# Quadrupole moments of baryons

A. J. Buchmann*
*Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-72076 Tübingen, Germany*

E. M. Henley†
*Department of Physics and Institute for Nuclear Theory, Box 351560, University of Washington, Seattle, Washington 98195*
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Quadrupole moments of decuplet baryons and the octet-decuplet transition quadrupole moments are calculated using Morpurgo's general QCD parametrization method. Certain relations among the decuplet and the octet to decuplet transition quadrupole moments are derived. These can be used to predict the $\Delta$ quadrupole moments which are difficult to measure.

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## I. INTRODUCTION

We use Morpurgo's QCD parametrization [1] to calculate all baryon octet-decuplet transition and decuplet quadrupole moments and to derive certain relations between them. The chosen method makes it clear from the outset that the results obtained do not depend on the particular quark-quark interaction model. We compare our relations with those obtained in a group theoretical analysis. Finally, we provide numerical estimates that can be compared with experiment.

Unfortunately, quadrupole moments of the decuplet baryons are still unknown. They are important for providing evidence for baryon nonsphericity. We believe that the electric quadrupole moment of the $\Omega^-$, as well as the octet-decuplet transition quadrupole moments, are amenable to measurement and we discuss some possible techniques in the summary.

## II. MORPURGO'S GENERAL PARAMETRIZATION METHOD FOR QUADRUPOLE MOMENTS

The general parametrization (GP) method, developed by Morpurgo, is based on the symmetries and the quark-gluon dynamics of the underlying field theory of quantum chromodynamics (QCD). Although noncovariant in appearance, all invariants that are allowed by Lorentz invariance are included in the operator basis (see below).

The basic idea is to *formally* define, for the observable at hand, a QCD operator $\Omega$ and QCD eigenstates $|B\rangle$ expressed explicitly in terms of quarks and gluons. The corresponding matrix element can, with the help of the unitary operator $V$, be reduced to an evaluation in the basis of auxiliary three-quark states $|\Phi_B\rangle$

$$\langle B|\Omega|B\rangle = \langle \Phi_B|V^\dagger \Omega V|\Phi_B\rangle = \langle W_B|\mathcal{O}|W_B\rangle. \tag{1}$$

The auxiliary states $|\Phi_B\rangle$ are pure three-quark states with orbital angular momentum $L=0$. The spin-flavor wave functions [2] contained in $|\Phi_B\rangle$ are denoted by $|W_B\rangle$. The operator $V$ dresses the pure three-quark states with $q\bar{q}$ components and gluons and thereby generates the exact QCD eigenstates $|B\rangle$. Furthermore, it is implied that $V$ contains a Foldy-Wouthuysen transformation allowing the auxiliary states to be written in terms of Pauli spinors.

One then writes the most general expression for the operator $\mathcal{O}$, in the present case for the electric quadrupole operator $\mathcal{Q}$, that is compatible with the space-time and inner QCD symmetries. The orbital and color space matrix elements[1] are absorbed into *a priori* unknown parameters, called $B$ and $C$, multiplying the spin-flavor invariants appearing in the expansion of $\mathcal{O}$. The method has been used to calculate various properties of baryons and mesons [1,3].

The electric quadrupole operator is composed of a two- and three-body term in spin-flavor space

$$\mathcal{Q} = B\sum_{i\neq j}^{3} e_i(3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j) + C\sum_{i\neq j\neq k}^{3} e_k(3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), \tag{2}$$

where $e_i=(1+3\tau_{iz})/6$ is the charge of the $i$th quark. More general operators containing second and third powers of the quark charge are conceivable [4] but are not considered here. Their contribution is suppressed by factors of $e^2/4\pi = 1/137$. The $z$-component of the Pauli spin (isospin) matrix $\boldsymbol{\sigma}_i$ ($\boldsymbol{\tau}_i$) is denoted by $\sigma_{iz}$ ($\tau_{iz}$).

Decuplet baryon quadrupole moments $Q_{B*}$ and octet-decuplet transition quadrupole moments $Q_{B\to B*}$ are obtained by calculating the matrix elements of the quadrupole operator in Eq. (2) between the three-quark spin-flavor wave functions $|W_B\rangle$

$$Q_{B*} = \langle W_{B*}|\mathcal{Q}|W_{B*}\rangle,$$

$$Q_{B\to B*} = \langle W_{B*}|\mathcal{Q}|W_B\rangle, \tag{3}$$

*Email address: alfons.buchmann@uni-tuebingen.de
†Email address: henley@phys.washington.edu

[1] Note that on the right-hand side of the last equality in Eq. (1) the integration over spatial and color degrees of freedom has been performed.

