Rephasing invariants of *CP* and *T* violation in the four-neutrino mixing models

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We calculate the rephasing invariants of CP and T violation in a favorable parametrization of the 4×4 lepton flavor mixing matrix. Their relations with the CP- and T-violating asymmetries in neutrino oscillations are derived, and the matter effects are briefly discussed.

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Since 1998, some robust evidence for atmospheric and solar neutrino oscillations has been accumulated from the Super-Kamiokande experiment [1]. In addition, $\nu_\mu\to\nu_e$ and $\bar{\nu}_\mu\to\bar{\nu}_e$ transitions have been observed by the Liquid Scintillation Neutrino Detector (LSND) Collaboration [2]. A simultaneous interpretation of solar, atmospheric and LSND data requires the introduction of a light sterile neutrino [3], because they involve three distinct mass-squared differences: $\Delta m^2_{\rm LSND}\gg\Delta m^2_{\rm atm}\gg\Delta m^2_{\rm sun}$. In the four-neutrino mixing models, CP and T symmetries are generally expected to be violated. The measurement of leptonic CP- and T-violating effects needs a new generation of accelerator neutrino experiments with very long baselines. So far some attention has been paid to the possibilities of observing four-neutrino mixing and CP (or T) violation in a variety of long-baseline neutrino oscillation experiments [4–8].

The violation of CP and T invariance in neutrino oscillations is attributed to the nontrivial complex phases in the 4×4 lepton flavor mixing matrix V. While the CP-violating phases of V can be assigned in many different ways, the observable effects of CP and T violation depend only upon some rephasing invariants of V [9]. It is therefore useful to investigate how those rephasing invariants are related to the flavor mixing angles and CP-violating phases in a specific parametrization of V, and how they are connected to the CP and T asymmetries in long-baseline neutrino oscillations. Although some attempts were made for this purpose [4–7], a complete and analytically exact result has not been achieved.

In this paper we establish the relationship between the CP and T asymmetries in neutrino oscillations and the rephasing invariants of CP and T violation defined in the four-neutrino mixing models. In particular, the exact expressions of those rephasing invariants are for the first time calculated on the basis of a favorable parametrization of the 4×4 lepton flavor mixing matrix V. Our results are very useful for a systematic and model-independent study of CP- and T-violating effects in various long-baseline neutrino oscillation experiments, and some of them are also applicable for the four-quark mixing models.

Let us begin with a generic ${\rm SU(2)_L\times U(1)_Y}$ model of electroweak interactions, in which there exist n charged leptons belonging to isodoublets, n active neutrinos belonging to isodoublets, and n' sterile neutrinos belonging to isosinglets. The charged-current weak interactions of leptons are then associated with a rectangular flavor mixing matrix of n rows and $(n+n')$ columns [10]. Without loss of generality, one may choose to identify the flavor eigenstates of charged leptons with their mass eigenstates. In this specific basis, the $n\times(n+n')$ lepton mixing matrix links the neutrino flavor eigenstates directly to the neutrino mass eigenstates. Although sterile neutrinos do not participate in normal weak interactions, they may oscillate among themselves and with active neutrinos. Once the latter is concerned we are led to a more general $(n+n')\times(n+n')$ lepton flavor mixing matrix [11], defined as V in the chosen flavor basis. For the flavor mixing of one sterile neutrino (ν_s) and three active neutrinos (ν_e,ν_μ,ν_τ), the explicit form of V can be written as

$$\begin{pmatrix}\nu_s\\ \nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}=\begin{pmatrix}V_{s0} & V_{s1} & V_{s2} & V_{s3}\\ V_{e0} & V_{e1} & V_{e2} & V_{e3}\\ V_{\mu0} & V_{\mu1} & V_{\mu2} & V_{\mu3}\\ V_{\tau0} & V_{\tau1} & V_{\tau2} & V_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_0\\ \nu_1\\ \nu_2\\ \nu_3\end{pmatrix}, \tag{1}$$

where ν_i (for $i=0,1,2,3$) denote the mass eigenstates of four neutrinos. If neutrinos are Dirac particles, V can be parametrized in terms of six mixing angles and three phase angles. If neutrinos are Majorana particles, however, three additional phase angles are needed to get a full parametrization of V. In either case, one may define the rephasing invariants of CP or T violation as follows:

