{D}\hat{D})^{ab}], \tag{41}$$

where $\hat{D}_\mu$ is defined with an arbitrary background field $\hat{A}_\mu$. Using the relation

$$G_{\mu\alpha}G_{\nu\beta}G_{\alpha\beta}=\tfrac{1}{2}G^2G_{\mu\nu}+\tfrac{1}{2}(G\tilde{G})\tilde{G}_{\mu\nu}$$

$$(\tilde{G}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}), \tag{42}$$

one can simplify the functional determinants of the valence gluon and the ghost loops to the following Abelian form:

$$\begin{aligned}\ln\det{}^{-1/2}[-g_{\mu\nu}(\hat{D}\hat{D})^{ab}-2gG_{\mu\nu}\epsilon^{abc}n^c]\\ =\ln\det[(-\tilde{D}^2+2a)(-\tilde{D}^2-2a)\\ \times(-\tilde{D}^2-2ib)(-\tilde{D}^2+2ib)],\end{aligned}$$

$$\ln\det M_{FP}=2\ln\det(-\tilde{D}^2), \tag{43}$$

where $\tilde{D}_\mu$ is defined with an arbitrary background $A_\mu+\tilde{C}_\mu$,

$$\tilde{D}_\mu=\partial_\mu+ig(A_\mu+\tilde{C}_\mu), \tag{44}$$

and

$$a=\frac{g}{2}\sqrt{\sqrt{G^4+(G\tilde{G})^2}+G^2},$$

$$b=\frac{g}{2}\sqrt{\sqrt{G^4+(G\tilde{G})^2}-G^2}. \tag{45}$$

Notice that in the Lorentz frame where the electric field becomes parallel to the magnetic field, a becomes purely magnetic and b becomes purely electric.

From this we have

$$\begin{aligned}\Delta S=i\ln\det[(-\tilde{D}^2+2a)(-\tilde{D}^2-2a)]+i\ln\det\\ \times[(-\tilde{D}^2-2ib)(-\tilde{D}^2+2ib)]-2i\ln\det(-\tilde{D}^2).\end{aligned} \tag{46}$$

We can evaluate the functional determinants, and for a general background with arbitrary a and b, the contribution of the gluon and ghost loops is given by [22]

$$\begin{aligned}\Delta\mathcal{L}=\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^3}\frac{abt^2/\mu^2}{\sinh(at/\mu^2)\sin(bt/\mu^2)}[\exp(-2at/\mu^2)\\ +\exp(2at/\mu^2)+\exp(2ibt/\mu^2)\\ +\exp(-2ibt/\mu^2)-2].\end{aligned} \tag{47}$$

The integral expression (47) of the effective action has been known for some time [8], but the actual integration of it is not easy to perform. Indeed, as far as we understand, the integration has never been evaluated correctly. This is because the integral contains (not only the usual ultraviolet divergence around $t\simeq0$) a severe infrared divergence around $t\simeq\infty$, which has to be regularized correctly. In the following, we will perform the integral for pure magnetic and pure electric backgrounds separately.

V. MONOPOLE CONDENSATION

For the pure monopole background, the integral (35) reduces to

$$\Delta\mathcal{L}=\Delta\mathcal{L}_{+}+\Delta\mathcal{L}_{-},$$

$$\Delta\mathcal{L}_{+}=\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^2}\,\frac{a/\mu^2}{\sinh(at/\mu^2)}\exp(-2at/\mu^2), \tag{48}$$

$$\Delta\mathcal{L}_{-}=\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^2}\,\frac{a/\mu^2}{\sinh(at/\mu^2)}\exp(2at/\mu^2),$$

where

$$a=\frac{gH}{\sqrt{2}}.$$

Notice that this is precisely the same integral that we obtain from Eq. (47) for the pure magnetic background (i.e., for $b=0$). This tells us that the evaluation of the effective action for an arbitrary magnetic background becomes mathematically identical to that for the pure monopole background.

As we have remarked, both integrals have the usual ultraviolet divergence at the origin, but the second integral has a severe infrared divergence at the infinity. To find the correct infrared regularization, one must understand the origin of the divergence. The infrared divergence can be traced back to the magnetic moment interaction of the gluons that we have in Eq. (10), which is also well known to be responsible for the asymptotic freedom [23]. This magnetic interaction generates negative eigenvalues in $\det K$ in the long-distance region, which cause the infrared divergence. More precisely, when the momentum k of the gluon parallel to the background magnetic field becomes smaller than the background field strength (i.e., when $k^2<a$), the lowest Landau-level gluon eigenfunction whose spin is parallel to the magnetic field acquires an imaginary energy and thus becomes tachyonic. It is these unphysical tachyonic states which cause the infrared divergence. So one must exclude these tachyonic modes in the calculation of the effective action, when one makes a proper infrared regularization. Including the tachyons in the physical spectrum will surely destablize QCD and make it ill-defined.