 

TABLE I. Two-quark ($B$) and three-quark ($C$) contributions to quadrupole moments of decuplet baryons in the SU(3) symmetry limit ($r=1$) and with broken flavor symmetry. SU(3)-flavor symmetry breaking is characterized by the ratio of $u$-quark and $s$-quark masses $r=m_u/m_s$. Two types (quadratic and cubic) of flavor symmetry breaking are considered.

| | $Q(r=1)$ | $Q$(quadratic) | $Q$(cubic) |
|---|---|---|---|
| $\Delta^-$ | $-4B-2C$ | $-4B-2C$ | $-4B-2C$ |
| $\Delta^0$ | 0 | 0 | 0 |
| $\Delta^+$ | $4B+2C$ | $4B+2C$ | $4B+2C$ |
| $\Delta^{++}$ | $8B+4C$ | $8B+4C$ | $8B+4C$ |
| $\Sigma^{*-}$ | $-4B-2C$ | $-(4B+2C)(1+2r)/3$ | $-(4B+2C)(1+r+r^2)/3$ |
| $\Sigma^{*0}$ | 0 | $2(B-C)(1-r)/3$ | $[2B(1+r-2r^2)-C(2-r-r^2)]/3$ |
| $\Sigma^{*+}$ | $4B+2C$ | $[4B(2+r)-2C(1-4r)]/3$ | $[4B(2+2r-r^2)-2C(1-2r-2r^2)]/3$ |
| $\Xi^{*-}$ | $-4B-2C$ | $-(4B+2C)(2r+r^2)/3$ | $-(4B+2C)(r+r^2+r^3)/3$ |
| $\Xi^{*0}$ | 0 | $4(B-C)(r-r^2)/3$ | $[4B(2r-r^2-r^3)-2C(r+r^2-2r^3)]/3$ |
| $\Omega^-$ | $-4B-2C$ | $-(4B+2C)r^2$ | $-(4B+2C)r^3$ |

where $B$ denotes a spin 1/2 octet baryon and $B^*$ a member of the spin 3/2 baryon decuplet.

### A. Missing one-quark quadrupole operator

In applications of the GP, a hierachy in the importance of one-, two-, and three-body operators is often found. One-body operators usually give a larger contribution to the matrix element than two-body operators, and two-body operators are usually more important than three-body operators [1]. This hierachy results from the additional gluon exchanges needed to generate two-quark and three-quark operators. The quark-gluon coupling $\alpha_s=g^2/4\pi$ is such that diagrams involving higher powers of $g$ are suppressed. In the GP method this is regarded as an empirical fact; in QCD with a large number ($N_c$) of colors this is because $g^2$ is inversely proportional to $N_c$, i.e., $g\propto 1/\sqrt{N_c}$, and diagrams involving higher powers of $g$ are suppressed.

However, for quadrupole moments, one-body operators containing rank 2 spherical harmonics in orbital space, e.g., $e_i Y^2(\hat{\mathbf{x}})$, do not contribute, because the GP method employs only $L=0$ wave functions. In Morpurgo's formulation, any possible $D$-state admixture in the QCD states $|B\rangle$ is moved from the wave function to the effective two- and three-quark operators $\mathcal{Q}$ acting in spin-flavor space. Even if $D$-waves were included in the Hilbert space, the orbital one-body contribution would be small due to the small $D$-state probability in the nucleon and $\Delta$ wave functions [5]. In any case, for quadrupole moments, we are left with two- and three-quark operators.

### B. Two- and three-quark quadrupole operators

The two- and three-body operators in Eq. (2) act in spin-flavor space. Although they formally operate on valence quark states, they are mainly a reflection of the $q\bar{q}$ and gluon degrees of freedom that have been eliminated from the Hilbert space, and which reappear as a quadrupole tensor in spin space [6,7]. As spin tensors of rank 2, they can induce spin 1/2→3/2 and 3/2→3/2 transitions.