$$J^{ij}_{\alpha\beta}\equiv{\rm Im}(V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}), \tag{2}$$

where the Greek subscripts run over (s,e,μ,τ) and the Latin superscripts run over $(0,1,2,3)$. Of course, $J^{ii}_{\alpha\beta}=J^{ij}_{\alpha\alpha}=0$ and $J^{ij}_{\alpha\beta}=-J^{ji}_{\alpha\beta}=-J^{ij}_{\beta\alpha}=J^{ji}_{\beta\alpha}$ hold by definition. The unitarity of V leads to the following correlation equations of $J^{ij}_{\alpha\beta}$:

$$\sum_i J^{ij}_{\alpha\beta}=\sum_j J^{ij}_{\alpha\beta}=\sum_\alpha J^{ij}_{\alpha\beta}=\sum_\beta J^{ij}_{\alpha\beta}=0. \tag{3}$$

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Hence there are totally nine independent $J^{ij}_{\alpha\beta}$, whose magnitudes depend only upon three of the six CP-violating phases or their combinations in a specific parametrization of V. Let us follow Ref. [4] to parametrize V as

$$V=\begin{pmatrix} c_{01}c_{02}c_{03} & c_{02}c_{03}\hat{s}^*_{01} & c_{03}\hat{s}^*_{02} & \hat{s}^*_{03} \\ \begin{array}{l} -c_{01}c_{02}\hat{s}_{03}\hat{s}^*_{13} \\ -c_{01}c_{13}\hat{s}_{02}\hat{s}^*_{12} \\ -c_{12}c_{13}\hat{s}_{01} \end{array} & \begin{array}{l} -c_{02}\hat{s}^*_{01}\hat{s}_{03}\hat{s}^*_{13} \\ -c_{13}\hat{s}^*_{01}\hat{s}_{02}\hat{s}^*_{12} \\ +c_{01}c_{12}c_{13} \end{array} & \begin{array}{l} -\hat{s}^*_{02}\hat{s}_{03}\hat{s}^*_{13} \\ +c_{02}c_{13}\hat{s}^*_{12} \end{array} & c_{03}\hat{s}^*_{13} \\ \begin{array}{l} -c_{01}c_{02}c_{13}\hat{s}_{03}\hat{s}^*_{23} \\ +c_{01}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} \\ -c_{01}c_{12}c_{23}\hat{s}_{02} \\ +c_{12}\hat{s}_{01}\hat{s}_{13}\hat{s}^*_{23} \\ +c_{23}\hat{s}_{01}\hat{s}_{12} \end{array} & \begin{array}{l} -c_{02}c_{13}\hat{s}^*_{01}\hat{s}_{03}\hat{s}^*_{23} \\ +\hat{s}^*_{01}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} \\ -c_{12}c_{23}\hat{s}^*_{01}\hat{s}_{02} \\ -c_{01}c_{12}\hat{s}_{13}\hat{s}^*_{23} \\ -c_{01}c_{23}\hat{s}_{12} \end{array} & \begin{array}{l} -c_{13}\hat{s}^*_{02}\hat{s}_{03}\hat{s}^*_{23} \\ -c_{02}\hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} \\ +c_{02}c_{12}c_{23} \end{array} & c_{03}c_{13}\hat{s}^*_{23} \\ \begin{array}{l} -c_{01}c_{02}c_{13}c_{23}\hat{s}_{03} \\ +c_{01}c_{23}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13} \\ +c_{01}c_{12}\hat{s}_{02}\hat{s}_{23} \\ +c_{12}c_{23}\hat{s}_{01}\hat{s}_{13} \\ -\hat{s}_{01}\hat{s}_{12}\hat{s}_{23} \end{array} & \begin{array}{l} -c_{02}c_{13}c_{23}\hat{s}^*_{01}\hat{s}_{03} \\ +c_{23}\hat{s}^*_{01}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13} \\ +c_{12}\hat{s}^*_{01}\hat{s}_{02}\hat{s}_{23} \\ -c_{01}c_{12}c_{23}\hat{s}_{13} \\ +c_{01}\hat{s}_{12}\hat{s}_{23} \end{array} & \begin{array}{l} -c_{13}c_{23}\hat{s}^*_{02}\hat{s}_{03} \\ -c_{02}c_{23}\hat{s}^*_{12}\hat{s}_{13} \\ -c_{02}c_{12}\hat{s}_{23} \end{array} & c_{03}c_{13}c_{23} \end{pmatrix}, \tag{4}$$

