Three-dimensional heterotic string theory: New approach and extremal solutions

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We develop a new formalism for the bosonic sector of low-energy heterotic string theory toroidally compactified to three dimensions. This formalism is based on the use of a single nonquadratic real matrix potential which transforms linearly under the action of a subgroup of the three-dimensional charging symmetries. We formulate a new charging symmetry invariant approach for the symmetry generation and straightforward construction of asymptotically flat solutions. Finally, using the developed approach and the established formal analogy between the heterotic and Einstein-Maxwell theories, we construct a general class of heterotic string theory extremal solutions of the Israel-Wilson-Perjes type. This class is asymptotically flat and charging symmetry complete; it includes the extremal solutions constructed before and possesses the nontrivial bosonic string theory limit.

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I. INTRODUCTION

Field theory limits of superstring theory include some concrete modifications of classical general relativity [1]. These effective field theories describe the dynamics of some set of the superstring excitation modes restricted by the corresponding limit conditions. In the low-energy limit of heterotic string theory one deals with the zero-mass modes, in which the bosonic sector includes the dilaton, Kalb-Ramond, Abelian gauge, and metric fields. These fields live in the multidimensional space-time and interact in a supergravity controlled form (see [2] and references therein).

A solution space of the low-energy limit of heterotic string theory has been extensively studied during the last several years [3–9]; its investigation actively continues at the present time. The main part of the physically and mathematically interesting solutions was found for the theory toroidally compactified to three and lower dimensions (see, however, some higher-dimensional examples in [10] and works referred to there). The reason for such activity generation is closely related to the fact of hidden symmetry enhancement in the process of toroidal compactification [11,12]. A qualitatively important new situation takes place in the case of toroidal compactification to three dimensions, when the heterotic string theory becomes the three-dimensional symmetric space model coupled to gravity [13]. Such a theory belongs to a class described in [14] and possesses the complete integrability property after the subsequent reduction to two dimensions [15,16]. In this sense the heterotic string theory is also of the same type as the Einstein, Kaluza-Klein, and Einstein-Maxwell theories [17–20].

In our approach we consider the three-dimensional heterotic string theory in a remarkable explicit Einstein-Maxwell form. In [21] a new form of the null-curvature matrix representation of the theory was established. This form is based on the use of the Ernst matrix potentials [22]. The Ernst matrix potential formulation is a straightforward matrix generalization of the conventional formulation of stationary Einstein-Maxwell theory [23] to the heterotic string theory case. In fact this generalization includes the Einstein-Maxwell theory as a special case (see [22] and this work below), and gives a convenient and simple method for the symmetry analysis of the theory. In [22] we classified the three-dimensional group of hidden symmetries in the matrix-valued Einstein-Maxwell form; namely, we separated it into the shift, electric-magnetic rotation, and Ehlers and Harrison type parts of the transformations (compare with the ones in [18,24]; see also [25]). The full subgroup of charging symmetries was also constructed, i.e., the total set of the symmetry transformations that preserve the asymptotic flatness property of the solutions (see references in [25] for the Einstein-Maxwell theory analogies). A representation of the theory was established that is linear with respect to the action of whole set of charging symmetry transformations.

In this article we continue and conclude this line of investigation of three-dimensional heterotic string theory. We develop in detail the above mentioned representation and formulate a new approach to the theory, which is based on the use of a new single nonquadratic matrix potential. This potential has the lowest possible matrix dimensionality compatible with the structure of the symmetric space model of the theory and gives a powerful and most compact tool for the theory investigation. We introduce three doublets of matrix potentials defined in terms of this underlying one. Any doublet consists of one scalar and one vector matrix potential; both these potentials undergo linear transformations when the charging symmetry subgroup acts. We formulate a new method for the symmetry generation and straightforward construction of the solutions which guarantees the charging symmetry completeness property of the result. Our approach is especially useful for work with asymptotically flat solutions, which attract the main interest in the physical applications of string theory (see [8] for string theory based black hole physics). We also establish a relation between our new representation and the null-curvature matrix one. This gives some new simplifications and promising possibilities in construction of the two-dimensional solutions using the inverse scattering transform method. We hope to illustrate this statement in the near future. In this article we apply the new

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formalism for straightforward construction of the charging symmetry invariant class of asymptotically flat extremal solutions of the Israel-Wilson-Perjes type. We "rediscover" the well-known corresponding Einstein-Maxwell theory solution class [26], and give its straightforward generalization to the heterotic string theory case. Our class of solutions is closely related to those known in the literature [7,27,28], and also includes the nontrivial subclass of extremal bosonic string theory solutions, i.e., solutions without multidimensional Abelian gauge fields.

II. THREE-DIMENSIONAL HETEROTIC STRING THEORY

The action for the bosonic sector of low-energy heterotic string theory reads [1]

$$S_D=\int d^DX|\det G_{MN}|^{1/2}e^{-\Phi}\left(R_D+\Phi_{,M}\Phi^{,M}-\frac{1}{12}H_{MNK}H^{MNK}-\frac{1}{4}F^I_{MN}F^{I\,MN}\right), \tag{2.1}$$

where $H_{MNK}=\partial_M B_{NK}-\frac{1}{2}A^I_M F^I_{NK}+$ cyclic$\{M,N,K\}$ and $F^I_{MN}=\partial_M A^I_N-\partial_N A^I_M$. Here X^M is the Mth $(M=1,\ldots,D)$ coordinate of the physical space-time of signature $(-,+,\ldots,+)$, G_{MN} is the metric, and Φ, B_{MN}, and A^I_M $(I=1,\ldots,n)$ are the dilaton, Kalb-Ramond, and Abelian gauge fields.

As declared in the Introduction, in this paper we consider the theory (2.1) toroidally compactified to three dimensions. Let us briefly describe this compactification [11–13]. First of all, let us put $D=d+3$ and denote $Y^M=X^m$ $(m=1,\ldots,d)$ and $x^\mu=X^{d+\mu}$, $\mu=1,2,3$. Second, let us separate all the field components into scalar, vector, and two-rank tensor quantities with respect to transformations of the coordinates x^μ. As a result one has three scalar matrices G, B, and A of dimensions $d\times d$, $d\times d$, and $d\times n$ constructed from the components G_{mk}, B_{nk}, and $A_{mI}=A^I_m$, respectively, and also the scalar function

$$\phi=\Phi-\ln|\det G|^{1/2}. \tag{2.2}$$

Then there are three column vector matrix columns $\vec{V}_1, \vec{V}_2$, and $\vec{V}_3$ of dimensions $d\times 1$, $d\times 1$, and $n\times 1$. They read

$$V_{1\,m\mu}=G^{-1}_{mk}G_{k\,d+\mu},$$

$$V_{2\,m\mu}=B_{m\,d+\mu}-B_{mk}V_{1\,k\mu}+\frac{1}{2}A^I_m V_{3\,I\mu},$$

$$V_{3\,I\mu}=-A^I_{d+\mu}+A^I_m V_{1\,m\mu}. \tag{2.3}$$

Finally, there are two tensor fields; they consist of the nonmatrix quantities

$$h_{\mu\nu}=e^{-2\phi}[G_{d+\mu\,d+\nu}-G_{mk}V_{1\,m\mu}V_{1\,k\nu}],$$

$$b_{\mu\nu}=B_{d+\mu\,d+\nu}-B_{mk}V_{1\,m\mu}V_{1\,k\nu}-\frac{1}{2}[V_{1\,m\mu}V_{2\,m\nu}-V_{1\,m\nu}V_{2\,m\mu}]. \tag{2.4}$$