Evaluating Eq. (2) between, e.g., $N$ and $\Delta$ spin-isospin wave functions leads to the following results for the $\Delta$ and the $N\to\Delta$ quadrupole moments

$$Q_{\Delta^+}=\langle W_{\Delta^+}|\mathcal{Q}_{[2]}+\mathcal{Q}_{[3]}|W_{\Delta^+}\rangle=4B+2C,$$

$$Q_{p\to\Delta^+}=\langle W_{\Delta^+}|\mathcal{Q}_{[2]}+\mathcal{Q}_{[3]}|W_p\rangle=2\sqrt{2}B-2\sqrt{2}C. \tag{4}$$

Similarly, the electric quadrupole moments for the other decuplet baryons and the octet-decuplet transition moments are calculated and listed in Tables I and II. In this way Morpurgo's method yields an efficient parameterization of baryon quadrupole moments in terms of few unknown parameters.

### C. Determination of the GP constants

In order to determine the two constants $B$ and $C$ we need two experimental inputs. From recent measurements of the ratio of electric quadrupole over magnetic dipole amplitudes in electromagnetic pionproduction ($E2/M1$ and $C2/M1$ ratios) [8,9], one can extract the $N\to\Delta$ transition quadrupole moment $Q_{p\to\Delta^+}$. For a first determination of $Q_{\Delta^+}$ from photo-pionproduction data in the $\Delta$ resonance region see Ref. [8]. Because the decuplet quadrupole moments or other octet-decuplet transition quadrupole moments are not yet very well known, we cannot fix the smaller constant $C$ with sufficient accuracy at this stage. Therefore, we assume $C\approx 0$ for the numerical evaluation. Our assumption that three-body ($C$) terms in the charge operator are smaller than two-body ($B$) terms is supported by work using the GP [3] and the $1/N_c$ expansion [4] methods. In both methods, $|C/B|$ is estimated to be at most 0.3.

We take the following approach in determining the constant $B$. In a quark model with exchange currents, it was found that the $N\to\Delta$ and $\Delta$ quadrupole moments receive the

TABLE II. Two-quark ($B$) and three-quark ($C$) contributions to the octet-decuplet transition quadrupole moments in the SU(3) symmetry limit ($r=1$) and with broken flavor symmetry. SU(3)-flavor symmetry breaking is characterized by the ratio of $u$-quark and $s$-quark masses $r=m_u/m_s$. Two types (quadratic and cubic) of flavor symmetry breaking are considered.

| | $Q(r=1)$ | $Q$(quadratic) | $Q$(cubic) |
|---|---|---|---|
| $p\rightarrow\Delta^+$ | $2\sqrt{2}(B-C)$ | $2\sqrt{2}(B-C)$ | $2\sqrt{2}(B-C)$ |
| $n\rightarrow\Delta^0$ | $2\sqrt{2}(B-C)$ | $2\sqrt{2}(B-C)$ | $2\sqrt{2}(B-C)$ |
| $\Sigma^-\rightarrow\Sigma^{*-}$ | 0 | $-2\sqrt{2}(2B+C)(1-r)/3$ | $-\sqrt{2}(2B+C)(2-r-r^2)/3$ |
| $\Sigma^0\rightarrow\Sigma^{*0}$ | $2\sqrt{2}(B-C)$ | $\sqrt{2}(B-C)(2+r)/3$ | $\sqrt{2}[2B(2-r+2r^2)-C(4+r+r^2)]/6$ |
| $\Lambda^0\rightarrow\Sigma^{*0}$ | $\sqrt{6}(B-C)$ | $\sqrt{6}(B-C)r$ | $\sqrt{6}[2Br-C(r+r^2)]/2$ |
| $\Sigma^+\rightarrow\Sigma^{*+}$ | $2\sqrt{2}(B-C)$ | $2\sqrt{2}[B(4-r)-C(1+2r)]/3$ | $2\sqrt{2}[B(4-2r+r^2)-C(1+r+r^2)]/3$ |
| $\Xi^-\rightarrow\Xi^{*-}$ | 0 | $-2\sqrt{2}(2B+C)(r-r^2)/3$ | $-\sqrt{2}(2B+C)(r+r^2-2r^3)/3$ |
| $\Xi^0\rightarrow\Xi^{*0}$ | $2\sqrt{2}(B-C)$ | $2\sqrt{2}(B-C)(r+2r^2)/3$ | $\sqrt{2}[2B(2r-r^2+2r^3)-C(r+r^2+4r^3)]/3$ |