where $c_{ij}\equiv\cos\theta_{ij}$ and $\hat{s}_{ij}\equiv s_{ij}e^{i\delta_{ij}}$ with $s_{ij}\equiv\sin\theta_{ij}$. Without loss of generality, the six mixing angles θ_{ij} can all be arranged to lie in the first quadrant. The six CP-violating phases δ_{ij} may take arbitrary values between 0 and 2π. After some lengthy but straightforward calculations, we obtain the explicit expressions of nine independent $J^{ij}_{\alpha\beta}$ defined in Eq. (2):

$$\begin{aligned} J^{02}_{\tau s}=&\, c^2_{01}c_{02}c^2_{03}c_{12}c_{13}c_{23}s_{02}s_{03}s_{23}\sin\phi_x + c_{01}c_{02}c^2_{03}c_{12}c_{13}c^2_{23}s_{01}s^2_{02}s_{03}s_{13}\sin\phi_y + (c^2_{23}s^2_{13}-s^2_{23})c_{01}c^2_{02}c^2_{03}c_{12}s_{01}s_{02}s_{12}\sin\phi_z \\ &- c_{01}c^2_{02}c^2_{03}c^2_{12}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) - c_{01}c_{02}c^2_{03}c_{13}c_{23}s_{01}s^2_{02}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z) \\ &+ c^2_{01}c_{02}c^2_{03}c_{13}c^2_{23}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z) - c_{01}c^2_{02}c^2_{03}c_{23}s_{01}s_{02}s^2_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+2\phi_z), \\ J^{03}_{\tau s}=&-c^2_{01}c_{02}c^2_{03}c_{12}c_{13}c_{23}s_{02}s_{03}s_{23}\sin\phi_x - c_{01}c_{02}c^2_{03}c_{12}c_{13}c^2_{23}s_{01}s_{03}s_{13}\sin\phi_y + c_{01}c_{02}c^2_{03}c_{13}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z) \\ &- c^2_{01}c_{02}c^2_{03}c_{13}c^2_{23}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z), \\ J^{23}_{\tau s}=&\, c_{02}c^2_{03}c_{12}c_{13}c_{23}s_{02}s_{03}s_{23}\sin\phi_x + c_{02}c^2_{03}c_{13}c^2_{23}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z), \end{aligned} \tag{5}$$

and

$$\begin{aligned} J^{02}_{se}=&\, c_{01}c_{02}c^2_{03}c_{12}c_{13}s_{01}s^2_{02}s_{03}s_{13}\sin\phi_y - c_{01}c^2_{02}c^2_{03}c_{12}c^2_{13}s_{01}s_{02}s_{12}\sin\phi_z + c^2_{01}c_{02}c^2_{03}c_{13}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z), \\ J^{13}_{se}=&\, c_{01}c_{02}c^2_{03}c_{12}c_{13}s_{01}s_{03}s_{13}\sin\phi_y - c_{02}c^2_{03}c_{13}s^2_{01}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z), \\ J^{23}_{se}=&\, c_{02}c^2_{03}c_{13}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z), \end{aligned} \tag{6}$$

