Maximally localized states and causality in noncommutative quantum theories

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We give simple representations for quantum theories in which the position commutators are nonvanishing constants. A particular representation reproduces results found using the Moyal star product. The notion of exact localization being meaningless in these theories, we adapt the notion of ‘‘maximally localized states’’ developed in another context. We find that Gaussian functions play this role in a $(2+1)$-dimensional model in which the noncommutation relations concern positions only. An interpretation of the wave function in this noncommutative geometry is suggested. We also analyze higher dimensional cases. A possible incidence on the causality issue for a quantum field theory with a noncommutating time is sketched.

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I. INTRODUCTION

Theories of noncommutative quantum mechanics have received wide attention once it was realized that they could be obtained as low energy limits of string theory in the presence of a B field [1,2]. However, the status of these theories is still plagued by conceptual challenges. In most of them, Lorentz invariance is explicitly broken. Actually, noncommutative quantum mechanics is not the only arena in which Lorentz symmetry is only approximate. For example, it has been suggested recently that the standard model itself may fit into this category [3,4]. The preferred frame was postulated to be the rest frame of the cosmic microwave background radiation. There is one major problem quantum noncommutative theories are believed to face when time and position do not commute: the lack of causality and unitarity [5]. The analysis which led to this result relies on the Weyl-Moyal correspondence which tells us how to handle theories with noncommuting positions. One simply works with functions of commuting variables but replaces any pointwise product by the Moyal star product which is nonlocal.

One of the main problems faced by this approach is that the meaning of the wave function is not clarified yet [6,7]. When two positions do not commute, they cannot be diagonalized simultaneously. The uncertainty relation prevents the wave function $\phi(\vec{x})$ appearing in noncommutative theories from being the probability for a particle to be localized at $\vec{x}$.

There is a model in which exact localization is also forbidden: the Kempf-Mangano-Mann (KMM) theory [8]. Inspired by what was done in this case and in [9], we will adapt the notion of ‘‘maximally localized state’’ to noncommutative quantum mechanics. This notion will be useful in the discussion of the causality issue.

The plan of the paper is as follows. In Sec. II we will briefly point out some characteristics of noncommutative quantum mechanics which have been obtained in specific cases using the Weyl-Moyal correspondence [10,11]. In Sec. III we will exhibit a representation of the positions and the momenta by operators acting on a usual space of functions. We will show that the results summarized in Sec. II are recovered. In Sec. IV, we first work in a $2+1$ dimensional model where the spatial coordinates do not commute. We give all the details leading to Gaussian functions as being maximally localized states. Considering a $3+1$ dimensional theory, we use the preceding construction to construct the appropriate states in the new context.

Rather than projecting on localized states, we now have to project on maximally localized states to gain viable information on positions. The last section is devoted to a brief reminder of the causality issue of a quantum field theory (QFT) possessing a noncommuting time. The way maximally localized states may alter the analysis is sketched.

II. QUANTUM MECHANICS USING THE MOYAL PRODUCT

The noncommutative quantum theories we are interested in obey the following relations:

$$[\hat{x}_k,\hat{x}_l]=i\,\theta_{kl},\quad [\hat{p}_k,\hat{p}_l]=0,\quad [\hat{x}_k,\hat{p}_l]=i\hbar\,\delta_{kl}. \tag{1}$$

The constant matrix θ has dimension L^2 and breaks explicitly the Lorentz invariance. The Weyl-Moyal correspondence is a map between the functions Φ of the operators $\hat{x}_k$ and the functions ϕ of the commuting variables x_i:

$$\Phi(\hat{x})=\int e^{i\alpha\cdot\hat{x}}\tilde{\phi}(\alpha)d\alpha,\quad \phi(x)=\int e^{-ix\beta}\tilde{\phi}(\beta)d\beta. \tag{2}$$

The usual product of two $\hat{x}$ valued functions is sent to the star product of the associated functions defined on commuting variables:

$$\Phi(\hat{x})\Psi(\hat{x})\rightarrow(\phi*\psi)(x), \tag{3}$$

with

$$(\phi*\psi)(x)=[e^{(i/2)\theta^{\mu\nu}\partial_{\xi^\mu}\partial_{\eta^\nu}}\phi(x+\xi)\psi(x+\eta)]_{\xi=\eta=0}. \tag{4}$$

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The theories with noncommuting positions are obtained from the usual actions in which all products become star products. For example, the action of a self-interacting scalar field reads

$$S=\int d^4x\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi-m^2\phi^*\phi-\frac{\lambda}{4!}\phi^*\phi^*\phi^*\phi\right). \tag{5}$$