largest contribution from two-body $q\bar{q}$ terms in the charge operator. The following relations between the neutron charge radius $r_n^2$, the $\Delta^+$, and the $p\rightarrow\Delta^+$ quadrupole moments were obtained [7]

$$\sqrt{2}Q_{p\rightarrow\Delta^+}=Q_{\Delta^+}=r_n^2. \tag{5}$$

The reasons for the existence of these quark model relations are: (i) one-quark and three-quark operators are suppressed for $r_n^2$, $Q_{\Delta^+}$, and $Q_{p\rightarrow\Delta^+}$, (ii) these observables are dominated by the same two-body charge operator $\rho_{[2]}$, (iii) there is a definite relation between the monopole ($C0$) term in $\rho_{[2]}$, which is responsible for the nonvanishing neutron charge radius, and the quadrupole ($C2$) term in $\rho_{[2]}$ that produces a nonzero quadrupole moment of the $\Delta^+$. In other words, all three observables are dominated by the cloud of $q\bar{q}$ pairs—effectively described by the two-quark exchange currents—and as a consequence are simply related.

Similar relations between $r_n^2$, $Q_{\Delta^+}$, and $Q_{p\rightarrow\Delta^+}$ were obtained in the pion cloud model [6] where the $q\bar{q}$ degrees of freedom enter in terms of an explicit pion contribution to the baryon wave functions, and in previous quark model calculations [10].

From Eq. (4) it is clear that the quark model relation between $Q_{\Delta^+}$ and $Q_{p\rightarrow\Delta^+}$ can only be exact if $C=0$. This is not the case. Nevertheless, it is well satisfied by the experimental neutron charge radius and the $p\rightarrow\Delta^+$ quadrupole moment extracted from the electromagnetic pionproduction data [8,9,11]. This provides some experimental evidence for the smallness of the constant $C$. By comparing Eq. (4) for the case $C=0$ with Eq. (5), we obtain

$$B=r_n^2/4.$$

## III. RESULTS FOR SPECTROSCOPIC QUADRUPOLE MOMENTS

In this section we present our results for baryon quadrupole moments and interpret them in terms of the higher SU(6) spin-flavor symmetry group and its SU(3) flavor and SU(2) isospin subgroups. The spin-flavor symmetry combines the spin 1/2 flavor octet baryons and the spin 3/2 flavor decuplet baryons into the symmetric **56** dimensional representation of the SU(6) spin-flavor group. If the spin-flavor symmetry was exact, octet and decuplet masses would be equal, the charge radii of neutral baryons would be zero, and all baryon quadrupole moments would vanish. In particular, $M_{\Delta^+}=M_p$, $r_{\Delta^0}^2=r_n^2=0$, and $Q_{\Delta^+}=Q_{p\rightarrow\Delta^+}=0$ [12].

### A. SU(6) spin-flavor symmetry breaking

SU(6) symmetry is only approximately realized in nature. It is broken by spin-dependent terms in the strong interaction Hamiltonian. Their presence explains why decuplet baryons are heavier than their octet member counterparts with the same strangeness. Likewise, it is broken by the spin-dependent electric quadrupole operators in Eq. (2). These have different matrix elements for spin 1/2 octet and spin 3/2 decuplet baryons, and give rise to nonzero quadrupole moments for decuplet baryons.

In the first column ($r=1$) of Table I we show our results for the decuplet quadrupole moments in terms of the GP constants $B$ and $C$ describing the contribution of two- and three-quark operators, assuming that SU(3)-flavor symmetry is exact. Table II lists the corresponding expressions for the octet-decuplet quadrupole transition moments. We observe that the decuplet quadrupole moments are proportional to their charge, and that the octet-decuplet transition moments between the negatively charged baryons are zero. Both results follow from the assumed flavor symmetry of the strong interaction.