The correct infrared regularization is dictated by the causality. To implement the causality in Eqs. (48), we first go to the Minkowski time with the Wick rotation, and find

$$\Delta\mathcal{L}_{+}=-\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^2}\,\frac{a/\mu^2}{\sin(at/\mu^2)}\exp(-2iat/\mu^2),$$

$$\Delta\mathcal{L}_{-}=-\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^2}\,\frac{a/\mu^2}{\sin(at/\mu^2)}\exp(+2iat/\mu^2). \tag{49}$$

In this form the infrared divergence has disappeared, but now we face the ambiguity in choosing the correct contours of the integrals in Eq. (49). Fortunately, this ambiguity can be resolved by the causality. To see this, notice that the two integrals $\Delta\mathcal{L}_{+}$ and $\Delta\mathcal{L}_{-}$ originate from the two determinants in Eq. (33), and the standard causality argument requires us to identify $2a$ in the first determinant as $2a-i\epsilon$ but in the second determinant as $2a+i\epsilon$. This tells us that the poles in the first integral in Eq. (49) should lie above the real axis, but the poles in the second integral should lie below the real axis. From this we conclude that the contour in $\Delta\mathcal{L}_{+}$ should pass below the real axis, but the contour in $\Delta\mathcal{L}_{-}$ should pass above the real axis. *With this causality requirement, the two integrals become complex conjugate to each other, which guarantees that $\Delta\mathcal{L}$ is explicitly real, without any imaginary part.* This removes the infrared divergence. We emphasize that this causality for the infrared regularization is precisely the same causality that determines the Feynman propagators in field theory. With this observation, we finally have

$$\Delta\mathcal{L}=\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^{2-\epsilon}}\,\frac{a/\mu^2}{\sinh(at/\mu^2)}[\exp(-2at/\mu^2)+\exp(-2at/\mu^2)], \tag{50}$$

where now ϵ is the ultraviolet cutoff which we have introduced to regularize the ultraviolet divergence.

Now we can perform the integral, and obtain

$$\Delta\mathcal{L}=\frac{11a^2}{48\pi^2}\left(\frac{1}{\epsilon}-\gamma\right)-\frac{11a^2}{48\pi^2}\left(\ln\frac{a}{\mu^2}-c\right),$$

$$c=1-\ln 2-\frac{24}{11}\zeta'(-1,\tfrac{3}{2})=0.945\,56\ldots, \tag{51}$$

where $\zeta(x,y)$ is the generalized Hurwitz zeta function. So with the ultraviolet regularization by modified minimal subtraction, we finally obtain the following effective Lagrangian [13,22]:

$$\mathcal{L}_{\rm eff}=-\frac{1}{2g^2}a^2-\frac{11}{48\pi^2}a^2\left(\ln\frac{a}{\mu^2}-c\right). \tag{52}$$

This completes our derivation of the one-loop effective Lagrangian of $SU(2)$ QCD in the presence of the monopole background. Notice that, as expected, the effective Lagrangian is explicitly invariant under the background gauge transformation (23) which involves only $\vec{C}_\mu$.

As we have indicated, there is another way to obtain the effective action which is more physical. Remember that the two integrals in Eq. (48) come from the two determinants in Eq. (33), and the infrared divergence in the second integral comes from the tachyonic modes contained in the second determinant in Eq. (33). So one can calculate the effective action by calculating the determinant correctly. Now, to evaluate the determinant, one is supposed to use a complete set of eigenfunctions which is made of the physical states. But obviously the tachyonic modes cannot be regarded as physical, because they violate the causality. A remarkable point is that by calculating the determinants with the physical states one can show that the second determinant become identical to the first one. This means that, by calculating the functional determinants correctly, one can obtain exactly the same effective action that we have obtained with the infrared regularization by causality. This provides another justification of our effective action (52).