as well as

$$
\begin{aligned}
J^{12}_{e\mu}= & -(c^2_{01}c^2_{12}s^2_{13}-c^2_{01}c^2_{13}s^2_{12}-s^2_{01}s^2_{03}s^2_{13}+s^2_{01}s^2_{12})c_{02}c_{12}c_{13}c_{23}s_{02}s_{03}s_{23}\sin\phi_x+(c^2_{02}c^2_{23}s^2_{12}-c^2_{12}c^2_{23}s^2_{02}+s^2_{02}s^2_{03}s^2_{23}\\
& -s^2_{12}s^2_{23})c_{01}c_{02}c_{12}c_{13}s_{01}s_{03}s_{13}\sin\phi_y+(c^2_{13}c^2_{23}-c^2_{13}s^2_{03}s^2_{23}-c^2_{23}s^2_{03}s^2_{13}+s^2_{03}s^2_{13}s^2_{23})c_{01}c^2_{02}c_{12}s_{01}s_{02}s_{12}\sin\phi_z+(c^2_{02}s^2_{13}\\
& -c^2_{13}s^2_{02})c_{01}c^2_{12}c_{23}s_{01}s_{02}s^2_{03}s_{13}s_{23}\sin(\phi_x-\phi_y)+(c^2_{12}-c^2_{02}s^2_{13}-c^2_{13}s^2_{02}+s^2_{02}s^2_{03}s^2_{13})c_{01}c_{02}c_{13}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\\
& -(c^2_{01}c^2_{12}c^2_{23}-c^2_{01}c^2_{12}s^2_{23}-c^2_{12}c^2_{23}s^2_{01}+s^2_{01}s^2_{03}s^2_{23}-s^2_{01}s^2_{12}s^2_{23})c_{02}c_{13}s_{02}s_{03}s_{12}s_{13}\sin(\phi_y-\phi_z)+(c^2_{01}c^2_{02}c^2_{13}-c^2_{01}c^2_{13}s^2_{02}s^2_{03}\\
& -c^2_{02}c^2_{13}s^2_{01}s^2_{03}+c^2_{13}s^2_{01}s^2_{02}s^2_{03})c_{12}c_{23}s_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+\phi_z)-(c^2_{02}c^2_{13}s^2_{12}-c^2_{02}s^2_{03}s^2_{12}s^2_{13}\\
& -c^2_{13}s^2_{02}s^2_{03}s^2_{12})c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y+2\phi_z)-c_{01}c^2_{02}c^2_{13}c_{23}s_{01}s_{02}s^2_{03}s_{13}s_{23}\sin(\phi_x+\phi_y)\\
& +c_{01}c_{02}c^2_{12}c_{13}c_{23}s_{01}s^2_{02}s_{03}s_{12}s_{23}\sin(\phi_x-\phi_z)+(c^2_{23}-s^2_{23})c_{01}c_{02}c_{12}c_{13}s_{01}s^2_{02}s_{03}s^2_{12}s_{13}\sin(\phi_y-2\phi_z)-(c^2_{01}\\
& -s^2_{01})c_{02}c_{12}c_{13}c_{23}s_{02}s_{03}s^2_{12}s^2_{13}s_{23}\sin(\phi_x-2\phi_y+2\phi_z)-c_{01}c_{02}c^2_{12}c_{13}c_{23}s_{01}s_{03}s_{12}s^2_{13}s_{23}\sin(\phi_x-2\phi_y+\phi_z)\\
& +c_{01}c_{02}c_{13}c_{23}s_{01}s^2_{02}s_{03}s^2_{12}s^2_{13}s_{23}\sin(\phi_x-2\phi_y+3\phi_z),\\
J^{13}_{e\mu}= & \, c_{02}c^2_{03}c_{12}c_{13}c_{23}s^2_{01}s_{02}s_{03}s^2_{13}s_{23}\sin\phi_x+c_{01}c_{02}c^2_{03}c_{12}c_{13}s_{01}s_{03}s_{13}s^2_{23}\sin\phi_y-c_{01}c^2_{03}c^2_{12}c^2_{13}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y)\\
& +c_{01}c_{02}c^2_{03}c_{13}c_{23}s_{01}s_{03}s_{12}s^2_{13}s_{23}\sin(\phi_x+\phi_z)-c_{02}c^2_{03}c_{13}s^2_{01}s_{02}s_{03}s_{12}s_{13}s^2_{23}\sin(\phi_y-\phi_z)\\
& +c_{01}c^2_{03}c^2_{13}c_{23}s_{01}s_{02}s^2_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+2\phi_z)-(c^2_{01}-s^2_{01}s^2_{02})c^2_{03}c_{12}c^2_{13}c_{23}s_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+\phi_z), \qquad (7)\\
J^{23}_{e\mu}= & -c_{02}c^2_{03}c_{12}c_{13}c_{23}s_{02}s_{03}s^2_{13}s_{23}\sin\phi_x+c_{02}c^2_{03}c_{13}s_{02}s_{03}s_{12}s_{13}s^2_{23}\sin(\phi_y-\phi_z)+c^2_{02}c^2_{03}c_{12}c^2_{13}c_{23}s_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+\phi_z),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_x &\equiv \delta_{03}-\delta_{02}-\delta_{23},\\
\phi_y &\equiv \delta_{03}-\delta_{01}-\delta_{13}, \qquad (8)\\
\phi_z &\equiv \delta_{02}-\delta_{01}-\delta_{12}.
\end{aligned}
$$