### B. SU(3) flavor symmetry breaking

In order to get an idea of the degree of SU(3) flavor symmetry breaking induced by the electromagnetic transition operator, we replace the spin-spin terms in Eq. (2) by expressions with a ''quadratic'' quark mass dependence

$$\sigma_i\sigma_j\rightarrow\sigma_i\sigma_j m_u^2/(m_i m_j)$$

as obtained from a one-gluon exchange interaction between the quarks. Flavor symmetry breaking is then characterized by the ratio $r=m_u/m_s$ of $u$ and $s$ quark masses, which is a known number. We use the same mass for $u$ and $d$ quarks to preserve the SU(2) isospin symmetry of the strong interaction, that is known to hold to a very good accuracy.

For comparison, we also use a flavor symmetry breaking of ‘‘cubic’’ quark mass dependence

$$\sigma_i\sigma_j \rightarrow \sigma_i\sigma_j m_u^3/(m_i^2 m_j),$$

that follows from the two-body gluon exchange charge density [13]. This leads to expressions for $Q_{B*}$ and $Q_{B\rightarrow B*}$ containing terms up to third order in $r$. No additional parameters are introduced in this way.

We emphasize that this treatment is not exact. The GP method of including SU(3) symmetry breaking is to introduce additional operators and parameters, which guarantees that flavor symmetry breaking is incorporated to all orders [14]. There are then so many undetermined constants that the theory can no longer make predictions. We expect that our approximate treatment includes the most important physical effect.

In the second and third columns of Tables I and II we present the analytic expressions for the decuplet and the octet-decuplet transition quadrupole moments with quadratic or cubic type of flavor symmetry breaking taken into account.

### C. Relations among quadrupole moments

Even though the SU(6) and SU(3) symmetries are broken, there exist—as a consequence of the underlying unitary symmetries—certain relations among the quadrupole moments. A relation is the stronger the weaker the assumptions required for its derivation. We are therefore interested in those relations that hold even when SU(3) symmetry breaking is included in the charge quadrupole operator. These are the ones, which are most likely satisfied in nature. The 18 quadrupole moments (10 diagonal decuplet and 8 decuplet-octet transition quadrupole moments) are expressed in terms of only two constants $B$ and $C$. Therefore, there must be 16 relations between them. Given the analytical expressions in Tables I and II, it is straightforward to verify that the following relations hold

$$0=Q_{\Delta^-}+Q_{\Delta^+}, \tag{6a}$$

$$0=Q_{\Delta^0}, \tag{6b}$$

$$0=2Q_{\Delta^-}+Q_{\Delta^{++}}, \tag{6c}$$

$$0=Q_{\Sigma*^-}-2Q_{\Sigma*^0}+Q_{\Sigma*^+}, \tag{6d}$$

$$0=3(Q_{\Xi*^-}-Q_{\Sigma*^-})-(Q_{\Omega^-}-Q_{\Delta^-}), \tag{6e}$$

$$0=Q_{p\rightarrow\Delta^+}-Q_{n\rightarrow\Delta^0}, \tag{6f}$$

$$0=Q_{\Sigma^-\rightarrow\Sigma*^-}-2Q_{\Sigma^0\rightarrow\Sigma*^0}+Q_{\Sigma^+\rightarrow\Sigma*^+}, \tag{6g}$$

$$0=Q_{\Delta^-}-Q_{\Sigma*^-}-\sqrt{2}Q_{\Sigma^-\rightarrow\Sigma*^-}, \tag{6h}$$

$$0=Q_{\Delta^+}-Q_{\Sigma*^+}+\sqrt{2}Q_{p\rightarrow\Delta^+}-\sqrt{2}Q_{\Sigma^+\rightarrow\Sigma*^+}, \tag{6i}$$

$$0=Q_{\Sigma*^0}-\frac{1}{\sqrt{2}}Q_{\Sigma^0\rightarrow\Sigma*^0}+\frac{1}{\sqrt{6}}Q_{\Lambda^0\rightarrow\Sigma*^0}, \tag{6j}$$

$$0=Q_{\Sigma*^-}-Q_{\Xi*^-}-\frac{1}{\sqrt{2}}Q_{\Xi^-\rightarrow\Xi*^-}-\frac{1}{\sqrt{2}}Q_{\Sigma^-\Sigma*^-}, \tag{6k}$$

$$0=Q_{\Xi*^0}+\frac{1}{\sqrt{2}}Q_{\Xi^0\rightarrow\Xi*^0}-\sqrt{\frac{2}{3}}Q_{\Lambda^0\rightarrow\Sigma*^0}. \tag{6l}$$