Now we are ready to establish the monopole condensation. To do this, we renormalize the effective action first. For this, notice that the effective action provides the following nontrivial effective potential:

$$V=\tfrac{1}{4}H^2\left[1+\frac{11g^2}{24\pi^2}\left(\ln\frac{gH}{\mu^2}-c_1\right)\right], \tag{53}$$

where

$$c_1=1-\tfrac{1}{2}\ln 2-\tfrac{24}{11}\zeta'(-1,\tfrac{3}{2})=1.292\,14\ldots .$$

So we can define the running coupling $\bar{g}$ by [7]

$$\left.\frac{\partial^2 V}{\partial H^2}\right|_{H=\bar{\mu}^2}=\frac{1}{2}\,\frac{g^2}{\bar{g}^2}. \tag{54}$$

With the definition, we obtain

$$\frac{1}{\bar{g}^2}=\frac{1}{g^2}+\frac{11}{24\pi^2}\left(\ln\frac{g\bar{\mu}^2}{\mu^2}-c_1+\frac{3}{2}\right), \tag{55}$$

from which we obtain the following β function:

$$\beta(\bar{\mu})=-\frac{11}{24\pi^2}\bar{g}^3. \tag{56}$$

This is exactly the same β function that one obtained from the perturbative QCD to prove the asymptotic freedom [23]. This confirms that our effective action is consistent with the asymptotic freedom.

The fact that the β function obtained from the effective action becomes identical to the one obtained by the perturbative calculation is really remarkable, because this is not always the case. In fact, in QED it has been demonstrated that the running coupling and the β function obtained from the effective action are different from those obtained from the perturbative method [24,25].

In terms of the running coupling, the renormalized potential is given by

$$V_{\rm ren}=\tfrac{1}{4}H^2\left[1+\frac{11}{24\pi^2}\bar{g}^2\left(\ln\frac{H}{\bar{\mu}^2}-\frac{3}{2}\right)\right], \tag{57}$$

which generates a nontrivial local minimum at

$$\langle H\rangle=\frac{\bar{\mu}^2}{g}\exp\left(-\frac{24\pi^2}{11\bar{g}^2}+1\right). \tag{58}$$

Notice that with $\bar{\alpha}_s=1$, we have

$$\frac{\langle H\rangle}{\bar{\mu}^2}=0.13819\ldots . \tag{59}$$

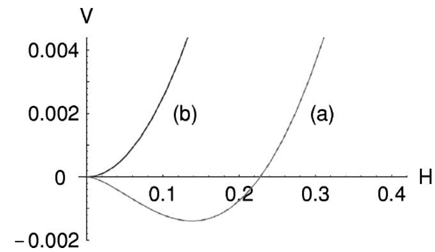

FIG. 1. The effective potential of $SU(2)$ QCD in the pure magnetic background. Here (a) is the effective potential and (b) is the classical potential.

This is nothing but the desired magnetic condensation. *This proves that the one-loop effective action of QCD in the presence of the constant magnetic background does generate a dynamical symmetry breaking through the monopole condensation* [13,22].

The corresponding effective potential is plotted in Fig. 1, where we have assumed $\bar{\alpha}_s=1$, $\bar{\mu}=1$. The effective potential clearly shows that there is indeed a dynamical symmetry breaking in QCD.

The renormalization-group invariance of the effective action is guaranteed by the Callan-Symanzik equation

$$\left(\bar{\mu}\frac{\partial}{\partial\bar{\mu}}+\beta\frac{\partial}{\partial\bar{g}}-\gamma(\vec{C}_\mu)\vec{C}_\mu\frac{\partial}{\partial\vec{C}_\mu}\right)V_{\rm ren}=0, \tag{60}$$

where $\gamma(\vec{C}_\mu)$ is the anomalous dimension for $\vec{C}_\mu$,

$$\gamma(\vec{C}_\mu)=-\frac{11}{48\pi^2}\bar{g}^2+O(\bar{g}^4). \tag{61}$$

This should be compared with that of the gluon field in perturbative QCD, $\gamma(\vec{A}_\mu)=-5\bar{g}^2/24\pi^2$ for $SU(2)$, in the absence of the quarks.

VI. ELECTRIC BACKGROUND

To make sure that our infrared regularization is indeed the correct one, it is necessary to have an independent confirmation of the above result. To do this, it is instructive to calculate the effective action with a pure electric background first.