Let us now perform the toroidal compactification of the first d dimensions. In fact, this procedure is equivalent to consideration of the special situation when all the field components are Y^m independent. Thus, below all the quantities are considered as functions of the coordinates x^μ. The use of the motion equations allows one to introduce the pseudoscalar fields u, v, and s (they are the $d\times 1$, $d\times 1$, and $n\times 1$ columns) and accordingly, the relations

$$\nabla\times\vec{V}_1=e^{2\phi}G^{-1}\left[\nabla u+\left(B+\frac{1}{2}AA^T\right)\nabla v+A\nabla s\right],$$

$$\nabla\times\vec{V}_2=e^{2\phi}G\nabla v-\left(B+\frac{1}{2}AA^T\right)\nabla\times\vec{V}_1+A\nabla\times\vec{V}_3.$$

$$\nabla\times\vec{V}_3=e^{2\phi}(\nabla s+A^T\nabla v)+A^T\nabla\times\vec{V}_1, \tag{2.5}$$

where all the vector operations are defined, respectively, by the three-metric $h_{\mu\nu}$. Thus the tensor field $b_{\mu\nu}$ is nondynamical; following [13] we put $b_{\mu\nu}=0$ in our analysis. This restriction does not put any new ones on the remaining dynamical quantities. The resulting effective three-dimensional system coincides with a symmetric space model coupled to gravity [13]. To express it in terms convenient for our consideration, let us introduce the following matrices $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{A}$ [21]:

$$\mathcal{G}=\begin{pmatrix}-e^{-2\phi}+v^TGv & v^TG\\ Gv & G\end{pmatrix},\quad \mathcal{B}=\begin{pmatrix}0 & -w^T\\ w & B\end{pmatrix},$$

$$\mathcal{A}=\begin{pmatrix}s^T+v^TA\\ A\end{pmatrix}, \tag{2.6}$$

where $w=u+Bv+\frac{1}{2}As$. Then, let us define the $[2(d+1)+n]\times[2(d+1)+n]$ matrix $\mathcal{M}$ as

$$\mathcal{M}=\begin{pmatrix} \mathcal{G}^{-1} & \mathcal{G}^{-1}(\mathcal{B}+\mathcal{T}) & \mathcal{G}^{-1}\mathcal{A} \\ (-\mathcal{B}+\mathcal{T})\mathcal{G}^{-1} & (\mathcal{G}-\mathcal{B}+\mathcal{T})\mathcal{G}^{-1}(\mathcal{G}+\mathcal{B}+\mathcal{T}) & (\mathcal{G}-\mathcal{B}+\mathcal{T})\mathcal{G}^{-1}\mathcal{A} \\ \mathcal{A}^T\mathcal{G}^{-1} & \mathcal{A}^T\mathcal{G}^{-1}(\mathcal{G}+\mathcal{B}+\mathcal{T}) & 1+\mathcal{A}^T\mathcal{G}^{-1}\mathcal{A} \end{pmatrix}, \tag{2.7}$$

where $\mathcal{T}=\frac{1}{2}\mathcal{A}\mathcal{A}^T$. This matrix satisfies the restrictions

$$\mathcal{M}^T=\mathcal{M}, \quad \mathcal{M}\mathcal{L}\mathcal{M}=\mathcal{M}, \tag{2.8}$$

where

$$\mathcal{L}=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.9}$$

so it parametrizes the symmetric space $O(d+1,d+1+n)/O(d+1)\times O(d+1+n)$. The resulting three-dimensional theory can be expressed in terms of the three-metric $h_{\mu\nu}$ and the matrix $\mathcal{M}$; the corresponding action reads

$$S_3=\int d^3x h^{1/2}(-R_3+L_3), \tag{2.10}$$

where

$$L_3=\frac{1}{8}\mathrm{Tr}(\nabla\mathcal{M}\,\mathcal{M}^{-1})^2. \tag{2.11}$$

previosuly the representation (2.10),(2.11) was considered in [13] with another form of the null-curvature matrix $\mathcal{M}$. In [29] a symplectic matrix representation was found for the special symmetric space model with $d=n=1$. In [30] one can find the unitary null-curvature matrix for the system with $d=1,n=2$.

Now let us use the motion equations and Eq. (2.8) and introduce on shell the vector matrix $\vec{\Omega}$ according to the relation

$$\nabla\times\vec{\Omega}=\nabla\mathcal{M}\,\mathcal{L}\,\mathcal{M}; \tag{2.12}$$

then from Eq. (2.8) it follows that $\vec{\Omega}^T=-\vec{\Omega}$. Then, from Eqs. (2.6), (2.7), and (2.12) one concludes that the matrices $\mathcal{M}$ and $\vec{\Omega}$ have equivalent natural block matrix structure: they are 5×5 block matrices. Using straightforward calculations one can check that

$$\vec{V}_1=\vec{\Omega}_{12}^T, \quad \vec{V}_2=\vec{\Omega}_{14}^T, \quad \vec{V}_3=\vec{\Omega}_{15}^T, \tag{2.13}$$

where the indices enumerate the corresponding matrix blocks. Let us also define the following set of scalar quantities:

$$S_0=-\mathcal{M}_{11}, \quad S_1=\mathcal{M}_{22}-\mathcal{M}_{11}^{-1}\mathcal{M}_{12}^T\mathcal{M}_{12},$$

$$S_2=\mathcal{M}_{24}-\mathcal{M}_{11}^{-1}\mathcal{M}_{12}^T\mathcal{M}_{14},$$

$$S_3=\mathcal{M}_{25}-\mathcal{M}_{11}^{-1}\mathcal{M}_{12}^T\mathcal{M}_{15}. \tag{2.14}$$