It has been suggested that quantum mechanics could be derived in this context by a similar replacement [10,11]. In the usual situation, the Schrödinger equation can be inferred from the action

$$S=\int dt\, d^2x\, \overline{\psi}\left[i\partial_t-\frac{\vec{p}^2}{2m}-V(x)\right]\psi \tag{6}$$

in a $2+1$ dimensional system. Introducing star products and using the relation

$$V(x)*\psi(x)=V\left(x-\frac{\tilde{p}}{2}\right)\psi(x) \tag{7}$$

(where $\tilde{p}^i=\theta^{ij}p_j$) which is obtained via a Fourier transform [10,11], one finds that the Weyl-Moyal correspondence reduces to the replacement

$$x\rightarrow x-\frac{1}{2}\tilde{p} \tag{8}$$

in the potential. Considering a central potential, the substitution

$$V(x^2+y^2)\rightarrow V\left(\frac{\theta^2}{4}p_x^2+\frac{\theta^2}{4}p_y^2+x^2+y^2-\theta L_z\right) \tag{9}$$

shows that the theory "looks like" one describing a particle of changed mass, with a nontrivial coupling to the angular momentum [10]. In the case of an asymmetric oscillator $V(x,y)=\frac{1}{2}ky^2$, the theory contains a term which looks like an interaction between a particle of charge q with a constant magnetic field B such that $qB=2/\theta$ [11]. At this point it is crucial to realize that although the Moyal product is written in terms of commuting variables x, these variables are simply a notation. There is no evidence that they represent physical coordinates, except in the undeformed case $\theta=0$. No argument has been presented which shows that the wave function ϕ gives a probability [6]. Moreover, the probability for a particle to be localized at a given position (x_1,x_2) is not a safe concept since these coordinates do not commute.

III. A REPRESENTATION OF THE COMMUTATION RELATIONS

The only modification to the usual theory introduced by Eq. (1) concerns the positions. It is therefore quite reasonable to look for a realization in which the momenta remain unchanged:

$$\hat{p}_i=-i\hbar\,\partial_{\xi_i}. \tag{10}$$

The introduction of the noncommutativity scale leads to the possible *Ansätze*

$$\hat{x}_i=\xi_i+\theta^{1/2}G_i(\theta^{1/2}\partial_{\xi^k}). \tag{11}$$

The functions G_i are taken analytic. Such an *ensätz* clearly fulfills the $[\hat{x},\hat{p}]$ commutation relations. The $[\hat{x},\hat{x}]$ commutators can then be used to constrain the coefficients of the Taylor expansions of the functions G_k.

As an illustration, let us consider a $2+1$ dimensional system, with spatial noncommutativity:

$$[\hat{x}_1,\hat{x}_2]=i\theta, \tag{12}$$

with θ positive. It is straightforward that one can take G linear and write simply

$$\begin{aligned}\hat{x}_1&=\xi_1+i\theta(a\partial_{\xi_1}+(1+c)\partial_{\xi_2}),\\ \hat{x}_2&=\xi_2+i\theta(c\partial_{\xi_1}+d\partial_{\xi_2}).\end{aligned} \tag{13}$$

At this stage, the constants a,c and d are arbitrary. The momenta $\hat{p}_i$ and the positions $\hat{x}_k$ act as operators on the space of functions of the variables (ξ_1,ξ_2). If we take the scalar product to be given by the usual formula

$$\langle\phi|\psi\rangle=\int d\xi_1\, d\xi_2\, \phi^*(\xi_1,\xi_2)\psi(\xi_1,\xi_2), \tag{14}$$

the $\hat{x}_i$ operators are symmetric provided that a,c and d are real.

Let us consider a harmonic oscillator in this theory:

$$\hat{H}=\frac{1}{2m}(\hat{p}_x^2+\hat{p}_y^2)+\frac{1}{2}k(\hat{x}^2+\hat{y}^2). \tag{15}$$

Using the representation given above, one obtains

$$\begin{aligned}\hat{H}=&-\left(\frac{\hbar^2}{2m}+\frac{1}{2}k\theta^2(a^2+c^2)\right)\partial_{\xi_1}^2-\left(\frac{\hbar^2}{2m}+\frac{1}{2}k\theta^2\right.\\ &\left.\times((1+c)^2+d^2)\right)\partial_{\xi_2}^2-k\theta^2(a(1+c)+cd)\partial_{\xi_1}\partial_{\xi_2}\\ &+\frac{1}{2}ki\theta(2a\xi_1\partial_{\xi_1}+2d\xi_2\partial_{\xi_2}+2(1+c)\xi_1\partial_{\xi_2}\\ &+2c\xi_2\partial_{\xi_1}+a+d)+\frac{1}{2}k(\xi_1^2+\xi_2^2).\end{aligned} \tag{16}$$

Let us forget for a moment the origin of this operator and treat it like in the usual, commutative theory. Can we reproduce the features shown in the last section, which come from an analysis based on the Moyal product? The answer is positive. The appearance of the "angular momentum operator" of the usual theory $i(\xi_1\partial_{\xi_2}-\xi_2\partial_{\xi_1})$, is guaranteed by the choice $c=-1/2$. The crossed derivative $\partial_{\xi_1}\partial_{\xi_2}$ vanishes if the relation $a=d$ holds and the terms $\xi_1\partial_{\xi_1}$ and $\xi_2\partial_{\xi_2}$ disappear if we also impose $a=0$. The final Hamiltonian reads

$$\hat{H}=-\left(\frac{\hbar^2}{2m}+\frac{1}{8}k\theta^2\right)\partial^2_{\xi_1}-\left(\frac{\hbar^2}{2m}+\frac{1}{8}k\theta^2\right)\partial^2_{\xi_2}+\frac{1}{2}k\theta L+\frac{1}{2}k(\xi_1^2+\xi_2^2). \tag{17}$$

This has been obtained in [10] and summarized in the previous section.