These twelve combinations of quadrupole moments do not depend on the flavor symmetry breaking parameter $r$. In fact, Eqs. (6a)–(6d) are already a consequence of the assumed SU(2) isospin symmetry of the strong interaction, and hold irrespective of the order of SU(3) symmetry breaking. Equation (6e) is the quadrupole moment counterpart of the ‘‘equal spacing rule’’ for decuplet masses. The latter was obtained from considering SU(3) invariance of the strong interaction with a second order symmetry breaking perturbation [15]. The remaining relations connect states in the octet and decuplet, and the assumption of SU(6) symmetric spin-flavor wave functions $|W_B\rangle$ in the auxiliary states is needed to derive them.

There are also four $r$-dependent relations,[2] which can be chosen as

$$0=\frac{1}{3}(2r+1)Q_{\Delta^+}+Q_{\Sigma*^-}, \tag{7a}$$

$$0=\frac{1}{6}\sqrt{2}(r-1)Q_{p\rightarrow\Delta^+}+Q_{\Sigma*^0}, \tag{7b}$$

$$0=\sqrt{2}r^2Q_{p\rightarrow\Delta^+}-\sqrt{2}Q_{\Xi^0\rightarrow\Xi*^0}+Q_{\Xi*^0}, \tag{7c}$$

$$0=r^2Q_{\Delta^-}-Q_{\Omega^-}. \tag{7d}$$

With the ‘‘cubic’’ SU(3) symmetry breaking, we obtain the same relations as in Eqs. (6a)–(6l) with the exception of Eqs. (6j) and (6l) involving the neutral baryons [16]. These no longer hold independently but their sum is again a valid relation. There are now five $r$-dependent relations which can be chosen as

$$0=\frac{1}{3}(1+r+r^2)Q_{\Delta^+}+Q_{\Sigma*^-}, \tag{8a}$$

$$0=(r-r^2)Q_{\Delta^+}-\sqrt{2}(2+r^2)Q_{p\rightarrow\Delta^+} +6\sqrt{2}Q_{\Sigma^0\rightarrow\Sigma*^0}, \tag{8b}$$

[2]The parameter combinations $Br$, $Cr$, $Br^2$, and $Cr^2$, when expressed in terms of quadrupole moments lead to four $r$-dependent relations among the quadrupole moments. We thank R. Lebed for pointing out the proper number of $r$-dependent relations.

$$0 = rQ_{\Sigma^{*-}} - Q_{\Xi^{*-}}, \tag{8c}$$

$$0 = (r - r^2)Q_{\Delta^+} + \sqrt{2}(r + r^3)Q_{p\to\Delta^+} - 3\sqrt{2}Q_{\Xi^0\to\Xi^{*0}}, \tag{8d}$$

$$0 = r^3 Q_{\Delta^-} - Q_{\Omega^-}. \tag{8e}$$

Other combinations of the expressions in Tables I and II can be written down if desirable.

### D. Comparison with Lebed's quadrupole moment relations

It is interesting to compare our results with a pure SU(6) symmetry analysis of baryon quadrupole moments [17]. After decomposing the product $\overline{\mathbf{56}} \otimes \mathbf{56} = \mathbf{3136} = \mathbf{1} \oplus \mathbf{35} \oplus \mathbf{405} \oplus \mathbf{2695}$ into its irreducible representations, the most general quadrupole operator $\mathcal{Q}$ is expressed in terms of operators transforming according to the **405** and **2695** dimensional representations of SU(6) spin-flavor symmetry:

$$\mathcal{Q} = \mathcal{Q}_{\mathbf{405}} + \mathcal{Q}_{\mathbf{2695}}. \tag{9}$$

The **1** and **35** dimensional representations, which contain only zero-body (constants) and one-body operators do not contribute for an angular momentum $J=2$ operator such as $\mathcal{Q}$. Lebed's twelve quadrupole moment relations [17] were derived by neglecting the $Q_{\mathbf{2695}}$ terms. The latter give numerically small matrix elements because they require a product of three SU(6) symmetry breaking operators, and are therefore suppressed. Omitting the $Q_{\mathbf{2695}}$ operators amounts to neglecting the three-quark terms $Q_{[3]}$ in the present approach.