With the help of Eq. (3), one may easily derive the expressions of all the other rephasing invariants of CP and T violation from Eqs. (5), (6) and (7). The results obtained above are new, and they are expected to be very useful for a systematic study of CP- and T-violating effects in the four-neutrino mixing models. The same results are also applicable for the discussion of CP and T violation in the four-quark mixing models [12].

Note that all CP- and T-violating observables in neutrino oscillations must be related linearly to $J^{ij}_{\alpha\beta}$. To see this point more clearly, we consider that a neutrino ν_α converts to another neutrino ν_β in vacuum. The probability of this conversion is given by

$$
\begin{aligned}
P(\nu_\alpha\to\nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}\,[\mathrm{Re}(V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i})\sin^2 F_{ji}]\\
-2\sum_{i<j}\,(J^{ij}_{\alpha\beta}\sin 2F_{ji}), \qquad (9)
\end{aligned}
$$

where $F_{ji}\equiv 1.27\Delta m^2_{ji}L/E$ with $\Delta m^2_{ji}\equiv m^2_j - m^2_i$, L stands for the baseline length (in units of km), and E is the neutrino beam energy (in units of GeV). CPT invariance assures that the transition probabilities $P(\nu_\beta\to\nu_\alpha)$ and $P(\overline{\nu_\alpha}\to\overline{\nu_\beta})$ are identical, and they can directly be read off from Eq. (9) through the replacement $J^{ij}_{\alpha\beta}\Rightarrow -J^{ij}_{\alpha\beta}$ (i.e., $V\Rightarrow V^*$). Thus the CP-violating asymmetries between $P(\nu_\alpha\to\nu_\beta)$ and $P(\overline{\nu_\alpha}\to\overline{\nu_\beta})$ are equal to the T-violating asymmetries between $P(\nu_\alpha\to\nu_\beta)$ and $P(\nu_\beta\to\nu_\alpha)$. The latter can be explicitly and compactly expressed as follows:

$$
\begin{aligned}
\Delta P_{\alpha\beta} &\equiv P(\nu_\beta\to\nu_\alpha) - P(\nu_\alpha\to\nu_\beta)\\
&= 16(J^{12}_{\alpha\beta}\sin F_{21}\sin F_{31}\sin F_{32}\\
&\quad + J^{01}_{\alpha\beta}\sin F_{10}\sin F_{30}\sin F_{31}\\
&\quad + J^{02}_{\alpha\beta}\sin F_{20}\sin F_{30}\sin F_{32}). \qquad (10)
\end{aligned}
$$

Equivalently, one may obtain

$$
\begin{aligned}
\Delta P_{\alpha\beta} = 16(&J^{23}_{\alpha\beta}\sin F_{21}\sin F_{31}\sin F_{32}\\
&- J^{02}_{\alpha\beta}\sin F_{10}\sin F_{20}\sin F_{21}\\
&- J^{03}_{\alpha\beta}\sin F_{10}\sin F_{30}\sin F_{31}), \qquad (11)
\end{aligned}
$$

or

$$
\begin{aligned}
\Delta P_{\alpha\beta} = 16(&J^{31}_{\alpha\beta}\sin F_{21}\sin F_{31}\sin F_{32}\\
&+ J^{01}_{\alpha\beta}\sin F_{10}\sin F_{20}\sin F_{21}\\
&- J^{03}_{\alpha\beta}\sin F_{20}\sin F_{30}\sin F_{32}). \qquad (12)
\end{aligned}
$$

In getting Eqs. (10)–(12), the equality $\sin 2F_{ij}+\sin 2F_{jk}+\sin 2F_{ki}=-4\sin F_{ij}\sin F_{jk}\sin F_{ki}$ and Eq. (3) have been used. Only three of the twelve asymmetries $\Delta P_{\alpha\beta}$ are independent, and they probe three of the six CP-violating phases (or their combinations) of V. Since only the transitions between active neutrinos can in practice be measured, we focus our interest on three independent asymmetries of CP and T violation: $\Delta P_{e\mu}$, $\Delta P_{\mu\tau}$ and $\Delta P_{\tau e}$.