So, for a particular choice of the free parameters appearing in the realization we chose for the noncommutative quantum theory, we can reproduce exactly some results derived using the Weyl-Moyal correspondence. The simplicity of the algebra will prove useful when tackling the interpretation of the wave function in this framework. In fact, even if the Hamiltonian written in Eq. (17) looks quite ordinary, one should keep in mind that once the wave equation is solved, the position operator along the first spatial coordinate is not simply the product by ξ_1. The energy eigenvalues have the same meaning as in the ordinary theory but the analysis concerning localization is much more involved. A similar situation occurs in another theory and has been exploited to handle the trans-Planckian problem of the black hole physics [12,13].

From the formula given in Eq. (1), one infers the uncertainty relation

$$\Delta x_1\Delta x_2\geqslant\frac{\theta}{2}. \tag{18}$$

This means that any state which is localized without any uncertainty in any of the two directions is unphysical.

IV. MAXIMALLY LOCALIZED STATES

A. The derivation

The uncertainty relation given in Eq. (18) puts a lower bound on localization. We shall look for states which saturate this bound. We will restrict ourselves to those displaying equal values of the uncertainties in the two directions:

$$\Delta x_1=\Delta x_2=\sqrt{\frac{\theta}{2}}. \tag{19}$$

This is motivated by the opinion that a state displaying a very small uncertainty in one direction and a large one in the remaining direction is undesirable; we adopt here a democratic treatment of the two coordinates.

We will say that a state is maximally localized at (λ_1,λ_2) *if it satisfies the equality of Eq. (19) and if* $\langle x_i\rangle=\lambda_i$.

These states are quite close to the coherent states in the usual quantum mechanics which verify $\Delta x\Delta p=\hbar/2$. As it stands, Eq. (19) is hardly tractable. The procedure we shall use is directly inspired by [8] and replaces these integral equations by a differential one. The uncertainty relation of Eq. (18) is obtained as a consequence of the inequality

$$\left\|\left(\hat{x}_1-\langle x_1\rangle+\frac{\langle[\hat{x}_1,\hat{x}_2]\rangle}{2(\Delta x_2)^2}(\hat{x}_2-\langle x_2\rangle)\right)|\phi\rangle\right\|\geqslant 0. \tag{20}$$

The vector whose norm is considered in the preceding formula vanishes for the states $|\phi\rangle$ which minimize the product of uncertainties. Using our expressions of the position operators, this is converted into a partial differential equation for the maximally localized states. We introduce the complex coordinates (u_1,u_2) by

$$u_1=(\alpha_1+i\beta_1)\xi_1+(\gamma_1+i\delta_1)\xi_2,$$
$$u_2=(\alpha_2+i\beta_2)\xi_1+(\gamma_2+i\delta_2)\xi_2, \tag{21}$$

where the following constants have been introduced to simplify future formulas:

$$\alpha_1=\frac{c}{D},\quad \beta_1=\frac{a}{D},\quad \gamma_1=\frac{d}{D},\quad \delta_1=\frac{1+c}{D},$$
$$\alpha_2=-\frac{d}{D},\quad \beta_2=\frac{1+c}{D},\quad \gamma_2=\frac{c}{D},\quad \delta_2=-\frac{a}{D},$$
$$D=-1-a^2-2c-2c^2-d^2. \tag{22}$$

One finds that the general solution to the partial differential equation is

$$\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)=f(u_2)\exp\left\{\frac{1}{2\theta}(2c+1-ia+id)u_1^2-\frac{1}{\theta}(d+a+i)u_2u_1+\frac{1}{\theta}(\lambda_1+i\lambda_2)u_1\right\} \tag{23}$$

with f an arbitrary function. At this level an important constraint comes from the fact that as $\theta\to 0$, we should reobtain usual quantum mechanics. The maximally localized states must in this limit coincide with position eigenstates which are delta functions. In formula, one should have

$$\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)\to\delta(\xi_1-\lambda_1)\,\delta(\xi_2-\lambda_2) \tag{24}$$

when $\theta\to 0$. In distribution theory one knows that the Dirac delta can be expressed as the limit of some appropriate functions, for example

$$\frac{1}{\sqrt{2\pi\theta}}\exp\left(-\frac{x^2}{\theta}\right),\quad \frac{\sqrt{\theta}}{\pi}\frac{\sin^2(x/\sqrt{\theta})}{x^2},\quad \frac{1}{\pi}\frac{\sqrt{\theta}}{x^2+\theta}. \tag{25}$$

Any combination of these functions with appropriate coefficients tends to the delta distribution. It can be conjectured that a maximally localized state may just be such a combination. Our expression for $\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)$ given in Eq. (23) involves exponentials; this makes it more reasonable to focus on the first element of the previous list. Our aim is to see if the function $f(u_2)$ can be chosen so that the maximally localized state is proportional to

$$\exp\left\{-\frac{(\xi_1-\lambda_1)^2}{\sigma_1^2\theta}-\frac{(\xi_2-\lambda_2)^2}{\sigma_2^2\theta}\right\}. \tag{26}$$