We state that these scalar and vector potentials can be effectively explored for the construction of all the multidimensional fields of the heterotic string theory (2.1). That is, for the $D=(d+3)$-dimensional line element one has

$$ds_{d+3}^2=(dY+V_{1\mu}dx^\mu)^TS_1^{-1}(dY+V_{1\nu}dx^\nu)+S_0ds_3^2, \tag{2.15}$$

where Y is the d-dimensional coordinate column with components Y^m and $ds_3^2=h_{\mu\nu}dx^\mu dx^\nu$. Then, for the matter fields one has the following expressions:

$$e^\Phi=|S_0\det S_1|^{1/2},$$

$$B_{mk}=\frac{1}{2}(S_1^{-1}S_2-S_2^TS_1^{-1})_{mk},$$

$$B_{m\,d+\nu}=\left\{V_{2\nu}+\frac{1}{2}(S_1^{-1}S_2-S_2^TS_1^{-1})V_{1\nu}-S_1^{-1}S_3V_{3\nu}\right\}_m,$$

$$B_{d+\mu\,d+\nu}=\frac{1}{2}[V_{1\mu}^T(S_1^{-1}S_2-S_2^TS_1^{-1})V_{1\nu}+V_{1\mu}^TV_{2\nu}-V_{1\nu}^TV_{2\mu}],$$

$$A_m^I=(S_1^{-1}S_3)_{mI},$$

$$A_{d+\mu}^I=(-V_{3\mu}+S_3^TS_1^{-1}V_{1\mu})_I. \tag{2.16}$$

Equations (2.13)–(2.16) allow one to translate any solution of the three-dimensional problem (2.10),(2.11) into the form of the physical fields of the heterotic string theory (2.1). They are especially useful in the framework of the solution approaches based on the use of the null-curvature matrix $\mathcal{M}$. This situation arises when one uses the Kramer-Neugebauer geodesic method [25] or the Belinsky-Zakharov inverse scattering transform technique [17]. Equations (2.13)–(2.16) also play an important role in our new approach to three-dimensional heterotic string theory, which is developed in the next section.

III. NEW APPROACH

For the following analysis it is necessary to introduce a pair of Ernst matrix potentials [22]. These are the matrices $\mathcal{X}$ and $\mathcal{A}$, where

$$\mathcal{X}=\mathcal{G}+\mathcal{B}+\frac{1}{2}\mathcal{A}\mathcal{A}^T, \tag{3.1}$$

and $\mathcal{A}$ is the potential introduced in the previous section. In terms of these potentials the three-dimensional Lagrangian L_3 takes the following form:

$$L_3 = \mathrm{Tr}\left[\frac{1}{4}(\nabla\mathcal{X} - \nabla\mathcal{A}\mathcal{A}^T)\mathcal{G}^{-1}(\nabla\mathcal{X}^T - \mathcal{A}\nabla\mathcal{A}^T)\mathcal{G}^{-1} + \frac{1}{2}\mathcal{G}^{-1}\nabla\mathcal{A}\nabla\mathcal{A}^T\right]; \tag{3.2}$$

here $\mathcal{G} = \frac{1}{2}(\mathcal{X} + \mathcal{X}^T - \mathcal{A}\mathcal{A}^T)$. The simplest solution of this three-dimensional theory, which corresponds to empty Minkowskian space-time, is given by the matrices $\mathcal{X}_0 = \Sigma = \mathrm{diag}(-1,-1;1,\ldots,1), \mathcal{A}_0 = 0$, and by the three-dimensional metric $h_{\mu\nu} = \delta_{\mu\nu}$ [see Eqs. (2.6),(3.1)]. Following the conventional terminology of general relativity [25] we call the solutions "asymptotically flat" if $\mathcal{X} \to \Sigma, \mathcal{A} \to 0$ when the three-point with the coordinates x^μ tends to spatial infinity (note that time is taken as one of the compactified dimensions). Let us now introduce the following pair of matrix potentials:

$$\mathcal{Z}_1 = 2(\mathcal{X} + \Sigma)^{-1} - \Sigma, \quad \mathcal{Z}_2 = \sqrt{2}(\mathcal{X} + \Sigma)^{-1}\mathcal{A}. \tag{3.3}$$

The map $(\mathcal{X},\mathcal{A}) \to (\mathcal{Z}_1, \mathcal{Z}_2)$ coincides with the inverse one; a similar substitution is familiar in stationary Einstein-Maxwell theory [31]. Our new approach is based on the use of the single $(d+1)\times(d+1+n)$ matrix potential $\mathcal{Z}$, where

$$\mathcal{Z} = (\mathcal{Z}_1 \;\; \mathcal{Z}_2). \tag{3.4}$$

This matrix potential is quadratic for bosonic string and nonquadratic for heterotic string theories. Using tedious but straightforward calculations and Eqs. (3.2)–(3.4), one can prove that in terms of the potential $\mathcal{Z}$ the three-dimensional matter Lagrangian reads

$$L_3 = \mathrm{Tr}[\nabla\mathcal{Z}(\Xi - \mathcal{Z}^T\Sigma\mathcal{Z})^{-1}\nabla\mathcal{Z}^T(\Sigma - \mathcal{Z}\Xi\mathcal{Z}^T)^{-1}], \tag{3.5}$$

where $\Xi = \mathrm{diag}(-1,-1;1,\ldots,1)$ is the $(d+1+n)\times(d+1+n)$ matrix. Equations (2.10) and (3.5) define the $\mathcal{Z}$-based formalism completely. The corresponding motion equations are

$$\nabla^2\mathcal{Z} + 2\nabla\mathcal{Z}\Xi\mathcal{Z}^T(\Sigma - \mathcal{Z}\Xi\mathcal{Z}^T)^{-1}\nabla\mathcal{Z} = 0,$$

$$R_{3\,\mu\nu} = \mathrm{Tr}[\mathcal{Z}_{,(\mu}(\Xi - \mathcal{Z}^T\Sigma\mathcal{Z})^{-1}\mathcal{Z}^T_{,\nu)}(\Sigma - \mathcal{Z}\Xi\mathcal{Z}^T)^{-1}]. \tag{3.6}$$