The formulas of $\Delta P_{\alpha\beta}$ will remarkably be simplified, if the hierarchy of neutrino mass-squared differences is taken into account. For illustration, we assume that the current data of solar, atmospheric and LSND neutrino oscillations can approximately be described by the well-known $(2+2)$ mixing scheme [3]. In this scheme the solar neutrino problem is attributed essentially to the $\nu_e \to \nu_s$ oscillation ($\Delta m^2_{\rm sun} \approx \Delta m^2_{10}$), the atmospheric neutrino anomaly arises dominantly from the $\nu_\mu \to \nu_\tau$ oscillation ($\Delta m^2_{\rm atm} \approx \Delta m^2_{32}$), and the LSND neutrino oscillation is governed by a bigger mass-squared difference ($\Delta m^2_{\rm LSND} \approx \Delta m^2_{21}$). Without loss of generality, we have taken $0 < m_0 < m_1 < m_2 < m_3$. The observed hierarchy $\Delta m^2_{\rm sun} \ll \Delta m^2_{\rm atm} \ll \Delta m^2_{\rm LSND}$ allows us to make an analytical approximation for Eq. (11):

$$\begin{aligned} \Delta P_{e\mu} &\approx 16(J^{23}_{e\mu}\sin F_{\rm atm} + J^{01}_{e\mu}\sin F_{\rm sun})\sin^2 F_{\rm LSND}, \\ \Delta P_{\mu\tau} &\approx 16(J^{23}_{\mu\tau}\sin F_{\rm atm} + J^{01}_{\mu\tau}\sin F_{\rm sun})\sin^2 F_{\rm LSND}, \\ \Delta P_{\tau e} &\approx 16(J^{23}_{\tau e}\sin F_{\rm atm} + J^{01}_{\tau e}\sin F_{\rm sun})\sin^2 F_{\rm LSND}, \end{aligned} \tag{13}$$

where $(F_{\rm sun}, F_{\rm atm}, F_{\rm LSND}) = 1.27(\Delta m^2_{\rm sun}, \Delta m^2_{\rm atm}, \Delta m^2_{\rm LSND})L/E$. If the magnitude of $J^{01}_{\alpha\beta}$ is comparable with or smaller than that of $J^{23}_{\alpha\beta}$ (for $\alpha,\beta = e,\mu,\tau$), then the asymmetries $\Delta P_{\alpha\beta}$ in Eq. (13) are associated primarily with the oscillating term $\sin F_{\rm atm}\sin^2 F_{\rm LSND}$. If $|J^{01}_{\alpha\beta}| \gg |J^{23}_{\alpha\beta}|$, however, the oscillation induced by $\sin F_{\rm sun}\sin^2 F_{\rm LSND}$ should not be neglected in $\Delta P_{\alpha\beta}$.

To get a feeling of the relative magnitudes of $J^{01}_{\alpha\beta}$ and $J^{23}_{\alpha\beta}$, we consider two special but instructive cases for the mixing angles of V:

(a) $s_{02} \to 0$ and $s_{03} \to 0$. With the help of Eqs. (5)–(7), we arrive at

$$\begin{aligned} J^{01}_{e\mu} &= J^{01}_{\mu\tau} = J^{01}_{\tau e} = 0, \\ J^{23}_{e\mu} &= J^{23}_{\mu\tau} = J^{23}_{\tau e} = c_{12}c^2_{13}c_{23}s_{12}s_{13}s_{23}\sin\delta, \end{aligned} \tag{14}$$

where $\delta \equiv \phi_x - \phi_y + \phi_z = \delta_{13} - \delta_{12} - \delta_{23}$. It turns out that the T-violating asymmetries in Eq. (13) amount to one another and measure a common CP-violating parameter, whose magnitude is identical to the well-known Jarlskog invariant defined in the three-neutrino mixing scheme [9].