The answer is that this can be done only when the constants appearing in Eq. (13) satisfy the relations $a=d=0$. To obtain an expression which looks like Eq. (26), one needs the function $f(u_2)$ to be quadratic. As the expression of $\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)$ given in Eq. (23) involves complex quantities, we choose

$$f(u_2)=N\exp\left\{\frac{1}{\theta}(A_1+iA_2)u_2^2+\frac{1}{\sqrt{\theta}}(B_1+iB_2)u_2\right\}, \tag{27}$$

the A_i,B_i being dimensionless real constants. We separate the real and the imaginary parts in the expression of the maximally localized state:

$$\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)=N\exp\{i(I_{11}\xi_1^2+I_{22}\xi_2^2+I_{12}\xi_1\xi_2+I_{10}\xi_1+I_{20}\xi_2)+(R_{11}\xi_1^2+R_{22}\xi_2^2+R_{12}\xi_1\xi_2+R_{10}\xi_1+R_{20}\xi_2)\}. \tag{28}$$

The expressions of the constants I_{ij} and R_{ij} in terms of the quantities ξ_k,σ_k,A_i,B_i are given in the Appendix. The preceding formula can be identified with Eq. (26) only if we can make all the I_{kl} vanish as well as R_{12}. The constants B_1 and B_2 are easily fixed by the requirement that

$$I_{10}=\frac{B_2\alpha_2+B_1\beta_2}{\sqrt{\theta}}+\frac{\beta_1\lambda_1+\alpha_1\lambda_2}{\theta}=0, \tag{29}$$

and

$$I_{20}=\frac{B_2\gamma_2+B_1\delta_2}{\sqrt{\theta}}+\frac{\delta_1\lambda_1+\gamma_1\lambda_2}{\theta}=0. \tag{30}$$

In a similar way, the vanishing of I_{11} and I_{22} fix A_1 and A_2. Simplifying, the remaining coefficients assume the following expressions:

$$I_{12}=\frac{a+d}{2(a+ac+cd)}\frac{1}{\theta}, \tag{31}$$

$$R_{11}=\frac{-a-2ac-ac^2+d-c^2d+ad^2+d^3}{4(-ac-2ac^2-ac^3+a^2d+a^2cd-c^2d-c^3d+acd^2)}\frac{1}{\theta}, \tag{32}$$

$$R_{22}=\frac{-a^3+2ac+ac^2-a^2d+c^2d}{4(-ac-2ac^2-ac^3+a^2d+a^2cd-c^2d-c^3d+acd^2)}\frac{1}{\theta}, \tag{33}$$

$$R_{12}=\frac{a-d}{2(-c-c^2+ad)}\frac{1}{\theta}, \tag{34}$$

$$R_{10}=\frac{-(1+c)\lambda_1-d\lambda_2}{c(1+c)-ad}\frac{1}{\theta}, \tag{35}$$

$$R_{20}=\frac{-(a\lambda_1+c\lambda_2)}{(-c(1+c)+d)}\frac{1}{\theta}. \tag{36}$$

The remaining term in the imaginary part vanishes only if $d=-a$ as is manifest in Eq. (31). The R_{kl} coefficients then simplify further:

$$R_{11}=\frac{a(1+c)}{2a(-a^2-c-c^2)}\frac{1}{\theta},\quad R_{12}=\frac{a}{-a^2-c(1+c)}\frac{1}{\theta}, \tag{37}$$

$$R_{22}=\frac{ac}{2a(-a^2-c-c^2)}\frac{1}{\theta}. \tag{38}$$

One sees that when $R_{12}=0$,R_{11} and R_{22} are only defined by their limits as $a\to 0$:

$$R_{11}=-\frac{1}{2c}\frac{1}{\theta},\quad R_{22}=-\frac{1}{2(1+c)}\frac{1}{\theta}, \tag{39}$$

$$R_{10}=\frac{\lambda_1}{c\theta},\quad R_{20}=\frac{\lambda_2}{(1+c)\theta}. \tag{40}$$

In summary, when $d=-a=0$, the choice of the constants A_1,A_2,B_1,B_2 explained earlier leads to a maximally localized state which takes the form

$$\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)=N\exp\left\{\frac{\xi_1^2}{2c\theta}-\frac{\xi_2^2}{2(1+c)\theta}-\frac{\lambda_1\xi_1}{c\theta}+\frac{\lambda_2\xi_2}{(1+c)\theta}\right\}. \tag{41}$$

This state is normalizable only if the quadratic terms are negative and this is realized provided that the constant c assumes values in the interval $]-1,0[$. One finally obtains that the wave function

$$\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)=\frac{1}{\pi\theta}\frac{1}{\sqrt{-2c}}\frac{1}{\sqrt{2(c+1)}}\exp\left\{\frac{1}{2c\theta}(\xi_1-\lambda_1)^2-\frac{1}{2(1+c)\theta}(\xi_2-\lambda_2)^2\right\} \tag{42}$$

represents a maximally localized state in the representation of the noncommutative quantum mechanics given by

$$\hat{x}_1=\xi_1+i\theta(1+c)\partial_{\xi_2},$$
$$\hat{x}_2=\xi_2+i\theta c\partial_{\xi_1}. \tag{43}$$

It is straightforward to verify, by the computation of integrals implying Gaussians multiplied by polynomials that in this state

$$\langle x_i\rangle=\lambda_i, \quad \Delta x_i=\sqrt{\frac{\theta}{2}}. \tag{44}$$