Our plan is to develop the $\mathcal{Z}$ formalism in detail; namely, we would like to obtain its explicit relation to the null-curvature matrix representation, the $\mathcal{Z}$-based scheme of calculation of the multidimensional field components, and also the representation of all the hidden symmetries in terms of $\mathcal{Z}$. To realize this program, let us express the matrix $\mathcal{M}$ from Eq. (2.7) in terms of the Ernst matrix potentials $\mathcal{X}$ and $\mathcal{A}$, and after that, using Eqs. (3.3),(3.4), let us translate the result into the $\mathcal{Z}$ language. After some nontrivial algebraic work one obtains that

$$\mathcal{M} = \mathcal{D}_1^T\mathcal{M}_1\mathcal{D}_1 + \mathcal{D}_1^T\mathcal{M}_2\mathcal{D}_2 + \mathcal{D}_2^T\mathcal{M}_2^T\mathcal{D}_1 + \mathcal{D}_2^T\mathcal{M}_3\mathcal{D}_2 - \mathcal{L}, \tag{3.7}$$

where

$$\mathcal{M}_1 = \mathcal{H}^{-1}, \quad \mathcal{M}_2 = \mathcal{H}^{-1}\mathcal{Z}, \quad \mathcal{M}_3 = \mathcal{Z}^T\mathcal{H}^{-1}\mathcal{Z}, \tag{3.8}$$

and

$$\mathcal{H} = \Sigma - \mathcal{Z}\Xi\mathcal{Z}^T. \tag{3.9}$$

In Eq. (3.7) the constant matrices $\mathcal{D}_1$ and $\mathcal{D}_2$ read

$$\mathcal{D}_1 = (\Sigma \;1\; 0), \quad \mathcal{D}_2 = \begin{pmatrix} 1 & -\Sigma & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{3.10}$$

Note that the block components of $\mathcal{D}_1$ are the $(d+1)\times(d+1)$, $(d+1)\times(d+1)$ and $(d+1)\times n$ matrices. The first row blocks of $\mathcal{D}_2$ have the same dimensionality; for the second row one has $n\times(d+1)$, $n\times(d+1)$, and $n\times n$ blocks. Now let us calculate the vector matrix $\vec{\Omega}$ according to Eq. (2.12). In this calculation it is convenient to use the following multiplication relations, which can be easily established for the matrices $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$\mathcal{D}_1\mathcal{L}\mathcal{D}_1^T = 2\Sigma, \quad \mathcal{D}_1\mathcal{L}\mathcal{D}_2^T = 0, \quad \mathcal{D}_2\mathcal{L}\mathcal{D}_2^T = -2\Xi. \tag{3.11}$$

The result reads

$$\vec{\Omega} = \mathcal{D}_1^T\vec{\Omega}_1\mathcal{D}_1 - \mathcal{D}_1^T\vec{\Omega}_2\mathcal{D}_2 + \mathcal{D}_2^T\vec{\Omega}_2^T\mathcal{D}_1 + \mathcal{D}_2^T\vec{\Omega}_3\mathcal{D}_2, \tag{3.12}$$

where

$$\nabla\times\vec{\Omega}_1 = \vec{J}, \quad \nabla\times\vec{\Omega}_2 = \mathcal{H}^{-1}\nabla\mathcal{Z} - \vec{J}\mathcal{Z},$$

$$\nabla\times\vec{\Omega}_3 = \nabla\mathcal{Z}^T\mathcal{H}^{-1}\mathcal{Z} - \mathcal{Z}^T\mathcal{H}^{-1}\nabla\mathcal{Z} + \mathcal{Z}^T\vec{J}\mathcal{Z}, \tag{3.13}$$

and the vector current $\vec{J}$ reads

$$\vec{J} = \mathcal{H}^{-1}(\mathcal{Z}\Xi\nabla\mathcal{Z}^T - \nabla\mathcal{Z}\Xi\mathcal{Z}^T)\mathcal{H}^{-1}. \tag{3.14}$$

Equations (3.7)–(3.10) and (3.12)–(3.14) give a translation of the solution expressed in terms of the $\mathcal{Z}$-related quantities $(\mathcal{M}_a, \vec{\Omega}_a)$, $a = 1,2,3$, to the $(\mathcal{M},\vec{\Omega})$ form. The inverse relations can easily be obtained using the operators

$$\Pi_1 = \frac{1}{2}\mathcal{L}\mathcal{D}_1^T\Sigma, \quad \Pi_2 = -\frac{1}{2}\mathcal{L}\mathcal{D}_2^T\Xi. \tag{3.15}$$

The $(\mathcal{M},\vec{\Omega}) \to (\mathcal{M}_a, \vec{\Omega}_a)$ map reads

$$\mathcal{M}_1 = \Pi_1^T\mathcal{M}\Pi_1 + \frac{1}{2}\Sigma, \quad \vec{\Omega}_1 = \Pi_1^T\vec{\Omega}\Pi_1,$$

$$\mathcal{M}_2 = \Pi_1^T\mathcal{M}\Pi_2, \quad \vec{\Omega}_2 = -\Pi_1^T\vec{\Omega}\Pi_2,$$

$$\mathcal{M}_3=\Pi_2^T\mathcal{M}\Pi_2-\frac{1}{2}\Xi,\quad \vec{\Omega}_3=\Pi_2^T\vec{\Omega}\Pi_2. \tag{3.16}$$

In proof of Eq. (3.17) one can use the following projective properties of Π_1 and Π_2:

$$\mathcal{D}_1\Pi_1=1,\quad \mathcal{D}_1\Pi_2=0,\quad \mathcal{D}_2\Pi_1=0,\quad \mathcal{D}_2\Pi_2=1;$$

$$\Pi_1^T\mathcal{L}\Pi_1=\frac{1}{2}\Sigma,\quad \Pi_1^T\mathcal{L}\Pi_2=0,\quad \Pi_2^T\mathcal{L}\Pi_2=-\frac{1}{2}\Xi. \tag{3.17}$$

In Eq. (3.16) the scalar and vector matrices are combined in three doublet "generations." The constituents of any doublet have the equivalent matrix dimensionalities and, moreover, the same natural block structure. This block structure is induced by Eqs. (2.6), (3.1), (3.3), (3.4), (3.8), (3.9), (3.13), and (3.14). One obtains that the constituents of the doublets $(\mathcal{M}_1,\vec{\Omega}_1),(\mathcal{M}_2,\vec{\Omega}_2)$, and $(\mathcal{M}_3,\vec{\Omega}_3)$ are the 2×2, 2×3, and 3×3 block matrices, respectively, with the first row of the following structure: $(1\times1,1\times d),(1\times1,1\times d,1\times n)$, and $(1\times1,1\times d,1\times n)$ (the structure of the remaining rows is completely defined by the given information). This means, for example, that the "13" block component of $\mathcal{M}_2$ (we denote it as $M_{2,13}$) is the $1\times n$ matrix. Performing the appropriate segmentation of the matrices D_1 and $\mathcal{D}_2$ (then $\mathcal{D}_1$ becomes 2×5, whereas $\mathcal{D}_2$ is the 3×5 block matrix) and applying Eqs. (3.7) and (3.12), one obtains the explicit relations between the $(\mathcal{M},\vec{\Omega})$ and $(\mathcal{M}_a,\vec{\Omega}_a)$ block components. The scalar part of the relations necessary in view of Eq. (2.14) reads