From Eqs. (46) and (47), we have for a pure electric background (i.e., for $a=0$)

$$\Delta S=i\ln\det(-\tilde{D}^2-2ib)+i\ln\det(-\tilde{D}^2+2ib) \tag{62}$$

and

$$\Delta\mathcal{L}=\frac{1}{16\pi^2}\int_0^\infty\frac{dt}{t^2}\,\frac{b/\mu^2}{\sin(bt/\mu^2)}[\exp(2ibt/\mu^2)+\exp(-2ibt/\mu^2)]. \tag{63}$$

There are different ways to evaluate the integral, but a simple and nice way of doing this follows from the observation that in the imaginary time (i.e., in the Minkowski time) the roles of the electric and magnetic fields are reversed. So with the

Wick rotation the above integral acquires the same form as Eq. (48). Indeed, with the Wick rotation Eq. (63) becomes

$$\Delta\mathcal{L}=-\frac{1}{16\pi^2}i^{\epsilon}\int_0^{\infty}\frac{dt}{t^{2-\epsilon}}\frac{b}{\sinh(bt)}[\exp(-2bt)+\exp(2bt)]. \tag{64}$$

Now, adopting the same infrared regularization as in the pure magnetic background, we obtain

$$\Delta\mathcal{L}=-\frac{11b^2}{48\pi^2}\left(\frac{1}{\epsilon}-\gamma\right)+\frac{11b^2}{48\pi^2}\left(\ln\frac{b}{\mu^2}-c\right)-i\frac{11b^2}{96\pi}. \tag{65}$$

So with the modified minimal subtraction we have (with the pure electric background)

$$\mathcal{L}_{\rm eff}=\frac{b^2}{2g^2}+\frac{11b^2}{48\pi^2}\left(\ln\frac{b}{\mu^2}-c\right)-i\frac{11b^2}{96\pi}. \tag{66}$$

We emphasize that in evaluating the above integral, the same infrared regularization is applied as in the pure magnetic background. With the pure electric background, the eigenfunctions of the second determinant in Eq. (62) become anticausal and thus unphysical in the long-distance region (i.e., for $k^2<b$), just like the eigenfunctions under the pure magnetic background become tachyonic and unphysical in the infrared region (i.e., for $k^2<a$). So we must again exclude these unphysical modes to evaluate the above integral.

Another way to perform the integral (63) is by choosing the proper contour. Notice that (unlike the pure magnetic background) the integrand here has poles on the real axis, so that we must specify the contour of the integral. To find the proper contour, first notice that the eigenvalues of the two determinants in Eq. (62) are complex conjugate to each other. This means that the contour of the two integrals in Eq. (63) should also be complex conjugate to each other. Secondly, one can make the first integral finite by choosing the contour to pass above the real axis and rotating it to the positive imaginary axis (i.e., by replacing t with it). This is justifiable, because the first integral is free of the controversial acausal states. With this, the contour of the second integral is fixed by complex conjugating the first contour. This means that the second contour must pass below the real axis, which one can rotate to the negative imaginary axis (by replacing t with $-it$). This makes the second integral finite. Finally, the causality requires us to replace b with $b+\epsilon$ in the first determinant but $b-\epsilon$ in the second determinant in Eq. (62). This means that the first contour should start from $0+\epsilon$, but the second one from $0-\epsilon$ in Eq. (63). From this, we conclude that half of the residue at the origin should contribute to the integral. This recipe reproduces Eq. (66), and justifies the result.

Notice that it is the causality that produces the imaginary part in Eq. (65). This is remarkable, because it was the same causality which has made Eq. (51) explicitly real. So in both pure magnetic and pure electric backgrounds, the causality determines the imaginary part of the effective action.

The contrast between the effective actions (52) and (66) is remarkable. First, the effective potential derived from Eq. (66) has no local minimum. This implies that the electric background does not generate a condensation. Secondly, Eq. (66) has an imaginary part

$$\mathrm{Im}\,\mathcal{L}=-\frac{11b^2}{96\pi}. \tag{67}$$

This implies that the electric background is unstable. But perhaps a more important point here is that the imaginary part is negative. *This means that the electric background generates the pair annihilation, rather than the pair creation, of the gluons.* This is because the negative imaginary part can be interpreted as the negative probability of the pair creation. This implies that the gluons in QCD, unlike the electrons in QED, tend to annihilate among themselves in the color electric field. This might sound strange, but actually it is not difficult to understand. Indeed this is precisely what the asymptotic freedom dictates. To understand this, remember that the gluon loop contributes positively but the quark loop contributes negatively to the asymptotic freedom [23]. Exactly for the same reason the gluon and quark loops contribute oppositely to the imaginary part of the effective action. But the quark loop in QCD, just like the electron loop in QED, generates a positive imaginary part [22]. This tells us that the gluon loop should generate a negative imaginary part in the effective action. This means that the asymptotic freedom, the antiscreening, and the pair annihilation all originate from the same physics. This is really remarkable.