This ensures that the state fulfills the condition not only in the limiting case $a\to 0$, but also in the case $a=0$ itself. The norm of this state is perfectly finite:

$$\langle\psi|\psi\rangle=-\frac{1}{c(1+c)}\frac{\pi}{8\theta}. \tag{45}$$

However, it goes to infinity as θ goes to zero. This agrees with the fact that the square of the Dirac distribution is not a mathematically well defined object. Like in the KMM theory [8], one finds that the mean momentum vanishes in this state:

$$\langle p_i\rangle=0. \tag{46}$$

The uncertainty in momentum reads

$$\Delta p_i=\left(-\frac{\hbar^2}{2c\theta}\right)^{1/2}, \tag{47}$$

and, along any direction, one has

$$\Delta x_i\Delta p_i=\frac{1}{2}\frac{\hbar}{(-c)^{1/2}}. \tag{48}$$

We cannot reach the lowest values allowed by the Heisenberg uncertainty relations since this corresponds to the value $c=-1$ which blows up the maximally localized state [see Eq. (42)].

It is quite surprising that the condition $a=d=0$ which was needed to recapture some behaviors found using the Weyl-Moyal correspondence in the last section is also the one leading to Gaussians for the maximally localized states. The condition $c=-1/2$ leads to a symmetric form of the maximally localized states in the variables ξ_1,ξ_2. This strongly suggests a way for the recovering of information on position from the Moyal-Weyl wave function.

B. The quasi-position representation

One can construct a new representation by projecting on the maximally localized states [8]:

$$\tilde{\alpha}(\xi_1,\xi_2)=\int d\lambda_1\, d\lambda_2\, \psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)\alpha(\lambda_1,\lambda_2). \tag{49}$$

For simplicity, we restrict ourselves to the case $c=-1/2$. The action of the position operators is given by

$$\hat{x}_1=\lambda_1+\frac{\theta}{2}(\partial_{\lambda_1}+i\partial_{\lambda_2}), \quad \hat{x}_2=\lambda_2+\frac{\theta}{2}(\partial_{\lambda_2}-i\partial_{\lambda_1}), \tag{50}$$

while the scalar product reads

$$\langle\tilde{\alpha}|\tilde{\beta}\rangle=\frac{1}{8\pi\theta}\int d\lambda_1\, d\lambda_2\, d\mu_1\, d\mu_2\, \tilde{\alpha}^*(\lambda_1,\lambda_2)\tilde{\beta}(\mu_1,\mu_2)\times\exp\left(-\frac{1}{2\theta}(\lambda_1-\mu_1)^2-\frac{1}{2\theta}(\lambda_2-\mu_2)^2\right). \tag{51}$$

As $\theta\to 0$, the Gaussian function tends to the delta distribution. This enables one to rewrite the preceding integral as based on two coordinates rather than four, recovering the well known situation of the position representation. The operators given in Eq. (50) also reduce to the common form in the same limit.

This representation is physically interesting because within it we know the interpretation of the wave function. Since it is obtained by the projection of maximally localized states, the square of the wave function $\tilde{\phi}(\xi_1,\xi_2)$ gives the probability for a particle to be localized in the intervals $[\xi_1-\frac{1}{2}\sqrt{\theta/2},\xi_1+\frac{1}{2}\sqrt{\theta/2}]$ on the first axis and a similar one centered around ξ_2 for the second axis.

The quasiposition representation obtained here is similar to the one found in the KMM theory in the sense that the scalar product involves functions defined at different points [8]. It differs from it by the fact that the operators are not given here by an infinite series in the deformation parameter.

C. The momentum representation

We worked in a representation which reduces to the position one in the undeformed limit. It is possible to carry similar calculations in the momentum representation. The parametrization

$$\hat{x}_1=i\partial_{p_1}-\frac{\theta}{2}p_2, \quad \hat{x}_2=i\partial_{p_2}+\frac{\theta}{2}p_1 \tag{52}$$

satisfies the commutation relations. One can construct maximally localized states which tend to exp(ipx) which is the Fourier transform of the Dirac delta.

D. Higher dimensions

The construction presented needs some modifications when addressing higher dimensions. Let us consider for example a $3+1$ dimensional model whose nonvanishing commutation relations are

$$[\hat{x}_1,\hat{x}_2]=[\hat{x}_2,\hat{x}_3]=[\hat{x}_3,\hat{x}_1]=i\theta. \tag{53}$$

It is realized by the following operators:

$$\hat{x}_1=\xi_1+i\theta(a_1\partial_{\xi_1}+(1+b_1)\partial_{\xi_2}+a_3\partial_{\xi_3}),$$

$$\hat{x}_2=\xi_2+i\theta(b_1\partial_{\xi_1}+b_2\partial_{\xi_2}+(1+c_2)\partial_{\xi_3}),$$

$$\hat{x}_3=\xi_3+i\theta((1+a_3)\partial_{\xi_1}+c_2\partial_{\xi_2}+c_3\partial_{\xi_3}). \tag{54}$$

The a_i,b_i,c_i are arbitrary but real constants. We want $x^2+y^2+z^2$ to be quadratic in the "momenta" and linear in the "angular momenta" like in Eq. (9) which is true for all dimensions. One needs the relations $a_1=b_2=c_3=0$ to cancel terms of the form $\xi_k\partial_{\xi_k}$ and we impose $b_1=a_3=c_2=-\frac{1}{2}$ to ensure the appearance of the "angular momenta." We end up with the following expressions for the position operators:

$$\hat{x}_1=\xi_1+i\frac{\theta}{2}(\partial_{\xi_2}-\partial_{\xi_3}),\quad \hat{x}_2=\xi_2+i\frac{\theta}{2}(\partial_{\xi_3}-\partial_{\xi_1}),$$

$$\hat{x}_3=\xi_3+i\frac{\theta}{2}(\partial_{\xi_1}-\partial_{\xi_2}). \tag{55}$$