$$\begin{aligned}
\mathcal{M}_{11}&=\mathcal{M}_{1,11}-2\mathcal{M}_{2,11}+\mathcal{M}_{3,11},\\
\mathcal{M}_{12}&=-\mathcal{M}_{1,12}G_0-\mathcal{M}_{2,12}+(\mathcal{M}_{2,21})^TG_0+\mathcal{M}_{3,12},\\
\mathcal{M}_{14}&=\mathcal{M}_{1,12}-\mathcal{M}_{2,12}G_0+(\mathcal{M}_{2,21})^T-\mathcal{M}_{3,12}G_0,\\
\mathcal{M}_{15}&=\sqrt{2}(\mathcal{M}_{2,13}+\mathcal{M}_{3,13}),\\
\mathcal{M}_{22}&=G_0\mathcal{M}_{1,22}G_0+G_0\mathcal{M}_{2,22}+(\mathcal{M}_{2,22})^TG_0+\mathcal{M}_{3,22},\\
\mathcal{M}_{24}&=G_0\mathcal{M}_{1,22}-G_0\mathcal{M}_{2,22}G_0+(\mathcal{M}_{2,22})^T-\mathcal{M}_{3,22}G_0-1,\\
\mathcal{M}_{25}&=\sqrt{2}(G_0\mathcal{M}_{2,23}+\mathcal{M}_{3,23});
\end{aligned} \tag{3.18}$$

whereas for the vector components [see Eq. (2.13)] one obtains

$$\begin{aligned}
\vec{\Omega}_{12}&=-\vec{\Omega}_{1,12}G_0+\vec{\Omega}_{2,12}+(\vec{\Omega}_{2,21})^TG_0+\vec{\Omega}_{3,12},\\
\vec{\Omega}_{14}&=-\vec{\Omega}_{1,12}G_0-\vec{\Omega}_{2,12}G_0+(\vec{\Omega}_{2,21})^T-\vec{\Omega}_{3,12}G_0,\\
\vec{\Omega}_{15}&=\sqrt{2}(\vec{\Omega}_{2,13}+\vec{\Omega}_{3,13}),
\end{aligned} \tag{3.19}$$

where $G_0=\mathrm{diag}(-1;1,\ldots,1)$ is the $d\times d$ matrix that gives the trivial value of the metric components corresponding to the compactified dimensions. Equations (2.13)–(2.16) and (3.18),(3.19) allow one to transform the result obtained in terms of $\mathcal{Z}$ to the form of the physical field components.

The last problem in our program is the $\mathcal{Z}$-based description of the hidden symmetries. First of all, from Eq. (3.5) it follows that the transformation

$$\mathcal{Z}\rightarrow C_1\mathcal{Z}C_2 \tag{3.20}$$

is a symmetry if

$$C_1^T\Sigma C_1=\Sigma,\quad C_2^T\Xi C_2=\Xi, \tag{3.21}$$

i.e., if $C_1\in O(2,d-1)$ and $C_2\in O(2,d-1+n)$. It is easy to show that the realization of this symmetry transformation on the set of $\mathcal{Z}$-related quantities $(\mathcal{M}_a,\vec{\Omega}_a)$ reads

$$\begin{aligned}
&\mathcal{M}_1\rightarrow C_1^{-1\,T}\mathcal{M}_1C_1^{-1},\quad \mathcal{M}_2\rightarrow C_1^{-1\,T}\mathcal{M}_2C_2,\\
&\mathcal{M}_3\rightarrow C_2^T\mathcal{M}_3C_2,\quad \vec{\Omega}_1\rightarrow C_1^{-1\,T}\vec{\Omega}_1C_1^{-1},\\
&\vec{\Omega}_2\rightarrow C_1^{-1\,T}\vec{\Omega}_2C_2,\quad \vec{\Omega}_3\rightarrow C_2^T\vec{\Omega}_3C_2.
\end{aligned} \tag{3.22}$$

Also it is possible to establish the action of the symmetry (3.20) on the null-curvature matrix $\mathcal{M}$; namely, using Eq. (3.7) and the projective relations (3.17), one concludes that

$$\mathcal{M}\rightarrow\mathcal{C}^T\mathcal{M}\mathcal{C}, \tag{3.23}$$

where $\mathcal{C}=\mathcal{C}_1\mathcal{C}_2$ and

$$\mathcal{C}_1=1+\Pi_1(C_1^{-1}-1)\mathcal{D}_1,\quad \mathcal{C}_2=1+\Pi_2(C_2-1)\mathcal{D}_2. \tag{3.24}$$

From Eq. (3.17) it follows that the matrices $\mathcal{C}_1$ and $\mathcal{C}_2$, which represent the $\mathcal{Z}$ transformations given by C_1 and C_2 in terms of $\mathcal{M}$, commute, and also that both these matrices satisfy the restriction for the group $O(d+1,d+1+n)$:

$$\mathcal{C}^T\mathcal{L}\mathcal{C}=\mathcal{L}. \tag{3.25}$$

Equation (3.25) contains all hidden symmetries for the theory under consideration [13]. Of course, the general $O(d+1,d+1+n)$ symmetry transformation does not coincide with the one derived above. In fact we have detected the subgroup $O(2,d-1)\times O(2,d-1+n)$ of the complete group of symmetry transformations. There is some "missing" symmetry from $O(d+1,d+1+n)$ whose action on $\mathcal{M}$ and $\mathcal{Z}$ must be established. For our purposes it will be sufficient to construct it in the infinitesimal form.

To do this, let us denote the generators of C_1 and C_2 as γ_1 and γ_2, i.e., let us put $C_1=1+\gamma_1, C_2=1+\gamma_2$ when $C_1,C_2\rightarrow1$. For the corresponding generators in the $\mathcal{M}$ representation from Eq. (3.24) one has

$$\Gamma_1=-\Pi_1\gamma_1\mathcal{D}_1,\quad \Gamma_2=\Pi_2\gamma_2\mathcal{D}_2. \tag{3.26}$$

These generators satisfy the algebra relation which follows from Eq. (3.25):

$$\Gamma^T=-\mathcal{L}\Gamma\mathcal{L}. \tag{3.27}$$

It is easy to prove that the general solution of Eq. (3.27) can be written in the form of

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3, \tag{3.28}$$

where

$$\Gamma_3 = 2(\Pi_2 \Xi \gamma_3^T \Pi_1^T - \Pi_1 \gamma_3 \Xi \Pi_2^T)\mathcal{L}. \tag{3.29}$$