VII. STABILITY OF MONOPOLE CONDENSATION

There have been many attempts to construct the effective action of QCD in the literature, and in appearance our vacuum (58) looks very much like the old Savvidy-Nielsen-Olesen (SNO) vacuum [7,8]. The major difference is that the effective action in the earlier approaches contained an imaginary part, which made the magnetic condensation unstable. In contrast, our effective action is explicitly real, which guarantees the stability of our monopole condensation. Indeed, it has been asserted that the SNO vacuum should be unstable, because the effective action which defines the vacuum develops an imaginary part [7,8],

$$\mathrm{Im}\,\mathcal{L}|_{\rm SNO}=\frac{1}{8\pi}a^2. \tag{68}$$

This destabilizes the vacuum through the pair creation of gluons. This assertion of the instability of the SNO vacuum, which comes from improper infrared regularizations, has been widely accepted and never convincingly revoked. As a consequence, it has been generally believed that the one-loop effective action cannot establish the monopole condensation in QCD. Our analysis tells us that this misleading belief has no foundation.

But since the absence of the absorptive part in our effective action is such a crucial point which distinguishes our effective action from the SNO action, one might like to have independent proof that our infrared regularization is indeed the correct one. Fortunately, there are various ways to make an independent confirmation of our effective actions (52) and (66). To see this, first notice that the imaginary part (68) of the SNO action as well as ours is quadratic in the background fields. This, with the definition (45), tells us that the imaginary part of the one-loop effective action is second order in the coupling constant g. So one can find the correct imaginary part of the effective action perturbatively, just by calculating the effective action up to second order in the coupling constant in the perturbative expansion. There are different ways of doing this. In fact, one can just calculate the relevant Feynmann diagrams of the perturbative expansion [26], or adopt Schwinger's method used in QED, to obtain the imaginary part [27]. Now, a remarkable point is that these perturbative calculations do reproduce a result which is identical to ours [28],

$$\mathrm{Im}\,\Delta\mathcal{L}=\begin{cases} 0, & b=0,\\ -\dfrac{11b^2}{96\pi}, & a=0. \end{cases} \tag{69}$$

This confirms that our infrared regularization is indeed correct. More importantly, this confirms that we do have the desired dynamical symmetry breaking and the magnetic condensation in QCD. It must be pointed out that the possibility that one could calculate the imaginary part of the effective action by the perturbative method, and that the SNO action could probably be incorrect, was first raised by Schanbacher [26]. Unfortunately, this remarkable work has been completely neglected so far, probably because it is also plagued by the defect that it is not gauge-independent.

We emphasize that this perturbative calculation of the imaginary part in QCD is justified precisely because the imaginary part of the effective action is second order in g. This is remarkable, because in general the one-loop effective action does not allow a perturbative expansion. For example, in QED the perturbative expansion of the imaginary (as well as the real) part of the effective action is divergent and does not make sense, because the point $e=0$ is singular [24,25]. This means that in QED the perturbative calculation does not reproduce the result of the one-loop effective action.

To reinforce our assertion, we now provide a third independent argument which supports our results. An important point to observe here is that the effective actions (52) and (66) are actually the mirror image of each other. To see this, notice that we can obtain Eq. (66) from Eq. (52) simply by replacing a with $-ib$, and similarly Eq. (52) from Eq. (66) by replacing b with ia. This is the first indication that there exists a fundamental symmetry which we call the duality in the effective action of QCD. *The duality states that the effective action must be invariant under the replacement*

$$a\rightarrow -ib, \quad b\rightarrow ia.$$

This type of duality was first established in the effective action of QED [24,25]. But we emphasize that exactly the same duality should also hold in our effective action of QCD, because we have already Abelianized it. An important point of the duality is that the duality provides a very useful tool to check the consistency of the effective action. In the present case, the duality indeed confirms the consistency of our effective actions (52) and (66). Obviously, this supports the idea that our calculation of the imaginary parts (69) is probably correct, or at least consistent with the duality. This tells us that the causality, the perturbative expansion, and the duality all strongly endorse the stability of our monopole condensation.

It must be emphasized that there are fundamental differences between the earlier attempts and the present approach. The earlier attempts had three problems. First, the separation between the classical background and the quantum field was not gauge-independent, which made it difficult to establish the gauge invariance of the one-loop effective action. Secondly, the origin of the magnetic background has never been clarified. As a consequence, the magnetic condensation could not be associated with the monopole background. These defects were serious enough, but perhaps the most serious problem was that the infrared divergence was not properly regularized in the earlier attempts. Because of this, the SNO effective action contained an imaginary part. This destabilizes the vacuum through the pair creation of gluons.