Our results in Tables I and II with only two-quark (B) terms retained satisfy all twelve Lebed relations independent of whether and how SU(3)-flavor symmetry is broken [18]. This shows that our analytic expressions for the electric quadrupole moments are compatible with a rigorous group theoretical approach.

When we include three-body ($C$) operators but no SU(3) symmetry breaking, ten of his relations are still satisfied, whereas the (8,0), (8,1)[3] relations are violated. This suggests that our three-body operators transform as flavor octets in the limit $r=1$.

Finally, if three-body operators are considered and SU(3) symmetry is explicitly broken, only the (64,3), (64,2), (35,2), and (27,2) relations with isospin $I \geqslant 2$ are satisfied. These can be obtained from linear combinations of our Eqs. (6a)–(6g). The restriction to quadrupole operators that are linear in the quark charge implies that one has only $I=0$ and $I=1$ operators, which cannot affect $I \geqslant 2$ combinations. Lebed's relations involve more quadrupole moments than our relations because isospin symmetry is not assumed in Ref. [17].

[3]The first number stands for the dimension of the irreducible SU(3)-flavor representation, and the second for the isospin $I$ and the dimension $2I+1$ of the particular SU(2)-isospin representation involved.

TABLE III. Numerical values for the quadrupole moments of decuplet baryons in units of (fm$^2$) according to the analytic expressions in Table I with $B = r_n^2/4$ and $C=0$. The experimental neutron charge radius [19], $r_n^2 = -0.113(3)$ fm$^2$, and the SU(3) symmetry breaking parameter [20], $r=0.6$, are used as input values.

| | $Q(r=1)$ | $Q$(quadratic) | $Q$(cubic) |
|---|---|---|---|
| $\Delta^-$ | 0.113 | 0.113 | 0.113 |
| $\Delta^0$ | 0 | 0 | 0 |
| $\Delta^+$ | −0.113 | −0.113 | −0.113 |
| $\Delta^{++}$ | −0.226 | −0.226 | −0.226 |
| $\Sigma^{*-}$ | 0.113 | 0.083 | 0.074 |
| $\Sigma^{*0}$ | 0 | −0.008 | −0.017 |
| $\Sigma^{*+}$ | −0.113 | −0.105 | −0.107 |
| $\Xi^{*-}$ | 0.113 | 0.059 | 0.044 |
| $\Xi^{*0}$ | 0 | −0.009 | −0.023 |
| $\Omega^-$ | 0.113 | 0.041 | 0.024 |

### E. Numerical results

After neglecting three-body operators ($C=0$), one can express the 18 quadrupole moments in terms of only one constant, $B$, which we have determined from the empirical neutron charge radius $r_n^2 = -0.113(3)$ fm$^2$ [19]. Numerical values are listed in Tables III and IV for the cases without ($r=1$) and with ($r=0.6$) flavor symmetry breaking. The electric quadrupole moments of the charged baryons are of the same order of magnitude as $r_n^2$, while those of the neutral baryons are considerably smaller. We expect that the inclusion of three-quark operators will not change the sign and the order of magnitude of the numerical results obtained.

With the cubic type of SU(3) symmetry breaking we obtain similar numerical values. For the quadrupole moments of most charged baryons and all transition quadrupole moments differences between both types of symmetry breaking are of the order of 20%. Larger differences occur for $Q_{\Sigma^{*0}}$,

TABLE IV. Numerical values for the octet-decuplet transition quadrupole moments in units of (fm$^2$) according to the analytic expressions in Table II with $B = r_n^2/4$ and $C=0$. The experimental neutron charge radius [19], $r_n^2 = -0.113(3)$ fm$^2$, and the SU(3) symmetry breaking parameter [20], $r=0.6$, are used as input values. The experimental $N\to\Delta$ transition quadrupole moments extracted from the measured $E2/M1$ ratios $Q^{exp}_{p\to\Delta^+} = -0.105(16)$ fm$^2$ [8], and $Q^{exp}_{p\to\Delta^+} = -0.085(13)$ fm$^2$ [9] agree well with our results using $C=0$.