Here γ_3 is an arbitrary constant parameter, whereas $\gamma_1^T = -\Sigma\gamma_1\Sigma, \gamma_2^T = -\Xi\gamma_2\Xi$ in view of Eq. (3.21). The generators Γ_a define transformations of the null-curvature matrix $\mathcal{M}$ of the form [see Eq. (3.24)]

$$\delta_a \mathcal{M} = \Gamma_a^T \mathcal{M} + \mathcal{M}\Gamma_a. \tag{3.30}$$

Then, using the $\mathcal{M}_a \leftrightarrow \mathcal{M}$ correspondence (3.16), it is easy to prove that the infinitesimal $\mathcal{Z}$ transformations read

$$\delta_1 \mathcal{Z} = \gamma_1 \mathcal{Z}, \quad \delta_2 \mathcal{Z} = \mathcal{Z}\gamma_2,$$
$$\delta_3 \mathcal{Z} = \gamma_3 - \mathcal{Z}\Xi\gamma_3^T \Sigma \mathcal{Z}. \tag{3.31}$$

Thus, the "missing" part of the hidden symmetries moves the trivial $\mathcal{Z}$ value, i.e., it does not preserve the trivial spatial asymptotics of the fields. From this it follows that Eq. (3.20) gives the general three-dimensional charging symmetry subgroup. The "missing" transformations must be removed from the procedure of symmetry generation of the asymptotically flat solutions.

Equations (3.26) and (3.29) show how the operators Π_1 and Π_2 realize a relation between the symmetry algebra realizations in the $\mathcal{Z}$ and $\mathcal{M}$ representations. Let $T(\gamma_a)$ denote the infinitesimal transformation in the arbitrary representation which corresponds to the generator γ_a. Then, as is easy to check, the commutation relations read

$$[T(\gamma_1'), T(\gamma_1'')] = T([\gamma_1', \gamma_1'']),$$
$$[T(\gamma_2'), T(\gamma_2'')] = T([\gamma_2', \gamma_2'']), \quad [T(\gamma_1), T(\gamma_2)] = 0,$$
$$[T(\gamma_1), T(\gamma_3)] = T(\gamma_3'), \quad [T(\gamma_2), T(\gamma_3)] = T(\gamma_3''),$$
$$\text{where} \quad \gamma_3' = -\gamma_1\gamma_3, \; \gamma_3'' = -\gamma_3\gamma_2,$$
$$[T(\gamma_3'), T(\gamma_3'')] = T(\gamma_1) + T(\gamma_2), \;\; \text{where}$$
$$\gamma_1 = (\gamma_3''\Xi\gamma_3'^T - \gamma_3'\Xi\gamma_3''^T)\Sigma,$$
$$\gamma_2 = \Xi(\gamma_3'^T \Sigma\gamma_3'' - \gamma_3''^T \Sigma\gamma_3'). \tag{3.32}$$

These relations will play an important role in the study of the infinite-dimensional symmetry group of heterotic string theory; this group arises after the subsequent reduction to two spatial dimensions. In this case the heterotic string theory becomes a completely integrable two-dimensional symmetric space model coupled to gravity [15].

IV. EXTREMAL SOLUTIONS

The new approach developed in the previous section gives all the necessary tools for generation of asymptotically flat solutions of the heterotic string theory (2.1) compactified to three dimensions on a torus. This generation procedure includes the following steps: (1) One must take some special asymptotically flat solution (or some consistent heterotic string theory subsystem) and represent it in terms of the potential $\mathcal{Z}$ and three-metric $h_{\mu\nu}$; (2) after that one must apply Eq. (3.20) with the matrices C_1 and C_2 which are the general solutions of the charging symmetry group relations (3.21); (3) one must calculate the charging symmetry transformed values of the doublets $(\mathcal{M}_a, \vec{\Omega}_a)$, and after that one must obtain the full set of multidimensional field components according to the scheme developed in the two previous sections.

Note that this simple program realizes the most general technique for generation of the asymptotically flat solutions in the theory under consideration; this program leads to generation of charging symmetry complete classes of solutions. However, the $\mathcal{Z}$ formalism can also be effectively used for the straightforward construction of the charging symmetry invariant and asymptotically flat solution families. In this paper we illustrate this statement by exploring one remarkable property of the three-dimensional heterotic string theory; namely, there is a close formal analogy between this theory and the classical stationary Einstein-Maxwell system. Let us clarify this question and give the corresponding example of application of the developed formalism. So let us consider the special subsystem with $d=1$ and $n=2$. Let us separate the 2×4 matrix potential $\mathcal{Z}$ into the two 2×2 blocks $\mathcal{Z}_\alpha$ [$\alpha=1,2$; see Eq. (3.4)], and define the subsystem under consideration by taking the ansatz

$$\mathcal{Z}_\alpha = \begin{pmatrix} z_\alpha' & z_\alpha'' \\ -z_\alpha'' & z_\alpha' \end{pmatrix}. \tag{4.1}$$

It is easy to prove that the motion equations (3.6) reduce to the system

$$\nabla^2 z + 2\frac{\nabla z\sigma_3 z^+}{1 - z\sigma_3 z^+}\nabla z = 0,$$
$$R_{3\,\mu\nu} = 2\frac{z_{,\mu}(\sigma_3 - z^+ z)^{-1} z^+_{,\nu}}{1 - z\sigma_3 z^+}, \tag{4.2}$$

where σ_3 is one of the Pauli matrices, $z=(z_1\, z_2)$, and $z_\alpha = z_\alpha' + iz_\alpha''$. Let us construct some special solution class for this system. This solution class arises in the framework of the ansatz

$$z = \lambda q, \tag{4.3}$$

where λ is the dynamical complex function and q is the complex constant 1×2 row. We state that the choice of λ, q, and $h_{\mu\nu}$ such that

$$\nabla^2 z = 0, \quad q\sigma_3 q^+ = 0, \quad h_{\mu\nu} = \delta_{\mu\nu} \tag{4.4}$$

gives a solution of the equations (4.2). In fact it is the well-known Israel-Wilson-Perjes class of solutions [26], and our subsystem is the conventional stationary Einstein-Maxwell theory. To prove this fact, let us introduce the potentials

$$E=\frac{1-z_1}{1+z_1},\quad F=\frac{\sqrt{2}z_2}{1+z_1}, \tag{4.5}$$

and take the row q in the form of

$$q=(1\; e^{i\delta}), \tag{4.6}$$

where δ is real without loss of any generality (λ is understood as the arbitrary complex harmonic function). Then for this solution one has

$$E=\frac{1-\lambda}{1+\lambda},\quad F=\frac{e^{i\delta}}{\sqrt{2}}(1-E), \tag{4.7}$$

i.e., the formulas defining the general stationary extremal solution class of the Einstein-Maxwell theory. Also the three-dimensional matter Lagrangian takes the following form in terms of the new potentials E and F:

$$L_3=L_{EM}=\frac{1}{2f^2}|\nabla E-\bar{F}\nabla F|^2-\frac{1}{f}|\nabla F|^2, \tag{4.8}$$

where $f=\frac{1}{2}(E+\bar{E}-|F|^2)$. Thus, E and F are actually the conventional complex Ernst potentials [23].