In contrast in our approach the separation of the monopole background from the quantum fluctuation is clearly gauge-independent. Moreover, our infrared regularization generates no imaginary part in the effective action. Because of these we obtain a stable vacuum made of monopole condensation which is both gauge- and Lorentz-invariant. Notice that the infrared regularization in Eq. (50) is not just to remove the infrared divergence (there are infinitely many ways to do this). The infrared divergence that we face here in QCD is also different from those one encounters in the effective action of the massless QED [24,25]. The infrared divergence in the massless QED comes from the zero modes. But these zero modes are physical modes, which should not be excluded in the calculation of the effective action. On the other hand, the infrared divergence that we have here comes from the unphysical modes, so that one must exclude these unphysical modes from the physical spectrum with a proper infrared regularization. Our analysis has provided ample reason why this has to be so (notice that in the earlier attempts these tachyonic modes are incorrectly identified as the "unstable" modes, but we emphasize that they are not just unstable but unphysical). And it is precisely these unphysical modes that generate the controversial imaginary part in the SNO action. So with the exclusion of the unphysical modes, the instability of the vacuum disappears completely. As importantly in our approach we can really claim that the magnetic condensation is a gauge-independent phenomenon. Furthermore, here we have demonstrated that it is precisely the Wu-Yang monopole that is responsible for the condensation.

VIII. QCD VERSUS SKYRME-FADDEEV ACTION

Recently, Faddeev and Niemi have discovered the knotlike topological solitons in the Skyrme-type nonlinear sigma model [9],

$$\mathcal{L}_{\mathrm{SF}}=-\frac{\mu^2}{2}(\partial_\mu\hat{n})^2-\tfrac{1}{4}(\partial_\mu\hat{n}\times\partial_\nu\hat{n})^2, \tag{70}$$

and made an interesting conjecture that the Skyrme-Faddeev action could be interpreted as an effective action for QCD in the low-energy limit [10]. But we emphasize that from our decomposition (1) it should have been evident that the above action is closely related to QCD. Indeed from the decomposition we have [14]

$$\mathcal{L}_{\rm SF} = -\tfrac{1}{4}\vec{H}_{\mu\nu}^2 - \frac{\mu^2}{2}\vec{C}_\mu^2. \tag{71}$$

This tells us that the Skyrme-Faddeev theory can be interpreted as a massive Yang-Mills theory where the gauge potential has the special form (7). Furthermore, we can claim that it is a theory of monopoles and at the same time a theory of confinement, where the monopole-antimonopole pairs are confined to form the knots [14,19]. But now with the effective action of QCD at hand we can discuss the connection between QCD and Skyrme-Faddeev theory in more detail.

Evidently the effective action (52) is invariant under both gauge and Lorentz transformations. On the other hand, we can express the effective action explicitly in terms of the monopole field strength $\vec{H}_{\mu\nu}$. This, of course, is not accidental. The background field method guarantees that the effective action should be expressed by the gauge-invariant form, invariant under the background gauge transformation (23). What is remarkable here is that, with Eq. (7), the background magnetic field $\vec{H}_{\mu\nu}$ can be expressed completely by the magnetic potential $\vec{C}_\mu$,

$$\vec{H}_{\mu\nu} = -g\vec{C}_\mu \times \vec{C}_\nu,$$

so that the effective potential (53) can actually be written completely in terms of $\vec{C}_\mu$,

$$V = \frac{g^2}{4}(\vec{C}_\mu \times \vec{C}_\nu)^2 \left\{ 1 + \frac{11g^2}{24\pi^2}\left(\ln\frac{g[(\vec{C}_\mu \times \vec{C}_\nu)^2]^{1/2}}{\mu^2} - c_1 \right) \right\}. \tag{72}$$

Now, just for a heuristic reason, suppose we choose a particular Lorentz frame and express the vacuum (58) by the vacuum expectation value of $\vec{C}_\mu$. In this case, the above effective potential generates the following mass matrix for $\vec{C}_\mu$:

$$M_{\mu\nu}^{ij} = \left\langle \frac{\delta^2 V}{\delta C_\mu^i \delta C_\nu^j} \right\rangle = m^2(\delta^{ij} - n^i n^j) g_{\mu\nu}, \tag{73}$$

where

$$m^2 = \frac{11g^4}{96\pi^2}\left\langle \frac{(\vec{C}_\mu \times \vec{H}_{\mu\nu})^2}{H^2} \right\rangle \tag{74}$$

can be interpreted as the "effective mass" for $\vec{C}_\mu$. This demonstrates that the magnetic condensation indeed generates the mass gap necessary for the dual Meissner effect and the confinement.