(b) $s_{02}, s_{03}, s_{12}, s_{13} \sim \epsilon \ll 1$ [4]. In this more realistic case, we obtain

$$\begin{aligned} J^{01}_{e\mu} \approx\; & c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z) + c_{01}c^2_{23}s_{01}s_{02}s_{12}\sin\phi_z \\ & - c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) \\ & + c_{01}s_{01}s_{03}s_{13}s^2_{23}\sin\phi_y, \\ J^{01}_{\mu\tau} \approx\; & 0, \\ J^{01}_{\tau e} \approx\; & c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z) - c_{01}s_{01}s_{02}s_{12}s^2_{23}\sin\phi_z \\ & - c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) \\ & - c_{01}c^2_{23}s_{01}s_{03}s_{13}\sin\phi_y, \end{aligned} \tag{15}$$

and

$$\begin{aligned} J^{23}_{e\mu} &\approx c_{23}s_{12}s_{13}s_{23}\sin\delta, \\ J^{23}_{\mu\tau} &\approx c_{23}s_{12}s_{13}s_{23}\sin\delta \\ & \quad + c_{23}s_{02}s_{03}s_{23}\sin\phi_x, \\ J^{23}_{\tau e} &\approx c_{23}s_{12}s_{13}s_{23}\sin\delta, \end{aligned} \tag{16}$$

where the corrections of $\mathcal{O}(\epsilon^3)$ or smaller have been neglected. Except $J^{01}_{\mu\tau} \sim 0$, the other five invariants of CP violation in Eqs. (15) and (16) are all suppressed by the factors of $\mathcal{O}(\epsilon^2)$. If $\delta \sim \mathcal{O}(1)$ holds, then the asymmetries $\Delta P_{\alpha\beta}$ in Eq. (13) are associated dominantly with the oscillating term $\sin F_{\rm atm}\sin^2 F_{\rm LSND}$. An interesting feature of $\Delta P_{\mu\tau}$ in case (b) is that it depends primarily upon $J^{23}_{\mu\tau}$, whose magnitude gets comparable contributions from the $\sin\delta$ and $\sin\phi_x$ terms. Therefore these two CP-violating phases could in principle be determined from the measurements of $\Delta P_{e\mu}$ and $\Delta P_{\mu\tau}$.

In practice, however, the matter effects on neutrino mixing parameters and neutrino oscillations must be taken into account. To express the pattern of neutrino oscillations in matter in the same form as that in vacuum, one may define the effective neutrino masses $\tilde{m}_i$ ($i=0,1,2,3$) and the effective lepton flavor mixing matrix $\tilde{V}$, in which the matter effects are already included [13]. Then the matter-corrected conversion probability of a neutrino ν_α to another neutrino ν_β can be written out in analogy to Eq. (9); and the counterpart of the T-violating asymmetry $\Delta P_{\alpha\beta}$ in matter is given, for instance, as

$$\begin{aligned} \Delta\tilde{P}_{\alpha\beta} = 16(& \tilde{J}^{23}_{\alpha\beta}\sin\tilde{F}_{21}\sin\tilde{F}_{31}\sin\tilde{F}_{32} - \tilde{J}^{02}_{\alpha\beta}\sin\tilde{F}_{10}\sin\tilde{F}_{20}\sin\tilde{F}_{21} \\ & - \tilde{J}^{03}_{\alpha\beta}\sin\tilde{F}_{10}\sin\tilde{F}_{30}\sin\tilde{F}_{31}), \end{aligned} \tag{17}$$