Inspired by what we have done in the previous subsection, we now look for the coefficients σ_1 which allow the function

$$\psi^{ml}_{\lambda_1,\lambda_2,\lambda_3}(\xi_1,\xi_2,\xi_3)=\left(\frac{1}{2\pi\theta}\right)^{3/2}\frac{1}{\sigma_1\sigma_2\sigma_3}\exp\left\{-\frac{(\xi_1-\lambda_1)^2}{\sigma_1^2\theta}-\frac{(\xi_2-\lambda_2)^2}{\sigma_2^2\theta}-\frac{(\xi_3-\lambda_3)^2}{\sigma_2^2\theta}\right\} \tag{56}$$

to satisfy the definition of the maximally localized state given in the fourth section. Using the representation specified in Eq. (55) and the three dimensional version of the scalar product given in Eq. (14) the conditions on the mean values of the positions are automatically satisfied. The ones concerning the uncertainties lead to the following conditions satisfied:

$$\Delta x_1\Delta x_2=\frac{\theta}{2}\Rightarrow\frac{\sigma_1^2}{4}+\frac{1}{4\sigma_2^2}+\frac{1}{4\sigma_3^2}=0, \tag{57}$$

$$\Delta x_1\Delta x_3=\frac{\theta}{2}\Rightarrow\frac{1}{4\sigma_1^2}+\frac{\sigma_2^2}{4}+\frac{1}{4\sigma_3^2}=0, \tag{58}$$

$$\Delta x_2\Delta x_3=\frac{\theta}{2}\Rightarrow\frac{1}{4\sigma_1^2}+\frac{1}{4\sigma_2^2}+\frac{1\sigma_3^2}{4}=0. \tag{59}$$

We can use Eqs. (57),(58) to express σ_1 and σ_3 in terms of σ_2:

$$\sigma_1^2=2-\frac{1}{\sigma_2^2}-\frac{4\sigma_2^4}{1+\sigma_2^4-2\sigma_2^2-\sqrt{1-4\sigma_2^2+6\sigma_2^4+4\sigma_2^6+\sigma_2^8}}, \tag{60}$$

$$\sigma_3^2=\frac{1}{4}+\frac{1}{4\sigma_2^4}-\frac{1}{2\sigma_2^2}-\frac{\sqrt{1-4\sigma_2^2+6\sigma_2^4+4\sigma_2^6+\sigma_2^8}}{4\sigma_2^4}. \tag{61}$$

Replacing, in the second part of Eq. (59), the variables σ_1 and σ_3 by the expressions obtained in the two preceding formulas, one obtains an equation in the parameter σ_2. Unfortunately, this equation does not admit a real solution, as a numerical treatment shows. One may think that a different choice of the parameters a_i,b_i,c_i may solve the problem; this is not the case.

The simplest way to understand this feature is the following. As shown in Eq. (20), the equality given in Eq. (19) for the directions x_1,x_2 is satisfied only by the eigenstates of the operator

$$\text{Ø}_{12}=\widehat{x_1}+i\widehat{x_2}. \tag{62}$$

So, a simultaneous solution of Eq. (19) for all couples of directions must be an eigenstate of the operators Ø_{12}, Ø_{13} and Ø_{23}. The commutators of these operators are nonvanishing:

$$[\text{Ø}_{12},\text{Ø}_{13}]=2\theta. \tag{63}$$

As we just pointed out, a state which saturates the three bounds is an eigenstate of the three operators; we denote its eigenvalues by λ_{12}, λ_{13} and λ_{23}. Using this, we infer from Eq. (63) the equation

$$[\text{Ø}_{12},\text{Ø}_{13}]|\psi\rangle=2\theta|\psi\rangle=(\lambda_{12}\lambda_{13}-\lambda_{13}\lambda_{12})|\psi\rangle=0 \tag{64}$$

so that the only state which saturates all the three bounds is the null vector of the Hilbert space *composed of the appropriate functions*. This means that we cannot saturate the three bounds by states *which are functions* for the geometry displayed in Eq. (53). Solving Eq. (57), one saturates the bound $\Delta x_1\Delta x_2$; the noncommutation of the operators forbids one to simultaneously satisfy the same relation for $\Delta x_2\Delta x_3$. This is the explanation of the failure to implement simultaneously the set of equations displayed in Eqs. (57),(58),(59).