With the above understanding, we can now study the possible connection between the Skyrme-Faddeev action and the effective action of QCD. To do this, we first expand the effective potential in terms of the monopole potential around the vacuum and make the following Taylor expansion:

$$\begin{aligned} V = V_0 &+ \frac{1}{2!}\left\langle \frac{\delta^2 V}{\delta C_\mu^i \delta C_\nu^j} \right\rangle \overline{C}_\mu^i \, \overline{C}_\nu^j \\ &+ \frac{1}{3!}\left\langle \frac{\delta^3 V}{\delta C_\mu^i \delta C_\nu^j \delta C_\rho^k} \right\rangle \overline{C}_\mu^i \overline{C}_\nu^j \overline{C}_\rho^k \\ &+ \frac{1}{4!}\left\langle \frac{\delta^4 V}{\delta C_\mu^i \delta C_\nu^j \delta C_\rho^k \delta C_\sigma^l} \right\rangle \overline{C}_\mu^i \overline{C}_\nu^j \overline{C}_\rho^k \overline{C}_\sigma^l + \cdots, \end{aligned} \tag{75}$$

where

$$\overline{C}_\mu^i = C_\mu^i - \langle C_\mu^i \rangle.$$

Now, near the vacuum we could neglect the higher-order terms and keep only the quartic polynomial in $\vec{C}_\mu$ for simplicity. In this approximation, the corresponding effective Lagrangian will acquire the form

$$\begin{aligned} \mathcal{L}_{\rm eff} &= -\tfrac{1}{2}m^2(\vec{C}_\mu)^2 - \frac{\alpha}{4}(\vec{C}_\mu \times \vec{C}_\nu)^2 \\ &\quad - \frac{\beta}{4}(\vec{C}_\mu \cdot \vec{C}_\nu)^2 - \frac{\gamma}{4}(\vec{C}_\mu)^4 + \cdots \\ &= -\frac{m^2}{2g^2}(\partial_\mu \hat{n})^2 - \frac{\alpha}{4g^2}(\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2 \\ &\quad - \frac{\beta}{4g^2}(\partial_\mu \hat{n} \cdot \partial_\nu \hat{n})^2 - \frac{\gamma}{4g^2}(\partial_\mu \hat{n})^4 + \cdots, \end{aligned} \tag{76}$$

where α, β, and γ are numerical parameters which can be fixed from Eq. (75). This is merely a generalized Skyrme-Faddeev Lagrangian [13,14]. This shows that one can indeed derive a generalized Skyrme-Faddeev action from QCD by expanding the effective potential around the vacuum. This, together with Eq. (71), establishes a firm connection between the Skyrme-Faddeev theory and QCD. In fact we can go further, and establish a deep connection between QCD and the Skyrme theory itself [14,19].

An important feature in our analysis is that the Skyrme-Faddeev action is intimately connected to the monopole condensation in QCD. In particular, our analysis makes it clear that the mass scale in the Skyrme-Faddeev action is directly related to the mass of the monopole potential, which determines the confinement scale in QCD. This is not surprising. Indeed, any attempt to relate the Skyrme-Faddeev action to QCD must produce the mass scale that the Skyrme-Faddeev action contains, and the only way to interpret this mass scale in QCD is through the confinement.

But it must be emphasized that our approximation (76) is by no means exact. There are two points that should be kept in mind here. First, we have kept only the quadratic part and neglected all the higher-order terms in Eq. (76). More seriously, in deriving the effective action we have neglected the derivatives of $\vec{H}_{\mu\nu}$ and thus the derivatives of $\vec{C}_\mu$, assuming that H is constant. Secondly, we had to choose a particular Lorentz frame to justify the expansion (75) of the effective action around the vacuum. So our derivation appears to have compromised the Lorentz invariance, although the generalized Skyrme-Faddeev action is obviously Lorentz-invariant. Consequently, our analysis establishes a possible connection between a ''generalized'' nonlinear sigma model of Skyrme-Faddeev type and QCD only in a limited sense. In particular, it does not assert that the simple-minded Skyrme-Faddeev action describes QCD in the infrared limit. In spite of these drawbacks, our analysis strongly endorses the fact that the Skyrme-Faddeev action has something in common with QCD, which is really remarkable.