In [32] one can find information about the charging symmetry invariant generation procedure for heterotic string theory with arbitrary values of d and n starting from this effective Einstein-Maxwell system. Below we use the material presented above as the pattern for straightforward construction of the general extremal solution class of Israel-Wilson-Perjes type in heterotic string theory. Actually, let us use the explicit similarity between the systems (3.6) and (4.2) and consider the heterotic string theory ansatz

$$\mathcal{Z}=\Lambda\mathcal{Q}, \tag{4.9}$$

where Λ is the dynamical $(d+1)\times\mathcal{K}$ real matrix and $\mathcal{Q}$ is the constant real $\mathcal{K}\times(d+1+n)$ matrix. It is easy to prove that the relations

$$\nabla^2\mathcal{Z}=0,\quad \mathcal{Q}\Xi\mathcal{Q}^T=0,\quad h_{\mu\nu}=\delta_{\mu\nu} \tag{4.10}$$

lead to solution of the motion equations (3.6). This solution is charging symmetry complete: an application of the transformation (3.20) is equivalent to the reparametrization $\Lambda\to C_1\Lambda,\mathcal{Q}\to\mathcal{Q}C_2$. This reparametrization is not important because there is no algebraic restriction on Λ and the restriction on $\mathcal{Q}$ is invariant with respect to this reparametrization [of course, we take the general matrix $\mathcal{Q}$ satisfying Eq. (4.10)].

Let us now obtain the heterotic string theory analogy of Eq. (4.6). In particular, we would like to clarify the question about the possible concrete values of the parameter $\mathcal{K}$. First of all, from Eq. (4.9) it follows that the $\mathcal{Q}$ rows can be taken as algebraically independent. Actually, if, for example, the $\mathcal{K}$th row is a linear combination of the others, i.e., if $\mathcal{Q}_{\mathcal{K}}=\Sigma_{l=1}^{\mathcal{K}-1}\beta_l\mathcal{Q}_l$ (β_l are the coefficients), then the removal of $\mathcal{Q}_{\mathcal{K}}$ from $\mathcal{Q}$ together with the replacement of the lth column Λ_l of Λ by the column $\Lambda_l+\beta_l\Lambda_{\mathcal{K}}$ effectively transforms Eq. (4.9) into the same ansatz but with the shift $\mathcal{K}\to\mathcal{K}-1$. Thus, we can actually take $\mathcal{Q}$ with independent rows without loss of generality. Then from this it follows that $\mathcal{K}<d+1+n$ [because the numbers of algebraically independent rows and columns coincide for any matrix, and in the case of $\mathcal{K}=d+1+n$ one has from Eq. (4.10) that det $\mathcal{Q}=0$, i.e., a contradiction with the proposed matrix $\mathcal{Q}$ row independence]. As a result parametrization of $\mathcal{Q}$ exists of the following form:

$$\mathcal{Q}=(\mathcal{Q}_1\;\mathcal{Q}_2), \tag{4.11}$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the $\mathcal{K}\times\mathcal{K}$ and $\mathcal{K}\times(d+1+n-\mathcal{K})$ matrices respectively. The matrix, $\mathcal{Q}_1$ must be nondegenerate ($\mathcal{Q}$ has independent rows), so one can represent $\mathcal{Q}$ as $\mathcal{Q}=\mathcal{Q}_1(1\;\mathcal{N})$, where $\mathcal{N}=\mathcal{Q}_1^{-1}\mathcal{Q}_2$. It is easy to see that the matrix $\mathcal{Q}_1$ can be absorbed by the reparametrization $\Lambda\mathcal{Q}_1\to\Lambda$ without loss of generality, so the pair $(\Lambda,\mathcal{N})$ defines our solution completely. Then, it is not difficult to establish that the algebraic $\mathcal{Q}$ restriction in Eq. (4.10), being written in terms of the established new form of the matrix $\mathcal{Q}$

$$\mathcal{Q}=(1\;\mathcal{N}), \tag{4.12}$$

is compatible only if $\mathcal{K}=1,2$. In the case of $\mathcal{K}=1$ the matrix $\Lambda=\Lambda^{(1)}$ is the $(d+1)\times 1$ column, whereas $\mathcal{N}=\mathcal{N}^{(1)}$ is the $1\times(d+n)$ row. This row satisfies the following algebraic restriction:

$$\bar{\mathcal{N}}^{(1)}\bar{\mathcal{N}}^{(1)\,T}=1+(\tilde{\mathcal{N}}^{(1)})^2, \tag{4.13}$$

where the parametrization $\mathcal{N}^{(1)}=(\tilde{\mathcal{N}}^{(1)}\bar{\mathcal{N}}^{(1)})$ is performed [here $\tilde{\mathcal{N}}^{(1)}$ is the number and $\bar{\mathcal{N}}^{(1)}$ is the $1\times(d-1+n)$ row]. In the case of $\mathcal{K}=2,\Lambda=\Lambda^{(2)}$ is the $(d+1)\times 2$ matrix, whereas $\mathcal{N}=\mathcal{N}^{(2)}$ consists of two $1\times(d-1+n)$ mutually orthogonal rows $\mathcal{N}_1^{(2)},\mathcal{N}_2^{(2)}$ of the unit norm:

$$\mathcal{N}^{(2)}=\begin{pmatrix}\mathcal{N}_1^{(2)}\\ \mathcal{N}_2^{(2)}\end{pmatrix},$$

$$\mathcal{N}_1^{(2)}\mathcal{N}_1^{(2)\,T}=\mathcal{N}_2^{(2)}\mathcal{N}_2^{(2)\,T}=1,\quad \mathcal{N}_1^{(2)}\mathcal{N}_2^{(2)\,T}=0. \tag{4.14}$$

The case of $\mathcal{K}=1$ exists for values of (d,n) such that $d+n\geqslant 2$ [this follows from Eq. (4.13)], whereas the case of $\mathcal{K}=2$ is consistent if $d+n\geqslant 3$. Thus, the first case is the only possible one for theories with $d=n=1$ (the four-dimensional Einstein-Maxwell dilaton-axion theory [33]) and with $d=2,n=0$ (the five-dimensional dilaton-Kalb-Ramond gravity or its analogous on-shell equivalent [34]). The special classes of extremal solutions of these theories can be found in [35]. In fact, only in these two special cases does the extremal solution with $\mathcal{K}=1$ exists as the original solution: we state that for all other situations, i.e. when the condition of the solution with $\mathcal{K}=2$ is satisfied, the $\mathcal{K}=1$ solution branch is a special case of the branch with $\mathcal{K}=2$. Actually, the submersion of the former branch into the latter has the extremely simple form

$$\Lambda_1^{(2)}=\Lambda^{(1)}, \quad \Lambda_2^{(2)}=\tilde{\mathcal{N}}^{(1)}\Lambda^{(1)}, \quad \mathcal{N}_1^{(2)}+\tilde{\mathcal{N}}^{(1)}\mathcal{N}_2^{(2)}=\bar{\mathcal{N}}^{(1)}. \tag{4.15}$$