where $\tilde{F}_{ji} \equiv 1.27\Delta\tilde{m}^2_{ji}L/E$, $\Delta\tilde{m}^2_{ji} \equiv \tilde{m}^2_j - \tilde{m}^2_i$, and $\tilde{J}^{ij}_{\alpha\beta} \equiv {\rm Im}(\tilde{V}_{\alpha i}\tilde{V}_{\beta j}\tilde{V}^*_{\alpha j}\tilde{V}^*_{\beta i})$. Note that $\Delta\tilde{m}^2_{ji}$ and $\tilde{J}^{ij}_{\alpha\beta}$ depend upon the matter parameters $a = \sqrt{2}G_{\rm F}N_e$ and $a' = \sqrt{2}G_{\rm F}N_n/2$, where N_e and N_n denote the respective background densities of electrons and neutrons [14]. It seems very difficult, if not impossible, to work out the analytically exact expressions of $\Delta\tilde{m}^2_{ji}$ and $\tilde{J}^{ij}_{\alpha\beta}$ in terms of a and a' as well as the neutrino mixing parameters in vacuum. Nevertheless, a relationship between $J^{ij}_{\alpha\beta}$ and $\tilde{J}^{ij}_{\alpha\beta}$ can be derived from the equality between the commutator of lepton mass matrices in vacuum and that in matter [7]. The result is

$$\Delta\tilde{m}^2_{10}\Delta\tilde{m}^2_{20}\Delta\tilde{m}^2_{30}\sum_{i=1}^{3}(\tilde{J}^{0i}_{\alpha\beta}|\tilde{V}_{\gamma i}|^2+\tilde{J}^{0i}_{\beta\gamma}|\tilde{V}_{\alpha i}|^2+\tilde{J}^{0i}_{\gamma\alpha}|\tilde{V}_{\beta i}|^2)$$
$$+\sum_{i=1}^{3}\sum_{j=1}^{3}[\Delta\tilde{m}^2_{i0}(\Delta\tilde{m}^2_{j0})^2(\tilde{J}^{ij}_{\alpha\beta}|\tilde{V}_{\gamma j}|^2+\tilde{J}^{ij}_{\beta\gamma}|\tilde{V}_{\alpha j}|^2+\tilde{J}^{ij}_{\gamma\alpha}|\tilde{V}_{\beta j}|^2)]$$
$$=\Delta m^2_{10}\Delta m^2_{20}\Delta m^2_{30}\sum_{i=1}^{3}(J^{0i}_{\alpha\beta}|V_{\gamma i}|^2+J^{0i}_{\beta\gamma}|V_{\alpha i}|^2+J^{0i}_{\gamma\alpha}|V_{\beta i}|^2)+\sum_{i=1}^{3}\sum_{j=1}^{3}[\Delta m^2_{i0}(\Delta m^2_{j0})^2(J^{ij}_{\alpha\beta}|V_{\gamma j}|^2+J^{ij}_{\beta\gamma}|V_{\alpha j}|^2+J^{ij}_{\gamma\alpha}|V_{\beta j}|^2)], \tag{18}$$

where (i,j,k) and (α,β,γ) run over (1,2,3) and (e,μ,τ), respectively. Parametrizing $\tilde{V}$ in terms of six effective mixing angles $\tilde{\theta}_{ij}$ and six effective *CP*-violating phases $\tilde{\delta}_{ij}$ (for $i,j=0,1,2,3$ and $i<j$) in analogy to Eq. (4), one may obtain the explicit expressions of $\tilde{J}^{ij}_{\alpha\beta}$ from Eqs. (5)–(7) and rewrite Eq. (18). If the mass of the sterile neutrino and its mixing with active neutrinos are "switched off" (i.e., $a'=0$, $m_0=0$, $\theta_{01}=\theta_{02}=\theta_{03}=0$, and $\delta_{01}=\delta_{02}=\delta_{03}=0$), then the analytically exact relations between the fundamental parameters in vacuum (m_i, θ_{12}, θ_{13}, θ_{23}, δ) and their counterparts in matter ($\tilde{m}_i$, $\tilde{\theta}_{12}$, $\tilde{\theta}_{13}$, $\tilde{\theta}_{23}$, $\tilde{\delta}$) can easily be obtained [13]. In this case, Eq. (18) will be simplified to an elegant form, the so-called Naumov identity [15].

In summary, we have calculated the rephasing invariants of *CP* and *T* violation in a favorable parametrization of the 4×4 lepton flavor mixing matrix and derived their relations with the *CP* and *T*-violating asymmetries in neutrino oscillations. The matter effects have been discussed to a limited extent. Our results are expected to be useful for a systematic and model-independent analysis of *CP* and *T* violation in the four-neutrino mixing scheme, in particular, when sufficient data become available from the forthcoming long-baseline neutrino oscillation experiments.

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