IX. DISCUSSION

In this paper, we have established the monopole condensation, which describes a stable vacuum of QCD. Furthermore, we have demonstrated the existence of a genuine dynamical symmetry breaking in QCD triggered by the monopole condensation. We were able to do this by calculating the one-loop effective action of $SU(2)$ QCD in the presence of a pure monopole background. There have been earlier attempts to calculate the effective action, but our result differs from the earlier results. The main difference with the earlier attempts was the controversial imaginary part in the effective action in the earlier attempts. This has made the SNO vacuum unstable. In contrast, with a proper infrared regularization, we have shown that the QCD vacuum made of monopole condensation is stable. We have provided three independent arguments to support our conclusion.

It is truly remarkable that the principles of the quantum field theory allow us to demonstrate the monopole condensation within the framework of the conventional quantum field theory. The assertion of the instability of the SNO vacuum has created a wrong impression that one cannot demonstrate monopole condensation with the one-loop effective action. Our analysis tells us that in fact one can demonstrate monopole condensation with the effective action. Notice, however, that this does not prove that monopole condensation is the true vacuum of QCD. To prove this, we have to calculate the effective action in an arbitrary color electromagnetic background, and show that indeed the monopole condensation is the true minimum of the effective potential. This is not an easy task. Even for the ''simple'' QED the calculation of the one-loop effective action in an arbitrary background has been completed only recently [24,25], 50 years after Schwinger's seminal work [27]. In a subsequent paper, we obtain the one-loop effective action of QCD for an arbitrary background, and demonstrate that indeed the monopole condensation is the true vacuum of QCD, at least at the one-loop level [22,29].

We conclude with the following remarks.

(i) It should be emphasized that the gauge-independent decomposition (1) of the non-Abelian gauge potential plays the crucial role in our analysis. The decomposition has been known for more than 20 years [2,3], but its physical significance appears to have been appreciated very little until recently. Now we emphasize that it is this decomposition which has made the gauge-independent separation of the classical background from the quantum field, and allows us to obtain the effective action of QCD without ambiguity. In particular, it is this decomposition which shows that vacuum condensation is indeed made of monopole condensation. Many of the earlier approaches had the critical defect that the decomposition of the non-Abelian gauge potential to the $U(1)$ potential and the charged vector field was not gauge-independent, which has made these approaches controversial. In particular, in these approaches one cannot make sure that the effective action (and the resulting magnetic condensation) obtained with the Abelian background really has a gauge-independent meaning.

(ii) There have been two competing proposals for the correct mechanism of the confinement in QCD, namely the one emphasizing the role of the instantons and the other emphasizing that of the monopoles. Our analysis strongly favors the monopoles as the physical source for the confinement. It provides a natural dynamical symmetry breaking, and generates the mass gap necessary for the confinement in QCD. Notice that the multiple vacua, even though it is an important characteristics of the non-Abelian gauge theory, did not play any crucial role in our calculation of the effective action. Moreover, our result shows that it is the monopole condensate, not the θ-vacuum, which describes the physical vacuum of QCD.

(iii) We have established a firm connection between the Skyrme-Faddeev action and QCD. On the other hand, the Skyrme-Faddeev theory (and the Skyrme theory itself) contains the topological knot states. If so, QCD could also likely admit such states, which might naturally be interpreted as ''glueballs.'' But these knots are not the ordinary glueballs made of valence gluons. They are made of the magnetic, not electric, flux. In this sense, they should be called ''magnetic'' glueballs [14]. The existence of such magnetic glueballs was predicted a long time ago [2,3]. Once monopole condensation sets in, one should expect the fluctuation of the condensed vacuum. But obviously the fluctuation modes have to be magnetic, which could be identified as magnetic glueballs. (A new feature here is that they have a topological stability. But this could be an artifact of the effective theory, not a genuine feature of QCD.) We can even predict that the mass of these glueballs starts from around 1.4 GeV [14]. If so, the remaining task is to look for convincing experimental evidence of the magnetic glueball states in the hadron spectrum [2,3].

Although we have concentrated to $SU(2)$ QCD in this paper, it must be clear from our analysis that the magnetic condensation is a generic feature of the non-Abelian gauge theory. A more detailed discussion which contains the calculation of the effective action in the presence of an arbitrary color electromagnetic background will be presented in an accompanying paper [29].

ACKNOWLEDGMENTS

One of the authors (Y.M.C.) thanks S. Adler and F. Dyson for the fruitful discussions, and Professor C. N. Yang for the continuous encouragements. The other (D.G.P.) thanks Professor C. N. Yang for financial support at Asia Pacific Center for Theoretical Physics, and appreciates Haewon Lee for numerous discussions. The work is supported in part by Korea Research Foundation (Grant KRF-2001-015-BP0085) and by the BK21 project of the Ministry of Education.

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