The construction of the preceding subsections can however prove useful. Let us consider the situation in which nonlocality is confined to the plane x_1,x_2:

$$[\hat{x}_1,\hat{x}_2]=i\theta,\quad [\hat{x}_1,\hat{x}_3]=[\hat{x}_2,\hat{x}_3]=0. \tag{65}$$

As the third direction does commute with the others, there is no Heisenberg uncertainty relation which prevents us from taking $\Delta x_3=0$. The situation in the noncommuting plane is exactly the one studied in the previous section. Taking a representation in which the supplementary operators are given by $\hat{x}_3=\xi_3$, $\hat{p}_3=-i\hbar\partial_{\xi_3}$, a maximally localized state can then be obtained as the product *of a function and a delta distribution*:

$$\psi^{ml}_{\lambda_1,\lambda_2}(\xi_1,\xi_2)=\frac{1}{\pi\theta}\frac{1}{\sqrt{-2c}}\frac{1}{\sqrt{2(c+1)}}\exp\left\{\frac{1}{2c\theta}(\xi_1-\lambda_1)^2 - \frac{1}{2(1+c)\theta}(\xi_2-\lambda_2)^2\right\}\delta(\xi_3-\lambda_3). \tag{66}$$

V. CAUSALITY OF A QFT WITH A NONCOMMUTING TIME

The analysis of [5] which led to the conclusion that quantum field theories with a noncommuting time were acausal relies on the interpretation of the wave function as giving the probability amplitude. This is valid in the ordinary ($\theta=0$) theory, but when noncommutativity sets in, one cannot simultaneously diagonalize the coordinates. The Heisenberg-like uncertainty forbids one to speak of an event happening at a time and a place known with infinite precision. What can we do to gain information on time and position in this context? The useful procedure was developed by KMM [8] in a different model: it is the projection on maximally localized states.

To give an idea of how our analysis may alter the causality issue, let us summarize the analysis of [5]. The theory under study is two dimensional and invariant under the Lorentz group:

$$[\hat{x}_\mu,\hat{x}_\nu]=i\theta\epsilon_{\mu\nu}. \tag{67}$$

In [5], the "time" coordinate is written t and the "space" coordinate x. One considers an incoming state of correlated pairs of particles with opposite momenta

$$|\phi\rangle_{in}=\int\frac{dk}{(2\pi)2E_k}\phi_{in}(k)|k,-k\rangle, \tag{68}$$

centered at two momenta po and $-po$:

$$\phi_{in}(p)=E_p\left[\exp\left(-\frac{(p-po)^2}{\lambda}\right)+\exp\left(-\frac{(p+po)^2}{\lambda}\right)\right], \tag{69}$$

with $E_p=\sqrt{p^2+m^2}$. One has that at "times" $t<0$, the two packets are well separated. At $t=0$, the wave function is concentrated at the "position" $x=x_1-x_2=0$ (x_i is the mean "position" of the ith wave packet). Then a collision takes place, due to the interaction. Considering a final state of the form

$$|\phi\rangle_{out}=\int\frac{dp}{(2\pi)2E_p}\phi_{out}(p)|p,-p\rangle, \tag{70}$$

[$\phi_{out}(p)$ is related to $\phi_{in}(p)$ by the S matrix] it is found that for the usual ϕ^4 theory, the outgoing wave function simply displays a small time delay. The corresponding noncommutative theory (i.e. with the interaction term $g\phi^*\phi^*\phi^*\phi$ using the Moyal product) behaves very differently. The wave packet, written in the "position" space, displays three peaks. The last peak leaves the collision point $x=0$ before the incoming wave packets given in Eq. (69) arrive there and this is interpreted as a violation of causality. The interested reader will find all the details in [5].

As we emphasized at the end of the second section and in the third one, the x,t variables appearing in the Moyal product are mere notations; they coincide with the physical position and time only when $\theta=0$. To obtain viable information on positions, one has to project on maximally localized states. *In these theories, a phenomenon cannot occur at a perfectly known time and a perfectly known position*. Any instant, any position is surrounded by a zone of fuzziness. Then, a more careful formulation should lead to a situation in which the collision between the ingoing packets is described as taking place during the time interval $[\lambda_{i0}-\frac{1}{2}\sqrt{\theta/2},\lambda_{i0}+\frac{1}{2}\sqrt{\theta/2}]$ in the position interval $[\lambda_{i1}-\frac{1}{2}\sqrt{\theta/2},\lambda_{i1}+\frac{1}{2}\sqrt{\theta/2}]$. The third outgoing packet will leave the region of collision at a time lying in an interval $[\lambda_{f0}-\frac{1}{2}\sqrt{\theta/2},\lambda_{f0}+\frac{1}{2}\sqrt{\theta/2}]$.

If these two time intervals are not disjoints, one cannot speak of an acausal process because of the fuzziness concerning time. The critical point concerns the calculation of the instants λ_{i0} and λ_{f0} which we do not have for the time being. One has to be especially careful since the status of the position operators in QFT is not exactly the one present in quantum mechanics. In the commutative case, this is embodied in the fact that the appropriate Newton-Wigner position operators are not simply the derivatives of the fields in the momentum representation [15]. So, one needs more to draw a definite conclusion.

The promising point in this picture is that the motion of the two incoming "particles" is not symbolized by two lines in the time-position plane but by two ribbons.

Nevertheless, some technical and conceptual problems must be addressed before a more elaborate treatment. One has to understand the Hamiltonian structure of the theories with noncommuting time better: the conjugate of the field is an infinite series, the energy momentum tensor and the current are not conserved, etc.

VI. CONCLUSION

In this work, we have shown how some results obtained using the Moyal product can be recovered by the choice of a particular representation of the position and momentum operators. We have shown that the maximally localized states associated with these representations can be chosen to be Gaussian functions which tend to the delta distribution as the parameter of noncommutativity is sent to zero. We have also suggested how these states may alter the analysis of the causality issue of a theory in which space and time do not commute.