These relations mean the following: $\Lambda^{(1)}$, $\tilde{\mathcal{N}}^{(1)}$, and $\bar{\mathcal{N}}^{(1)}$ are arbitrary parameters (they parametrize the branch with $\mathcal{K}=1$), whereas $\Lambda_1^{(2)}$, $\Lambda_2^{(2)}$, $\mathcal{N}_1^{(2)}$, and $\mathcal{N}_2^{(2)}$ must be taken to satisfy Eq. (4.15). Note that the last relation in Eq. (4.15) can always be realized; for the proof one can use some simple geometrical reasons. Namely, $\bar{\mathcal{N}}^{(1)}$ is the vector in the (at least) two-dimensional Euclidean space $(d-1+n\geqslant 2)$ with the norm $\sqrt{1+(\tilde{\mathcal{N}}^{(1)})^2}$. It is clear that one can always take the two-dimensional plane in this space and two unit mutually orthogonal vectors $\mathcal{N}_1^{(2)}$ and $\mathcal{N}_2^{(2)}$ in it in such a way that the vector $\bar{\mathcal{N}}^{(1)}$ has unit projection on $\mathcal{N}_1^{(2)}$ and projection on $\mathcal{N}_2^{(2)}$ equal to $\tilde{\mathcal{N}}^{(1)}$. This choice exactly corresponds to the third relation in Eq. (4.15). Both the critical heterotic and bosonic string theories are consistent representatives for our extremal solution class with $\mathcal{K}=2$.

Let us now briefly discuss the solution decoding procedure in this case, i.e., the translation of the extremal solution found from the $\mathcal{Z}$ form to the language of multidimensional string theory fields. In this procedure the single qualitative step is related to calculation of the matrices $\mathcal{M}_a$ and $\vec{\Omega}_a$; all the remaining work is algebra that is simple in principle but technically tedious. The calculation gives

$$\mathcal{M}_1=\Sigma, \quad \mathcal{M}_2=\Sigma\Lambda\mathcal{Q}, \quad \mathcal{M}_3=\mathcal{Q}^T\Lambda^T\Sigma\Lambda\mathcal{Q}, \tag{4.16}$$
$$\vec{\Omega}_1=0, \quad \vec{\Omega}_2=\Sigma\vec{\Theta}_1\mathcal{Q}, \quad \vec{\Omega}_3=\mathcal{Q}^T\vec{\Theta}_2\mathcal{Q},$$

where the vector fields $\vec{\Theta}_1$ and $\vec{\Theta}_2$ are defined by the relations

$$\nabla\times\vec{\Theta}_1=\nabla\Lambda, \quad \nabla\times\vec{\Theta}_2=\nabla\Lambda^T\Sigma\Lambda-\Lambda^T\Sigma\nabla\Lambda. \tag{4.17}$$

For the solutions with $\mathcal{K}=1$ the term $\vec{\Theta}_2$ vanishes; for the solutions with $\mathcal{K}=2$ this term is the 2×2 antisymmetric matrix; its "12" component is defined by the relation

$$\nabla\times\vec{\Theta}_{2,12}=\nabla\Lambda_1^{(2)\,T}\Sigma\Lambda_2^{(2)}-\nabla\Lambda_2^{(2)\,T}\Sigma\Lambda_1^{(2)}. \tag{4.18}$$

We will not continue the calculation of the heterotic string theory field components in this paper. Let us only note that the constructed solution class is asymptotically flat if the harmonic matrix function Λ vanishes at spatial infinity. In the case when Λ has Coulomb asymptotic behavior the vector matrix $\vec{\Omega}_1$ generates Dirac string peculiarities, whereas $\vec{\Omega}_2$ leads to the dipole moments of the fields. However, this simple picture is not complete; there are some nuances and lesser effects. In a forthcoming publication we hope to continue the analysis of the above established Israel-Wilson-Perjes type solution of heterotic string theory and also give some new analytic material related to its supersymmetric properties.

V. CONCLUSION

In this paper we have developed a new formalism for low-energy heterotic string theory compactified to three dimensions on a torus. This formalism is extremely compact and based on the use of a single matrix potential $\mathcal{Z}$. The formalism includes three pairs of $\mathcal{Z}$ related quantities $(\mathcal{M}_a,\vec{\Omega}_a)$; they play an important role in the translation of the $\mathcal{Z}$ expressed solution to the language of heterotic string theory field components. Using our formalism, one can generate asymptotically flat solutions of the theory that possess the charging symmetry completeness property. The positive feature of the new approach is related to the fact that the charging symmetry subgroup of transformations of the theory acts as a linear homogeneous map on the matrix potential $\mathcal{Z}$. Thus the straightforward construction of new solutions seems at least really promising in this approach. We have illustrated this statement by the straightforward construction of a general extremal solution of Israel-Wilson-Perjes type for heterotic string theory with arbitrary numbers d and n of toroidally compactified dimensions and the original multidimensional Abelian gauge fields. For the case of $d=n=1$ our class is defined by two arbitrary harmonic functions and one constant parameter; for $d=2,n=0$ we have three harmonic and one constant parametric degrees of freedom. For the remaining heterotic string theory cases one has $2(d+1)$ basic arbitrary harmonic functions and $2(d+n)-5$ independent constant parameters. We state that for the special subset of $d=1$ and $n\geqslant 2$ theories our solution exactly coincides with the solution obtained in [7], if one removes all the spatial field asymptotics from the latter solution. Of course, we can also generate them in our solution using a shift transformation (with the generator γ_3 in $\mathcal{Z}$ representation).

Concerning the nearest perspectives of activity based on the use of the new formalism, we hope to apply it in combination with the inverse scattering transform method for construction of the general two-dimensional soliton solution of the theory. Three-dimensional generation using the total subgroup of the charging symmetry transformations according to the plan formulated in the previous section is also in our plans. Also, it is necessary to perform the supersymmetric analysis of the Israel-Wilson-Perjes type solution constructed in this article. It seems that there is a compact supersymmetric generalization of the $\mathcal{Z}$ based formalism, but the corresponding work is now just beginning.

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