| | $Q(r=1)$ | $Q$(quadratic) | $Q$(cubic) |
|---|---|---|---|
| $p\to\Delta^+$ | −0.080 | −0.080 | −0.080 |
| $n\to\Delta^0$ | −0.080 | −0.080 | −0.080 |
| $\Sigma^-\to\Sigma^{*-}$ | 0 | 0.021 | 0.028 |
| $\Sigma^0\to\Sigma^{*0}$ | −0.080 | −0.035 | −0.028 |
| $\Lambda^0\to\Sigma^{*0}$ | −0.069 | −0.042 | −0.042 |
| $\Sigma^+\to\Sigma^{*+}$ | −0.080 | −0.090 | −0.084 |
| $\Xi^-\to\Xi^{*-}$ | 0 | 0.013 | 0.014 |
| $\Xi^0\to\Xi^{*0}$ | −0.080 | −0.035 | −0.034 |

$Q_{\Xi^{*0}}$, and $Q_{\Omega^-}$. Here, the details of how flavor symmetry is broken are important. For example, from Eq. (8e) we obtain $Q_{\Omega^-}=0.024\ \mathrm{fm}^2$ compared to $Q_{\Omega^-}=0.041\ \mathrm{fm}^2$ following from Eq. (7d). Nevertheless, if a quadrupole moment of this order of magnitude is measured, one could distinguish between various theoretical approaches [21].

In the SU(3) limit, the quadrupole moments of the neutral baryons are exactly zero. In addition, the transition moments involving the negatively charged baryons are zero, because of *U*-spin conservation, which forbids such transitions if flavor symmetry is exact [22]. Furthermore, the sum of all decuplet quadrupole moments is zero in this limit.

## IV. SUMMARY

Morpurgo's general QCD parametrization method is used to relate the spectroscopic quadrupole moments of all members of the baryon decuplet and the corresponding octet-decuplet transition quadrupole moments. Our analysis includes two- and three-quark currents. Because the method relies mainly on the symmetries of QCD, our predictions are to a large extent model independent. The model dependence resides in our approximate treatment of flavor-symmetry breaking. A more rigorous approach will mainly affect the numerically small quadrupole moments.

We have found eleven relations between the quadrupole moments that are independent of the way SU(3)-flavor symmetry is broken. Among them are four relations, which follow already from the assumption of SU(2) isospin symmetry of the strong interaction and the linearity of the electric quadrupole moment operator in the quark charge. These should be very well satisfied in nature.

We have compared our results with a purely group theoretical analysis of quadrupole moments by Lebed. Our theory reproduces all Lebed relations. In addition, we find that some of his relations are more general, and hold even with certain types of three-quark operators included.

Measurements of baryon quadrupole moments are difficult but within reach. They are important for determining the geometric shape of baryons. For example, the $\Omega^-$ baryon lives sufficiently long to observe the x rays emitted when it is captured into an outer Bohr orbit of a heavy nucleus and cascades down to a state with a lower principal quantum number [23]. The x ray frequency depends on whether the charge distribution of the $\Omega^-$ is spherically symmetric or deformed. If it is deformed, the additional interaction energy between its quadrupole moment and the electric field gradient of the nucleus will lead to a frequency shift proportional to $Q_{\Omega^-}$. It has also been suggested to measure the degree of longitudinal spin polarization of an initially transversely polarized $\Omega^-$ beam when passing through a crystal [24]. The interaction between the crystal electric field and the $\Omega^-$ quadrupole moment results in a longitudinal polarization, which is again proportional to $Q_{\Omega^-}$.

In the Primakoff reaction $Y+Pb\rightarrow Y^*+Pb$, a high energy octet hyperon $Y$ is inelastically scattered in the electromagnetic field of, e.g., a $Pb$ nucleus, and a decuplet hyperon $Y^*$ is produced in the final state [25]. An octet-decuplet transition quadrupole moment will affect the $Y^*$ production rate and its size may be extracted from the Primakoff cross section. Transition quadrupole moments can also be obtained from the cross section for kaon photoproduction. In the cross section for $\gamma p\rightarrow K^+\Sigma^{*0}\rightarrow K^+\Lambda^0\gamma$ [26] the radiative decay width for $\Sigma^{*0}\rightarrow\Lambda^0\gamma$ with its magnetic dipole $(M1)$ and electric quadrupole $(E2)$ contributions enters. The $E2$ contribution is a measure of the transition quadrupole moment.

With the help of the present theory the experimentally inaccessible quadrupole moments can be obtained from those that can be measured, and the geometric shape of baryons can be calculated in various models [6].

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