One may ask if the representation we chose reproduces the results which can be obtained using the Moyal product for any physical system. We do not know the answer yet.

The method we used here to study noncommutative quantum mechanics is closer to [8] than to [14] in the sense that we did not introduce a differential calculus compatible with the commutation relations between the coordinates. This structure is usually used to construct an invariant action which leads to the field equations. Our procedure may not be applicable to curved spaces, contrary to the method used in [14].

It should be noted that the noncommutation of the positions raises a supplementary ordering problem. We did not address this because we studied central potentials only.

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APPENDIX: COEFFICIENTS

We give the explicit form of the coefficients appearing in Eq. (28):

$$I_{11}=\frac{1}{\theta}\left(-\frac{a\alpha_1^2}{2}+\frac{d\alpha_1^2}{2}-\alpha_1\alpha_2+A_2\alpha_2^2+\alpha_1\beta_1+2c\alpha_1\beta_1 -a\alpha_2\beta_1-d\alpha_2\beta_1+\frac{a\beta_1^2}{2}-\frac{d\beta_1^2}{2}-a\alpha_1\beta_2-d\alpha_1\beta_2 +2A_1\alpha_2\beta_2+\beta1\beta_2-A2\beta_2^2\right), \tag{A1}$$

$$I_{22}=\frac{1}{\theta}\left(-\frac{a\gamma_1^2}{2}+\frac{d\gamma_1^2}{2}-\gamma_1\gamma_2+A2\gamma_2^2+\gamma_1\delta_1+2c\gamma_1\delta_1 -a\gamma_2\delta_1-d\gamma_2\delta_1+\frac{a\delta_1^2}{2}-\frac{d\delta_1^2}{2}-a\gamma_1\delta_1-d\gamma_1\delta_2 +2A_1\gamma_2\delta_2+\delta_1\delta_2-A_2\delta_2^2\right), \tag{A2}$$

$$I_{12}=\frac{1}{\theta}(-a\alpha_1\gamma_1+d\alpha_1\gamma_1-\alpha_2\gamma_1+\beta_1\gamma_1+2c\beta_1\gamma_1 -a\beta_2\gamma_1-d\beta_2\gamma_1-a_1\gamma_2+2A_2\alpha_2\gamma_2-a\beta_1\gamma_2 -d\beta_1\gamma_2+2A_1\beta_2\gamma_2+\alpha_1\delta_1+2c\alpha_1\delta_1-a\alpha_2\delta_1 -d\alpha_2\delta_1+a\beta_1\delta_1-d\beta_1\delta_1+\beta_2\delta_1-a\alpha_1\delta_2-d\alpha_1\delta_2 +2A_1\alpha_2\delta_2+\beta_1\delta_2-2A_2\beta_2\delta_2), \tag{A3}$$

$$R_{11}=\frac{1}{\theta}\left(\frac{\alpha_1^2}{2}+c\alpha_1^2-a\alpha_1\alpha_2-d\alpha_1\alpha_2+A_1\alpha_2^2+a\alpha_1\beta_1 -d\alpha_1\beta_1+\alpha_2\beta_2-\frac{\beta_1^2}{2}-c\beta_1^2+\alpha_1\beta_2-2A2\alpha_2\beta_2 +a\beta_1\beta_2+d\beta_1\beta_2-A_1\beta_2^2\right), \tag{A4}$$

$$R_{22}=\frac{1}{\theta}\left(\frac{\gamma_1^2}{2}+c\gamma_1^2-a\gamma_1\gamma_2-d\gamma_1\gamma_2+A_1\gamma_2^2+a\gamma_1\delta_1 -d\gamma_1\delta_1+\gamma_2\delta_1-\frac{\delta_1^2}{2}-c\delta_1^2+\gamma_1\delta2-2A_2\gamma_2\delta_2 +a\delta_1\delta_2+d\delta_1\delta_2-A_1\delta_2^2\right), \tag{A5}$$

$$R_{12}=\frac{1}{\theta}(-a\alpha_1\gamma_1+2c\alpha_1\gamma_1-a\alpha_2\gamma_1+a\beta_1\gamma_1-d\beta_1\gamma_1 +\beta_2\gamma_1-a\alpha_1\gamma_2-d\alpha_1\gamma_2+2A_1\alpha_2\gamma_2+\beta_1\gamma_2 -2A_2\beta_2\gamma_2+a\alpha_1\delta_1-d\alpha_1\gamma_1+\alpha_2\delta_1-\beta_1\delta_1 -2c\beta_1\delta_1+a\beta_2\delta_1+d\beta_2\delta_1+\alpha_1\delta_2-2A_2\alpha_2\delta_2 +a\beta_1\delta_2+d\beta_1\delta_2-2A_1\beta_2\delta_2), \tag{A6}$$

$$R_{10}=\frac{B_1\alpha_2-B_2\beta_2}{\sqrt{\theta}}+\frac{\alpha_1\lambda_1-\beta_1\lambda_2}{\theta}, \tag{A7}$$

$$R_{20}=\frac{B_1\gamma_2-B_2\delta_2}{\sqrt{\theta}}+\frac{\gamma_1\lambda_1-\delta_1\lambda_2}{\theta}. \tag{A8}$$

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